Thermal field theory and generalized light front quantization

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The dependence of thermal field theory on the surface of quantization and on the velocity of the heat bath is investigated by working in general coordinates that are arbitrary linear combinations of the Minkowski coordinates. In the general coordinates the metric tensor $g_{\overline{\mu\nu}}$ is nondiagonal. The Kubo-Martin-Schwinger condition requires periodicity in thermal correlation functions when the temporal variable changes by an amount $-i/(T\sqrt{g_{\overline{00}}})$. Light-front quantization fails since $g_{\overline{00}}=0$; however, various related quantizations are possible.

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I. INTRODUCTION

A. The light front and thermal field theory

For many years light-front quantization has been applied to deep inelastic scattering and the Wilson operator product expansion. More recently it has been used in QCD to compute hadron structure [1–4]. Light-front quantization brings both conceptual and computational simplifications to certain hadronic processes [5,6].

Much of the computational simplification occurs because the mass shell condition expressed in terms of the momentum variables $p^{\pm}=(p^0\pm p^3)/\sqrt{2}$ is

$$p^{-}=\frac{\boldsymbol{p}_{\perp}^2+m^2}{2p^{+}}. \tag{1.1a}$$

The operator $P^{-}=(P^0-P^3)/\sqrt{2}$ plays the role of the Hamiltonian in that it generates the evolution of the fields in the coordinate $x^{+}=(x^0+x^3)/\sqrt{2}$:

$$[P^{-},\phi(x)]=-i\frac{\partial\phi}{\partial x^{+}}. \tag{1.1b}$$

Recently Brodsky suggested [7] that the computational advantages of quantizing in light-front coordinates might carry over to statistical mechanics and thermal field theory done on the light front and proposed as the appropriate partition function

$$\mathrm{Tr}[\exp\{-P^{-}/T_{\mathrm{LC}}\}]. \tag{1.1c}$$

The relation of T_{LC} to the usual invariant temperature T was unspecified.

The suggestion was pursued by Alves, Das, and Perez [8]. For a free field theory they found an immediate problem that results from the vanishing of the on-shell energy, Eq. (1.1a), as $p_{+}\to\infty$. The breakdown is not specific to the canonical ensemble. In the microcanonical ensemble the entropy is a measure of the multiparticle phase space available to a gas of particles whose total energy is fixed. To any configuration with a fixed value of the total P^{-} one can add any number of zero-energy particles each having an infinite value of p^{+}. The entropy of such a configuration diverges.

Alves, Das, and Perez [8] showed that a thermal average performed in the rest frame but using light-front variables does work. Using $P^0=(P^{+}+P^{-})/\sqrt{2}$ the partition function is

$$\mathrm{Tr}[\exp\{-(P^{+}+P^{-})/\sqrt{2}T\}], \tag{1.2}$$

instead of Eq. (1.1c). They performed one-loop calculations of the self-energy in scalar field theories with either a ϕ^4 interaction or a ϕ^3 interaction. The final results for both calculations were exactly the same as the conventional answers.

B. Thermal field theory in generalized coordinates

In order to explore the various possible options it is most efficient to consider quantization in a general set of spacetime coordinates and later examine light-front quantization as a special case. The metric signature is $(+,-,-,-)$.

The conventional approach is to impose quantization conditions on fields at a fixed value of x^0. The operator that generates time evolution is P_0. The partition function is

$$\mathrm{Tr}[\exp\{-P_0/T\}], \tag{1.3}$$

as is appropriate for a heat bath at rest. If this system is viewed from a Lorentz frame with velocity v and four-velocity $u^{\alpha}=(\gamma,0,0,\gamma v)$, then the quantization will be at a fixed value of $x^{\alpha}u_{\alpha}$; the evolution operator will be $P_{\alpha}u^{\alpha}$, and the partition function will be $\mathrm{Tr}[\exp(-\beta P_{\alpha}u^{\alpha})]$. The fact that u^{α} serves as both the vector normal to the surface of quantization and the velocity vector of the heat bath is a unique feature of Lorentz boosted coordinates and is not true in more general coordinates.

The present paper investigates the dependence of thermal field theory on the surface of quantization and on the velocity of the heat bath. Subsequent analysis will show that for any vector n_{α} that is timelike or lightlike, it is possible to quantize at a fixed value of $n_{\alpha}x^{\alpha}$,

$$[\pi(x),\phi(0)]\ \delta(n_{\alpha}x^{\alpha}-c)=-i\ \delta^4(x), \tag{1.4a}$$

and employ the partition function

$$\mathrm{Tr}[\exp\{-P_{\alpha}u^{\alpha}/T\}], \tag{1.4b}$$

appropriate to a heat bath moving with four-velocity $u^{\alpha} = (\gamma, 0, 0, \gamma v)$, provided only that n_{α} and u^{α} satisfy

$$u^{\alpha} n_{\alpha} > 0. \tag{1.4c}$$

The conventional rest frame choice is $n^0 = u^0 = 1$ and $n^j = u^j = 0$. A Lorentz boost from the rest frame still gives $n^{\alpha} = u^{\alpha}$. The equality of these two vectors is not general, as various examples will show. One can see that light-front quantization using the partition function in Eq. (1.1c) fails, even apart from the divergence issues mentioned previously, because it requires $v \to 1$ in order that $P_{\alpha} u^{\alpha}$ be proportional to P_-. This makes $T_{\rm LC} = T\sqrt{1-v^2}/\sqrt{2} \to 0$. In contrast, the successful approach of Alves, Das, and Perez [8] in Eq. (1.2) fits into the general framework above with $n^{\alpha} = (1,0,0,1)$ and $u^{\alpha} = (1,0,0,0)$.

The purpose of this paper is to investigate thermal field theory formulated in general coordinates that are arbitrary linear combinations of the Cartesian t, x, y, and z and to determine if there are any computational advantages to other formulations. The method is conservative in that the new coordinates are restricted to be linear combinations of the Cartesian coordinates [9], a restriction which guarantees that physical consequences will be the same for the following reason. The Lagrangians of fundamental field theories are invariant under arbitrary nonlinear coordinate transformations in accord with the principle of equivalence, and thus are trivially invariant under the linear coordinate transformations considered here. Physically interesting quantities in thermal field theories are either scalars, spinors, or tensors. Whether calculated in Cartesian coordinates or more general linear coordinates, the scalar quantities should be identical and the spinor and tensor quantities should be simply related by the chosen transformations.

Section II develops the formalism of thermal field theory in the general coordinates in order to determine the possible choices for the surface of quantization and the velocity of the heat bath. Section III treats several examples that are specifically related to light-front quantization, and it may be read independently of Sec. II. Section IV provides some conclusions.

II. THERMAL FIELD THEORY IN OBLIQUE COORDINATES

A. Transformed coordinates and metric

From the Cartesian coordinates $x^0 = t$, x^1, x^2, and x^3 a new set of coordinates $x^{\bar{0}}$, $x^{\bar{1}}$, $x^{\bar{2}}$, and $x^{\bar{3}}$ can be formed by taking linear combinations:

$$x^{\bar{\mu}} = A^{\bar{\mu}}_{\alpha} x^{\alpha}, \tag{2.1}$$

where the $A^{\bar{\mu}}_{\alpha}$ are a set of 16 real constants. The notation used is due to Schouten [10]. It expresses the fact that the space-time point specified by the four-vector x does not change under a coordinate transformation; only the labels used to indentify the components change. For example, the light-front choice is expressed as $x^{\bar{0}} = (x^0 + x^3)/\sqrt{2}$. It is also convenient that for $A^{\bar{\mu}}_{\alpha}$ the superscript $\bar{\mu}$ is neither a first index nor a second index since $\bar{\mu}$ cannot be confused with α.

It is misleading to refer to the general coordinate transformations as a change in the reference frame. A change in the physical reference frame can only be done by rotations and Lorentz boosts.

The 16 real numbers $A^{\bar{\mu}}_{\alpha}$ must have a nonzero determinant. To guarantee that $x^{\bar{0}}$ is a time coordinate, it is necessary to require that the covariant Lorentz vector $A^{\bar{0}}_{\alpha}$ be timelike or lightlike:

$$A^{\bar{0}}_{\alpha} A^{\bar{0}}_{\beta} \; g^{\alpha\beta} \geqslant 0, \tag{2.2}$$

where $g_{\alpha\beta} = {\rm diag}(1,-1,-1,-1)$ is the Minkowski metric. It is sometimes useful to express the transformation matrix in terms of partial derivatives:

$$A^{\bar{\mu}}_{\alpha} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\alpha}}. \tag{2.3}$$

The relation (2.1) can be inverted to find the Cartesian coordinates x^{β} as linear combinations of the new coordinates $x^{\bar{\nu}}$. These partial derivatives are denoted by

$$A^{\beta}_{\bar{\nu}} = \frac{\partial x^{\beta}}{\partial x^{\bar{\nu}}}. \tag{2.4}$$

The transformations satisfy

$$A^{\bar{\mu}}_{\alpha} A^{\alpha}_{\bar{\nu}} = \delta^{\bar{\mu}}_{\bar{\nu}}, \quad A^{\bar{\mu}}_{\alpha} \; A^{\beta}_{\bar{\mu}} = \delta^{\beta}_{\alpha}. \tag{2.5}$$

The differential length element in Cartesian coordinates is $g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$. In the new coordinates there is a new metric $g_{\bar{\mu}\bar{\nu}}$ given by

$$g_{\bar{\mu}\bar{\nu}} = A^{\alpha}_{\bar{\mu}} \; A^{\beta}_{\bar{\nu}} \; g_{\alpha\beta}, \tag{2.6}$$

which satisfies $g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = g_{\bar{\mu}\bar{\nu}}\, dx^{\bar{\mu}} dx^{\bar{\nu}}$. If all 12 of the off-diagonal entries in $g_{\bar{\mu}\bar{\nu}}$ vanish, the new coordinates are orthogonal. This occurs, for example, if the A are Lorentz transformations. In general the $g_{\bar{\mu}\bar{\nu}}$ are not diagonal and in general the new coordinates are oblique. It will be convenient for later purposes to denote the determinant of the covariant metric by

$$g = {\rm Det}[g_{\bar{\mu}\bar{\nu}}] < 0. \tag{2.7}$$

The Jacobian of the coordinate transformation

$$dx^0 dx^1 dx^2 dx^3 = J\, dx^{\bar{0}} dx^{\bar{1}} dx^{\bar{2}} dx^{\bar{3}}$$

is therefore

$$J = {\rm Det}[A^{\alpha}_{\bar{\mu}}] = \sqrt{-g}. \tag{2.8}$$

It will also be necessary to use the contravariant metric $g^{\bar{\mu}\bar{\nu}}$, related to the Minkowski metric by

$$g^{\bar{\mu}\bar{\nu}}=A^{\bar{\mu}}_{\alpha}\,A^{\bar{\nu}}_{\beta}\,g^{\alpha\beta}. \tag{2.9}$$

The requirement in Eq. (2.2) can be stated as

$$g^{\bar{0}\bar{0}}\geqslant 0. \tag{2.10}$$

B. Four-momentum operators

In the conventional quantization at fixed x^0 the momentum operators P_λ are the generators of space-time translations. For a scalar field operator $\phi(x)$ they satisfy

$$[P_\lambda,\phi]=-i\frac{\partial\phi}{\partial x^\lambda}.$$

The linear combination of these generators defined by

$$P_{\bar{\mu}}=A^{\lambda}_{\bar{\mu}}\,P_\lambda, \tag{2.11}$$

with $A^{\lambda}_{\bar{\mu}}$ given in Eq. (2.4), will satisfy

$$[P_{\bar{\mu}},\phi]=-i\frac{\partial x^\lambda}{\partial x^{\bar{\mu}}}\,\frac{\partial\phi}{\partial x^\lambda}=-i\frac{\partial\phi}{\partial x^{\bar{\mu}}}. \tag{2.12}$$

In particular, $P_{\bar{0}}$ generates the evolution in the variable $x^{\bar{0}}$. Appendix B shows that if one quantizes at fixed $x^{\bar{0}}$ then the Hamiltonian is this same operator $P_{\bar{0}}$.

Note that the temporal evolution is in the variable $x^{\bar{0}}=A^{\bar{0}}_{\alpha}\,x^\alpha$ but the evolution operator is in a different linear combination: $P_{\bar{0}}=A^{\alpha}_{\bar{0}}P_\alpha$. These two vectors are reciprocal in the sense that $A^{\bar{0}}_{\alpha}\,A^{\alpha}_{\bar{0}}=1$. Lorentz transformations are special in that the metrics are the same, $g_{\bar{\alpha}\bar{\beta}}=g_{\alpha\beta}$, and the two vectors are the same, $A^{\bar{0}}_{\alpha}=g_{\alpha\beta}A^{\beta}_{\bar{0}}$.

C. Density operator

Field theory at finite temperature requires a density operator and exactly what the density operator should be is not obvious when using general oblique coordinates. For conventional quantization at fixed x^0 the density operator has the form

$$\hat{\varrho}=\exp\{-\beta P_\alpha u^\alpha\}, \tag{2.13}$$

where u^α is some four-velocity. Rewriting this in general coordinates gives $\hat{\varrho}=\exp\{-\beta P_{\bar{\sigma}}u^{\bar{\sigma}}\}$. The appropriate value of $u^{\bar{\sigma}}$ is undetermined but since the Minkowski four-velocity satisfies $u^\alpha u^\beta g_{\alpha\beta}=1$, the velocity in oblique coordinates must satisfy

$$u^{\bar{\sigma}}u^{\bar{\mu}}g_{\bar{\sigma}\bar{\mu}}=1. \tag{2.14}$$

1. Kubo-Martin-Schwinger relation

With a partition function of the above form, the thermal Wightman function is

$$\mathcal{D}_>(x)=\mathrm{Tr}[\hat{\varrho}\;\;\phi(x)\phi(0)]/\mathrm{Tr}[\hat{\varrho}].$$

Consider a shift in the oblique time, i.e., $x^{\bar{0}}\to x^{\bar{0}}+\delta x^{\bar{0}}$ with fixed values of $x^{\bar{1}},x^{\bar{2}},x^{\bar{3}}$. In terms of the Cartesian coordinates, $x^\alpha\to x^\alpha+\delta x^\alpha$ with $\delta x^\alpha=A^{\alpha}_{\bar{0}}\,\delta x^{\bar{0}}$. Using $P_\alpha\delta x^\alpha=P_{\bar{0}}\delta x^{\bar{0}}$ the shifted field operator is

$$\begin{aligned}\phi(x+\delta x)&=\exp(iP_\alpha\delta x^\alpha)\,\phi(x)\exp(-iP_\alpha\delta x^\alpha)\\&=\exp(iP_{\bar{0}}\delta x^{\bar{0}})\,\phi(x)\exp(-iP_{\bar{0}}\delta x^{\bar{0}}).\end{aligned}$$

For evolution in imaginary values of $\delta x^{\bar{0}}$ to be equivalent to thermal averaging requires that the density operator involve only $P_{\bar{0}}$ and not $P_{\bar{j}}$. Therefore the spatial components of the contravariant velocity must vanish: $u^{\bar{1}}=u^{\bar{2}}=u^{\bar{3}}=0$. This describes a heat bath that is at rest in the oblique coordinates. The normalized velocity vector is

$$u^{\bar{\sigma}}=\left(\frac{1}{\sqrt{g_{\bar{0}\bar{0}}}},0,0,0\right). \tag{2.15}$$

Note that this imposes a new requirement on the metric,

$$g_{\bar{0}\bar{0}}>0, \tag{2.16}$$

that is different from Eq. (2.10). This condition prevents standard light-front quantization since $g_{\bar{0}\bar{0}}$ would vanish. In all subsequent discussions it will be assumed that Eq. (2.16) is satisfied. The density operator is

$$\hat{\varrho}=\exp\{-\beta P_{\bar{0}}/\sqrt{g_{\bar{0}\bar{0}}}\}, \tag{2.17}$$

or equivalently, Eq. (2.13) if the velocity vector is expressed in Cartesian coordinates:

$$u^\alpha=A^{\alpha}_{\bar{\mu}}u^{\bar{\mu}}=A^{\alpha}_{\bar{0}}/\sqrt{g_{\bar{0}\bar{0}}}. \tag{2.18}$$

The vector normal to the quantization surface is $n_\alpha=A^{\bar{0}}_{\alpha}$ and therefore $n_\alpha u^\alpha=1/\sqrt{g_{\bar{0}\bar{0}}}>0$, as stated in Eq. (1.4c).

The density operator is used to define the thermal averaged Wightman function:

$$\mathcal{D}_>(x)=\mathrm{Tr}[\exp\{-\beta P_{\bar{0}}/\sqrt{g_{\bar{0}\bar{0}}}\}\phi(x)\phi(0)]/Z, \tag{2.19}$$

where $Z=\mathrm{Tr}[\hat{\varrho}]$ is the partition function. Under imaginary shifts in the oblique time of the form $\delta x^{\bar{0}}=i\alpha/\sqrt{g_{\bar{0}\bar{0}}}$, the behavior is

$$\begin{aligned}\mathcal{D}_>(x^{\bar{0}}+i\alpha/\sqrt{g_{\bar{0}\bar{0}}},x^{\bar{j}})=\mathrm{Tr}\Bigg[&\exp\left\{-\frac{(\beta+\alpha)P_{\bar{0}}}{\sqrt{g_{\bar{0}\bar{0}}}}\right\}\phi(x)\\&\times\exp\left\{\frac{\alpha P_{\bar{0}}}{\sqrt{g_{\bar{0}\bar{0}}}}\right\}\phi(0)\Bigg]/Z.\end{aligned} \tag{2.20}$$

The spectrum of $P_{\bar{0}}$ is bounded from below but will always have arbitrarily large positive eigenvalues. If the two exponents are negative then the infinitely large energies will be suppressed. In other words, Eq. (2.20) is analytic for $-\beta \leqslant \alpha \leqslant 0$. For the choice $\alpha = -\beta$ or equivalently $\delta x^{\alpha} = -i\beta u^{\alpha}$ the result is

$$\mathcal{D}_{>}(x^{\bar{0}} - i\beta/\sqrt{g_{\bar{0}\bar{0}}}, x^{\bar{j}}) = \mathcal{D}_{>}(-x^{\bar{0}}, -x^{\bar{j}}). \tag{2.21}$$

This is the Kubo-Martin-Schwinger relation [11] expressed in oblique coordinates.

2. Tolman's law

The dependence of the partition function (2.17) on the combination $P_{\bar{0}}/T\sqrt{g_{\bar{0}\bar{0}}}$ merits further discussion. One explanation comes from perturbation theory in which at each order the eigenvalues of the operator $P_{\bar{0}}$ are the sum of single particle energies $p_{\bar{0}}$, each satisfying a mass shell condition $g^{\bar{\mu}\bar{\nu}} p_{\bar{\mu}} p_{\bar{\nu}} = m^2$, for various masses. Among the allowed coordinate transformations are scale transformations. Rescaling the contravariant time by a factor λ, as in $x^{\bar{0}} \to \lambda\, x^{\bar{0}}$, would rescale the covariant energy by $p_{\bar{0}} \to p_{\bar{0}}/\lambda$. The combination $P_{\bar{0}}/T$ would not be invariant under this transformation. However, the scale transformation changes the covariant metric, $g_{\bar{0}\bar{0}} \to g_{\bar{0}\bar{0}}/\lambda^2$, and the combination $P_{\bar{0}}/T\sqrt{g_{\bar{0}\bar{0}}}$ is invariant.

There is another way to understand why the partition function (2.17) depends on the combination $T\sqrt{g_{\bar{0}\bar{0}}}$. A condition necessary for thermal equilibrium in inertial coordinates is that the temperature should be uniform in space and time. Tolman [12] investigated the conditions for thermal equilibrium in a gravitational field and showed that a temperature gradient is necessary to prevent the flow of heat from regions of higher gravitational potential to regions of lower gravitational potential. The quantitative result is summarized by the statement that the product $T\sqrt{g_{\bar{0}\bar{0}}}$ must be constant and it is known at Tolman's law [13].

An alternative derivation is given by Landau and Lifshitz [14], who discussed how to compute the entropy in the microcanonical ensemble in a general curvilinear coordinate system (with or without gravity). In the microcanonical ensemble, the temperature is computed by differentiating the entropy with respect to the energy and this leads to the result that $T\sqrt{g_{\bar{0}\bar{0}}}$ must be constant.

3. Covariant density operator

For later purposes it is convenient to express the density operator (2.17) in a covariant form in terms of the conserved energy-momentum operator $T^{\bar{\nu}}_{\cdot\ \bar{\mu}}$. The generators of translations are

$$P_{\bar{\mu}} = \sqrt{-g}\int dx^{\bar{1}} dx^{\bar{2}} dx^{\bar{3}}\ T^{\bar{0}}_{\cdot\ \bar{\mu}}.$$

We rewrite this in terms of the energy-momentum operator in Cartesian coordinates, $T^{\alpha}_{\cdot\ \lambda}$, as

$$P_{\bar{\mu}} = \int dS_{\alpha}\ T^{\alpha}_{\cdot\ \lambda}\ A^{\lambda}_{\bar{\mu}}, \tag{2.22}$$

in accordance with Eq. (2.11). The three-dimensional, differential surface element is

$$dS_{\alpha} = \sqrt{-g}\ \ A^{\bar{0}}_{\alpha} dx^{\bar{1}} dx^{\bar{2}} dx^{\bar{3}}. \tag{2.23}$$

This surface of quantization is orthogonal to three contravariant vectors: $A^{\alpha}_{\bar{j}} dS_{\alpha} = 0$ for $\bar{j} = \bar{1}, \bar{2}$, and $\bar{3}$. Contracting $P_{\bar{\mu}}$ with $u^{\bar{\mu}}$ and using Eq. (2.15) gives the density operator in covariant form:

$$\hat{\varrho} = \exp\left\{-\beta\int dS_{\alpha} T^{\alpha}_{\cdot\lambda} u^{\lambda}\right\}. \tag{2.24}$$

D. Thermal field theory in real time

To quantize a field theory at a fixed value of $x^{\bar{0}}$ is straightforward but there are some unfamiliar aspects that originate from the oblique metric $g_{\bar{\mu}\bar{\nu}}$ being nondiagonal. Appendix B performs the explicit quantization for a scalar field theory. It is most natural to deal with contravariant space-time coordinates $x^{\bar{\mu}}$ and covariant momentum variables $p_{\bar{\mu}}$ so that the solutions of the field equations are superpositions of plane waves of the form $\exp(\pm i p_{\bar{\mu}} x^{\bar{\mu}})$. For spinor particles the Dirac matrices are $\gamma^{\bar{\mu}} = A^{\bar{\mu}}_{\alpha}\gamma^{\alpha}$ and they satisfy $\{\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}\} = 2g^{\bar{\mu}\bar{\nu}}$. For gauge boson propagators there are natural generalizations of Coulomb gauge, axial gauge, and covariant gauges. This section will summarize some familiar results expressed in oblique coordinates. Either canonical quantization or functional integration may be used [16,17].

1. Propagators

At zero temperature the free propagator for a scalar field is

$$D(p_{\bar{0}}, p_{\bar{j}}) = \frac{1}{g^{\bar{\mu}\bar{\nu}} p_{\bar{\mu}} p_{\bar{\nu}} - m^2 + i\epsilon}. \tag{2.25}$$

The integration measure over loop momenta is

$$\int \frac{dp_{\bar{0}} dp_{\bar{1}} dp_{\bar{2}} dp_{\bar{3}}}{\sqrt{-g}(2\pi)^4}. \tag{2.26}$$

At nonzero temperature the propagator has the usual 2×2 matrix structure [16,17], and the Bose-Einstein or Fermi-Dirac functions become

$$n = \frac{1}{\exp(\beta|p_{\bar{0}}|/\sqrt{g_{\bar{0}\bar{0}}}) \mp 1}. \tag{2.27}$$

As discussed later, the most interesting possibility is to choose a coordinate transformation such that $g^{\bar{0}\bar{0}} = 0$. This makes the denominator of the propagator linear in $p_{\bar{0}}$ and therefore there is only one pole in Eq. (2.25).

If $g^{\bar{0}\bar{0}} \neq 0$ then the propagator has poles at two values of $p_{\bar{0}}$ but the positive and negative values of $p_{\bar{0}}$ will have different magnitudes. It is sometimes convenient to express the propagator in a mixed form, in terms of the covariant energy $p_{\bar{0}}$ but with the contravariant momenta $p^{\bar{j}}$. To do this, we use the identity

$$g_{\bar{\mu}\bar{\nu}}p^{\bar{\mu}}p^{\bar{\nu}}=\frac{1}{g_{\bar{0}\bar{0}}}(g_{\bar{0}\bar{0}}\,p^{\bar{0}}+g_{\bar{0}\bar{i}}\,p^{\bar{i}})^2+\left(g_{\bar{i}\bar{j}}-\frac{g_{\bar{0}\bar{i}}g_{\bar{0}\bar{j}}}{g_{\bar{0}\bar{0}}}\right)p^{\bar{i}}p^{\bar{j}}$$

$$=\frac{(p_{\bar{0}})^2}{g_{\bar{0}\bar{0}}}-\gamma_{\bar{i}\bar{j}}p^{\bar{i}}p^{\bar{j}},$$

where $\gamma_{\bar{i}\bar{j}}$ is given by

$$\gamma_{\bar{i}\bar{j}}=-g_{\bar{i}\bar{j}}+\frac{g_{\bar{0}\bar{i}}g_{\bar{0}\bar{j}}}{g_{\bar{0}\bar{0}}}. \tag{2.28}$$

The determinant is $\mathrm{Det}[\gamma_{\bar{i}\bar{j}}]=-g/g_{\bar{0}\bar{0}}>0$ [18]. We define the effective energy to be

$$E=[m^2+\gamma_{\bar{i}\bar{j}}\ p^{\bar{i}}p^{\bar{j}}]^{1/2}. \tag{2.29}$$

The same propagator in these variables is

$$D(p_{\bar{0}},p^{\bar{j}})=\frac{g_{\bar{0}\bar{0}}}{(p_{\bar{0}})^2-g_{\bar{0}\bar{0}}\ E^2+i\epsilon}, \tag{2.30}$$

and the poles are now at $p_0=\pm\sqrt{g_{\bar{0}\bar{0}}}\,E$; the Bose-Einstein or Fermi-Dirac functions become

$$n=\frac{1}{\exp(\beta E)\mp 1}. \tag{2.31}$$

The integration over covariant energy and contravariant three-momenta is

$$\sqrt{\frac{-g}{g}}\int\ \frac{dp_{\bar{0}}\ dp^{\bar{1}}dp^{\bar{2}}dp^{\bar{3}}}{(2\pi)^4}. \tag{2.32}$$

2. Statistical mechanics

The partition function provides a direct calculation of the thermodynamic functions via the Helmholtz free energy F:

$$\exp\{-\beta F/\sqrt{g_{\bar{0}\bar{0}}}\}=\mathrm{Tr}[\exp\{-\beta P_{\bar{0}}/\sqrt{g_{\bar{0}\bar{0}}}\}]. \tag{2.33}$$

The free energy $F(T,V)$ then allows computation of the pressure and entropy:

$$P=-\left(\frac{\partial F}{\partial V}\right)_T, \tag{2.34a}$$

$$S=-\left(\frac{\partial F}{\partial T}\right)_V, \tag{2.34b}$$

and the energy is $U=F+TS$.

3. Covariant statistical mechanics

The prescriptions in Eq. (2.34) can be derived rather elegantly from the partition function if it is formulated covariantly [13,15]. In global thermal equilibrium considered here, the temperature T and the velocity $u^{\bar{\mu}}$ of the heat bath are independent of space-time and so there is no heat flux or viscosity. The thermal average of the energy-momentum operator has the perfect fluid form:

$$\frac{\mathrm{Tr}[\hat{\varrho}T^{\overline{\nu\mu}}]}{\mathrm{Tr}[\hat{\varrho}]}=(\rho+P)g^{\overline{\nu\mu}}-Pu^{\bar{\nu}}u^{\bar{\mu}}, \tag{2.35}$$

where ρ is the energy density. The first law of thermodynamics guarantees that there is an additional state function, the entropy density σ, related to energy density and pressure by

$$T\sigma=\rho+P. \tag{2.36}$$

This relation is equivalent to the more familiar differential relation $TdS=dU+PdV$.

It is convenient to express the right-hand side of Eq. (2.33) as the trace of the covariant density operator (2.24). The left-hand side of Eq. (2.33) can be written covariantly in terms of the differential surface element (2.23) using $\sqrt{-g}\ dx^{\bar{1}}dx^{\bar{2}}dx^{\bar{3}}/\sqrt{g_{\bar{0}\bar{0}}}=dS_\alpha u^\alpha$ as an integral over the free energy density Φ:

$$F/\sqrt{g_{\bar{0}\bar{0}}}=\int\ dS_\alpha u^\alpha\Phi.$$

The manifestly covariant statement of Eq. (2.33) is

$$\exp\left\{-\beta\int\ dS_\alpha\Phi u^\alpha\right\}=\mathrm{Tr}\left[\exp\left\{-\beta\int\ dS_\alpha T^{\alpha}{}_{\cdot\lambda}u^\lambda\right\}\right]. \tag{2.37}$$

Now we apply this to two different equilibrium states with infinitesimally different β and u^α. The difference gives the differential relation

$$d(\beta\Phi u^\alpha)=\frac{\mathrm{Tr}[\hat{\varrho}\ T^{\alpha}{}_{\cdot\lambda}]}{\mathrm{Tr}[\hat{\varrho}\ ]}d(\beta u^\lambda). \tag{2.38}$$

The differential on the left-hand side is

$$d(\beta\Phi)u^\alpha+(\beta\Phi)du^\alpha.$$

Using the thermal average of the energy-momentum tensor (2.35), the right-hand side is

$$[(\rho+P)u^\alpha u_\lambda-P\delta^\alpha_\lambda](d\beta u^\lambda+\beta du^\lambda)=\rho d\beta u^\alpha-P\beta du^\alpha,$$

where $u_\lambda du^\lambda=0$ has been used. Equating the left- and right-hand sides and noting that u^α and du^α are orthogonal vectors gives the two relations

$$\Phi=-P, \tag{2.39a}$$

$$d(\beta\Phi)=\rho d\beta. \tag{2.39b}$$

The first relation is the same as Eq. (2.34a). The second relation implies $d\Phi=(\Phi-\rho)dT/T=-(P+\rho)dT/T=-\sigma dT$. Therefore $\sigma=-\partial\Phi/\partial T$, which is the same as Eq. (2.34b), but expressed in terms of densities.

E. Thermal field theory in imaginary time

It is also interesting to quantize in imaginary oblique time by letting $x^{\bar{0}}=-i\tau$ with $0\leqslant\tau\leqslant\beta/\sqrt{g_{\bar{0}\bar{0}}}$. As shown in Eq. (2.20) the thermal Wightman function is analytic for τ in this region. Since the Cartesian coordinates are linear combinations of the oblique coordinates, $x^{\alpha}=A^{\alpha}_{\bar{\mu}}x^{\bar{\mu}}$, making $x^{\bar{0}}$ purely imaginary makes some of the Cartesian coordinates complex. In other words, going to imaginary time does not commute with the coordinate transformations.

The Euclidean propagator comes from the replacement $p_{\bar{0}}\rightarrow -i\omega_n$ in Eq. (2.25) where $\omega_n=2\pi nT\sqrt{g_{\bar{0}\bar{0}}}$:

$$D_E(p)=\frac{-1}{g^{\bar{0}\bar{0}}\omega_n^2+2ig^{\bar{0}\bar{j}}\omega_n\, p_{\bar{j}}-g^{\bar{i}\bar{j}}p_{\bar{i}}p_{\bar{j}}+m^2}. \qquad (2.40)$$

Note that the denominator has an imaginary part because of the nondiagonal metric but the real part of the denominator is positive as always. In perturbation theory the summation/integration over loop momenta is

$$\frac{T\sqrt{g_{\bar{0}\bar{0}}}}{\sqrt{-g}}\sum_{n=-\infty}^{\infty}\int\frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{(2\pi)^3}.$$

One can canonically quantize in imaginary time, in which case the field operators obey equations of motion. Alternatively one can use a Euclidean functional integral [16,17,19] in which case the fields are periodic under $\tau\rightarrow\tau+\beta/\sqrt{g_{\bar{0}\bar{0}}}$. The partition function is

$$Z=\int_{\text{periodic}}[D\phi]\exp\left\{\int d^4x_E\ \mathcal{L}\right\}, \qquad (2.41)$$

where the integration element in four-dimensional Euclidean space-time is given by

$$\int d^4x_E=\sqrt{-g}\int_0^{\beta/\sqrt{g_{\bar{0}\bar{0}}}}d\tau\int dx^{\bar{1}}dx^{\bar{2}}dx^{\bar{3}}.$$

Thermodynamics is computed from the Helmholtz free energy $F(T,V)$,

$$\exp\{-\beta F/\sqrt{g_{\bar{0}\bar{0}}}\}=Z, \qquad (2.42)$$

using Eq. (2.34).

III. EXAMPLES

It is easy to see that time-independent transformations will not yield anything new. More specifically, if $x^0=x^{\bar{0}}$ then the quantization is at fixed x^0 and if x^1, x^2, and x^3 are independent of $x^{\bar{0}}$ then $A^{j}_{\bar{0}}=0$ so that $u^j=0$ and the density operator will be $\exp\{-\beta P_0\}$.

This section deals with a general class of examples based on transformations of the forms

$$x^{\bar{0}}=ax^0+bx^3,$$

$$x^{\bar{3}}=cx^0+dx^3, \qquad (3.1a)$$

with $x^{\bar{1}}=x^1$ and $x^{\bar{2}}=x^2$ always understood. All the examples in this section will result from special choices of $a,b,c,$ and d. If $|a|>|b|$ and $|d|>|c|$ then $x^{\bar{0}}$ is a true time coordinate and $x^{\bar{3}}$ is a true space coordinate. Light-front coordinates violate this, viz., $|a|=|b|$ and $|c|=|d|$. The inverse transformation to Eq. (3.1a) is

$$x^0=(dx^{\bar{0}}-bx^{\bar{3}})/N,$$

$$x^3=(-cx^{\bar{0}}+ax^{\bar{3}})/N, \qquad (3.1b)$$

where $N=ad-bc\neq 0$. The two conditions

$$a>0,\quad \frac{d}{N}>0 \qquad (3.1c)$$

guarantee that increasing values of $x^{\bar{0}}$ correspond to increasing x^0.

(i) Metric. The contravariant metric is

$$g^{\bar{\mu}\bar{\nu}}=\begin{pmatrix} a^2-b^2 & 0 & 0 & ac-bd\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ ac-bd & 0 & 0 & c^2-d^2\end{pmatrix}, \qquad (3.2a)$$

and the covariant metric is

$$g_{\bar{\mu}\bar{\nu}}=\begin{pmatrix} d^2-c^2 & 0 & 0 & ac-bd\\ 0 & -N^2 & 0 & 0\\ 0 & 0 & -N^2 & 0\\ ac-bd & 0 & 0 & b^2-a^2\end{pmatrix}\frac{1}{N^2}. \qquad (3.2b)$$

The necessary requirement $g_{\bar{0}\bar{0}}\neq 0$ implies $|d|\neq|c|$; however, there is nothing wrong with choosing $|a|=|b|$. The 3×3 matrix defined in Eq. (2.28) is

$$\gamma_{\bar{i}\bar{j}}=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & (d^2-c^2)^{-1}\end{pmatrix}. \qquad (3.3)$$

(ii) Momenta. The contravariant form of the oblique momenta is $p^{\bar{\nu}}=A^{\bar{\nu}}_{\alpha}\,p^{\alpha}$, parallel to Eq. (3.1a). From this, the covariant momenta are obtained by applying the metric, $p_{\bar{\mu}}=g_{\bar{\mu}\bar{\nu}}\,p^{\bar{\nu}}$, with the result

$$p_{\bar{0}}=(dp^0+cp^3)/N,$$

$$p_{\bar{3}}=-(bp^0+ap^3)/N, \qquad (3.4)$$

and, of course, $p_{\bar{1}}=-p^1$, $p_{\bar{2}}=-p^2$.

(iii) Density operator. The time evolution operator is

$$P_{\bar{0}}=\frac{\partial x^{\alpha}}{\partial x^{\bar{0}}}P_{\alpha}=\frac{dP_0-cP_3}{N},$$

and therefore the density operator is

$$\exp\{-\beta P_{\bar{0}}/\sqrt{g_{\bar{0}\bar{0}}}\}=\exp\left\{-\beta\frac{|d|P_0-cP_3\epsilon(d)}{\sqrt{d^2-c^2}}\right\},$$

after using Eq. (3.1c). Here $\epsilon(d)=\pm1$ is the sign function. As expected, $|d|=|c|$ is excluded. One can understand the form of the density operator in a more physical way. Since the heat bath is at rest in the $x^{\bar{3}}$ coordinate, its laboratory velocity is $v=dx^3/dx^0=-c/d$ from Eq. (3.1a). Using $\gamma=(1-v^2)^{-1/2}$ gives the four-velocity

$$U^{\alpha}=(\gamma,0,0,\gamma v)=\left[\frac{|d|}{\sqrt{d^2-c^2}},0,0,\frac{-c\ \ \epsilon(d)}{\sqrt{d^2-c^2}}\right]. \quad (3.5)$$

Thus the density operator is $\hat{\varrho}=\exp(-\beta P_{\alpha}U^{\alpha})$.

A. Lorentz transformations and generalizations

(i) Quantization at rest but with a moving heat bath. The most intuitive situation physically is to quantize conventionally at fixed x^0 but to have a velocity v for the heat bath. This is easily accomplished by the choices $a=1,b=0,c=\gamma v$, and $d=\gamma$ so that

$$x^{\bar{0}}=x^0,$$

$$x^{\bar{3}}=\gamma(x^3+vx^0). \quad (3.6)$$

The density operator is $\exp\{-\beta\gamma(P_0-vP_3)\}$.

(ii) Quantization surface and heat bath moving differently. A more general option is to give the quantization surface a velocity v' and the heat bath a velocity v:

$$x^{\bar{0}}=\gamma'(x^0+v'x^3),$$

$$x^{\bar{3}}=\gamma(x^3+vx^0). \quad (3.7)$$

However, only the velocity v enters into the density operator: $\hat{\varrho}=\exp\{-\beta\gamma(P_0-vP_3)\}$. Note that the metric will depend on v and v'.

B. Light front and generalizations

As stated previously, strict light-front coordinates in which $|d|=|c|$ are forbidden.

(i) Front moving at $v<1$. An interesting case is the transformation

$$x^{\bar{0}}=x^0\cos(\theta/2)+x^3\sin(\theta/2),$$

$$x^{\bar{3}}=x^0\sin(\theta/2)-x^3\cos(\theta/2), \quad (3.8)$$

where $-\pi/2<\theta<\pi/2$. These would be light-front coordinates if θ were allowed to take the value $\pm\pi/2$. The covariant and contravariant metrics are equal:

$$g_{\bar{\mu}\bar{\nu}}=\begin{pmatrix}\cos\theta & 0 & 0 & \sin\theta\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ \sin\theta & 0 & 0 & -\cos\theta\end{pmatrix}=g^{\bar{\mu}\bar{\nu}}. \quad (3.9)$$

The density operator is

$$\hat{\varrho}=\exp\left\{-\beta\frac{P_0\cos(\theta/2)+P_3\sin(\theta/2)}{\cos\theta}\right\}, \quad (3.10)$$

and it obviously fails at $\theta=\pi/2$. Alternatively, one can use $v=\tan(\theta/2)$ and express the transformation as

$$x^{\bar{0}}=(x^0+vx^3)/\sqrt{1+v^2},$$

$$x^{\bar{3}}=(vx^0-x^3)/\sqrt{1+v^2}, \quad (3.11)$$

so that the density operator is $\exp\{-\beta\gamma(P_0+vP_3)\}$.

(ii) Choice of Alves, Das, and Perez (ADP). The calculations in Ref. [8] can be stated as the choices $a=b=d=1,c=0$:

$$x^{\bar{0}}=x^0+x^3,$$

$$x^{\bar{3}}=x^3. \quad (3.12)$$

The density operator becomes $\exp\{-\beta P_0\}$. The covariant metric is

$$g_{\bar{\mu}\bar{\nu}}=\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}, \quad (3.13)$$

and so the covariant coordinates are $x_{\bar{0}}=g_{\bar{0}\bar{\nu}}x^{\bar{\nu}}=x^0$ and $x_{\bar{3}}=g_{\bar{3}\bar{\nu}}x^{\bar{\nu}}=-(x^0+x^3)=-\sqrt{2}x^+$. The corresponding covariant momentum components are

$$p_{\bar{0}}=p^0,$$

$$p_{\bar{3}}=-(p^0+p^3)=-\sqrt{2}p^+.$$

The contravariant metric is

$$g^{\bar{\mu}\bar{\nu}}=\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ -1 & 0 & 0 & -1\end{pmatrix}. \quad (3.14)$$

Because $g^{\bar{0}\bar{0}}$ vanishes, the momentum space propagator is linear in $p_{\bar{0}}$:

$$g^{\bar{\mu}\bar{\nu}}p_{\bar{\mu}}p_{\bar{\nu}}=-2p_{\bar{0}}p_{\bar{3}}-(p_{\bar{1}})^2-(p_{\bar{2}})^2-(p_{\bar{3}})^2$$

$$=2\sqrt{2}p^0p^+-(p^1)^2-(p^2)^2-2(p^+)^2.$$

In the Euclidean formulation the contravariant time becomes negative and imaginary: $x^{\bar{0}}\rightarrow -i\tau$; the covariant energy becomes discrete and imaginary: $p_{\bar{0}}\rightarrow -i\omega_n$ with $\omega_n=2\pi nT$. (Note $g_{\bar{0}\bar{0}}=1$.) The Euclidean propagator used in Ref. [8] is

$$\frac{1}{2i\sqrt{2}\ \omega_n\ p^+-(p^1)^2-(p^2)^2-2(p^+)^2-m^2}. \qquad (3.15)$$

(iii) ADP with a moving heat bath. It is simple to modify the previous case to allow for a moving heat bath. We choose $a=b=1$, $c=\gamma v$, and $d=\gamma$ so that

$$x^{\bar{0}}=x^0+x^3,$$

$$x^{\bar{3}}=\gamma(vx^0+x^3). \qquad (3.16)$$

The density operator is $\exp\{-\beta\gamma(P_0-vP_3)\}$, corresponding to a moving heat bath. As before $g^{\bar{0}\bar{0}}=0$ but now $g_{\bar{0}\bar{0}}=\sqrt{1+v}/\sqrt{1-v}$.

IV. CONCLUSIONS

In standard light-front quantization $g_{\bar{0}\bar{0}}=g_{++}=0$ and this makes it impossible to formulate statistical mechanics and thermal field theory. Physically, the problem is the infinite velocity of the light front.

The most interesting possibility is to choose oblique coordinates which satisfy $g_{\bar{0}\bar{0}}\neq 0$ but $g^{\bar{0}\bar{0}}=0$ as in Eq. (3.12), which is the case studied by Alves, Das, and Perez [8]. The advantage of choosing $g^{\bar{0}\bar{0}}=0$ is that the denominator of the propagator, $g^{\bar{\mu}\bar{\nu}}p_{\bar{\mu}}p_{\bar{\nu}}-m^2$, will be linear in the energy variable $p_{\bar{0}}$. Consequently the propagator will have only one pole and not two. This reduces the computational effort required for multiloop diagrams. Any diagram for which the kinematics allows N propagators to be on shell would normally produce 2^N contributions. However if $g^{\bar{0}\bar{0}}=0$, there will be only one contribution.

A very straightforward application would be to compute the quark and gluon propagators in the hard thermal loop approximation using Eq. (3.12) and verify the rotational invariance of the dispersion relations [17]. A more ambitious task would be to compute the vertex functions in the hard thermal loop approximation.

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APPENDIX A: LORENTZ INVARIANCE OF $g_{\bar{\mu}\bar{\nu}}$

The general nondiagonal metric $g_{\bar{\mu}\bar{\nu}}$ is always invariant under three rotations and three Lorentz boosts. However, the representation of these six transformations depends on the coordinate system $x^{\bar{\mu}}$.

The usual representation of a Lorentz transformation from one set of Cartesian coordinates to another, $x^{\alpha'}=\Lambda^{\alpha'}_{\alpha}x^{\alpha}$, leaves invariant the Minkowski metric tensor:

$$\Lambda^{\alpha'}_{\alpha}\Lambda^{\beta'}_{\beta}\ g_{\alpha'\beta'}=g_{\alpha\beta}. \qquad (A1)$$

As before, the index notation of Schouten [10] is used. Here α' runs over $0',1',2'$, and $3'$. The Minkowski metric is invariant: $g_{0'0'}=g_{00}=1$, $g_{1'1'}=g_{11}=-1$, etc.

Each Lorentz transformation of the Cartesian coordinates induces a Lorentz transformation of the oblique coordinates, $x^{\bar{\mu}}=W^{\bar{\mu}}_{\bar{\rho}}x^{\bar{\rho}}$, where

$$W^{\bar{\mu}}_{\bar{\rho}}=A^{\bar{\mu}}_{\alpha'}\Lambda^{\alpha'}_{\alpha}A^{\alpha}_{\bar{\rho}}. \qquad (A2)$$

Because Λ keeps the Cartesian metric invariant, W automatically keeps the oblique metric invariant:

$$W^{\bar{\mu}}_{\bar{\rho}}W^{\bar{\nu}}_{\bar{\sigma}}g_{\bar{\mu}\bar{\nu}}=g_{\bar{\rho}\bar{\sigma}}. \qquad (A3)$$

APPENDIX B: QUANTIZATION IN OBLIQUE COORDINATES

This section will show how to perform the explicit quantization in an arbitrary oblique coordinate system for the free scalar field and then calculate the thermal average of the free energy-momentum tensor.

1. Equation of motion

The action expressed as an integral over contravariant coordinates is $\sqrt{-g}\int dx^{\bar{0}}dx^{\bar{1}}dx^{\bar{2}}dx^{\bar{3}}\mathcal{L}$ with Lagrangian density

$$\mathcal{L}=\frac{1}{2}g^{\bar{\mu}\bar{\nu}}\frac{\partial\phi}{\partial x^{\bar{\mu}}}\frac{\partial\phi}{\partial x^{\bar{\nu}}}-\frac{1}{2}m^2\phi^2. \qquad (B1)$$

The field equation that follows from the Lagrangian density is

$$g^{\bar{\mu}\bar{\nu}}\frac{\partial^2\phi}{\partial x^{\bar{\mu}}\partial x^{\bar{\nu}}}=m^2\phi. \qquad (B2)$$

The solution to this will be a superposition of plane waves of the form $\exp(-ip_{\bar{\alpha}}x^{\bar{\alpha}})$, with the phase expressed in terms of contravariant spatial coordinates and covariant momentum coordinates. The equation of motion gives the mass shell condition

$$g^{\bar{\mu}\bar{\nu}}p_{\bar{\mu}}p_{\bar{\nu}}=m^2. \qquad (B3)$$

This is a quadratic equation for $p_{\bar{0}}$ with two solutions,

$$p_{\bar{0}\pm}=-\frac{g^{\bar{0}\bar{j}}p_{\bar{j}}}{g^{\bar{0}\bar{0}}}\pm\left[\frac{m^2+p_{\bar{i}}p_{\bar{j}}\Gamma^{\bar{i}\bar{j}}}{g^{\bar{0}\bar{0}}}\right]^{1/2}, \qquad (B4)$$

where $\Gamma^{\bar{i}\bar{j}}$ is the 3×3 matrix

$$\Gamma^{\bar{i}\bar{j}}=\frac{g^{\bar{0}\bar{i}}g^{\bar{0}\bar{j}}}{g^{\bar{0}\bar{0}}}-g^{\bar{i}\bar{j}}. \tag{B5}$$

The two solutions are $\exp(-ip_{\bar{0}\pm}x^{\bar{0}}-ip_{\bar{j}}x^{\bar{j}})$. The energies $p_{\bar{0}\pm}$ are not invariant under momentum inversion, but rather $p_{\bar{0}-}\to -p_{\bar{0}+}$ when $p_{\bar{j}}$ changes sign. Therefore one can use for the second plane wave the negative momentum solution $\exp(ip_{\bar{0}+}x^{\bar{0}}+ip_{\bar{j}}x^{\bar{j}})$. The solution to the field equation can be expanded as

$$\phi(x)=\int \frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{\sqrt{-g}(2\pi)^3 2|p^{\bar{0}}|}[a(p)e^{-ip\cdot x}+a(p)^{\dagger}e^{ip\cdot x}], \tag{B6}$$

where $p\cdot x=p_{\bar{0}+}x^{\bar{0}}+p_{\bar{j}}x^{\bar{j}}$. The contravariant energies $p^{\bar{0}\pm}$ are equal in magnitude:

$$\begin{aligned} p^{\bar{0}\pm}&=g^{\bar{0}\bar{0}}p_{\bar{0}\pm}+g^{\bar{0}\bar{j}}p_{\bar{j}}\\ &=\pm\sqrt{g^{\bar{0}\bar{0}}}\;[m^2+p_{\bar{i}}p_{\bar{j}}\Gamma^{\bar{i}\bar{j}}]^{1/2}, \end{aligned} \tag{B7}$$

and $|p^{\bar{0}}|$ will be denoted simply by $p^{\bar{0}}$ [18].

2. Canonical quantization

For quantization on the surfaces of constant $x^{\bar{0}}$, the canonical momentum is

$$\pi(x)=\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x^{\bar{0}})}=g^{\bar{0}\mu}\,\frac{\partial\phi}{\partial x^{\mu}}=\frac{\partial\phi}{\partial x_{\bar{0}}}. \tag{B8}$$

The explicit mode expansion is

$$\pi(x)=-i\int \frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{\sqrt{-g}(2\pi)^3 2}[a(p)e^{-ip\cdot x}-a(p)^{\dagger}e^{ip\cdot x}].$$

If the mode operators are required to satisfy

$$[a(p),a^{\dagger}(p')]=\sqrt{-g}\,2p^{\bar{0}}(2\pi)^3\prod_{j=1}^{3}\;\delta(p_{\bar{j}}-p'_{\bar{j}}), \tag{B9}$$

then the equal time commutator has the correct value:

$$\begin{aligned} &[\pi(x),\phi(x')]_{x^{\bar{0}}=x'^{\bar{0}}}\\ &\quad=-i\int \frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{(2\pi)^3 2}[e^{-ip_{\bar{j}}(x^{\bar{j}}-x'^{\bar{j}})}+e^{ip_{\bar{j}}(x^{\bar{j}}-x'^{\bar{j}})}]\\ &\quad=-\frac{i}{\sqrt{-g}}\;\delta(x^{\bar{1}}-x'^{\bar{1}})\,\delta(x^{\bar{2}}-x'^{\bar{2}})\,\delta(x^{\bar{3}}-x'^{\bar{3}}). \end{aligned}$$

3. Microcausality

It is easy to verify microcausality. The commutator of two fields is

$$[\phi(x),\phi(0)]=\int \frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{\sqrt{-g}(2\pi)^3 2p^{\bar{0}}}[e^{-ip_{\bar{\alpha}}x^{\bar{\alpha}}}-e^{ip_{\bar{\alpha}}x^{\bar{\alpha}}}].$$

We change to Minkowski integration variables by defining $p_{\bar{\alpha}}=p_{\lambda}\partial x^{\lambda}/\partial x^{\bar{\lambda}}$ so that $p_{\bar{\alpha}}x^{\bar{\alpha}}=p_{\lambda}x^{\lambda}$. The integration measure is invariant and therefore

$$[\phi(x),\phi(0)]=\int \frac{dp_1dp_2dp_3}{(2\pi)^3 2p^0}[e^{-ip\cdot x}-e^{ip\cdot x}].$$

This is the conventional answer for the commutator. It vanishes for spacelike separations $x_{\lambda}x^{\lambda}<0$. Since $x_{\bar{\alpha}}x^{\bar{\alpha}}=x_{\lambda}x^{\lambda}$ it vanishes for $x_{\bar{\alpha}}x^{\bar{\alpha}}<0$.

4. Hamiltonian

The canonical Hamiltonian density is

$$\mathcal{H}=\pi\frac{\partial\phi}{\partial x^{\bar{0}}}-\mathcal{L}. \tag{B10}$$

It is convenient to express this in terms of mixed contravariant and covariant derivatives:

$$\mathcal{H}=\frac{1}{2}\left[\frac{\partial\phi}{\partial x_{\bar{0}}}\frac{\partial\phi}{\partial x^{\bar{0}}}-\frac{\partial\phi}{\partial x_{\bar{1}}}\frac{\partial\phi}{\partial x^{\bar{1}}}-\frac{\partial\phi}{\partial x_{\bar{2}}}\frac{\partial\phi}{\partial x^{\bar{2}}}-\frac{\partial\phi}{\partial x_{\bar{3}}}\frac{\partial\phi}{\partial x^{\bar{3}}}+m^2\phi^2\right].$$

The Hamiltonian requires integrating over the contravariant three-volume,

$$P_{\bar{0}}=\sqrt{-g}\int dx^{\bar{1}}dx^{\bar{2}}dx^{\bar{3}}\mathcal{H}.$$

Working this out explicitly gives

$$P_{\bar{0}}=\int \frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{\sqrt{-g}(2\pi)^3 2p^{\bar{0}}}\,\frac{p_{\bar{0}}}{2}[a^{\dagger}(p)a(p)+a(p)a^{\dagger}(p)]. \tag{B11}$$

Note that the covariant energy $p_{\bar{0}}$ in the numerator does not cancel the contravariant energy $p^{\bar{0}}$ in the denominator. The commutation relation

$$[P_{\bar{0}},\phi(x)]=-i\frac{\partial\phi}{\partial x^{\bar{0}}}$$

verifies that the Hamiltonian is the generator of translations in the contravariant time variable $x^{\bar{0}}$.

5. Energy and momentum

The canonical energy-momentum tensor is

$$T^{\bar{\nu}}_{\cdot\,\bar{\mu}}=\frac{\partial\phi}{\partial x_{\bar{\nu}}}\frac{\partial\phi}{\partial x^{\bar{\mu}}}-\delta^{\bar{\nu}}_{\bar{\mu}}\mathcal{L} \tag{B12}$$

and satisfies the conservation laws $\partial T^{\bar{\nu}}_{\cdot\,\bar{\mu}}/\partial x^{\bar{\nu}}=0$. The $\bar{\mu}=\bar{0}$ and $\bar{\mu}=\bar{m}$ components of this equation are

$$0=\frac{\partial T^{\bar{0}}_{\cdot\,\bar{0}}}{\partial x^{\bar{0}}}+\frac{\partial T^{\bar{n}}_{\cdot\,\bar{0}}}{\partial x^{\bar{n}}}, \tag{B13}$$

$$0=\frac{\partial T^{\bar{0}}_{\cdot\,\bar{m}}}{\partial x^{\bar{0}}}+\frac{\partial T^{\bar{n}}_{\cdot\,\bar{m}}}{\partial x^{\bar{n}}}. \tag{B14}$$

From Eq. (B10), $\mathcal{H}=T^{\bar{0}}_{\cdot\,\bar{0}}$ and thus $T^{\bar{0}}_{\cdot\,\bar{0}}$ is the energy density. The first of the above equations indentifies $T^{\bar{n}}_{\cdot\,\bar{0}}$ as the energy flux. From the second, $T^{\bar{0}}_{\cdot\,\bar{m}}$ is the momentum density and $T^{\bar{n}}_{\cdot\,\bar{m}}$ is the momentum flux. Integrating the energy and momentum densities over a contravariant three-volume gives

$$P_{\bar{\mu}}=\sqrt{-g}\int dx^{\bar{1}}dx^{\bar{2}}dx^{\bar{3}}T^{\bar{0}}_{\cdot\,\bar{\mu}}. \tag{B15}$$

These integrals are independent of the contravariant time: $\partial P_{\bar{\mu}}/\partial x^{\bar{0}}=0$. They generate translations in the contravariant coordinates:

$$[P_{\bar{\mu}},\phi(x)]=-i\frac{\partial\phi}{\partial x^{\bar{\mu}}}. \tag{B16}$$

The explicit form for the three-momentum operators is

$$P_{\bar{j}}=\int\frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{\sqrt{-g}(2\pi)^3 2p^{\bar{0}}}\frac{p_{\bar{j}}}{2}[a^\dagger(p)a(p)+a(p)a^\dagger(p)]. \tag{B17}$$

6. Thermal averages

This section will show that despite the somewhat complicated dispersion relations in Eqs. (B4) and (B7), the thermal average of the energy-momentum tensor can be computed directly to give the conventional answer.

Bose-Einstein statistics gives for the thermal average of the energy-momentum tensor of a free gas of scalar particles,

$$\langle T_{\bar{\mu}\bar{\nu}}\rangle=\int\frac{dp_{\bar{1}}dp_{\bar{2}}dp_{\bar{3}}}{\sqrt{-g}(2\pi)^3p^{\bar{0}}}\frac{p_{\bar{\mu}}p_{\bar{\nu}}}{\exp(\beta p_{\bar{0}}/\sqrt{g_{\bar{0}\bar{0}}})-1}. \tag{B18}$$

To perform this integration, we change to Minkowski momenta k_α, where

$$p_{\bar{\mu}}=A^{\alpha}_{\bar{\mu}}k_\alpha. \tag{B19}$$

The mass shell condition requires $k_0=(\boldsymbol{k}^2+m^2)^{1/2}$, and $p_{\bar{0}}/\sqrt{g_{\bar{0}\bar{0}}}=k_\lambda u^\lambda$ with $u^\lambda=A^{\lambda}_{\bar{0}}u^{\bar{0}}$, the oblique velocity given by Eq. (2.15). The change of variables gives

$$\langle T_{\bar{\mu}\bar{\nu}}\rangle=A^{\alpha}_{\bar{\mu}}A^{\beta}_{\bar{\nu}}\int\frac{dk_1dk_2dk_3}{(2\pi)^3k_0}\frac{k_\alpha k_\beta}{\exp(\beta k_\lambda u^\lambda)-1}, \tag{B20}$$

whose evaluation is standard:

$$\begin{aligned}\langle T_{\bar{\mu}\bar{\nu}}\rangle&=A^{\alpha}_{\bar{\mu}}A^{\beta}_{\bar{\nu}}((\rho+P)u_\alpha u_\beta-Pg_{\alpha\beta})\\&=(\rho+P)u_{\bar{\mu}}u_{\bar{\nu}}-Pg_{\bar{\mu}\bar{\nu}}.\end{aligned} \tag{B21}$$

The final result is expressed in terms of the oblique velocity vector and the oblique metric tensor. The most physical quantity is the mixed tensor,

$$\langle T^{\bar{\mu}}_{\cdot\,\bar{\nu}}\rangle=\begin{pmatrix}\rho & 0 & 0 & 0\\ (\rho+P)g_{\bar{0}\bar{1}}/g_{\bar{0}\bar{0}} & -P & 0 & 0\\ (\rho+P)g_{\bar{0}\bar{2}}/g_{\bar{0}\bar{0}} & 0 & -P & 0\\ (\rho+P)g_{\bar{0}\bar{3}}/g_{\bar{0}\bar{0}} & 0 & 0 & -P\end{pmatrix}, \tag{B22}$$

where Eq. (2.15) has been used. The off-diagonal entries in the metric give a nonzero momentum density: $\langle T^{\bar{0}}_{\cdot\,\bar{n}}\rangle=(\rho+P)g_{\bar{0}\bar{n}}/g_{\bar{0}\bar{0}}$.

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