Finite temperature renormalization group effective potentials for the linear sigma model

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We derive an approximate renormalization group (RG) solution for the linear sigma model of Gell-Mann and Levy at finite temperature. For our purposes, the Fermionic degrees of freedom of the model are interpreted as quarks which interact via the σ and π mesons to provide a phenomenological description of hot nuclear matter at temperatures from 1 GeV to approximately 100 MeV. Our solution consists of two coupled, nonlinear flow equations for the Yukawa coupling, g, of the model and the Bosonic effective potential, U. These equations are solved numerically to study the behavior of the model as it evolves from high to low energy scales at finite temperature T. This allows us to determine the critical temperature, T_c, at which chiral symmetry breaking occurs, and to assess the relative sensitivity of this quantity with respect to variations in the input parameters of the model. Our results are consistent with values for T_c obtained from other theoretical approaches such as lattice gauge theory or other RG techniques.

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I. INTRODUCTION

In recent years, considerable effort has gone into the study of relativistic quantum field theories at finite temperature. Issues addressed in these studies range from the evolution of the early universe to the deconfinement phase transition of QCD. A powerful tool for understanding the structure of such theories is the renormalization group (RG) as originally formulated by Wilson and co-workers [1]. Much recent work on this subject (see, e.g., Ref. [2] for a collection of many relevant references) has been based on the local effective potential approximation originally proposed in Ref. [3]. In the present work, we extend our RG local effective potential approach for finite temperature systems [4] to treat the linear sigma model at finite temperature.

The linear sigma model of Gell-Mann and Levy was originally formulated by Schwinger [5] in 1958. Since that time, it has been thoroughly studied using a variety of techniques, and has recently enjoyed a resurgence of popularity as the "chiral quark meson" model, [6,7] in which the Fermion fields correspond to valence quarks. For our purposes, we will consider only the case of finite temperature spontaneous chiral symmetry breaking for the model, leaving the more complex case of finite temperature dynamical symmetry breaking for future work.

It is well known that the strong interactions in thermal equilibrium are significantly different at high temperature than at low temperature. For example, a phase transition at some critical temperature may separate high and low temperature physics [8]. In particular, it was realized early on that such a transition should be closely connected to a qualitative change in the chiral condensate $\langle\bar{\psi}\psi\rangle$ in accordance with the observation that spontaneous symmetry breaking tends to be absent at sufficiently high temperatures. In addition, it has been pointed out that for small values of the up and down quark masses, the chiral transition is expected to share the universality class of the $O(4)$ Heisenberg model [9,10].

Our intent here is to determine the temperature, T_c, at which chiral symmetry is spontaneously broken in the linear sigma model, to determine the order of the associated phase transition, and to analyze the RG flow of the couplings as the temperature is raised. For our purposes, we assume the meson fields of the linear sigma model are approximations to degrees of freedom which would result from integrating out the gluon degrees of freedom of QCD and introducing composite operators for the mesonic bound states [11,12]. Our Fermionic fields ψ may then be identified as quark degrees of freedom which acquire a mass M by virtue of the appearance of a nonzero vacuum expectation value of the scalar field, ϕ, of the model. The interesting part of the flow of the resulting action is then due to quark and meson fluctuations about the tree-level ground state, as well as the thermal fluctuations which appear at nonzero temperature.

Such techniques may prove to be particularly useful in, e.g., the study of high temperature phase transitions in the Higgs sector of the standard model and their possible connection to the generation of baryonic asymmetry in the universe. Lattice Monte Carlo methods have been the primary means used to determine the nonperturbative content of the effective potential in such problems to date. We expect that our finite temperature RG techniques will be a useful complement to lattice MC calculations especially since, in the analysis which follows, we take pains to make careful comparisons between RG and lattice calculations throughout.

II. REVIEW OF LINEAR SIGMA MODEL

Our model involves a Fermionic isodoublet field ψ with zero bare mass, a triplet of pseudoscalar pions, and a scalar field ϕ. The Lagrangian is written as

$$\mathcal{L}=\frac{1}{2}[(\partial_\mu\phi)^2+(\partial_\mu\vec{\pi})^2]-U(\rho)+\bar{\psi}[iZ(\rho)\gamma^\mu\partial_\mu-V(\rho)]\psi, \tag{1}$$

where

$$U(\rho)=\frac{\mu^2}{2}(\phi^2+\vec{\pi}^2)+\frac{\lambda}{4}(\phi^2+\vec{\pi}^2)^2 \tag{2}$$

and

$$V(\rho)=g(\rho)\Gamma=g(\rho)(\phi+i\vec{\tau}\cdot\vec{\pi}\gamma^5). \tag{3}$$

Here, we have defined

$$\rho^2\equiv\phi^2+\vec{\pi}^2, \tag{4}$$

and explicitly included the Fermionic wave-function renormalization $Z(\rho)$. This Lagrangian is invariant under the operations of parity, charge conjugation and time reversal, and it is renormalizable [13–15]. In addition, the model has symmetries under the groups $SU_V(2)$ and $SU_A(2)$.

We now briefly examine the issue of spontaneously broken symmetry. When the quantity μ^2 of Eq. (2) is negative, the potential $U(\rho)$ has a minimum for a nonzero value of ρ, namely

$$\rho^2=v^2\equiv\phi^2+\pi^2=-\frac{2\mu^2}{\lambda}. \tag{5}$$

We can now rewrite the Lagrangian of Eq. (1) in terms of the shifted field $\phi'=\phi-v$ as

$$\mathcal{L}=\bar{\psi}[iZ(\rho)\gamma^\mu\partial_\mu+gv+g(\phi+i\vec{\tau}\cdot\vec{\pi}\gamma^5)]\psi+\frac{1}{2}[(\partial_\mu\phi')^2+(\partial_\mu\vec{\pi})^2]+\frac{1}{2}(\mu^2+3\lambda v^2)\phi'^2+\frac{1}{2}(2\mu^2+\lambda v^2)\vec{\pi}^2+\lambda v\phi'(\phi'^2+\vec{\pi}^2)-\frac{\lambda}{4}(\phi'^2+\vec{\pi}^2)^2+\phi'(\mu^2 v+\lambda v^3). \tag{6}$$

Inspection of this familiar expression reveals that the pion field remains massless, while the fermion and scalar fields have acquired masses given by

$$m_\psi=gv, \tag{7}$$

$$m^2=\mu^2+3\lambda v^2, \tag{8}$$

respectively. Thus, even though our Lagrangian is invariant under $SU_L(2)\times SU_R(2)$, the ϕ and $\vec{\pi}$ fields no longer form a multiplet with equal masses. This chiral symmetry has been "spontaneously" broken. It is our objective to determine how fluctuations alter this simple tree-level picture of the broken symmetry ground state.

We note that chiral symmetry may also be broken "dynamically" by the addition of a new term $c\phi$ to the linear sigma model Lagrangian. The application of zero temperature RG techniques (which constitute the starting point of our finite temperature RG approach) to the linear sigma model with dynamical symmetry breaking has recently been undertaken by Johnson, Shepard and McNeil [16]. However, in the present work we focus solely on spontaneous chiral symmetry breaking. Extension of the formulation presented here to include dynamical symmetry breaking is straightforward but results in a considerable increase in algebraic complexity. Moreover, the presence of dynamical symmetry breaking should have little influence on the properties of the ground state at temperatures near T_c, the chiral transition temperature.

III. DERIVATION OF THE RG FLOW EQUATIONS

We begin the derivation of our finite temperature RG flow equations with the Euclidean action for the linear sigma model at finite temperature, obtained from Eq. (1) by means of the usual Wick rotation:

$$S_E=\int_0^\beta d\tau\int d^3x\left\{\frac{1}{2}(\partial_E\phi)^2+\frac{1}{2}(\partial_E\vec{\pi})^2+U(\rho)+\bar{\psi}\left[Z(\rho)\left(\gamma^0\frac{\partial}{\partial\tau}-i\vec{\gamma}\cdot\nabla\right)+V(\rho)\right]\psi\right\}. \tag{9}$$

Here we have employed the Euclidean, or imaginary time, approach of Matsubara [18,19], with the boson (fermion) fields periodic (antiperiodic) in the τ direction with period β.

The fields $\phi(x)$ and $\psi(x)$ are expanded in terms of their Fourier components as

$$\phi(\vec{x},\tau)=\phi_0+\varphi(\vec{x},\tau)=\phi_0+\left(\frac{\beta}{V}\right)^{1/2}\sum_{n,\vec{q}_i\neq 0}e^{i(\vec{q}_i\cdot\vec{x}+2n\pi/\beta\tau)}\phi_n(\vec{q}_i), \tag{10}$$

and

$$\psi(\vec{x},\tau)=\chi_+e^{i\pi/\beta\tau}+\frac{1}{\sqrt{V}}\sum_{n,\vec{q}_i\neq 0}e^{i[\vec{q}_i\cdot\vec{x}+(2n+1)\pi/\beta\tau]}\chi_n(\vec{q}_i), \tag{11}$$

while $\vec{\pi}(x)$ is expanded similarly as

$$\vec{\pi}(\vec{x},\tau)=\vec{\pi}_0+\left(\frac{\beta}{V}\right)^{1/2}\sum_{\vec{q}_i,n\neq 0}\vec{\xi}_n(\vec{q}_i)e^{i(\vec{q}_i\cdot\vec{x}+2n\pi/\beta\tau)}. \tag{12}$$

Normalization of the fields is chosen so that the Fourier amplitudes are dimensionless, and so that standard thermodynamic identities are recovered using the present path integral formalism.

The potentials are also expanded via a Taylor series. The bosonic potential U becomes

$$U(\rho)=U(\rho_0)+U_\phi|_{\phi_0,\vec{\pi}_0}\phi+U_{\vec{\pi}}|_{\phi_0,\vec{\pi}_0}\vec{\xi}+\frac{1}{2!}U_{\phi\phi}\Big|_{\phi_0,\vec{\pi}_0}\phi^2+\frac{1}{2!}U_{\phi\pi_i}\Big|_{\phi_0,\vec{\pi}_0}\phi\xi_i+\frac{1}{2!}U_{\pi_i\phi}\Big|_{\phi_0,\vec{\pi}_0}\xi_i\phi+\frac{1}{2!}U_{\pi_i\pi_j}\Big|_{\phi_0,\vec{\pi}_0}\xi_i\xi_j+\dots, \tag{13}$$

with a similar expression for V.

After a bit of algebra, we find the following expressions for derivatives of the potentials U and V:

$$\frac{\partial U}{\partial \phi}=\mu^2\phi+\lambda\phi\rho^2, \tag{14}$$

$$\frac{\partial U}{\partial \pi_i}=\mu^2\pi_i+\lambda\pi_i\rho^2, \tag{15}$$

$$\frac{\partial^2 U}{\partial \phi^2}=\mu^2+\lambda(\phi^2+\rho^2), \tag{16}$$

$$\frac{\partial^2 U}{\partial \pi_i\partial\pi_j}=\mu^2\delta_{i,j}+\lambda\delta_{i,j}\rho+2\lambda\pi_i\pi_j, \tag{17}$$

$$\frac{\partial^2 U}{\partial\phi\partial\pi_i}=\frac{\partial^2 U}{\partial\pi_i\partial\phi}=2\lambda\phi\pi_i, \tag{18}$$

and

$$\frac{\partial V}{\partial\phi}=g'\frac{\phi}{\rho}\Gamma+g, \tag{19}$$

$$\frac{\partial V}{\partial\pi_i}=g'\frac{\pi_i}{\rho}\Gamma+i\tau_i\gamma^5 g, \tag{20}$$

$$\frac{\partial^2 V}{\partial\phi^2}=\left[g''\left(\frac{\phi}{\rho}\right)^2+\frac{g'}{\rho}-\frac{g'\phi^2}{\rho^3}\right]\Gamma+2\frac{g'\phi}{\rho}, \tag{21}$$

$$\frac{\partial^2 V}{\partial\pi_i\partial\pi_j}=\left[g''\frac{\pi_i\pi_j}{\rho^2}+\frac{g'\delta_{i,j}}{\rho}-\frac{g'\pi_i\pi_j}{\rho^3}\right]\Gamma+\frac{g'}{\rho}(\pi_i\tau_j+\pi_j\tau_i)i\gamma^5, \tag{22}$$

$$\frac{\partial^2 V}{\partial\phi\partial\pi_i}=\frac{\partial^2 V}{\partial\pi_i\partial\phi}=\left[g''\frac{\pi_i\phi}{\rho^2}-\frac{g'\phi\pi_i}{\rho^3}\right]\Gamma+\frac{g'\phi}{\rho}i\tau_i\gamma^5+\frac{g'\pi_i}{\rho}, \tag{23}$$

where it is understood that in what follows, all the derivatives are to be evaluated at $\phi=\phi_0$ and $\vec{\pi}=\vec{\pi}_0$, as indicated in Eq. (13), and that primes denote derivatives with respect to ρ.

By virtue of the expansions of Eqs. (10)–(23) and the orthogonality of the field components, the Bosonic portion of the action,

$$S_{E_B}=\int_0^\beta d\tau\int d^3x\left\{\frac{1}{2}(\partial_E\phi)^2+\frac{1}{2}(\partial_E\vec{\pi})^2+U(\rho)\right\}, \tag{24}$$

may be written as

$$\beta VU(\rho_0)+\frac{\beta V}{2}\sum_{\vec{q}_i,n\neq 0}[\phi_n^*(\vec{q}_i)\xi_{1n}^*(\vec{q}_i)\quad \xi_{2n}^*(\vec{q}_i)\quad \xi_{3n}^*(\vec{q}_i)]\times\begin{pmatrix} q_i^2+U_{\phi\phi} & U_{\phi\pi_1} & U_{\phi\pi_2} & U_{\phi\pi_3}\\ U_{\pi_1\phi} & q_i^2+U_{\pi_1\pi_1} & U_{\pi_1\pi_2} & U_{\pi_1\pi_3}\\ U_{\pi_2\phi} & U_{\pi_2\pi_1} & q_i^2+U_{\pi_2\pi_2} & U_{\pi_2\pi_3}\\ U_{\pi_3\phi} & U_{\pi_3\pi_1} & U_{\pi_3\pi_2} & q_i^2+U_{\pi_3\pi_3}\end{pmatrix}\times\begin{pmatrix}\phi_n(\vec{q}_i)\\ \xi_{1n}(\vec{q}_i)\\ \xi_{2n}(\vec{q}_i)\\ \xi_{3n}(\vec{q}_i)\end{pmatrix} \tag{25}$$

to second order in the field amplitudes. Here, $q_i^2\equiv\vec{q}_i^{\,2}+\omega_n^2$ and $\omega_n\equiv 2n\pi/\beta$ is the n^{th} Matsubara frequency. We will henceforth refer to the above matrix of derivatives of U as $\mathbf{U}$.

In a similar fashion, we now consider the Fermionic portion of Eq. (9), consisting of

$$S_{E_F}=\int_0^\beta d\tau\int d^3x\left\{\bar{\psi}\left[Z(\rho)\left(\gamma^0\frac{\partial}{\partial\tau}-i\vec{\boldsymbol{\gamma}}\cdot\nabla\right)+V(\rho)\right]\psi\right\}. \tag{26}$$

Proceeding as in the case of Eq. (24), the kinetic term of this expression may be written as

$$\beta VZ\left\{\overline{\chi_+}\gamma^0\chi_+ i\frac{\pi}{\beta}+\frac{1}{V}\sum_{\vec{q}_i,n\neq 0}\overline{\chi_n}(\vec{q}_i)\times\left[i\gamma^0\left(\omega_n+\frac{\pi}{\beta}\right)+\vec{\gamma}\cdot\vec{q}_i\right]\chi_n(\vec{q}_i)\right\}, \tag{27}$$

where we now omit the ρ dependence of Z. In addition, we have terms quadratic in the Bosonic field amplitudes which involve the uniform Fermionic amplitudes $\overline{\chi_+}$ and χ_+ from the expansion of the Fermionic potential V. We denote the resulting matrix of these terms as $\mathbf{W}$, with

$$\mathbf{W}=\overline{\chi_+}(\mathbf{W}_1+\mathbf{W}_2+\mathbf{W}_3+\mathbf{W}_4)\chi_+ \tag{28}$$

where

$$\mathbf{W}_1=\frac{\beta^2}{2}\frac{g''}{\rho^2}\Gamma\begin{pmatrix}\phi^2 & \phi\pi_1 & \phi\pi_2 & \phi\pi_3\\ \pi_1\phi & \pi_1^2 & \pi_1\pi_2 & \pi_1\pi_3\\ \pi_2\phi & \pi_2\pi_1 & \pi_2^2 & \pi_2\pi_3\\ \pi_3\phi & \pi_3\pi_1 & \pi_3\pi_2 & \pi_3^2\end{pmatrix}, \tag{29}$$

$$\mathbf{W}_2=\frac{\beta^2}{2}\frac{g'}{\rho}\Gamma\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \tag{30}$$

$$\mathbf{W}_3=-\frac{\beta^2}{2}\frac{g'}{\rho^3}\Gamma\begin{pmatrix}\phi^2 & \phi\pi_1 & \phi\pi_2 & \phi\pi_3\\ \pi_1\phi & \pi_1^2 & \pi_1\pi_2 & \pi_1\pi_3\\ \pi_2\phi & \pi_2\pi_1 & \pi_2^2 & \pi_2\pi_3\\ \pi_3\phi & \pi_3\pi_1 & \pi_3\pi_2 & \pi_3^2\end{pmatrix}, \tag{31}$$

and

$$\mathbf{W}_4=-\frac{\beta^2}{2}\frac{g'}{\rho}\begin{pmatrix}2\phi & i\phi\tau_1\gamma^5+\pi_1 & i\phi\tau_2\gamma^5+\pi_2 & i\phi\tau_3\gamma^5+\pi_3\\ i\phi\tau_1\gamma^5+\pi_1 & 3i\pi_1\tau_1\gamma^5 & \pi_1\tau_3+\pi_3\tau_1 & \pi_1\tau_3+\pi_3\tau_1\\ i\phi\tau_2\gamma^5+\pi_2 & \pi_1\tau_2+\pi_2\tau_1 & 3i\pi_2\tau_2\gamma^5 & \pi_2\tau_3+\pi_3\tau_2\\ i\phi\tau_3\gamma^5+\pi_3 & \pi_1\tau_3+\pi_3\tau_1 & \pi_2\tau_3+\pi_3\tau_2 & 3i\pi_3\tau_3\gamma^5\end{pmatrix}. \tag{32}$$

We may now integrate over the nonuniform Fermionic field amplitudes $\overline{\chi_n}(\vec{q}_i)$ and $\chi_n(\vec{q}_i)$, via the following RG flow relation:

$$e^{-S_E(\Lambda-\Delta\Lambda)}=\prod_{\vec{q}_i,n}{}'^{(\Lambda)}\int d\phi_n^*(\vec{q}_i)d\phi_n(\vec{q}_i)d\vec{\xi}_n^*(\vec{q}_i)\times d\vec{\xi}_n(\vec{q}_i)d\overline{\chi_n}(\vec{q}_i)d\chi_n(\vec{q}_i)e^{-S_E(\Lambda)}. \tag{33}$$

Making use of Eq. (A17) of Appendix A, and noting that here

$$\mathbf{M}=\beta[iZ\gamma^0(\omega_n+\pi/\beta)+Z\vec{\gamma}\cdot\vec{q}_i+g(\rho)(\phi+i\vec{\tau}\cdot\vec{\pi}\gamma^5)], \tag{34}$$

$$\begin{aligned}J=-\beta^{3/2}\bigg[&\phi_n(\vec{q}_i)\left(g'\phi\frac{\Gamma}{\rho}+g\right)\chi_+\\&+\xi_{1n}(\vec{q}_i)\left(g'\pi_1\frac{\Gamma}{\rho}+gi\tau_1\gamma^5\right)\chi_+\\&+\xi_{2n}(\vec{q}_i)\left(g'\pi_2\frac{\Gamma}{\rho}+gi\tau_2\gamma^5\right)\chi_+\\&+\xi_{3n}(\vec{q}_i)\left(g'\pi_3\frac{\Gamma}{\rho}+gi\tau_3\gamma^5\right)\chi_+\bigg],\end{aligned} \tag{35}$$

and

$$\begin{aligned}\overline{J}=-\beta^{3/2}\bigg[&\overline{\chi_+}\left(g'\phi\frac{\Gamma}{\rho}+g\right)\phi_n^*(\vec{q}_i)\\&+\overline{\chi_+}\left(g'\pi_1\frac{\Gamma}{\rho}+gi\tau_1\gamma^5\right)\xi_{1n}^*(\vec{q}_i)\chi_+\\&+\overline{\chi_+}\left(g'\pi_2\frac{\Gamma}{\rho}+gi\tau_2\gamma^5\right)\xi_{2n}^*(\vec{q}_i)\\&+\overline{\chi_+}\left(g'\pi_3\frac{\Gamma}{\rho}+gi\tau_3\gamma^5\right)\xi_{3n}^*(\vec{q}_i)\bigg],\end{aligned} \tag{36}$$

we may write that

$$\det\mathbf{M}=\beta^{N_fC_d}\left\{Z^2\left[\vec{q}_i^{\,2}+\left(\omega_n+\frac{\pi}{\beta}\right)^2\right]+g^2\rho^2\right\}^{N_fC_d/2}, \tag{37}$$

and

$$\begin{aligned}\mathbf{M}^{-1}&=\frac{1}{\beta}\left(\frac{g(\phi-i\vec{\tau}\cdot\vec{\pi}\gamma^5)-iZ\gamma^0(\omega_n+\pi/\beta)-Z\vec{\gamma}\cdot\vec{q}_i}{Z^2[\vec{q}_i^{\,2}+(\omega_n+\pi/\beta)^2]+g^2\rho^2}\right)\\&=\frac{1}{\beta D_F}\{g\Gamma^\dagger-Z[i\gamma^0(\omega_n+\pi/\beta)-\vec{\gamma}\cdot\vec{q}_i]\},\end{aligned} \tag{38}$$

where

$$D_F \equiv Z^2[\vec{q}_i^{\,2} + (\omega_n + \pi/\beta)^2] + g^2\rho^2$$
$$= Z^2[\Lambda^2 + (4n+1)\pi^2/\beta^2] + g^2\rho^2 \tag{39}$$

since we are integrating over Fermionic modes satisfying the shell constraint defined by

$$q_i^2 \equiv \vec{q}_i^{\,2} + \left(\frac{2n\pi}{\beta}\right)^2 = \Lambda^2. \tag{40}$$

We may now write the product $\bar{J}\mathbf{M}^{-1}J$ as $\mathbf{N}+\mathbf{K}$ where $\mathbf{N}$ denotes the matrix formed with $g\Gamma^\dagger/D_F$ and $\mathbf{K}$ identifies the matrix containing the quantity $-iZ\gamma^0(\omega_n+\pi/\beta)/D_F$. We note that the term $\vec{\gamma}\cdot\vec{q}_i/D_F$ may be disregarded here since the spherical symmetry of the $\vec{q}_i$ will cause this term to vanish when we integrate over the spatial wave vectors $\vec{q}_i$ in the shell.

In order to facilitate computation and the increase the ease with which the equations may be written down, we break the matrices $\mathbf{N}$ and $\mathbf{K}$ into sums:

$$\mathbf{N} = \beta^2 \frac{\bar{\chi}_+}{D_F}(\mathbf{N}_1+\mathbf{N}_2+\mathbf{N}_3)\chi_+, \tag{41}$$

$$\mathbf{K} = -\beta^2\bar{\chi}_+(\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_3+\mathbf{K}_4+\mathbf{K}_5)\frac{iZ\gamma^0(\omega_n+\pi/\beta)}{D_F}\chi_+. \tag{42}$$

Here,

$$\mathbf{N}_1 = gg'^2\Gamma\begin{pmatrix} \phi^2 & \phi\pi_1 & \phi\pi_2 & \phi\pi_3 \\ \pi_1\phi & \pi_1^2 & \pi_1\pi_2 & \pi_1\pi_3 \\ \pi_2\phi & \pi_2\pi_1 & \pi_2^2 & \pi_2\pi_3 \\ \pi_3\phi & \pi_3\pi_1 & \pi_3\pi_2 & \pi_3^2 \end{pmatrix}, \tag{43}$$

$$\mathbf{N}_2 = ig^2g'\rho\begin{pmatrix} -2i\phi & \phi\tau_1\gamma^5 - i\pi_1 & \phi\tau_2\gamma^5 - i\pi_2 & \phi\tau_3\gamma^5 - i\pi_3 \\ \phi i\tau_1\gamma^5 - i\pi_1 & 3\pi_1\tau_1\gamma^5 & (\pi_1\tau_2+\pi_2\tau_1)\gamma^5 & (\pi_1\tau_3+\pi_3\tau_1)\gamma^5 \\ \phi\tau_2\gamma^5 - i\pi_2 & (\pi_1\tau_2+\pi_2\tau_1)\gamma^5 & 3\pi_2\tau_2\gamma^5 & (\pi_2\tau_3+\pi_3\tau_2)\gamma^5 \\ \phi\tau_3\gamma^5 - i\pi_3 & (\pi_3\tau_3+\pi_3\tau_1)\gamma^5 & (\pi_2\tau_3+\pi_3\tau_2)\gamma^5 & 3\pi_3\tau_3\gamma^5 \end{pmatrix}, \tag{44}$$

$$\mathbf{N}_3 = g^3\begin{pmatrix} \Gamma^\dagger & \Gamma^\dagger i\tau_1\gamma^5 & \Gamma^\dagger i\tau_2\gamma^5 & \Gamma^\dagger i\tau_3\gamma^5 \\ i\tau_1\gamma^5\Gamma^\dagger & -\tau_1\Gamma^\dagger\tau_1 & -\tau_1\Gamma^\dagger\tau_2 & \tau_1\Gamma^\dagger\tau_3 \\ i\gamma^5\tau_2\Gamma^\dagger & -\tau_2\Gamma^\dagger\tau_1 & \tau_2\Gamma^\dagger\tau_2 & -\tau_2\Gamma^\dagger\tau_3 \\ -i\gamma^5\tau_3\Gamma^\dagger & -\tau_3\Gamma^\dagger\tau_1 & -\tau_3\Gamma^\dagger\tau_2 & -\tau_3\Gamma^\dagger\tau_3 \end{pmatrix}, \tag{45}$$

with the explicit form of the more complex $\mathbf{K}$ matrices given in Appendix B.

Completing the integration over the Fermionic field amplitudes, we arrive at the expression

$$e^{-S(\Lambda-\Delta\Lambda)} = \exp\left\{-\beta V\left[\bar{\chi}_+ g(\rho)\Gamma\chi_+ + \bar{\chi}_+ Z\gamma^0\frac{i\pi}{\beta}\chi_+\right]\right\}\left[Z^2\left(\Lambda^2+(4n+1)\frac{\pi^2}{\beta^2}\right)+g^2\rho^2\right]^{N_fC_d/2}$$
$$\times\prod_{\vec{q}_i,n}{}'^{(\Lambda)}\int d\phi_n^*(\vec{q}_i)d\phi_n(\vec{q}_i)d\vec{\xi}_n^*(\vec{q}_i)d\vec{\xi}_n(\vec{q}_i)$$
$$\times\exp\left\{-\beta V\left[U(\rho)+\sum_{\vec{q}_i,n\neq 0}[\phi_n^*(\vec{q}_i)\xi_{1n}^*(\vec{q}_i)\xi_{2n}^*(\vec{q}_i)\xi_{3n}^*(\vec{q}_i)]\left(\mathbf{U}+\mathbf{W}-\frac{2\beta}{V}(\mathbf{N}+\mathbf{K})\right)\begin{pmatrix}\phi_n(\vec{q}_i)\\ \xi_{1n}(\vec{q}_i)\\ \xi_{2n}(\vec{q}_i)\\ \xi_{3n}(\vec{q}_i)\end{pmatrix}\right]\right\}. \tag{46}$$

We now perform the remaining integrations over the amplitudes of the $\vec{\pi}$ and ϕ fields. By means of the formulas of Appendix A, we find

$$e^{-S(\Lambda-\Delta\Lambda)}=\exp\left\{-\beta V\left(\overline{\chi_+}g(\rho)\Gamma\chi_+ +\overline{\chi_+}Z\gamma^0\frac{i\pi}{\beta}\chi_+\right)\right\}\times\prod_{\vec{q}_i,n}{}'^{(\Lambda)}\left[Z^2\left(\Lambda^2+(4n+1)\frac{\pi^2}{\beta^2}\right)+g^2\rho^2\right]^{N_fC_d/2}\times\{\det[\mathbf{U}+\mathbf{W}-2(\mathbf{N}+\mathbf{K})]\}^{-1/2},\quad(47)$$

which leads to

$$U^{(\Lambda-\Delta\Lambda)}+\overline{\chi_+}\left(g^{(\Lambda-\Delta\Lambda)}\Gamma+Z\gamma^0\frac{i\pi}{\beta}\right)\chi_+ =U^{(\Lambda)}+\overline{\chi_+}\left(g^{(\Lambda)}\Gamma Z\gamma^0\frac{i\pi}{\beta}\right)\chi_+ -\frac{1}{\beta V}\sum_{\vec{q}_i,n\neq 0}{}'^{(\Lambda)}\ln(D_F)^{N_fC_d/2}+\frac{1}{2\beta V}\times\ln\det[\mathbf{U}+\mathbf{W}-2(\mathbf{N}+\mathbf{K})].\quad(48)$$

Now, in order to facilitate the evaluation of the above determinant of the matrices $\mathbf{U}$, $\mathbf{W}$, $\mathbf{N}$ and $\mathbf{K}$, we perform a similarity transformation on them, as outlined in Appendix C. The determinant of the transformed matrices, $\mathbf{U}'$, $\mathbf{W}'$, $\mathbf{N}'$ and $\mathbf{K}'$ may be written as

$$\det[\mathbf{U}'+\mathbf{W}'-2(\mathbf{N}'+\mathbf{K}')] =\det\mathbf{U}'\cdot\det\{\mathbf{1}+\mathbf{U}'^{-1}[\mathbf{W}'-2(\mathbf{N}'+\mathbf{K}')]\},\quad(49)$$

and, by virtue of the Grassmann character of $\mathbf{W}'$, $\mathbf{N}'$ and $\mathbf{K}'$,

$$\det\{\mathbf{1}+\mathbf{U}'^{-1}[\mathbf{W}'-2(\mathbf{N}'+\mathbf{K}')]\} =1+\sum_{m=1}^{4}U'^{-1}_{mm}[\mathbf{W}'-2(\mathbf{N}'+\mathbf{K}')]_{mm}.\quad(50)$$

Consequently, we may evaluate the determinant of $\mathbf{U}'$ and expand the log of the above expression to arrive at

$$\ln\det[\mathbf{U}'+\mathbf{W}'-2(\mathbf{N}'+\mathbf{K}')] =\ln\det\mathbf{U}'+\overline{\chi_+}\left\{\frac{1}{D_\phi}\left[\left(g''+2\frac{g'}{\rho}\right)-2\frac{(gg'^2\rho^2+2g^2g'\rho+g^3)}{D_F}+2(g'^2\rho^2+2gg'\rho+g^2)\frac{iZ\gamma^0(\omega_n+\pi/\beta)}{D_F}\right]+\frac{3}{D_{\vec{\pi}}}\left[\left(\frac{g'}{\rho}+\frac{2g^3}{D_F}\right)\Gamma+2g^2\frac{iZ\gamma^0(\omega_n+\pi/\beta)}{D_F}\right]\right\}\chi_+,\quad(51)$$

where we have retained only terms to lowest order in $\overline{\chi_+}\chi_+$ and have defined

$$D_\phi=q_i^2+U''^{(\Lambda)}=\Lambda^2+U''^{(\Lambda)},\quad(52)$$

$$D_{\vec{\pi}}=q_i^2+U'^{(\Lambda)}/\rho=\Lambda^2+U'^{(\Lambda)}/\rho.\quad(53)$$

Now, equating coefficients of like powers of $\overline{\chi_+}\chi_+$, we obtain the following system of coupled equations:

$$U^{(\Lambda-\Delta\Lambda)}=U^{(\Lambda)}+\frac{1}{2\beta V}\sum_{\vec{q}_i,n}{}'^{(\Lambda)}[\ln(\Lambda^2+U''^{(\Lambda)})+3\ln(\Lambda^2+U'^{(\Lambda)}/\rho)]-\frac{1}{\beta V}\sum_{\vec{q}_i,n}{}'^{(\Lambda)}\times\ln\{Z^2[\Lambda^2+(4n+1)\pi^2/\beta^2]+g^{(\Lambda)2}\rho^2\}^{N_fC_d/2}\quad(54)$$

$$g^{(\Lambda-\Delta\Lambda)}=g^{(\Lambda)}+\frac{1}{2\beta V}\sum_{\vec{q}_i,n}{}'^{(\Lambda)}\left[\frac{1}{D_\phi}\left(g''+2\frac{g'}{\rho}-2\right)\right]\times\frac{(gg'^2\rho^2+2g^2g'\rho+g^3)}{D_F}+\frac{3}{D_{\vec{\pi}}}\left[\left(\frac{g'}{\rho}+\frac{2g^3}{D_F}\right)\right].\quad(55)$$

$$Z^{(\Lambda-\Delta\Lambda)}=Z^{(\Lambda)}\left\{1-\frac{1}{2\beta V}\sum_{\vec{q}_i,n}{}'^{(\Lambda)}\times\left[\frac{1}{D_\phi}(2g'^2\rho^2+2gg'\rho+g^2)+\frac{6g^2}{D_{\vec{\pi}}}\right]\times\frac{(2n+1)}{D_F}\right\}.\quad(56)$$

Here, in Eqs. (54)–(56), we have omitted the superscript Λ for couplings and potentials in the summands, as well as their explicit ρ dependence, for the sake of brevity. This convention is observed in the following expressions.

We disregard the flow of $Z^{(\Lambda)}$ and set it equal to unity. We do this because it simplifies the numerical solution of the flow equations, and also because our explicit calculations including the flow of Z show its influence to be of the order of 1–2%, with little if any effect on the critical temperature at which chiral symmetry breaking occurs. Then, in the continuum limit, the flow equations for the $O(N)$ linear sigma model at finite temperature are:

$$\frac{dU^{(\Lambda)}}{d\Lambda} = -\frac{A_{d-1}}{2\beta}\sum_{n=-N}^{N}\Lambda(\Lambda^2-\omega^2)^{1/2} \times\left\{N_f C_d \ln\left[\Lambda^2+(4n+1)\frac{\pi^2}{\beta^2}+g^{(\Lambda)2}\rho^2\right] - \ln(\Lambda^2+U''^{(\Lambda)}) - (N-1)\ln(\Lambda^2+U'^{(\Lambda)}/\rho)\right\} \tag{57}$$

$$\frac{dg^{(\Lambda)}}{d\Lambda} = -\frac{A_{d-1}}{2\beta}\sum_{n=-N}^{N}\Lambda(\Lambda^2-\omega^2)^{1/2}\left\{\frac{1}{\Lambda^2+U''^{(\Lambda)}} \times\left(g''+2\frac{g'}{\rho}-2\frac{(gg'^2\rho^2+2g^2g'\rho+g^3)}{D_F}\right) + \frac{(N-1)}{\Lambda^2+U^{(\Lambda)\prime}/\rho}\left[\frac{g'}{\rho}+\frac{2g^3}{D_F}\right]\right\}. \tag{58}$$

Now, as $\beta\to 0$, the fermions decouple from the theory since the lowest lying fermion mode will then lie outside the sphere of integration of radius Λ. In that case, only the Bosonic potential U flows, according to

$$\frac{dU^{(\Lambda)}}{d\Lambda} = -\frac{A_{d-1}}{2}\Lambda^{d-2}[\ln(\Lambda^2+U''^{(\Lambda)}) - (N-1)\ln(\Lambda^2+U'^{(\Lambda)}/\rho)], \tag{59}$$

a result consistent with the phenomenon of dimensional reduction [17]. Moreover, as $\beta\to\infty$, we recover the expected zero temperature result of Johnson, Shepard and McNeil [16], which is

$$\frac{dU^{(\Lambda)}}{d\Lambda} = \frac{A_d}{2}\Lambda^{d-1}[N_f C_d \ln(\Lambda^2+g^2\rho^2) - \ln(\Lambda^2+U''^{(\Lambda)}) - (N-1)\ln(\Lambda^2+U^{(\Lambda)\prime}/\rho)] \tag{60}$$

$$\frac{dg^{(\Lambda)}}{d\Lambda} = -\frac{A_d}{2}\Lambda^{d-1}\left[\frac{1}{D_\phi}\left(g''+2\frac{g'}{\rho}\right) - 2\frac{(gg'^2\rho^2+2g^2g'\rho+g^3)}{D_F}\right] + \frac{(N-1)}{D_{\vec{\pi}}}\left[\left(\frac{g'}{\rho}+\frac{2g^3}{D_F}\right)\right]. \tag{61}$$

So, we see that our equations readily reduce to the appropriate expressions in the limit of both high and low temperatures.

IV. EVALUATION OF T_c

Having derived the RG flow equations for the model in accordance with the local potential approximation, we now proceed to calculate the temperature T_c at which chiral symmetry breaking spontaneously occurs. To accomplish this, we numerically integrate Eqs. (57) and (58) using a power series expansion for $U(\phi)$ of the form

$$U^{(\Lambda)}(\phi) = \sum_{j=1}^{M}\frac{\mathcal{U}_{2j}(\Lambda)}{2n}\phi^{2j}, \tag{62}$$

with a similar expression for the Yukawa coupling g. Boundary conditions for U are specified by

$$\mathcal{U}_2(\Lambda_0) = m_0^2, \quad \mathcal{U}_4(\Lambda_0) = \lambda_0,$$

and

$$\mathcal{U}_{2j}(\Lambda_0) = 0 \quad \text{for} \quad j \geqslant 3, \tag{63}$$

while $g^{(\Lambda_0)}$ is simply g_0. Dividing the range of the Λ integration into N equal length intervals, we have

$$\Lambda_n = \Lambda_0 - n\Delta\Lambda, \tag{64}$$

with

$$\Delta\Lambda = \frac{(\Lambda_0-\Lambda_{IR})}{N}. \tag{65}$$

The field variable, ϕ, is discretized in a similar fashion, ranging over the $I+1$ values ϕ_0 to ϕ_I. Integration is then accomplished by fitting the quantities U and g with expansions of the form of Eq. (62), and computing g', g'', U'', etc., from the expansion coefficients. Then, U and g at the new momentum scale Λ_{n+1} are computed for each ϕ_i. This process is iterated until Λ_{IR} is reached and $U^{(\Lambda_{IR})}(\phi_i)$ and $g^{(\Lambda_{IR})}(\phi_i)$ are determined for each of the $I+1$ values of ϕ_i. For the calculations shown here, the number of fitting terms used is 10, with 4000 integration steps along Λ. The independent variable ρ ranges from 0 to 1.5 and is divided into 50 subintervals.

Once $U^{(\Lambda_{IR})}(\phi)$ and $g^{(\Lambda_{IR})}(\phi)$ have been determined numerically, we extract observables as follows: dropping the (Λ_{IR}) superscripts, the expectation value of the scalar field, $\langle\phi\rangle$, satisfies

$$U'(\phi=\langle\phi\rangle) = 0. \tag{66}$$

The effective scalar mass m is determined by

$$m^2 = U''(\phi=\langle\phi\rangle) \tag{67}$$

while the quartic scalar self-coupling is

$$\lambda = U''''(\phi=\langle\phi\rangle). \tag{68}$$

The effective Yukawa coupling is given by

$$g = g(\phi=\langle\phi\rangle). \tag{69}$$

We have yet to specify the value of any of the input parameters of the model and thereby set the scale for our calculations. Clearly, it is necessary to do so to make physically meaningful predictions. Therefore, due to the dearth of theoretical or experimental constraints at the scale of the UV cutoff, we look to observables at the IR scale to constrain the free parameters of the model. It may be argued that one of the parameters most rigidly constrained by low energy data is the value of the pion decay constant, f_π. For this reason, we tune the input bare scalar mass μ_0 so that at $T=0$, the vacuum expectation value of the scalar field is 92.4 MeV, i.e., $\langle\phi\rangle=f_\pi$.

For our UV cutoff, we select a value of 1 GeV since at this energy the dynamics of the strong interactions begin to be dominated by nonlocal mesonlike effective four quark interactions which are not computable by means of perturbative methods [20]. Moreover, below this scale, at approximately 600 MeV, it is thought that quarks are supplemented by additional degrees of freedom in the form of mesonic bound states. We note that this is well above the confinement scale of $\Lambda_{QCD}\simeq 200$ MeV, implying that at this scale there exists a mixture of quarks and mesons. Presumably, the meson dynamics may be described by light quarks with a Yukawa coupling, with leading order gluon effects accounted for by the formation of the mesonic bound states. It is expected that other hadronic bound states provide only subleading contributions to the dynamics since their masses are greater than those of the light scalar mesons. Consequently, we believe that our implementation of the linear sigma model constitutes an effective description of some of the dynamics of QCD at scales below 1 GeV. In addition, due to the decoupling of the quarks, and the entire colored sector of the theory with them, the matter of confinement likely has little influence on the meson dynamics at scales less than 300 MeV. We therefore set our IR scale at 135 MeV since the flow of the couplings effectively stops at scales below the pion mass.

This leaves us with two bare parameters still unfixed: the quartic scalar coupling λ_0 and the Yukawa coupling g_0. Since preliminary work with scalar only models [22] suggests that the values of λ_0 and μ_0 are not entirely independent, we calculate T_c for a variety of values of λ_0, tuning μ_0 for each case to ensure that at $T=0$, $\langle\phi\rangle=f_\pi$. In this way we assess the sensitivity of our value of T_c with respect to λ_0.

Finally, although our zero-density model does not include nucleons, we know that the Goldberger-Treiman relation asserts that $g\simeq M_N/3f_\pi\simeq 3.7$. Thus, since we are aware that the quark-meson coupling is strong at Λ_{IR}, our calculations use values of g_0 such that renormalized values of g lie in the range $2.0<g<4.0$ (see, e.g., Ref. [16] for more discussion). We may therefore assess the variation of T_c with g_0 for a physically relevant range of values of the renormalized coupling g. This is similar to the approach of Berges [20] which fixes g_0 and uses M_N as phenomenological input.

In this way, we have computed the expectation value of the scalar field $\langle\phi\rangle$ and the renormalized values of the scalar mass m, the quartic coupling λ, the Yukawa coupling g for temperatures ranging from 60 to 180 MeV. Figures 1–4 display these four quantities as functions of temperature over this range. These plots correspond to the following values of the input parameters: $\lambda_0=10.0$; $g_0=3.0$; $\mu_0=8.5$. They are representative of the results obtained for a wide range of values of λ_0 $(0\leqslant\lambda_0\leqslant 30)$ and a physically relevant range of values of g_0 $(2\leqslant g_0\leqslant 3.5)$. Renormalized parameters were calculated for a series of values of T.

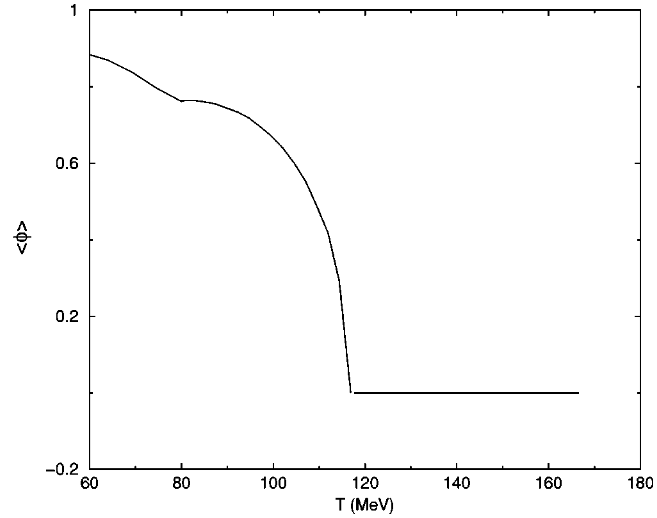

FIG. 1. Vacuum expectation value of ϕ vs T.

We have plotted in Fig. 1 the renormalized expectation value of the scalar field as a function of temperature. It is apparent that this quantity goes to zero continuously as we approach the critical temperature, thereby characterizing the phase transition as second order. Calculation of this quantity using a very fine mesh over the interval containing T_c confirms this assertion. According to Wilczek [23], this is something of a special case, occurring only when two flavors of quarks are modeled. For three or more quark flavors, the chiral phase transition appears to be discontinuous. This is supported by a number of numerical [24] and analytical [25] works addressing this problem. A likely interpretation of this is that for $N_f\geqslant 3$, fluctuations grow so large near T_c that they induce a first order transition [26].

Further examination of Fig. 1 clearly shows that the value of T_c is approximately 118 MeV, which compares favorably with the result of Berges, *et al.* [20], who have obtained a

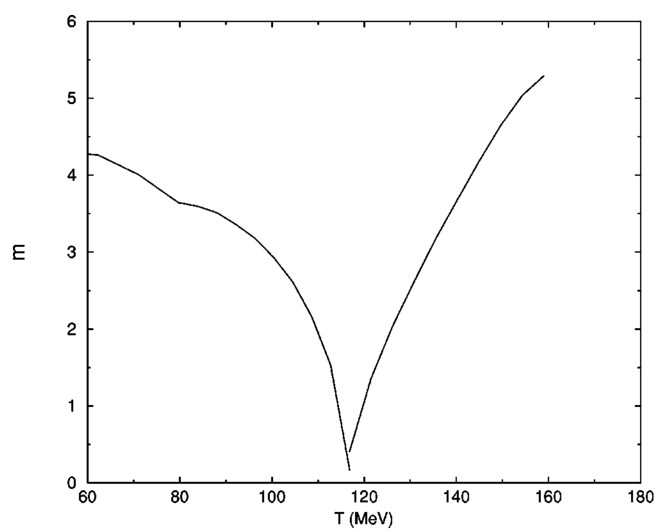

FIG. 2. Renormalized scalar mass vs T.

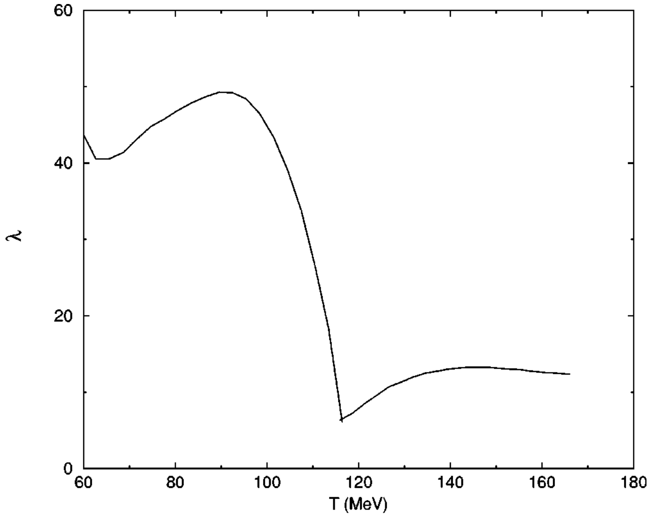

FIG. 3. Renormalized quartic coupling vs T.

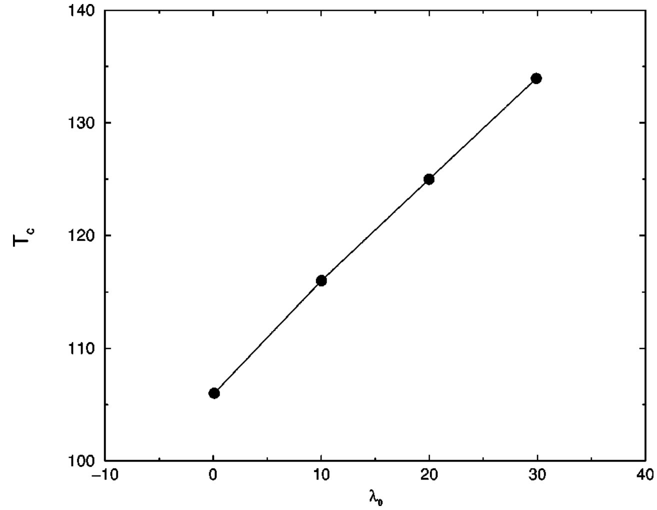

FIG. 5. Variation of T_c with λ_0.

value of 115 MeV for a calculation similar to ours, but involving a smooth cutoff approach. Our result is also reasonably consistent with the results of lattice gauge theory. For example, Karsch [27], using Monte Carlo techniques, finds that for $N_f=2$, $T_c \simeq 150$ MeV.

We also note the appearance of a slight cusp in the figure at about 80 MeV, as well as in Figs. 2–4. This is a consequence of the last Fermionic mode decoupling from our system as the temperature rises, implying that the Fermionic sector of the theory has decoupled before the critical temperature has been reached.

In Fig. 2, we have plotted the renormalized scalar mass m as a function of temperature. We see that the signature of the phase transition is clearly visible at the minimum of 118 MeV. That the value of m does not reach zero in the graph is due to the fact that the values of T for which m was calculated did not include the exact value of T for which $m=0$. However, a calculation of the value of m in the neighborhood of T using a fine mesh shows that m does actually go to zero at $T=T_c$, as we expect. Due to the temperature dependence of $\langle \phi \rangle$, we observe a monotonic increase of m in the symmetric phase. This is expected since m corresponds to a relevant operator of the RG flow.

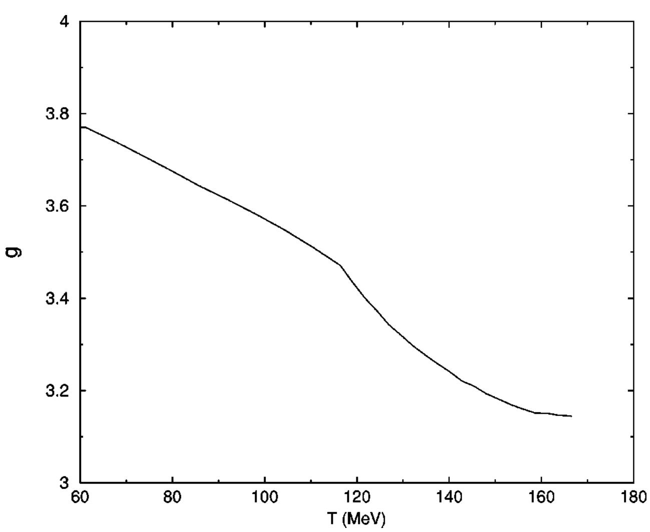

FIG. 4. Renormalized Yukawa coupling vs T.

Figure 3 displays the renormalized quartic coupling λ as a function of temperature. We observe that the value of λ rises rapidly with T up to the temperature at which the fermions decouple from the theory. We expect lambda to be a relevant operator at T not equal to zero since our four-dimensional theory becomes effectively three-dimensional for nonzero T. The signature of the phase transition is readily evident, with the value of λ leveling off in the symmetric phase.

In Fig. 4 we see the temperature dependence of the renormalized Yukawa coupling, g. The value of this coupling is monotonically decreasing as the system heats up, and the signature of the phase transition is also visible as an abrupt change in the slope of g. However, the running Yukawa coupling depends only modestly on temperature for $T \leqslant 120$ MeV, losing less than 10% of its value between $T=60$ MeV and $T=120$ MeV. Moreover, a sizeable Yukawa coupling improves the predictive capability of our method since it implies the rapid approach of the running couplings to infrared fixed points [28]. For this reason, the specific form of $U(\rho)$ at the UV cutoff scale is irrelevant, with the exception of the value of μ_0^2, the one relevant parameter of the model. These results compare quite well with the work of Berges [20], particularly when we correct for the wave-function renormalization by which he rescales this coupling.

Variation of the bare quartic coupling λ_0 of the model for fixed g_0 shows a modest dependence of the value of T_c on λ_0. This dependence is shown in Fig. 5 for $\lambda_0 = 0.0, 10.0, 20.0$ and 30.0, with $g_0=3.0$ for all values of λ_0. Clearly, the value of T_c increases monotonically with λ_0. However, this increase is relatively small given the sizeable range of λ_0 over which it occurs. This is consistent with results obtained by similar methods [20]. Plots for other values of g_0 reveal approximately the same dependence of T_c on λ_0.

In Fig. 6, we see the dependence of T_c on a physically relevant range of values of g_0, with λ_0 fixed at a typical

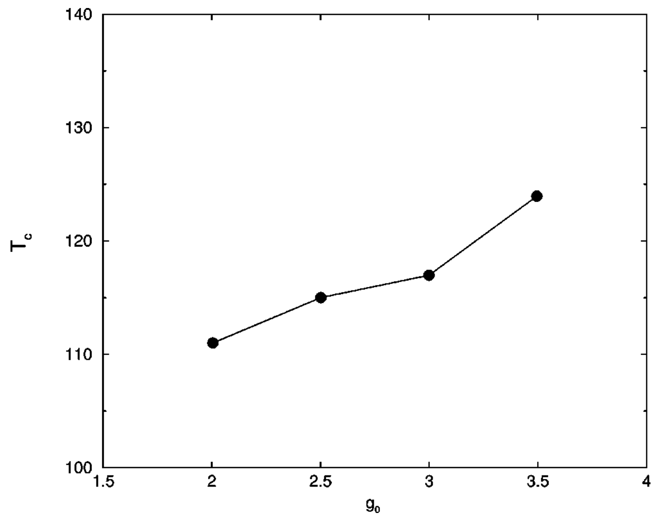

FIG. 6. Variation of T_c with g_0.

value of 10.0. Here, as in Fig. 5, we observe that T_c increases with g_0, though the change is relatively minor. We have found this behavior to be consistent for different values of λ_0. For smaller values of λ_0, the curve is effectively translated down by a few MeV. Conversely, for $\lambda_0 > 10.0$, the curve is translated up by a corresponding amount.

Given the relatively limited sensitivity of our value of T_c to changes in the quartic and Yukawa couplings, we believe that our method provides a reasonably robust determination of this quantity. However, the absence of any specific confinement mechanism in our treatment of the problem may need to be addressed. Specifically, it has been remarked that the effective QCD gauge coupling increases more rapidly at high temperature [21], leading to an increase of Λ_{QCD} with this parameter. Thus, as the renormalization scale Λ gets closer to Λ_{QCD}, the effects of quark sector confinement should be increasingly significant. This observation may be tempered by the recollection that the fermions decouple from our RG flow as T reaches the value Λ/π; i.e., the decoupling obviously occurs earlier in the flow for higher temperatures. In the end, though, we cannot be certain that significant confinement related corrections to the meson physics of our model can be ignored if Λ_{QCD} exceeds the value Λ/π.

V. SUMMARY AND OUTLOOK

The focus of this work has been the application of the local potential RG approximation to the linear sigma model at finite temperature. We have shown that this technique enables us to account for the essential dynamics of the theory while effectively including the effects of thermal fluctuations. In particular, we have observed the existence of a thermally induced phase transition, and have been able to determine the critical temperature T_c at which chiral symmetry breaking spontaneously occurs, which as we have explained, is of some relevance to the real world of QCD. Our efforts yielded values for T_c ranging from approximately 105 to 135 MeV, with a weak dependence upon the bare quartic and Yukawa couplings of the model. These observations are consistent with results obtained by means of similar RG methods [20], as well as lattice gauge techniques [27].

Although our consideration of the linear sigma model in the limit of exact $SU(2)$ symmetry constitutes an effective test to assess the application of our approach to a theory relevant to modern nuclear physics, the work presented in this paper is only a beginning for the application of the present RG technique using a sharp momentum cutoff. It is easy to see that our work could readily be extended in several ways. The most obvious extension is the consideration of dynamically broken chiral symmetry; i.e., the addition of explicit quark mass terms to the Lagrangian of Eq. (1). This would allow us to make calculations involving a nonzero pion mass, and would therefore permit the calculation of $\pi-\pi$ scattering lengths at finite temperature. In fact, this work has been done for the case of zero temperature by Johnson, Shepard and McNeil [16], and is of considerable algebraic complexity, even at $T=0$. It is therefore likely that consideration of this problem at $T\neq 0$ would constitute a major research effort, although it is almost certainly feasible and has been investigated by others using somewhat different methods [20].

Another possibility is the inclusion of finite density in our calculation, permitting an analysis of nuclear phenomena. We speculate that it might be possible to do this by phasing out the flow of Fermionic modes in the neighborhood of the Fermi momentum, k_F, in accordance with the Fermi-Dirac distribution, with Bosonic modes continuing to flow. In this way, we could presumably directly account for finite temperature effects.

It may also be possible to include strangeness in our work by extending it to $SU(3)\times SU(3)$. As remarked by Jungnickel and Wetterich [28], this would require the addition of two more field variants. Due to the significantly larger mass of the strange quark, this symmetry is much more broken in nature though it is still of real phenomenological utility.

Finally, also worth consideration is the application of our technique to NJL models at finite temperature [29], the well known Skyrme Lagrangian [30], or, in the interest of including specific mechanisms for the confinement characteristic of QCD, the MIT bag model [31]. Thus, there are a considerable number of phenomenological models, and individual variations of them, which may be investigated using our technique.

ACKNOWLEDGMENT

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APPENDIX A: GAUSSIAN INTEGRALS

This appendix provides detailed derivations of Gaussian integral formulas used to formally evaluate path integral expressions in the text.

It is a familiar fact that

$$\int_{\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}. \tag{A1}$$

This result may be generalized to a Gaussian integral of n real variables:

$$I(b)=\int_{\infty}^{\infty}dx_1\ldots dx_n\exp\{-x_ib_{ij}x_j\},\qquad\text{(A2)}$$

where the b_{ij} are elements of a matrix $\mathbf{B}$. If $I(b)$ is to be real, then $\mathbf{B}$ must be Hermitian. We assume that this is the case in what follows. We may perform a similarity transformation by means of the unitary matrix $\mathbf{U}$, to the diagonal matrix

$$\mathbf{B}'=\mathbf{U}^{\dagger}\mathbf{B}\mathbf{U}.\qquad\text{(A3)}$$

The Hermiticity of $\mathbf{B}$ implies that the matrix $\mathbf{U}$ may always be constructed from the eigenvectors of $\mathbf{B}$. We now define new variables of interest: $x_i=U_{ij}z_j$. Our integral becomes

$$I(b)=\int_{\infty}^{\infty}dz_1\ldots dz_nJ(z)\exp\left\{-\sum_{i=1}^{n}\lambda_iz_i^2\right\},\qquad\text{(A4)}$$

with λ_i the i^{th} eigenvalue of $\mathbf{B}$ and $J(z)$ the Jacobian associated with the unitary transformation. However, the Jacobian of a unitary transformation is simply unity (up to an irrelevant phase which may be absorbed into the definition of the transformation matrix $\mathbf{U}$). By virtue of this fact, $I(b)$ may now be evaluated as a product of simple Gaussian integrals of the form of Eq. (A1):

$$I(b)=\pi^{n/2}\prod_{i=1}^{n}(\lambda_i)^{1/2}=\left(\frac{\pi^n}{\det\mathbf{B}}\right)^{1/2}.\qquad\text{(A5)}$$

We note, however, that each of the simple Gaussian integrals contributing to the above expression is well defined only if the associated eigenvalue is nonzero and positive.

The preceding example may be extended to include in the exponential a term linear in x_i. To do this, we translate x_i by a constant $\alpha_i/2$, so that α defines an n-dimensional vector. Since the limits of integration are infinite, this leaves the value of the integral unchanged, and we have

$$I(b)=\int_{\infty}^{\infty}dx_1\ldots dx_n\exp\left(-x_ib_{ij}x_j\pm\alpha_ia_{ij}x_j-\frac{1}{4}\alpha_ib_{ij}\alpha_j\right).\qquad\text{(A6)}$$

Assuming that the matrix $\mathbf{B}$ has an inverse $\mathbf{B}^{-1}$ and defining $c_i=\alpha_jb_{ji}$, it is apparent that $\alpha_ib_{ij}\alpha_j=c_i(\mathbf{B}^{-1})_{ij}c_j$. Inserting this result into Eq. (A6) yields

$$I(b)=\int_{\infty}^{\infty}dx_1\ldots dx_n\exp(-\mathbf{x}^T\mathbf{B}\mathbf{x}\pm\mathbf{c}^T\mathbf{x})$$
$$=\left(\frac{\pi^n}{\det\mathbf{B}}\right)^{1/2}\exp\left(\frac{1}{4}\mathbf{c}^T\mathbf{B}^{-1}\mathbf{c}\right).\qquad\text{(A7)}$$

If we wish to consider a Gaussian integral over complex variables of the form $z_n=x_n+y_n$, the preceding results may be readily extended to give

$$\int_{\infty}^{\infty}dz_1dz_1^*\ldots dz_ndz_n^*\exp(ix_j^*b_{jk}z_k+id_j^*z_j+id_jz_j^*)$$
$$=\frac{(2\pi)^n}{\det\mathbf{B}}\exp(i\mathbf{d}^{\dagger}\mathbf{B}^{-1}\mathbf{d}),\qquad\text{(A8)}$$

where $\mathbf{B}$ is again assumed to be an invertible Hermitian matrix, and d_i is a complex constant.

Finally, we must consider Gaussian integrals of Grassmann variables, i.e., variables obeying the relation

$$\{\eta_a,\eta_b\}=0.\qquad\text{(A9)}$$

This, together with the definition of Grassman integration

$$\int d\eta\eta=\int d\eta^*\eta^*=1,\qquad\text{(A10)}$$

$$\int d\eta=\int d\eta^*=0,\qquad\text{(A11)}$$

leads to

$$\int d\eta^*d\eta e^{-\eta^*\eta}=1\qquad\text{(A12)}$$

for a complex Grassmann variable η. This is readily obtained by means of a power series expansion of the exponential, together with the above definition of Grassmann integration.

For the case of N complex Grassmann variables, we have

$$I=\int d\eta_1^*d\eta_1\ldots d\eta_N^*d\eta_N\exp(-\eta_i^*M_{ij}\eta_j),\qquad\text{(A13)}$$

where M_{ij} are the elements of an $n\times nc$-number matrix $\mathbf{M}$. We assume that $\mathbf{M}$ is Hermitian so that I is a real c number. The Hermiticity of $\mathbf{M}$ implies that it may be diagonalized by means of some unitary matrix $\mathbf{V}$ so that the matrix $\mathbf{P}=\mathbf{V}\mathbf{M}\mathbf{V}^{\dagger}$ is diagonal with elements

$$P_{ij}=\lambda^{(i)}\delta_{ij}.\qquad\text{(A14)}$$

The Grassmann variables are then transformed to $\eta_i=V_{ij}\xi_k$ and, as above, by virtue of the unitarity of $\mathbf{V}$, the Jacobian of this transformation is unity. The integral may then be written as

$$I=\prod_{r=1}^{N}\int d\xi_r^*d\xi_r\exp(-\lambda^{(r)}\xi_r^*\xi_r).\qquad\text{(A15)}$$

By means of Eq. (A12) this becomes

$$I=\prod_{r=1}^{N}\lambda^{(r)}=\det\mathbf{M}.\qquad\text{(A16)}$$

This result may be readily extended to show that

$$\int d\eta_1^* d\eta_1 \ldots d\eta_N^* d\eta_N \exp(-\eta^\dagger \mathbf{M}\eta + J^\dagger \eta + \eta^\dagger J) = \det \mathbf{M} \exp(J^\dagger \mathbf{M}^{-1} J), \tag{A17}$$

for a Hermitian matrix **M** and complex Grassmann constants $\{J_i\}$.

APPENDIX B: THE K MATRICES

This appendix shows the specific form of the **K** matrices which are part of the derivation of the RG flow equations.

From Eq. (42) we have the matrix **K** defined as

$$\mathbf{K} = -\beta^2 \overline{\chi_+}(\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 + \mathbf{K}_4 + \mathbf{K}_5) \frac{i\gamma^0(\omega_n + \pi/\beta)}{D_F} \chi_+ , \tag{B1}$$

where the matrices $\mathbf{K}_i$ are given by the following expressions:

$$\mathbf{K}_1 = \begin{pmatrix} g'^2\phi^2 & g'^2\phi\pi_1 & g'^2\phi\pi_2 & g'^2\phi\pi_3 \\ g'^2\pi_1\phi & g'^2\pi_1^2 & g'^2\pi_1\pi_2 & g'^2\pi_1\pi_3 \\ g'^2\pi_2\phi & g'^2\pi_2\pi_1 & g'^2\pi_2^2 & g'^2\pi_2\pi_3 \\ g'^2\pi_3\phi & g'^2\pi_3\pi_1 & g'^2\pi_3\pi_2 & g'^2\pi_3^2 \end{pmatrix}, \tag{B2}$$

$$\mathbf{K}_2 = \begin{pmatrix} \frac{2gg'}{\rho}\phi^2 + g^2 & \frac{2gg'}{\rho}\phi\pi_1 & \frac{2gg'}{\rho}\phi\pi_2 & \frac{2gg'}{\rho}\phi\pi_3 \\ \frac{2gg'}{\rho}\pi_1\phi & \frac{2gg'}{\rho}\pi_1^2 + g^2 & \frac{2gg'}{\rho}\pi_1\pi_2 & \frac{2gg'}{\rho}\pi_1\pi_3 \\ \frac{2gg'}{\rho}\pi_2\phi & \frac{2gg'}{\rho}\pi_2\pi_1 & \frac{2gg'}{\rho}\pi_2^2 + g^2 & \frac{2gg'}{\rho}\pi_2\pi_3 \\ \frac{2gg'}{\rho}\pi_3\phi & \frac{2gg'}{\rho}\pi_3\pi_1 & \frac{2gg'}{\rho}\pi_3\pi_2 & \frac{2gg'}{\rho}\pi_3^2 + g^2 \end{pmatrix}, \tag{B3}$$

$$\mathbf{K}_3 = \frac{gg'}{\rho} i\tau_1\gamma^5 \begin{pmatrix} 0 & -(\phi^2+\pi_1^2)-a^2 & -\pi_1\pi_2 & \pi_1\pi_3 \\ (\phi^2+\pi_1^2)+a^2 & 0 & \phi\pi_2 & \pi_3\phi \\ \pi_1\pi_2 & -\phi\pi_2 & 0 & -(\pi^2+\pi_3^2)-a^2 \\ -\pi_1\pi_3 & -\pi_3\phi & (\pi^2+\pi_3^2)-a^2 & 0 \end{pmatrix}, \tag{B4}$$

$$\mathbf{K}_4 = \frac{gg'}{\rho} i\tau_2\gamma^5 \begin{pmatrix} 0 & -\pi_1\pi_2 & -(\phi^2+\pi_2^2)-a^2 & \pi_2\pi_3 \\ \pi_1\pi_2 & 0 & -\phi\pi_1 & -(\pi_1^2+\pi_3^2)-a^2 \\ (\phi^2+\pi_2^2)+a^2 & \phi\pi_1 & 0 & \pi_3\phi \\ -\pi_2\pi_3 & (\pi_1^2+\pi_3^2)-a^2 & -\pi_3\phi & 0 \end{pmatrix}, \tag{B5}$$

$$\mathbf{K}_5 = \frac{gg'}{\rho} i\tau_3\gamma^5 \begin{pmatrix} 0 & -\phi\pi_2 & -\phi\pi_1 & (\phi^2+\pi_3^2)+a^2 \\ \phi\pi_2 & 0 & (\pi_1^2+\pi_2^2)+a^2 & \phi\pi_1 \\ \phi\pi_1 & -(\pi_1^2+\pi_2^2)-a^2 & 0 & -\pi_2\phi \\ -(\phi^2+\pi_3^2)+a^2 & -\phi\pi_1 & \pi_2\phi & 0 \end{pmatrix}, \tag{B6}$$

with $a^2 \equiv g^2\rho/gg'$.

APPENDIX C: SIMILARITY TRANSFORMATION

This appendix contains details of the similarity transformation used to facilitate the derivation of the RG flow equations.

From Eq. (49), we must evaluate

$$\det \mathbf{U}\cdot \det\{\mathbf{1}+\mathbf{U}^{-1}[\mathbf{W}-2(\mathbf{N}+\mathbf{K})]\}, \tag{C1}$$

which, owing to the Grassman nature of the matrices **W**, **N** and **K**, only requires explicit evaluation of the diagonal terms of each of these three matrices. However, calculation of $\det \mathbf{U}$ is more complex, and for this reason we employ a similarity transformation S, given by

$$\mathbf{S}\equiv \frac{1}{\rho}\begin{pmatrix} \phi & \pi_1 & \pi_2 & \pi_3 \\ -\pi_1 & \phi & 0 & 0 \\ -\pi_2 & 0 & \phi & 0 \\ -\pi_3 & 0 & 0 & \phi \end{pmatrix}, \tag{C2}$$

which diagonalizes **U** and leaves the value of $\det \mathbf{U}$ unchanged. The transformed matrices, denoted by primes, are then given by, e.g.,

$$\mathbf{U}' = \mathbf{SUS}^{-1} = \begin{pmatrix} q_i^2+U'' & 0 & 0 & 0 \\ 0 & q_i^2+U'/\rho & 0 & 0 \\ 0 & 0 & q_i^2+U'/\rho & 0 \\ 0 & 0 & 0 & q_i^2+U'/\rho \end{pmatrix}, \tag{C3}$$

with the other matrices **W**, **N** and **K**, transforming in an identical fashion.

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