Domain of validity of the evolution of perturbations in Newtonian cosmology with pressure

R. R. R. Reis
Universidade Federal do Rio de Janeiro—UFRJ Instituto de Física, Cep 21945-970, Brazil
(Received 12 December 2002; published 14 April 2003)

It has been argued that by removing a pressure gradient term in the continuity equation, it is possible to obtain, with a semiclassical formulation, the same expression for the density contrast growing mode, as obtained in a full relativistic treatment. In this context, we reinvestigate the evolution of perturbations in an expanding Newtonian universe with pressure, but we consider a general scenario in which the equation of state parameter is time dependent and the perturbations are not necessarily adiabatic. We verify that, in this case, the suggested modification of the continuity equation does not provide equivalence between the relativistic and Newtonian descriptions.

DOI: 10.1103/PhysRevD.67.087301 PACS number(s): 98.80.Jk

I. INTRODUCTION

In 1934, McCrea and Milne [1], by generalizing a previous study of Milne [2], developed what later has been called Newtonian cosmology [3]. By considering the cosmological principle, assuming that pressure is negligibly small and using Newtonian dynamics and gravitation, they showed how the correct relativistic equations for the evolution of the universe could be obtained with a mathematically more simple treatment. Later, McCrea [4] modified their basic equations to take into account pressure. He showed that the analogy between Newtonian and relativistic approaches is completely restored if two physical notions from relativity theory are adopted: the convertibility of mass and energy, mediated by the factor c^2, and the possibility of distinction between inertial and gravitational mass [5]. In 1965, Harrison [6] obtained the same result, without any concepts from general relativity. The significance, limitations, and validity of Newtonian cosmology have been discussed by many authors [7]. As described above, here we will follow the more traditional and widespread approach and will not consider more geometrical and formal ones [8].

The basic equations of Newtonian cosmology are

$$\left(\frac{\partial \vec{u}}{\partial t}\right)_r + (\vec{u}\cdot\vec{\nabla}_r)\vec{u} = -\vec{\nabla}_r\phi - (\rho+P)^{-1}\vec{\nabla}_r P, \tag{1.1}$$

$$\nabla_r^2\phi = 4\pi G(\rho+3P), \tag{1.2}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_r + \vec{\nabla}_r\cdot(\rho+P)\vec{u} = 0, \tag{1.3}$$

where ρ, P, $\vec{u}$, and ϕ are, respectively, the density, pressure, velocity field, and gravitational potential of the cosmic fluid, and now we are considering $c=1$. Equations (1.1), (1.2), and (1.3) are the Euler equation, the Poisson equation, and the continuity equation, respectively.

The above set of equations lead to the correct relativistic equations for the cosmic evolution, if spatial curvature is zero, $K=0$. However, if we study perturbation theory in this framework, the results do not agree with the relativistic approach [9]. The equivalence is obtained only for $P=0$.

To circumvent this problem, in Ref. [10] it was proposed a modification of the continuity equation such that

$$\left(\frac{\partial \rho}{\partial t}\right)_r + \vec{\nabla}_r\cdot(\rho\vec{u}) + P\vec{\nabla}_r\cdot\vec{u} = 0. \tag{1.4}$$

However, to get the correct result they restricted their analysis to adiabatic perturbations and, in addition, considered only a constant equation of state.

The goal of this paper is to reanalyze this problem without these restrictions. The recent observational evidence for acceleration in the cosmic expansion makes it important to investigate the evolution of perturbations in a universe with a exotic component described by a time-varying equation of state.

This paper is organized as follows. In Sec. II, we obtain the evolution equations of the perturbations, in the linear regime, for the Newtonian approach. In Sec. III, we study the problem from the relativistic point of view, still in the linear regime, and considering only scalar perturbations. In Sec. IV, we discuss our results and conclude.

II. COSMOLOGICAL PERTURBATIONS IN AN EXPANDING NEWTONIAN UNIVERSE

We start with the basic hydrodynamical equations that describe the motion of a Newtonian cosmic fluid: the Euler's equation (1.1), the Poisson's equation (1.2), and the modified continuity equation (1.4),

$$\left(\frac{\partial \vec{u}}{\partial t}\right)_r + (\vec{u}\cdot\vec{\nabla}_r)\vec{u} = -\vec{\nabla}_r\phi - (\rho+P)^{-1}\vec{\nabla}_r P, \tag{2.1}$$

$$\nabla_r^2\phi = 4\pi G(\rho+3P), \tag{2.2}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_r + \vec{\nabla}_r\cdot(\rho\vec{u}) + P\vec{\nabla}_r\cdot\vec{u} = 0. \tag{2.3}$$

As usual in perturbation theory, we assume small perturbations around the homogeneous background solution in the form

$$\rho = \rho_0 + \delta\rho, \tag{2.4}$$

$$P=P_0+\delta P,$$

$$\phi=\phi_0+\varphi,$$

$$\vec{u}=\vec{u}_0+\vec{v}.$$

We use a subscript "0" to denote background quantities. Placing Eqs. (2.4) in the Euler equation (2.1) and keeping only first order terms, we obtain

$$\left(\frac{\partial\vec{v}}{\partial t}\right)_r+(\vec{u}_0\cdot\vec{\nabla}_r)\vec{v}+(\vec{v}\cdot\vec{\nabla}_r)\vec{u}_0=-\vec{\nabla}_r\varphi-(\rho_0+P_0)^{-1}\vec{\nabla}_r(\delta P). \tag{2.5}$$

Doing the same with the Poisson and the continuity equations we have

$$\nabla^2\varphi=4\pi G(\delta\rho+3\delta P), \tag{2.6}$$

$$\left(\frac{\partial\delta\rho}{\partial t}\right)_r+(\rho_0+P_0)\vec{\nabla}_r\cdot\vec{v}+\vec{\nabla}_r\cdot(\delta\rho\vec{u}_0)+\delta P\vec{\nabla}_r\cdot\vec{u}_0=0. \tag{2.7}$$

For convenience, we make a change to comoving coordinates such that

$$\vec{r}=R\vec{x}, \tag{2.8}$$

$$\nabla_x=R\nabla_r, \tag{2.9}$$

$$\left(\frac{\partial}{\partial t}\right)_x=\left(\frac{\partial}{\partial t}\right)_r+\frac{\dot R}{R}(\vec{x}\cdot\nabla_x), \tag{2.10}$$

where the overdot denotes time derivative.

Using Eqs. (2.9) and (2.10) in Eqs. (2.5), (2.6), and (2.7), and introducing the density contrast $\delta\equiv\delta\rho/\rho_0$, we obtain the following system:

$$\dot{\vec{v}}+\frac{\dot R}{R}\vec{v}=-\frac{1}{R}\vec{\nabla}\varphi-\frac{1}{R}\frac{1}{\rho_0+P_0}\vec{\nabla}(\delta P), \tag{2.11}$$

$$\nabla^2\varphi=4\pi GR^2(\delta\rho+3\delta P), \tag{2.12}$$

$$\rho_0\dot\delta-3\frac{\dot R}{R}(\rho_0+P_0)\delta=-\frac{\rho_0+P_0}{R}\vec{\nabla}\cdot\vec{v}-3\frac{\dot R}{R}(\delta\rho+\delta P). \tag{2.13}$$

We can define the quantities: $w=P_0/\rho_0$, $c_{eff}^2=\delta P/\delta\rho$. The quantity c_{eff}^2 is what gives the critical scale for stabilization of perturbations in the general case [11].

Now, we can rewrite the above equations as

$$\dot{\vec{v}}+\frac{\dot R}{R}\vec{v}+\frac{1}{R}\vec{\nabla}\varphi+\frac{c_{eff}^2}{R}\frac{1}{1+w}\vec{\nabla}(\delta)=0, \tag{2.14}$$

$$\nabla^2\varphi-4\pi GR^2\rho_0\delta(1+3c_{eff}^2)=0, \tag{2.15}$$

$$\dot\delta+3\frac{\dot R}{R}(c_{eff}^2-w)\delta+\frac{1+w}{R}\vec{\nabla}\cdot\vec{v}=0, \tag{2.16}$$

where we considered that $c_{eff}=c_{eff}(t)$.

Time differentiating Eq. (2.16), taking the divergence of Eq. (2.14), and using Eq. (2.15), we obtain the evolution equation for the density contrast:

$$\begin{aligned}\ddot\delta+\dot\delta\left\{3H(c_{eff}^2-w)-\frac{\dot w}{1+w}+2H\right\}+\delta\left\{3\frac{\ddot R}{R}(c_{eff}^2-w)\right.\\ +3H^2(c_{eff}^2-w)+3H[(\dot{c_{eff}^2})-\dot w]-4\pi G(1+3c_{eff}^2)\\ \left.\times(1+w)\rho_0-3H\dot w\frac{c_{eff}^2-w}{1+w}\right\}-c_{eff}^2\frac{\nabla^2\delta}{R^2}=0.\end{aligned} \tag{2.17}$$

By using now the following equations (valid for the background):

$$\frac{\ddot R}{R}=-\frac{4\pi G}{3}\rho_0(1+3w), \tag{2.18}$$

$$H^2=\left(\frac{\dot R}{R}\right)^2=\frac{8\pi G}{3}\rho_0, \tag{2.19}$$

$$\frac{\dot w}{1+w}=-3H(c_s^2-w), \tag{2.20}$$

we can rewrite Eq. (2.17) as

$$\begin{aligned}\ddot\delta-[3(2w-c_s^2-c_{eff}^2)-2]H\dot\delta+3H^2\\ \times\left\{\left[\frac{3}{2}w^2-4w-\frac{1}{2}+3c_s^2\right]\right.\\ \left.+3c_{eff}^2(3c_s^2-6w-1)+\frac{\dot{c_{eff}^2}}{H}\right\}\delta\\ =\frac{c_{eff}^2}{R^2}\nabla^2\delta.\end{aligned} \tag{2.21}$$

III. RELATIVISTIC COSMOLOGICAL PERTURBATION EQUATIONS

To treat the problem through general relativistic framework, we have to introduce perturbations of the metric and the energy-momentum tensor. Following the work of Kodama and Sasaki [12], we have

$$\tilde g_{00}=-a^2[1+2AY], \tag{3.1}$$

$$\tilde g_{0j}=-a^2BY_j, \tag{3.2}$$

$$\tilde g_{ij}=a^2[\gamma_{ij}+2H_LY\gamma_{ij}+2H_TY_{ij}], \tag{3.3}$$

$$\tilde{T}^0_0 = -\rho[1+\delta Y], \tag{3.4}$$

$$\tilde{T}^0_j = (\rho+P)(v-B)Y_j, \tag{3.5}$$

$$\tilde{T}^j_0 = -(\rho+P)vY^j, \tag{3.6}$$

$$\tilde{T}^i_j = P[\delta_j{}^i + \pi_L \delta_j{}^i + \pi_T Y_j{}^i], \tag{3.7}$$

where Y are scalar harmonic functions satisfying the equation $(\mathbf{\Delta}+k^2)Y=0$, A, B, H_L, H_T are the perturbations in the lapse function, the displacement vector, the isotropic, and the anisotropic parts of the metric, respectively; and δ, v, π_L, and π_T are the relative perturbations in the energy density, the velocity, the isotropic, and the anisotropic pressures.

The quantities that represent the perturbations, as written above, are not gauge invariants. However, we can build invariant quantities from them, such as

$$\Psi = A + k^{-1}\left(\frac{a'}{a}\right)(B - k^{-1}H_T') + k^{-1}(B' - k^{-1}H_T''), \tag{3.8}$$

$$\Pi = \pi_T, \tag{3.9}$$

$$\Gamma = \pi_L - \frac{c_s^2}{w}\delta = (c_{eff}^2 - c_s^2)\frac{\delta}{w}, \tag{3.10}$$

$$V = v - k^{-1}H_T', \tag{3.11}$$

$$\Delta = \delta + 3(1+w)\left(\frac{a'}{a}\right)k^{-1}(v-B). \tag{3.12}$$

In this section, we keep the notation of [12], in particular the prime denotes differentiation with respect to the conformal time $(d\eta = dt/R)$.

Equation (3.10) defines the difference between $c_{eff}^2 = \delta P/\delta\rho$ and $c_s^2 = \dot{P}/\dot{\rho}$. These two quantities are equal only in the case of adiabatic pressure perturbations, characterized by $\Gamma=0$.

To obtain the relativistic equivalent of Eq. (2.21), we have to study perturbations in the Einstein equations

$$\delta G_\nu{}^\mu = 8\pi G\,\delta T_\nu{}^\mu. \tag{3.13}$$

Using Eqs. (3.1)–(3.7) and the definitions (3.8)–(3.12) we obtain, for the gauge invariant V and Δ, the following system [12]:

$$V' + \frac{a'}{a}V = -k\left(4\pi G\frac{a^2\rho}{k^2} - \frac{c_s^2}{1+w}\right)\Delta + \frac{w}{1+w}k\Gamma, \tag{3.14}$$

$$\Delta' - 3w\frac{a'}{a}\Delta = -(1+w)kV, \tag{3.15}$$

where we took, both, the spatial curvature (K) and the anisotropic pressure perturbation (Π) equal to zero.

A second-order equation for Δ is obtained by eliminating V from Eqs. (3.14) and (3.15):

$$\Delta'' - [3(2w - c_s^2) - 1]\frac{a'}{a}\Delta' + 3\left\{\left[\frac{3}{2}w^2 - 4w - \frac{1}{2} + 3c_s^2\right]\times\left(\frac{a'}{a}\right)^2 + \frac{k^2}{3}c_s^2\right\}\Delta = -k^2 w\Gamma. \tag{3.16}$$

IV. COMPARING THE RESULTS

In order to compare Eq. (3.16) with the corresponding Newtonian second-order equation for the density contrast (2.21), we use Eq. (3.10), and reexpress Eq. (3.16) as

$$\ddot{\Delta} - [3(2w - c_s^2) - 2]\frac{\dot{a}}{a}\dot{\Delta} + 3\left\{\left[\frac{3}{2}w^2 - 4w - \frac{1}{2} + 3c_s^2\right]\times\left(\frac{\dot{a}}{a}\right)^2\right\}\Delta = -\frac{k^2}{a^2}c_{eff}^2\Delta, \tag{4.1}$$

where now the derivatives are taken with respect to the time t.

Comparing Eq. (4.1) with Eq. (2.21), using the properties of the harmonic functions Y [12], and considering that, if $K=0$, the operator $\mathbf{\Delta} \equiv \gamma^{ij}\,{}^s\vec{\nabla}_i {}^s\vec{\nabla}_j$ is equivalent to ∇^2, it is clear that there is only one case where those equations are equivalent. It occurs when $c_{eff}^2 = c_s^2 = w$, that is, the case in which there is only adiabatic pressure perturbation and the equation of state parameter is constant. This is the case investigated in [10]. For a discussion of the Newtonian limits of the relativistic cosmological perturbations in the case of adiabatic pressure perturbations but with $w=w(t)$, see [13].

In summary, we have shown that the modification in the continuity equation, suggested in [10], to obtain equivalence between Newtonian and relativistic approaches only works in the very special case, where adiabatic pressure perturbations and constant equation of state parameter are considered. In addition, the equivalence is complete, in this case, for every gauge in which $\Delta=\delta$.

ACKNOWLEDGMENTS

The idea of the present work was suggested to me by Ioav Waga during his cosmology course. I would also like to thank him and Sergio Jorás for the discussions which were essential for the conclusion of this paper. I also thank Marcos Freaza for his useful help, and ‘‘Coordenação de Aperfeiçoamento de Pessoal de Nível Superior’’ (CAPES) for the financial support.

[1] W.H. McCrea and E.A. Milne, Quarterly J. Math. **5**, 73 (1934); republished in Gen. Relativ. Gravit. **32**, 1949 (2000).
[2] E.A. Milne, Quarterly J. Math. **5**, 64 (1934); republished in Gen. Relativ. Gravit. **32**, 1939 (2000).
[3] D. Layzer, Astron. J. **59**, 170 (1954).
[4] W.H. McCrea, Proc. R. Soc. London **206**, 562 (1951).
[5] E.T. Whittaker, Proc. R. Soc. London **149**, 384 (1935).
[6] E.R. Harrison, Ann. Phys. (N.Y.) **35**, 437 (1965).
[7] W.H. McCrea, Nature (London) **175**, 466 (1955); H. Bondi, *Cosmology*, 2nd ed. (Cambridge University Press, Cambridge, England, 1960); P.T. Landsberg and A.D. Evans, *Mathematical Cosmology* (Clarendon, Oxford, 1977); C. Callan, R.H. Dicke, and P.J.E. Peebles, Am. J. Phys. **33**, 105 (1965); E.R. Harrison, Rev. Mod. Phys. **39**, 863 (1967); E.L. Schücking, Texas Q. **10**, 270 (1967); E.R. Harrison, *Cosmology: The Science of the Universe*, 2nd ed. (Cambridge University Press, Cambridge, England, 2000).
[8] C. Rueden and N. Straumann, Helv. Phys. Acta **70**, 318 (1997).
[9] R.K. Sachs and A.M. Wolfe, Astrophys. J. **147**, 73 (1967).
[10] J.A.S. Lima, V. Zanchin, and R. Brandenberger, Mon. Not. R. Astron. Soc. **291**, L1 (1997).
[11] W. Hu, Astrophys. J. **506**, 485 (1998).
[12] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[13] Jai-chan Hwang and Hyerim Noh, astro-ph/9701137.