Self-dual cosmic strings and gravitating vortices in gauged sigma models

Y. Verbin*

Department of Natural Sciences, The Open University of Israel, P.O.B. 39328, Tel Aviv 61392, Israel

S. Madsen[†] and A. L. Larsen[‡]

Department of Physics, University of Odense, Campusvej 55, 5230 Odense M, Denmark

Cosmic strings are considered in two types of gauged sigma models, which generalize the gravitating Abelian Higgs model. The two models differ by whether the U(1) kinetic term is of the Maxwell or Chern-Simons form. We obtain the self-duality conditions for a general two-dimensional target space defined in terms of field dependent "dielectric functions." In particular, we analyze analytically and numerically the equations for the case of O(3) models (two-sphere as target space), and find cosmic string solutions of several kinds as well as gravitating vortices. We classify the solutions by their flux and topological charge. We note an interesting connection between the Maxwell and Chern-Simons type models, which is responsible for simple relations between the self-dual solutions of both types. There is however a significant difference between the two systems, in that only the Chern-Simons type sigma model gives rise to spinning cosmic vortices.

I. INTRODUCTION

Topological defects are generally believed to have been formed during a series of phase transitions in the early universe. In particular, many field-theoretical models suggest the formation of cosmic strings (for a review, see for instance Vilenkin and Shellard [1]).

The BOOMERANG results [2], especially concerning the location of the acoustic peaks [3,4], essentially rule out grand unified theory (GUT) scale ($\sim 10^{16}$ GeV) topological defects being the *sole* source of the cosmic microwave background (CMB) anisotropy, which reflects the early universe density fluctuations, eventually leading to galaxy formation. However, recent analysis [5–9] shows that a mixture of cosmic strings and inflation is consistent with current CMB data.

Independently of a possible role in the large scale structure formation, and even if their mass scale is lower than the GUT scale, cosmic strings may have also a number of significant observable astrophysical effects. Just to mention a few: Cosmic strings could be sources of double images [1], of gravitational waves [10,11] and of ultrahigh energy cosmic rays [12]. Recently it was even suggested [13] that they can serve as gamma ray burst engines. Thus, cosmic strings are still of considerable interest in cosmology and astrophysics.

The most popular field-theoretical system, used to model cosmic string generation [1], is the Abelian Higgs model. There exist, however, other systems which may be used for the same purpose and are not more complicated. As long as we model cosmic strings as static cylindrically symmetric sources coupled to gravity, which is essentially a two dimensional system, we may borrow circular solutions of other well-known two dimensional systems, namely nonlinear sigma models.

Nonlinear sigma models [14] are very much used as effective theories describing various systems such as low energy effective QCD, and are also used as "toy models" for gauge theories. A generalization of these, where the global symmetry group or its subgroup is promoted to be local, are gauged sigma models which appear naturally in supersymmetric field theories.

All these models have been extensively studied for decades. Most of the results were obtained in flat (Minkowski) spacetime, but more recently gravitating solutions were studied too. However, the gravitating sigma model solutions are usually taken to be spherically symmetric as in the cases of gravitating Skyrmions and textures (see e.g. Volkov and Gal'tsov [15]), although some authors gave attention to cylindrically symmetric solutions of the non-gauged [16,17] as well as the gauged models with either Maxwell [18] or Chern-Simons terms [19–22].

Here we will analyze the gravitating gauged sigma models with a general two-dimensional target space defined in terms of two field dependent "dielectric functions," which may be viewed as generalized Abelian Higgs models: Maxwell type and Chern-Simons type, according to the gauge field term in the Lagrangian. We will find the general conditions for self-dual cosmic string solutions and get several kinds of interesting solutions.

We will start by considering, in the next section, static and translationally invariant solutions of a generic gauged gravitating non-linear sigma model of a Maxwell type, and find simple conditions for the existence of self-dual solutions. In Secs. III and IV, we concentrate in the O(3) model and add rotational symmetry to obtain self-dual cosmic string solutions. In Sec. V, we move to the Chern-Simons type of model, and consider stationary (not static) solutions. First we obtain the self-duality conditions for this kind of solutions, and then we specialize to spinning cosmic vortices. In Sec. VI, we discuss the relation between the two types of theories

*Electronic address: verbin@oumail.openu.ac.il

[†]Electronic address: Soeren@Madsen2TheMax.dk

[‡]Electronic address: all@fysik.sdu.dk

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and solutions, and we end with a discussion of the topological charges in Sec. VII.

II. THE GENERALIZED ABELIAN HIGGS MODEL

We start by considering a generalization of the Abelian Higgs model defined by the action

$$S = \int d^4x \sqrt{|g|} \left(\frac{1}{2}\mathcal{E}_1(|\Phi|)(D_\mu\Phi)^*(D^\mu\Phi) - U(|\Phi|) \right.$$
$$\left. - \frac{1}{4}\mathcal{E}_2(|\Phi|)F_{\mu\nu}F^{\mu\nu} + \frac{1}{16\pi G}R \right) \quad (2.1)$$

where $\mathcal{E}_1(|\Phi|)$ and $\mathcal{E}_2(|\Phi|)$ are non-negative dimensionless functions, which may be interpreted as Weyl factors of a conformally-flat target space metric of a nonlinear sigma model with local U(1) symmetry [23]. The function $\mathcal{E}_2(|\Phi|)$ plays further the role of a dielectric function [24,25]. In what follows, we use the collective notation $\mathcal{E}_a$, ($a=1,2$), and refer to both as "dielectric functions." The action (2.1) also generalizes the ones considered by Lohe [26,27]. Furthermore, the action (2.1) arises in various extended supergravity theories [23]. In the general case, we may also add the term $\mathcal{E}_3(|\Phi|)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, but this term vanishes identically for the configurations considered in the present paper and has no effect on the field equations.

The field equations derived from the action (2.1) are

$$\mathcal{E}_1(|\Phi|)D_\mu D^\mu\Phi + \frac{\Phi^*}{2|\Phi|}\frac{d\mathcal{E}_1}{d|\Phi|}D_\mu\Phi D^\mu\Phi + \frac{\Phi}{|\Phi|}\frac{dU}{d|\Phi|}$$
$$+ \frac{\Phi}{4|\Phi|}\frac{d\mathcal{E}_2}{d|\Phi|}F_{\mu\nu}F^{\mu\nu} = 0 \quad (2.2)$$

$$\nabla_\mu(\mathcal{E}_2(|\Phi|)F^{\mu\nu}) = j^\nu$$
$$= -\frac{i}{2}e\mathcal{E}_1(|\Phi|)(\Phi^*(D^\nu\Phi) - \Phi(D^\nu\Phi)^*) \quad (2.3)$$

$$\frac{1}{8\pi G}R_{\mu\nu} + \frac{1}{2}\mathcal{E}_1(|\Phi|)((D_\mu\Phi)^*(D_\nu\Phi) + (D_\nu\Phi)^*(D_\mu\Phi))$$
$$- U(|\Phi|)g_{\mu\nu} + \mathcal{E}_2(|\Phi|)\left(g^{\kappa\lambda}F_{\mu\kappa}F_{\lambda\nu} + \frac{1}{4}F_{\kappa\lambda}F^{\kappa\lambda}g_{\mu\nu} \right)$$
$$= 0. \quad (2.4)$$

Conventions: $D_\mu = \nabla_\mu - ieA_\mu$, signature $(+,-,-,-)$ and $R^\kappa_{\lambda\mu\nu} = \partial_\nu\Gamma^\kappa_{\lambda\mu} - \partial_\mu\Gamma^\kappa_{\lambda\nu} + \cdots$.

Usually (but not always) we will be interested in potentials, which ensure spontaneous symmetry breaking and which lead to a massive gauge field. That is to say, potentials with a circle of degenerate minima at (say) $|\Phi|=v$ and with positive second derivative at v. We further normalize the potential such that it vanishes in the vacuum. Thus, altogether

$$U = 0, \quad U' = \frac{dU}{d|\Phi|} = 0, \quad \text{for } |\Phi| = v. \quad (2.5)$$

With these normalizations, the Higgs boson mass and gauge boson mass are, respectively,

$$m_H^2 = \frac{U''(v)}{\mathcal{E}_1(v)}, \quad m_A^2 = \frac{e^2v^2\mathcal{E}_1(v)}{\mathcal{E}_2(v)}. \quad (2.6)$$

In order to study cosmic string solutions, we assume the metric and matter fields to be static and symmetric under z translations. Thus we assume that A_μ and Φ depend only on the two transverse coordinates x^k, and for the metric we use the following form:

$$ds^2 = N^2(x^k)dt^2 - \gamma_{ij}(x^k)dx^i dx^j - K^2(x^k)dz^2. \quad (2.7)$$

We also require the presence of a magnetic field only, i.e. that the gauge potential will have the form

$$A_\mu dx^\mu = A_i(x^k)dx^i \quad (2.8)$$

such that the Maxwell tensor will contain a single magnetic component B:

$$F_{\mu\nu}dx^\mu \wedge dx^\nu = -B\sqrt{|\gamma|}\epsilon_{ij}dx^i \wedge dx^j \quad (2.9)$$

where $|\gamma| = |\det(\gamma_{ij})|$. In order to get the field equations for static solutions, we compute also the components of the Ricci tensor:

$$R_{00} = -\frac{N}{K}\nabla_i(K\nabla^i N)$$

$$R_{33} = \frac{K}{N}\nabla_i(N\nabla^i K)$$

$$R_{ij} = R_{ij}(\gamma) + \frac{1}{N}\nabla_i\nabla_j N + \frac{1}{K}\nabla_i\nabla_j K$$

$$R_{i0} = R_{i3} = 0, \quad R_{03} = 0 \quad (2.10)$$

where ∇_i is the covariant derivative with respect to the two-dimensional metric γ_{ij} and $R_{ij}(\gamma)$ the corresponding Ricci tensor.

A significant simplification of this system is obtained if self-duality conditions are satisfied, i.e. if the system admits a Bogomol'nyi limit [28]. It is well known [16] that in the usual Higgs model, the flat space considerations can be carried over to curved background if $N(x^i)$ and $K(x^i)$ are constants, say, 1. We will see now that the present generalized Higgs model has also a Bogomol'nyi limit if we keep $N(x^i) = K(x^i) = 1$. If we use these conditions, we find that the (00) and (33) components of Einstein equations will be satisfied only if

$$U(|\Phi|) = \frac{1}{2}\mathcal{E}_2(|\Phi|)B^2. \quad (2.11)$$

Now we turn to the (ij) components of the Einstein equations, or even better to G_{ij} which vanish identically. Consequently, $T_{ij}=0$ as well and we get

$$\frac{1}{2}\mathcal{E}_1(|\Phi|)((D_i\Phi)^*(D_j\Phi)+(D_j\Phi)^*(D_i\Phi))$$

$$-\left(\frac{1}{2}\mathcal{E}_1(|\Phi|)\gamma^{kl}(D_k\Phi)^*(D_l\Phi)+U(|\Phi|)\right.$$

$$\left.-\frac{1}{2}\mathcal{E}_2(|\Phi|)B^2\right)\gamma_{ij}=0. \tag{2.12}$$

Using Eq. (2.11), this condition simplifies further and it follows that it is equivalent to the curved spacetime version of the self-duality condition:

$$D_i\Phi=i\,\eta\sqrt{|\gamma|}\epsilon_{ij}\gamma^{jk}D_k\Phi \tag{2.13}$$

where $\eta=\pm1$ corresponds to self-dual or anti-self-dual solutions. This is a first order equation for the Higgs field. Analogously, Eq. (2.11) is a first order equation for the gauge potential.

We also find the following expression for the two dimensional Ricci scalar:

$$R(\gamma)=-8\pi G(\mathcal{E}_1(|\Phi|)\gamma^{ij}(D_i\Phi)^*(D_j\Phi)+4U(|\Phi|)) \tag{2.14}$$

which serves as the Einstein equation for the two metric γ_{ij}.

Equation (2.2) for the Higgs field becomes a consistency condition, which constrains the form of the potential $U(|\Phi|)$:

$$\eta e|\Phi|\mathcal{E}_1(|\Phi|)B+\frac{dU}{d|\Phi|}+\frac{B^2}{2}\frac{d\mathcal{E}_2}{d|\Phi|}=0. \tag{2.15}$$

Maxwell equations (2.3) give

$$\partial_j(\mathcal{E}_2(|\Phi|)B)=-\frac{\eta e}{2}\mathcal{E}_1(|\Phi|)\partial_j|\Phi|^2 \tag{2.16}$$

which is a second order equation for the gauge potential, but it is not an independent one, as is easily shown: If we think of B as a function of $|\Phi|$, we may use Eq. (2.11) and get from Eq. (2.15) the following relation, which we will refer to as the "Bogomol'nyi constraint:"

$$\frac{d}{d|\Phi|}(\mathcal{E}_2(|\Phi|)B)+\eta e|\Phi|\mathcal{E}_1(|\Phi|)=0. \tag{2.17}$$

This is actually again the Maxwell equation (2.16) in disguise, and can be used together with Eq. (2.11) to get the function $U(|\Phi|)$ for any set of given "dielectric functions" $\mathcal{E}_a$, $a=1,2$. In order to do it directly, we need to express the magnetic field B in terms of the potential U. If B has a definite sign, we infer from the Bogomol'nyi constraint that $\mathcal{E}_2(|\Phi|)B(|\Phi|)\geq0$ for $\eta=+1$, and similarly for negative values, so we may take a square root of Eq. (2.11):

$$B=\eta\sqrt{2U/\mathcal{E}_2} \tag{2.18}$$

and get a simple relation between the potential and the two dielectric functions, which is a necessary condition for the action (2.1) to have self-dual solutions, i.e. a Bogomol'nyi limit:

$$\frac{d}{d|\Phi|}\sqrt{2\mathcal{E}_2(|\Phi|)U(|\Phi|)}+e|\Phi|\mathcal{E}_1(|\Phi|)=0. \tag{2.19}$$

The simplest case where we can apply Eq. (2.19) is of course $\mathcal{E}_a=1$, which reproduces immediately the standard Higgs potential,

$$U(|\Phi|)=\frac{a}{4}(v^2-|\Phi|^2)^2 \tag{2.20}$$

where the Bogomol'nyi relation between the coupling constants holds

$$\alpha\equiv e^2/a=2. \tag{2.21}$$

Note that here, the vacuum expectation value v enters into the potential as an integration constant, and has also a role of the value of $|\Phi|$ for which the magnetic field vanishes.

The next case is the nonlinear O(3) sigma model, which is obtained by taking $\mathcal{E}_1$ to be the usual conformal factor for S^2 with a radius $\mu/2$ [14]:

$$\mathcal{E}_1(|\Phi|)=1/(1+|\Phi|^2/\mu^2)^2. \tag{2.22}$$

The parameter μ sets a second energy scale in the system and renders $\mathcal{E}_1$ dimensionless. It is very simple to integrate the Bogomol'nyi constraint (2.17), or equivalently Eq. (2.19), also in this case and get the following form of $B(|\Phi|)$ and potential, which allows a Bogomol'nyi limit in the O(3) generalized Abelian Higgs model:

$$B(|\Phi|)=\frac{\eta e}{2(1+v^2/\mu^2)}\frac{1}{\mathcal{E}_2(|\Phi|)}\frac{v^2-|\Phi|^2}{1+|\Phi|^2/\mu^2} \tag{2.23}$$

$$U(|\Phi|)=\frac{e^2}{8(1+v^2/\mu^2)^2}\frac{1}{\mathcal{E}_2(|\Phi|)}\frac{(v^2-|\Phi|^2)^2}{(1+|\Phi|^2/\mu^2)^2}. \tag{2.24}$$

Here again v is an integration constant, which parametrizes the minimum of the potential and the field value for which the magnetic field vanishes. Generally there is no relation between this scale and the scale μ. We therefore have a one-parameter family of potentials in this O(3) generalized Abelian Higgs model for any given $\mathcal{E}_2(|\Phi|)$, which we may take at our will. Some special cases with $\mathcal{E}_2(|\Phi|)=1$ were already analyzed by several authors mainly in flat space [29,30], but also with coupling to gravity [18].

A more "symmetric" picture may be obtained by using the angular variable on S^2 defined by

$$|\Phi|=\mu\tan(\Theta/2). \tag{2.25}$$

Generically, the potential exhibits a U(1)-symmetry breaking minimum at $|\Phi|=v$, but there are two special limits $v=0,\infty$, where the ground state is only a point in target space

(the north or south poles of S^2); namely, no symmetry breaking occurs. This pattern of symmetry breaking is reflected by the Higgs and gauge boson masses, which turn out to have the following equal (due to the self-duality) values:

$$m_H = m_A = \frac{ev}{1 + v^2/\mu^2}. \qquad (2.26)$$

The two cases of symmetric vacuum correspond to vanishing Higgs and gauge masses.

Just as a final check we note that the Higgs potential [Eq. (2.20) with (2.21)] is obtained in the limit $\mu \rightarrow \infty$ from Eq. (2.24) with $\mathcal{E}_2(|\Phi|) = 1$. The masses which are obtained in this limit are the usual ones.

Now we return to the Einstein equation (2.14) and notice that for self-dual solutions, its right-hand side is actually a two-dimensional divergence (it must be as we will see in Sec. VII), and may be simplified to

$$R(\gamma) = \frac{16\pi G}{\eta e} \nabla_i(\gamma^{ij}\mathcal{E}_2(|\Phi|)B\partial_j\log|\Phi|). \qquad (2.27)$$

Without loss of generality, we may take the two-dimensional transverse part of the metric tensor to be conformally flat, i.e. $\gamma_{ij} = H^2(x^k)\delta_{ij}$ so the Einstein equation (2.14) reduces further to

$$\delta^{ij}\partial_i\left(\partial_j\log H - \frac{8\pi G}{\eta e}\mathcal{E}_2(|\Phi|)B\partial_j\log|\Phi|\right) = 0. \qquad (2.28)$$

This equation is not of first order, but it may get the form of a two dimensional Laplace equation, if we introduce a "super dielectric function" $\mathcal{A}(|\Phi|)$, which solves the following equation:

$$|\Phi|\frac{d\mathcal{A}}{d|\Phi|} = \eta\mathcal{E}_2(|\Phi|)B(|\Phi|) = \sqrt{2\mathcal{E}_2(|\Phi|)U(|\Phi|)}$$
$$(2.29)$$

or equivalently the second order equation:

$$\frac{1}{|\Phi|}\frac{d}{d|\Phi|}\left(|\Phi|\frac{d\mathcal{A}}{d|\Phi|}\right) + e\mathcal{E}_1(|\Phi|) = 0. \qquad (2.30)$$

Now we can use the function $\mathcal{A}$, in order to give Eq. (2.28) the form of a Laplace equation:

$$\delta^{ij}\partial_i\partial_j\left(\log H - \frac{8\pi G}{e}\mathcal{A}\right) = 0. \qquad (2.31)$$

Special solutions will be discussed in Sec. IV.

III. FLAT SPACE SOLUTIONS

First we discuss flat space solutions within this general framework. As mentioned above, some of the solutions are already known but new ones can be easily obtained.

In order to study a single cosmic string solution, we take the usual cylindrically symmetric Nielsen-Olesen ansatz for n flux units:

$$\Phi = \mu f(r)e^{in\varphi}, \quad A_\mu dx^\mu = A(r)d\varphi \qquad (3.1)$$

where μ is a second energy scale, which is generally independent of v. We will also assume that the dielectric functions depend on $|\Phi|$ through the dimensionless ratio $|\Phi|/\mu = f$ only. For further use we define here $v/\mu \equiv \beta$.

As for the boundary conditions, we will see that they will not be the same for all systems, but will rather have to be adapted to the specific system. However, since we are interested in solutions with finite energy per unit length and finite flux, the boundary conditions at infinity should ensure asymptotically vanishing energy density. The usual Nielsen-Olesen conditions should be generalized such that all three contributions (scalar gauged kinetic term, potential and Maxwell term) will vanish asymptotically. That is

$$\lim_{r\rightarrow\infty} U(f(r)) = 0, \quad \lim_{r\rightarrow\infty}\mathcal{E}_2(f(r))B^2(r) = 0,$$

$$\lim_{r\rightarrow\infty} f^2(r)\mathcal{E}_1(f(r))(eA(r)-n)^2 = 0. \qquad (3.2)$$

In order to proceed, we will concentrate in the O(3) model. The field equations (with $\eta = +1$ which we will mostly use from now on) are the following simple first order set:

$$\frac{rf'}{f} = eA(r) - n \qquad (3.3)$$

$$\frac{A'}{r} = -\frac{e\mu^2}{2(1+\beta^2)}\frac{\beta^2 - f^2}{\mathcal{E}_2(f)(1+f^2)} \qquad (3.4)$$

where a prime denotes differentiation with respect to r.

We still have a freedom in the function $\mathcal{E}_2$, but we choose until further notice $\mathcal{E}_2(f) = 1$. The simplest case, which yields the closest to the Abelian Higgs flux tube, is the case $\beta = 1$ which corresponds to a potential with a minimum along the equator of S^2. We thus impose the following additional boundary conditions:

$$f(0) = 0, \quad A(0) = 0 \qquad (3.5)$$

and the two last conditions in Eq. (3.2) may be replaced by

$$\lim_{r\rightarrow\infty} A(r) = n/e. \qquad (3.6)$$

Note that the boundary conditions we took enforce n to be negative and $A(r)$ to be a non-positive decreasing function. The magnetic field $B(r)$ is non-negative and the flux (in units of $2\pi/e$) is $-eA(\infty) = -n$, which is positive.

There is no known analytical solution of this system, but it is very easy to get numerical solutions. We will comment a little more about it at the end of this section. Figure 1 contains the field variables (Θ, A and B) in this $\beta = 1$ case. This is actually the solution discussed already by Mukherjee [30]. There is another related solution, which is obtained by re-
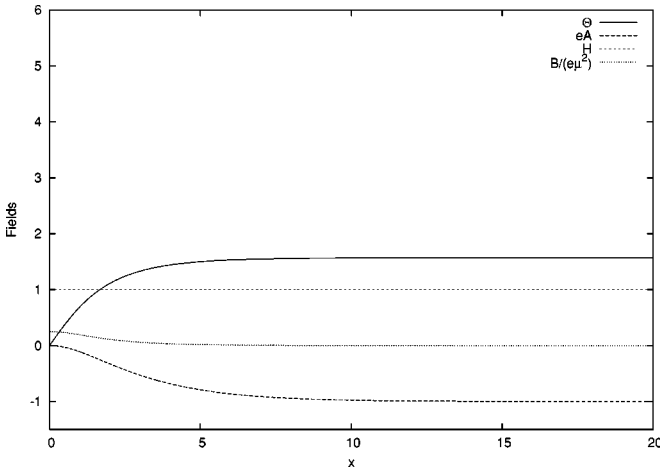
FIG. 1. The solution to Eqs. (3.3), (3.4), for $\mathcal{E}_2 = 1$. Θ is the angular coordinate on S^2. We also show the dimensionless magnetic field. The metric field, which is constant in this case, is included for comparison with the following figures. The parameters used are $n = -1$, $\beta = 1$. The dimensionless length coordinate is defined by $x = e\mu r$.

flecting this solution in the S^2 target space with respect to the equator, i.e. starting at $f = \infty$ on the z axis ($r = 0$) and decreasing with r to $f = 1$:

$$\lim_{r \to 0} f(r) = \infty, \quad \lim_{r \to \infty} f(r) = 1. \tag{3.7}$$

The reflection property is much more transparent in terms of the angular field variable Θ, Eq. (2.25), in terms of which the boundary conditions on the scalar field are

$$\Theta(0) = \pi, \quad \lim_{r \to \infty} \Theta(r) = \pi/2. \tag{3.8}$$

This reflected solution has obviously negative magnetic flux.

Another solution, which was discussed already by Schroers [29], corresponds to the limiting case of potential with a minimum which does not break the U(1) symmetry. This corresponds, in our terminology, to $\beta = 0$ while μ stays finite so $v = 0$. In this case, the minimum of the potential is on the north pole of S^2. Finite energy (per unit length) solutions still exist, but they have now different boundary conditions, namely

$$\Theta(0) = \pi, \quad A(0) = 0 \tag{3.9}$$

$$\lim_{r \to \infty} \Theta(r) = 0, \quad \lim_{r \to \infty} B(r) = 0. \tag{3.10}$$

The last boundary condition in Eq. (3.2) is automatically satisfied as $f \to 0$ asymptotically. This gives a solution, which maps the z axis ($r = 0$) to the south pole of S^2 and the circle $r \to \infty$ to the north pole. Since there is no symmetry breaking at spatial infinity, the magnetic flux is not quantized but rather takes continuous values.

In order to compare these solutions with the $\beta = 1$ ones, it is more instructive to present the "anti-Schroers" solutions,

which are solutions to the same problem with the reflected potential ($\beta \to \infty$). The boundary conditions should be therefore also reflected:

$$\Theta(0) = 0, \quad A(0) = 0 \tag{3.11}$$

$$\lim_{r \to \infty} \Theta(r) = \pi, \quad \lim_{r \to \infty} B(r) = 0 \tag{3.12}$$

where again the last boundary condition in Eq. (3.2) is automatically satisfied, as now $f^2 \mathcal{E}_1 \to 0$ asymptotically. There are two unusual features of this family of solutions: First, as mentioned above, there is no flux quantization, and second, there does not exist a solution with $|n| = 1$ but with $|n| \geqslant 2$ [29]. Consequently, the energy density is maximal not on the symmetry axis, but rather on a cylindrical surface whose radius is of the order of $1/e\mu$. Similar behavior is seen in the analogous $|n| \geqslant 2$ Nielsen-Olesen flux tubes [1].

Another kind of interesting potential is a triple well potential, whose minima are at the north and south pole and at the equator of the target space. In terms of the angular field Θ, it has the following simple form:

$$U(\Theta) = \frac{e^2 \mu^4}{128} \sin^2(2\Theta). \tag{3.13}$$

This potential is obtained by using $\mathcal{E}_2(|\Phi|) = \frac{1}{4}(|\Phi|/\mu + \mu/|\Phi|)^2$ and $v = \mu$ in Eq. (2.24). Actually, $\mathcal{E}_2$ may be multiplied by an arbitrary constant, thus having the height of the potential free. Since the "polar" minima (at $\Theta = 0, \pi$) do not break the U(1) symmetry, we do not expect flux quantization for solutions that approach asymptotically one of these minima. On the other hand, solutions which have $\Theta = \pi/2$ as their asymptotic value, will have of course quantized magnetic flux.

There are thus several kinds of finite energy (per unit length) solutions in this system. For $\Theta(0) = 0$, all three kinds with asymptotic values $\Theta = 0, \pi/2, \pi$ are realized. The fields Θ, $\bar{A}$ and $\bar{B}$ in Figs. 3, 5, 7 which actually correspond to the Chern-Simons type system analyzed in Sec. V depict also the solutions of the present system with $\Theta(\infty) = \pi/2, \pi, 0$ respectively. We will say more about it in Sec. VI.

Similar solutions exist with $\Theta(0) = \pi$. A different kind of solution starts with a "non-polar" value [i.e. $\Theta(0) \neq 0, \pi$], but for continuity must have $n = 0$. The asymptotic values correspond to each of the three possible vacua.

The numerical procedure we have used is essentially the same as in previous works [31,32], and we will not elaborate about it more than the following remark regarding the treatment of boundary conditions. A common practice in numerical analysis is to replace the boundary conditions at infinity with the same boundary conditions at some finite point, r_∞. This works well on the system (3.3), (3.4), provided that β is not 0 or ∞. If we try to shoot from r_∞ to 0 in these two cases, it is easy to see (from the equations in terms of the angular field, Θ), that the solutions become constant. We are therefore forced to replace the boundary condition on Θ with $\Theta(\infty) = \beta \pm \delta$ where δ is a (positive) parameter, which is to

be determined numerically. It turns out, that the continuum of solutions is found by giving δ slightly different values.

Another remark concerns the classification of solutions according to the possible choices of boundary conditions. As is already clear from the flat space solutions, the O(3) system has a reflection symmetry with respect to the S^2 equatorial plane (i.e. $\Theta \rightarrow \pi - \Theta$ or $f \rightarrow 1/f$) together with $\beta \rightarrow 1/\beta$. We may therefore limit ourselves to solutions with $f(0) = 0$ (which for $\eta = +1$ are also $n < 0$ solutions) and classify them according to the asymptotic values $f(\infty)$, which are the various minima of the potential function. The $n > 0$ solutions for the same η and β (and hence the same potential), which start at the "south pole" of target space, are easily obtained from the $n < 0$ solutions with $\beta \rightarrow 1/\beta$ and further obvious changes as $n \rightarrow -n$ and $A \rightarrow -A$.

IV. GRAVITATING SOLUTIONS

In this section we consider gravitating O(3) cosmic strings. It is in fact very easy to get the gravitating self-dual cylindrical solutions (i.e. single cosmic string solutions) in the Bogomol'nyi limit, since the field equations for the matter (scalar and gauge) fields are very similar to those in flat spacetime. More precisely, they are given by the following first order equations:

$$\frac{rf'}{f} = \eta(eA(r) - n) \qquad (4.1)$$

which is identical with the Minkowskian one [for $\eta = +1$ compare Eq. (3.3)], and

$$\frac{A'}{H^2 r} = -\frac{\eta e \mu^2}{2(1 + \beta^2)} \frac{\beta^2 - f^2}{\mathcal{E}_2(f)(1 + f^2)}. \qquad (4.2)$$

Einstein equations for the metric field (2.28) get in the cylindrical case the following form:

$$\left[r \left((\log H)' - \frac{8\pi G}{\eta e} B \mathcal{E}_2(f)(\log f)' \right) \right]' = 0. \qquad (4.3)$$

Since we are interested here in cases where the matter fields tend asymptotically to their vacuum values, the geometry of space will evidently be conic with a deficit angle given by the usual relation:

$$\frac{\delta \varphi}{2\pi} = -\lim_{r \to \infty} (r(\log H)'). \qquad (4.4)$$

We may obtain more information about the deficit angle by integrating equation (4.3) once and using the result at both ends, $r = 0$ and $r \to \infty$:

$$\frac{\delta \varphi}{2\pi} = \frac{8\pi G}{e} |nB(0)| \mathcal{E}_2(f(0)) \qquad (4.5)$$

where we used the boundary conditions (3.5) and (3.6) supplemented by $H(0) = 1$, $H'(0) = 0$, and the fact that $-nB(0)$ is always (i.e. for $\eta = \pm 1$) positive. The factor
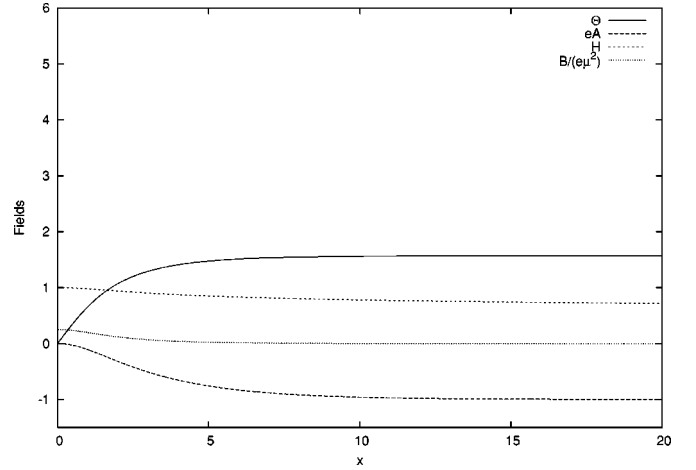

FIG. 2. The solution to Eqs. (4.1)–(4.3), for $\mathcal{E}_2 = 1$. The parameters used are $n = -1$, $\beta = 1$, $8\pi G \mu^2 = 0.5$. Compare with Fig. 1.

$B(0)\mathcal{E}_2(f(0))$ in the last equation can be expressed in terms of the dielectric function $\mathcal{E}_1(f)$ using Eq. (2.17) and assuming a vanishing magnetic field as $f \rightarrow \beta$:

$$\frac{\delta \varphi}{2\pi} = 8\pi G \mu^2 |n| \int_0^\beta f \mathcal{E}_1(f) df. \qquad (4.6)$$

For the O(3) sigma model $\mathcal{E}_1(f)$ under consideration here, this integral can be explicitly calculated, or we may rather use the already given expression of the magnetic field—Eq. (4.2). Both ways we get

$$\frac{\delta \varphi}{2\pi} = \frac{4\pi G v^2 |n|}{1 + \beta^2}. \qquad (4.7)$$

Note that these two last equations are valid for the $f(0) = 0$ solutions—see final note of previous section. The well-known usual Higgs result [33] is obtained for $\beta = 0$, while holding v fixed. On the other hand, we may take another limit to obtain the deficit angle for the "anti-Schroers" solutions by $\beta \rightarrow \infty$, while holding μ fixed. This gives $(\delta \varphi / 2\pi)_S = 4\pi G \mu^2 |n|$. This result applies also to the Schroers solutions.

In Fig. 2, we show the fields (Θ, A, B, H) for the case $\beta = 1$, $8\pi G \mu^2 = 0.5$; compare with Fig. 1.

V. CHERN-SIMONS TYPE OF THE GENERALIZED HIGGS MODEL

The possibility of a second field dependent "dielectric function" $\mathcal{E}_2$, used at the end of Sec. III, is much more interesting from another aspect, which is a connection with $D = 3$ Chern-Simons theory. It turns out that the self-dual solutions for a $D = 3$ system of a gauged sigma model coupled to pure Chern-Simons theory, are related to those of our generalized Higgs system. In other words, we may replace the Maxwell term with a Chern-Simons term in the $D = 3$ version of the action (2.1) to get a generalized Chern-Simons-Higgs (GCSH) model. We will find interesting relations between the self-dual solutions of this GCSH system

and the previous one, which from now on we call generalized Maxwell-Higgs (GMH) model.

There is of course a physical difference between time-independent solutions in both models, which is the presence of electric charge and electric field in the GCSH model. However, due to the linear relation between the magnetic field and the electric charge density, a $|\Phi|$-dependent magnetic energy density appears, which in flat space mimics exactly the Maxwellian energy density in the GMH system, as will be shown below. This sheds new light on the well-known self-dual solutions of the Chern-Simons gauged O(3) sigma model [34–37]. The self-dual solutions of the Chern-Simons type of the Abelian Higgs system [38,39], fit also to this framework by the obvious choice $\mathcal{E}_1(|\Phi|)=1$. In order to demonstrate these relations we write the $D=3$ action:

$$S=\int d^3x\sqrt{|g|}\left(\frac{1}{2}\mathcal{E}_1(|\Phi|)(D_\mu\Phi)^*(D^\mu\Phi)-U(|\Phi|)\right.$$

$$\left.+\frac{\kappa}{4}\frac{\epsilon^{\lambda\mu\nu}}{\sqrt{|g|}}A_\lambda F_{\mu\nu}+\frac{1}{16\pi G_3}R\right)\quad(5.1)$$

where all the geometrical terms here are 3-dimensional, and κ is a positive parameter (it cannot be a field-dependent function, as in the GMH case, because it would break gauge invariance) with dimensions of 1/length. Actually, the fields here have different dimensionalities than in $D=4$, but the self-dual solutions of the GMH system are identical in $D=3$ and $D=4$, so the comparison becomes trivial.

The field equations of this system are

$$\mathcal{E}_1(|\Phi|)D_\mu D^\mu\Phi+\frac{\Phi^*}{2|\Phi|}\frac{d\mathcal{E}_1}{d|\Phi|}D_\mu\Phi D^\mu\Phi+\frac{\Phi}{|\Phi|}\frac{dU}{d|\Phi|}=0$$
$$(5.2)$$

$$\frac{\kappa}{2}\frac{\epsilon^{\lambda\mu\nu}}{\sqrt{|g|}}F_{\lambda\mu}=j^\nu=-\frac{i}{2}e\mathcal{E}_1(|\Phi|)(\Phi^*(D^\nu\Phi)-\Phi(D^\nu\Phi)^*)$$
$$(5.3)$$

$$\frac{1}{8\pi G_3}R_{\mu\nu}+\frac{1}{2}\mathcal{E}_1(|\Phi|)((D_\mu\Phi)^*(D_\nu\Phi)+(D_\nu\Phi)^*(D_\mu\Phi))$$
$$-2U(|\Phi|)g_{\mu\nu}=0.\quad(5.4)$$

Spontaneous symmetry breaking occurs if the potential has a circle of degenerate minima, just as for the GMH system. The mass of the Higgs field is given by the same expression as for GMH, Eq. (2.6), but the mass (squared) of the gauge field is now[1]

$$m_A^2=\left(\frac{e^2v^2\mathcal{E}_1(v)}{\kappa}\right)^2.\quad(5.5)$$

In order to find static self-dual solutions in flat space, one may proceed by dropping the curvature terms and look for

---

[1]Compare Deser and Yang [40].

conditions for Eq. (2.13) to be satisfied. However, we prefer to take the more general approach of dealing with the gravitating system from the beginning. We therefore try to repeat what we did in the previous sections and require $g_{00}=1$ in a $D=3$ version of the static metric (2.7). It turns out that this cannot be done "naively," but some modifications are required. The reason is the well-known property of Chern-Simons vortices, which should carry also electric charge so the Maxwell tensor must contain in this case also an electric field, and the vortex carries angular momentum as well. This addition has important consequences if we treat the gravitational field as dynamical, since the angular momentum of the Chern-Simons field forces the gravitational field to be stationary and not simply static. We therefore parametrize the $D=3$ metric by

$$ds^2=N^2(x^k)(dt+L_i(x^l)dx^i)^2-\gamma_{ij}(x^k)dx^idx^j\quad(5.6)$$

and write for the gauge field

$$A_\mu dx^\mu=A_0(x^k)dt+A_i(x^k)dx^i.\quad(5.7)$$

In order to get the field equations for stationary solutions, we introduce also the following notation:

$$\bar{A}_i=A_i-A_0L_i$$

$$\bar{D}_i=\partial_i-ie\bar{A}_i$$

$$L_{ij}=\partial_iL_j-\partial_jL_i=\sqrt{|\gamma|}\epsilon_{ij}\ell$$

$$\bar{F}_{ij}=\partial_i\bar{A}_j-\partial_j\bar{A}_i=-\sqrt{|\gamma|}\epsilon_{ij}\bar{B}\quad(5.8)$$

and compute the components of the Ricci tensor:

$$R_{00}=-N\nabla_i\nabla^iN-\frac{N^4}{2}\ell^2$$

$$R_0^i=-\frac{1}{2N\sqrt{|\gamma|}}\epsilon^{ij}\partial_j(N^3\ell)\quad(5.9)$$

$$R^{ij}=\frac{1}{N}\nabla^i\nabla^jN+\frac{1}{2}(R(\gamma)-N^2\ell^2)\gamma^{ij}$$

where ∇_i is the covariant derivative with respect to the two-dimensional metric γ_{ij} and $R(\gamma)$ the corresponding Ricci scalar.

Now we impose $N=1$ and find that the (00) component of Einstein equations (5.4) will be satisfied only with the following condition on the potential:

$$U(|\Phi|)=\frac{e^2}{2}|\Phi|^2\mathcal{E}_1(|\Phi|)(A_0)^2-\frac{\ell^2}{32\pi G_3}.\quad(5.10)$$

From the (ij) Einstein equations (or better from the G^{ij} equations), we find the form of the self-duality equation in this case

$$\bar{D}_i\Phi=i\eta\sqrt{|\gamma|}\epsilon_{ij}\gamma^{jk}\bar{D}_k\Phi\quad(5.11)$$

as well as the following expression for the two dimensional Ricci scalar:

$$R(\gamma) = \ell^2 - 8\pi G_3(\mathcal{E}_1(|\Phi|)\gamma^{ij}(\bar{D}_i\Phi)^*(\bar{D}_j\Phi) + 4U(|\Phi|)). \tag{5.12}$$

Due to the self-duality, the other field equations simplify as follows. The non-diagonal Einstein equations give

$$\partial_i\ell = 16\pi G_3\,\eta e|\Phi|\mathcal{E}_1(|\Phi|)A_0\partial_i|\Phi|. \tag{5.13}$$

The spatial Chern-Simons equations give

$$\partial_i A_0 = -\frac{\eta e}{\kappa}|\Phi|\mathcal{E}_1(|\Phi|)\partial_i|\Phi| \tag{5.14}$$

which is used in the time component of the Chern-Simons equations to obtain an expression for $\bar{B}$:

$$\bar{B} = \left(\ell + \frac{e^2}{\kappa}|\Phi|^2\mathcal{E}_1(|\Phi|)\right)A_0. \tag{5.15}$$

Finally, the equation for the Higgs field gives another functional relation between the potential and the other quantities:

$$\eta e|\Phi|\mathcal{E}_1(|\Phi|)\bar{B} + \frac{dU}{d|\Phi|} - \frac{e^2}{2}(A_0)^2\frac{d}{d|\Phi|}(|\Phi|^2\mathcal{E}_1) = 0. \tag{5.16}$$

Actually it is not an independent equation, since by substitution of Eq. (5.10) in Eq. (5.16), one may get back Eq. (5.15).

The two differential equations for $\ell(x^k)$ and $A_0(x^k)$ may be also converted into equations for $\ell(|\Phi|)$ and $A_0(|\Phi|)$:

$$\frac{d\ell}{d|\Phi|} = 16\pi G_3\,\eta e|\Phi|\mathcal{E}_1(|\Phi|)A_0, \tag{5.17}$$

$$\frac{dA_0}{d|\Phi|} = -\frac{\eta e}{\kappa}|\Phi|\mathcal{E}_1(|\Phi|) \tag{5.18}$$

and we find a simple expression of ℓ in terms of A_0:

$$\ell = 8\pi G_3\kappa(c - A_0^2). \tag{5.19}$$

where c is an integration constant, which should be non-negative for solutions with finite angular momentum (where ℓ vanishes asymptotically).

As in the gravitating GMH model, the right hand side of Eq. (5.12) is a two-dimensional divergence and we find

$$R(\gamma) = 16\pi G_3\kappa\nabla_i\left(\frac{1}{\eta e}\gamma^{ij}A_0\partial_j\log|\Phi| + \frac{c\,\epsilon^{ij}}{\sqrt{|\gamma|}}L_j\right). \tag{5.20}$$

If we use a conformally flat metric, Eq. (5.20) can be rewritten as

$$\delta^{ij}\partial_i\left(\partial_j\log H - \frac{8\pi G_3\kappa}{\eta e}A_0\partial_j\log|\Phi|\right) = 8\pi G_3\kappa cH^2\ell. \tag{5.21}$$

Now we consider some special cases that allow self-dual solutions. We will then concentrate in rotationally symmetric solutions, i.e. gravitating vortices.

A. $\mathcal{E}_1(\Phi|) = 1$

This is the Chern-Simons version of the usual Higgs model, which we may consider either in flat space or coupled to gravity. Integration of Eq. (5.18) is trivial and gives

$$A_0 = \frac{\eta e}{2\kappa}(v^2 - |\Phi|^2) \tag{5.22}$$

where v is an integration constant. In flat space $\ell = 0$ and Eq. (5.10) gives immediately the well-known [38,39] sixth order potential:

$$U(|\Phi|) = \frac{e^4}{8\kappa^2}|\Phi|^2(v^2 - |\Phi|^2)^2. \tag{5.23}$$

In the case of a gravitating system, we solve for $\ell(|\Phi|)$ and find

$$\ell(|\Phi|) = \frac{2\pi G_3 e^2}{\kappa}(\sigma - (v^2 - |\Phi|^2)^2) \tag{5.24}$$

where $\sigma = 4\kappa^2 c/e^2$. The potential is easily found to be

$$U(|\Phi|) = \frac{e^4}{8\kappa^2}[|\Phi|^2(v^2 - |\Phi|^2)^2 - \pi G_3(\sigma - (v^2 - |\Phi|^2)^2)^2] \tag{5.25}$$

which has an unbounded additional term, thus rendering the whole system possibly unstable. Notwithstanding the possible instability, this potential was discussed by several authors [19–22] and vortex solutions were also obtained. There is a simple and natural way to cure the ill-behaved potential, which is generalizing this system to the O(3) sigma model.

B. $\mathcal{E}_1(\Phi|) = 1/(1 + |\Phi|^2/\mu^2)^2$

This is the Chern-Simons gauged O(3) sigma model considered in flat background by some authors [34–37,41]. Within the present framework the analysis is straightforward. First we integrate Eq. (5.18) and find

$$A_0 = \frac{\eta e\mu^2}{2\kappa(1 + \beta^2)}\frac{\beta^2 - |\Phi|^2/\mu^2}{1 + |\Phi|^2/\mu^2}. \tag{5.26}$$

In flat space $\ell = 0$ and Eq. (5.10) gives immediately the following potential:

$$U(|\Phi|) = \frac{e^4\mu^4}{8\kappa^2(1 + \beta^2)^2}\frac{|\Phi|^2(\beta^2 - |\Phi|^2/\mu^2)^2}{(1 + |\Phi|^2/\mu^2)^4}. \tag{5.27}$$

Some authors have already discussed the self-dual solutions in flat space for special cases of the potential, which from this point of view are just special values of β: $\beta = 0$ [34,36] and $\beta = 1$ [37]. This general form of the potential appeared

already (in a different parametrization) in Ref. [36], but only the case $\beta=0$ was discussed there. Kimm et al. [35] have studied the general case in flat space and found three kinds of flux tube solutions characterized by the asymptotic value of $f(r)$, which (for $f(0)=0$) may be either 0, β or ∞.

If this Chern-Simons gauged O(3) model is coupled to gravity, we find that the ill-behaved potential from the Higgs system becomes now bounded from below. First we integrate Eq. (5.17) either directly or using Eqs. (5.19) and (5.26) to get

$$\ell(|\Phi|) = \frac{2\pi G_3 e^2 \mu^4}{\kappa(1+\beta^2)^2}\left(\lambda - \frac{(\beta^2-|\Phi|^2/\mu^2)^2}{(1+|\Phi|^2/\mu^2)^2}\right) \quad (5.28)$$

where the role of the integration constant is played by λ defined by $\lambda=(1+\beta^2)^2\sigma/\mu^4$. Then we find from Eq. (5.10) a new potential with λ as an additional (non-negative) free parameter:

$$U(|\Phi|) = \frac{e^4\mu^4}{8\kappa^2(1+\beta^2)^2}\left[\frac{|\Phi|^2(\beta^2-|\Phi|^2/\mu^2)^2}{(1+|\Phi|^2/\mu^2)^4}\right.$$
$$\left. - \frac{\pi G_3\mu^4}{(1+\beta^2)^2}\left(\lambda - \frac{(\beta^2-|\Phi|^2/\mu^2)^2}{(1+|\Phi|^2/\mu^2)^2}\right)^2\right].$$
$$(5.29)$$

This potential is clearly bounded from below, thus restoring the stability. Moreover, it has always a local minimum at $|\Phi|=v=\beta\mu$, which breaks the U(1) local symmetry. The extremal points at $|\Phi|=0$ and $|\Phi|\to\infty$, which are U(1)-symmetric, may be either minima or maxima depending on the following conditions:

$$\beta^2(1+\beta^2)+4\pi G_3\mu^2(\beta^4-\lambda)>0 \quad \text{minimum at } |\Phi|=0$$
$$(5.30)$$
$$1+\beta^2+4\pi G_3\mu^2(1-\lambda)>0 \quad \text{minimum at } |\Phi|\to\infty.$$

The potential of the gravitating Chern-Simons type of the usual Higgs system, is obtained in the limit $\mu\to\infty$ (with $\lambda\mu^4$ and $\beta\mu$ kept finite).

Actually a special case of this potential for $\lambda=0$ was recently obtained by Abou-Zeid and Samtleben [42] from a different direction of three dimensional $N=2$ supergravity theory. We will see that $\lambda\neq0$ yields interesting solutions as well.

Now we concentrate in rotationally symmetric solutions, i.e. we use the Nielsen-Olesen ansatz including the time component of the gauge potential:

$$\Phi=\mu f(r)e^{in\varphi}, \quad A_\mu dx^\mu = A_0(r)dt + A(r)d\varphi \quad (5.31)$$

with the additional requirement that all the metric components depend on r only.

The field equations for self-dual rotationally symmetric solutions are quite similar to the ones of the GMH model with two additional equations: For A_0 and for the metric component $L_\varphi(r)$ (L_r now vanishes). The dependence of A_0 on f is obvious from Eq. (5.26). For $L_\varphi(r)$ we have

$$\frac{L'_\varphi}{H^2 r} = \ell \quad (5.32)$$

where the dependence of ℓ on f is obtained from Eq. (5.28). The self-duality equation for the scalar field is now ($\eta=+1$ as usual):

$$\frac{rf'}{f} = e\bar{A}(r) - n \quad (5.33)$$

where for brevity we denote $\bar{A}=A-A_0L_\varphi$. The equation for $\bar{A}(r)$ may be easily obtained from Eq. (5.15), expressing ℓ and A_0 in terms of f:

$$\frac{\bar{A}'}{H^2 r} = -\frac{e^3\mu^4}{2\kappa^2(1+\beta^2)}\frac{\beta^2-f^2}{1+f^2}\left[\frac{2\pi G_3\mu^2}{(1+\beta^2)^2}\left(\lambda - \frac{(\beta^2-f^2)^2}{(1+f^2)^2}\right)\right.$$
$$\left. + \frac{f^2}{(1+f^2)^2}\right]. \quad (5.34)$$

Einstein equations for the metric field (5.21) reduce in this case to the following:

$$\left[r\left((\log H)' - \frac{8\pi G_3\kappa}{e}A_0(\log f)'\right) - 8\pi G_3\kappa c L_\varphi\right]' = 0. \quad (5.35)$$

The reflection symmetry observed in the GMH system exists in the GCSH system as well, provided λ is also rescaled according to its β dependence. We may therefore limit ourselves to solutions with $f(0)=0$ as before. Clearly there are three kinds of solutions classified by the three possible values which $f(\infty)$ may take, according to the different minima of the potential. However, these three values $U(0)$, $U(v)$ and $U(\infty)$ are generally different from each other, and each of them vanishes for a different value of λ. Since vanishing value of the potential minimum is an essential property of our localized solutions, we encounter here a situation which differs from that in flat space [35], where all three possible boundary conditions are realized for the same potential. In the gravitating case, each kind is realized for a different value of λ. The flux tube solutions with quantized flux and the usual boundary condition $f(\infty)=\beta$ exist for $\lambda=0$ only. Representative solutions are depicted in Figs. 3 and 4. The two other solution types with $f(\infty)=0,\infty$ exist for $\lambda=\beta^4,1$, respectively and are depicted in Figs. 5–8. We may therefore summarize the situation by the following relations:

$$\lambda=\beta^4 \Rightarrow \lim_{r\to\infty} f(r)=0$$

$$\lambda=0 \Rightarrow \lim_{r\to\infty} f(r)=\beta \quad (5.36)$$

$$\lambda=1 \Rightarrow \lim_{r\to\infty} f(r)=\infty.$$

A special case is $\lambda=\beta=1$ which allows both boundary conditions [$f(\infty)=0,\infty$] for the same potential.
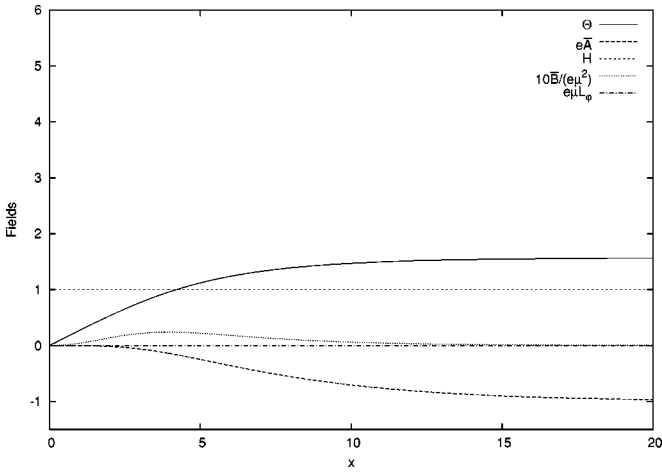
FIG. 3. The solution to Eqs. (5.32)–(5.35), with the boundary condition $f(\infty)=\beta$. The parameters used are $n=-1$, $\beta=1$, $\kappa/e\mu=1$, $8\pi G_3\mu^2=0$, i.e. gravity is not included. Notice that we magnify the magnetic field by a factor 10 here and in the subsequent figures. The gauge field tends to its asymptotic value slower than in the previous case as is seen from the value of $e\bar{A}$ at $x=20$ which is somewhat above -1. The curves in this figure represent also the corresponding solution for the GMH system with the triple well potential—see remark below Eq. (6.2).

Note that all three kinds of boundary conditions ensure also a vanishing asymptotic value of the angular momentum density ℓ, which is a necessary condition for a finite angular momentum. The gauge potential A_0 need not always vanish asymptotically. It does only in the $f(\infty)=\beta$ case.

For all these three kinds of boundary conditions, the matter fields tend asymptotically to their vacuum values. Thus asymptotically, the geometry of space-time will be a rotating conic metric. The angular deficit may be easily obtained from Eq. (5.35) to be

$$\frac{\delta\varphi}{2\pi}=-\frac{8\pi G_3\kappa}{e}[nA_0(0)+A_0(\infty)(eA(\infty)-n)].$$
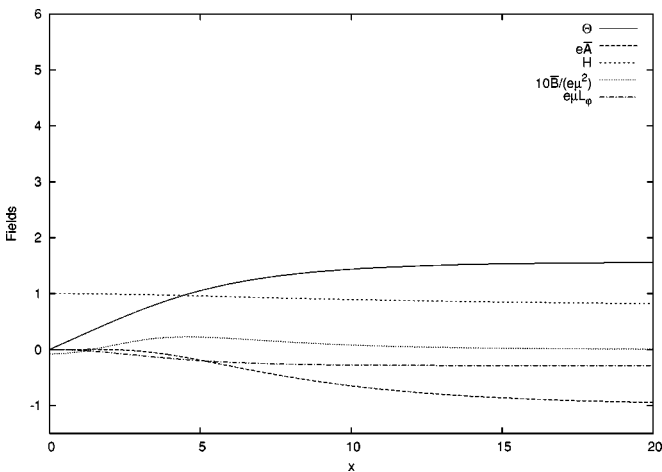
$$(5.37)$$

FIG. 4. The solution to Eqs. (5.32)–(5.35), with the boundary condition $f(\infty)=\beta$. The parameters used are $n=-1$, $\beta=1$, $\kappa/e\mu=1$, $8\pi G_3\mu^2=0.5$, i.e. gravity is included. Compare with Fig. 3.
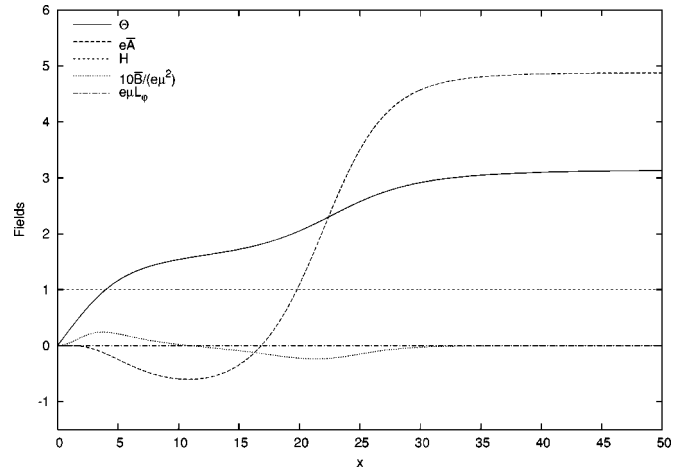

FIG. 5. The solution to Eqs. (5.32)–(5.35), with the boundary condition $f(\infty)=\infty$. The parameters used are $n=-1$, $\beta=1$, $\kappa/e\mu=1$, $8\pi G_3\mu^2=0$, i.e. gravity is not included. The curves in this figure represent also the corresponding solution for the GMH system with the triple well potential—see remark below Eq. (6.2).

For vanishing c [or $A_0(\infty)$ or λ, i.e. quantized flux] we find

$$\frac{\delta\varphi}{2\pi}=-\frac{8\pi G_3\kappa}{e}nA_0(0)=8\pi G_3\mu^2|n|\int_0^\beta f\mathcal{E}_1(f)df$$

$$(5.38)$$

that is the same as the result (4.6) for the GMH model. For the O(3) case under consideration here, we find the same expression as we had in the case of O(3)-GMH:

$$\frac{\delta\varphi}{2\pi}=\frac{4\pi G_3 v^2|n|}{1+\beta^2}, \quad \lim_{r\to\infty}f(r)=\beta. \quad (5.39)$$

The angular deficit for the other solutions without flux quantization (i.e. non-vanishing c or λ) can be easily calculated in the O(3) model from Eq. (5.37):
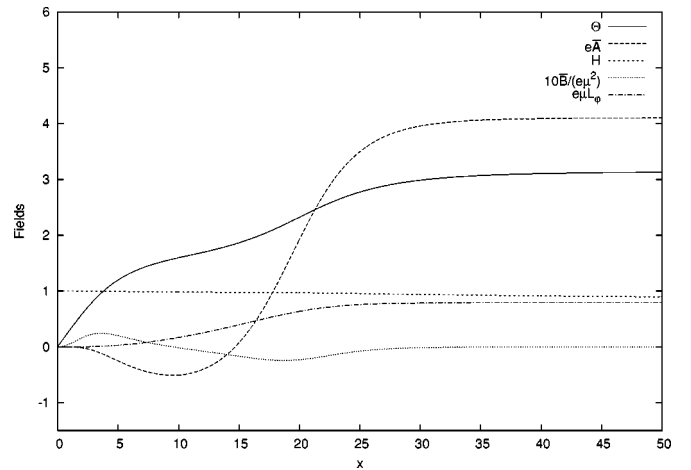

FIG. 6. The solution to Eqs. (5.32)–(5.35), with the boundary condition $f(\infty)=\infty$. The parameters used are $n=-1$, $\beta=1$, $\kappa/e\mu=1$, $8\pi G_3\mu^2=0.0625$, i.e. gravity is included. Compare with Fig. 5.
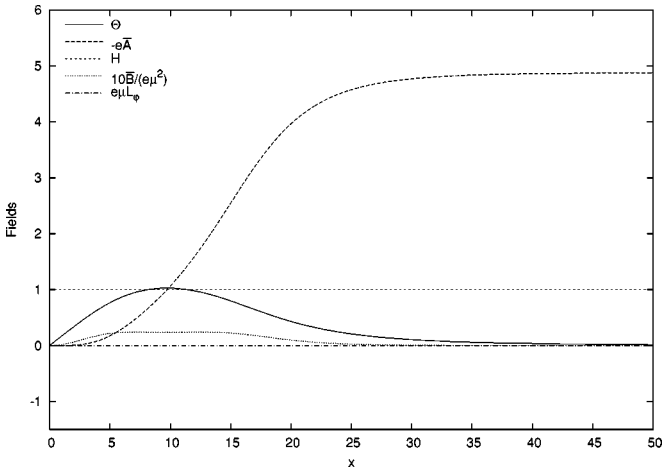
FIG. 7. The solution to Eqs. (5.32)–(5.35), with the boundary condition $f(\infty)=0$. The parameters used are $n=-1$, $\beta=1$, $\kappa/e\mu =1$, $8\pi G_3\mu^2=0$, i.e. gravity is not included. Notice that we show $-e\bar{A}$. The curves in this figure represent also the corresponding solution for the GMH system with the triple well potential—see remark below Eq. (6.2).

$$\frac{\delta\varphi}{2\pi}=\frac{4\pi G_3\mu^2\beta^2 e|A(\infty)|}{1+\beta^2}, \quad \lim_{r\to\infty}f(r)=0 \quad (5.40)$$

$$\frac{\delta\varphi}{2\pi}=4\pi G_3\mu^2\left|\frac{eA(\infty)}{1+\beta^2}-n\right|, \quad \lim_{r\to\infty}f(r)=\infty. \quad (5.41)$$

Note the dependence on the unquantized magnetic flux represented here by $A(\infty)$. Thus we have in these cases a continuum of values for the angular deficit for a given potential and n and $f(\infty)$ values.

We end this section by computing the angular momentum J of the rotationally symmetric solutions, which we get by
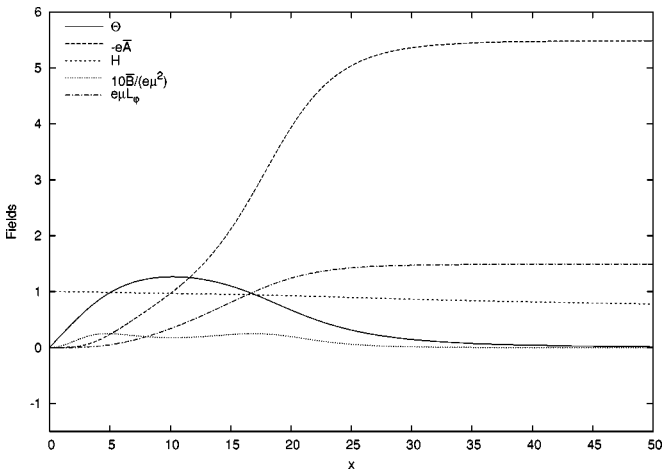

FIG. 8. The solution to Eqs. (5.32)–(5.35), with the boundary condition $f(\infty)=0$. The parameters used are $n=-1$, $\beta=1$, $\kappa/e\mu =1$, $8\pi G_3\mu^2=0.15$, i.e. gravity is included. Compare with Fig. 7.

integrating T^0_φ over all two-space. The angular momentum also determines the asymptotic value of the metric component $L_\varphi(r)$ such that

$$\left(1-\frac{\delta\varphi}{2\pi}\right)L_\varphi(\infty)=-4GJ. \quad (5.42)$$

This relation is most easily obtained in a Gaussian normal coordinate system, in which the line element is written as: $ds^2=-d\rho^2+h_{\alpha\beta}dx^\alpha dx^\beta$ ($x^\alpha=t,\varphi$). Since $h_{\alpha\beta}$ depends only on ρ, the components of the Ricci tensor have the following simple form in terms of the extrinsic curvature K^α_β:

$$R^\alpha_\beta=\frac{1}{\sqrt{|h|}}\frac{\partial}{\partial\rho}(\sqrt{|h|}K^\alpha_\beta), \quad K^\alpha_\beta=-\frac{1}{2}h^{\alpha\gamma}\frac{\partial h_{\gamma\beta}}{\partial\rho}. \quad (5.43)$$

It turns out that we can compute the angular momentum directly without even the self-duality assumption, due to the simple identity which holds for solutions of the flux tube form (5.31):

$$eT^0_\varphi=(n-eA(r))j^0. \quad (5.44)$$

Now we use the time component of the Chern-Simons equation (5.3), to write j^0 in terms of B and get

$$J=\frac{2\pi\kappa}{e}\int_0^\infty(n-eA(r))A'dr=\frac{\pi\kappa}{e}(2n-eA(\infty))A(\infty). \quad (5.45)$$

Solutions which break the U(1) symmetry asymptotically have quantized angular momentum of

$$J=\frac{\pi\kappa}{e^2}n^2 \quad (5.46)$$

while otherwise, there is no flux quantization and we cannot say about the angular momentum more than the right hand side of Eq. (5.45). Note that the angular momentum in Eq. (5.46) is a very general result independent of all details of the theory, except the existence of symmetry breaking vacuum in the potential. We note further that all solutions have the same sign of angular momentum, which is a manifestation of parity violation.

VI. CORRESPONDENCE BETWEEN MAXWELL AND CHERN-SIMONS TYPE OF THE GENERALIZED HIGGS MODEL

We note that in flat space, the electric energy density of the Chern-Simons field plays an equivalent role to that of the magnetic energy density in the GMH model. Moreover, due to Eq. (5.15), there is a relation between the magnetic and electric terms, thus enabling us to eliminate the scalar (electric) potential from the equations and to give them a form identical to those of the GMH model.

Comparison of the Bogomol'nyi constraints in both cases, Eqs. (2.17) and (5.18), gives a relation between the magnetic

field in the GMH model and the time component of the gauge potential in the GCSH model:

$$\mathcal{E}_2 B = \kappa A_0. \tag{6.1}$$

Further comparison of the magnetic field appearing in this equation with the one in Eq. (5.15) (with $\ell = 0$) or the potentials [Eqs. (2.11) and (5.10) (with $\ell = 0$)], yields the following characteristic relation for the generalized Maxwell-Higgs model:

$$\frac{\kappa^2}{\mathcal{E}_2(|\Phi|)} = e^2 |\Phi|^2 \mathcal{E}_1(|\Phi|). \tag{6.2}$$

Thus all the flat space self-dual solutions to the $D=3$ system, studied in Sec. V, are also self-dual solutions to the $D=4$ GMH model of Sec. III, provided we use $\mathcal{E}_2$ consistent with Eq. (6.2). Consequently, the curves in Figs. 3, 5, 7 represent also the corresponding fields of the GMH model with the triple well potential (3.13) and the appropriate dielectric function mentioned together with that potential.

However, the presence of an electric field in the $D=3$ Chern-Simons theory, in addition to the magnetic field, is responsible for the existence of angular momentum, which calls for corrections to this simple correspondence when gravity is considered dynamical. It is straightforward (although tedious) to show that there does not exist spinning gravitating self-dual flux tubes in the GMH model. Therefore, we have only static versus stationary correspondence, and all we can hope for is equivalence of the two-dimensional spatial metrics in addition to the scalar and vector correspondence. This turns out to be the case, provided we also impose $c = \sigma = \lambda = 0$.

Equation (6.1) is evidently still valid in the gravitating case and should be accompanied by $B = \bar{B}$ where B is the GMH magnetic field defined by Eq. (2.9) and $\bar{B}$—in Eq. (5.8). Thus we get the relation instead of Eq. (6.2):

$$\frac{\kappa^2}{\mathcal{E}_2(|\Phi|)} = e^2 |\Phi|^2 \mathcal{E}_1(|\Phi|) + \kappa \ell. \tag{6.3}$$

Note that now the potentials are not equal in both cases, but there is an ℓ-dependent difference:

$$U_M = U_{CS} - \frac{\ell^2}{32\pi G_3}. \tag{6.4}$$

As for the metric tensor, it is easy to see that both expressions for the Ricci scalar, Eqs. (2.27) and (5.20), become identical due to Eq. (6.1), thus resulting in equal 2-metrics in the two cases.

Actually, those ℓ-dependent modifications should be expressed in terms of functions that appear in the Lagrangian, and should be considered as functions of $|\Phi|$. We will not do this explicitly, since it is straightforward and does not add any more insight into the physical picture.

VII. TOPOLOGICAL CHARGE

The quantization of magnetic flux is intimately connected with the fact that the minimum of the potential breaks the local U(1) symmetry. Consequently, the magnetic flux number is the index, or winding number, of the map defined by the scalar fields from infinite distance from the flux tube to the vacuum manifold.

If however, the target space is compact, there exists a further possibility of homotopy classification of the maps defined by the fields from "real" space to target space. We can use, as a ground of the present discussion, the well-known notions from sigma models defined in three space-time dimensions [14]. The GCSH model is naturally three dimensional, but the GMH model should be thought of here as three dimensional as well. In this case, the scalar fields map the two dimensional physical space into target space. We have therefore an identically conserved topological current defined by

$$K^\lambda = -\frac{\epsilon^{\lambda\mu\nu}}{2\Omega_T \sqrt{|g|}} (i\mathcal{E}_1(|\Phi|)(D_\mu\Phi)^*(D_\nu\Phi) + e\mathcal{F}(|\Phi|)F_{\mu\nu}) \tag{7.1}$$

where Ω_T is the target space volume, and $\mathcal{F}(|\Phi|)$ is defined by

$$\frac{d\mathcal{F}}{d|\Phi|} = |\Phi|\mathcal{E}_1(|\Phi|). \tag{7.2}$$

This is a gauge invariant generalization of the standard non-gauged sigma model current, whose time component integral over all space is the winding number, or index, of the map defined by the sigma model fields. Note that $\mathcal{F}(|\Phi|)$ is connected to $\mathcal{A}(|\Phi|)$ [defined in Eq. (2.30)] by $e\mathcal{F}(|\Phi|) = -|\Phi|d\mathcal{A}/d|\Phi|$, up to a possible additive constant. By the field equations of the GCSH model, it is also related to A_0 by $\kappa A_0(|\Phi|) = -\eta e\mathcal{F}(|\Phi|)$, up to an additive constant. Unlike previous authors [30,35–37], we fix the additive constant of $\mathcal{F}$ such that $\mathcal{F}$ vanishes as $|\Phi| \to \infty$, irrespective of the boundary conditions imposed on the solutions. This way, we can treat uniformly all sigma models differing only by the potential term. For the special O(3) "dielectric function," given in Eq. (2.22), we may write the topological charge (and current) in terms of $\mathcal{E}_1(|\Phi|)$ only, since $\mathcal{F}(|\Phi|) = -\mu^2\sqrt{\mathcal{E}_1(|\Phi|)}/2$. In this case, we have also $\Omega_T = \pi\mu^2$ and the apparent dependence on the scale μ disappears.

The fact that this gauge invariant current has still a topological meaning is clearly seen by the fact that it is actually the non-gauged sigma model current with an addition, which is a divergence of an anti-symmetric tensor:

$$K^\lambda = -\frac{\epsilon^{\lambda\mu\nu}}{2\Omega_T \sqrt{|g|}} \left\{ (i\mathcal{E}_1(|\Phi|)(\partial_\mu\Phi)^*(\partial_\nu\Phi) \right.$$
$$\left. + 2e\nabla_\mu[\mathcal{F}(|\Phi|)A_\nu] \right\}. \tag{7.3}$$

Indeed, the first term of Eq. (7.3) can also be written as a divergence of an anti-symmetric tensor, so we have the alternative form:

$$K^\nu = \nabla_\mu K^{\mu\nu},$$

$$K^{\mu\nu} = \frac{\mathcal{F}(|\Phi|)\epsilon^{\mu\nu\lambda}}{2\Omega_T\sqrt{|g|}}\left(\frac{i}{|\Phi|^2}(\Phi^*\partial_\lambda\Phi - \Phi\partial_\lambda\Phi^*) + 2eA_\lambda\right).$$
(7.4)

Integration of the time component of the topological current provides us with the topological charge T, which characterizes the solutions from the homotopy point of view. For our stationary solutions, we write the two equivalent expressions:

$$T = -\frac{1}{2\Omega_T}\int d^2x(i\epsilon^{ij}\mathcal{E}_1(|\Phi|)(D_i\Phi)^*(D_j\Phi)$$

$$-2e\sqrt{|\gamma|}B\mathcal{F}(|\Phi|))$$

$$= -\frac{i}{2\Omega_T}\int d^2x\epsilon^{ij}\mathcal{E}_1(|\Phi|)(\partial_i\Phi)^*(\partial_j\Phi)$$

$$-\frac{e}{\Omega_T}\mathcal{F}(w)\oint_{r\to\infty}A_i dx^i$$
(7.5)

where w is the asymptotic value of $|\Phi|$. We stress the difference between w, which corresponds to a property of a solution, and v which is a parameter in the potential. w need not be equal to v (although it may be), as happens for solutions of the GCSH model discussed in Sec. V, or solutions to the GMH model with the potential (3.13), which may have all three asymptotic values $w = 0, v, \infty$.

Note that T cannot be expressed as a surface integral (actually line integral in $D = 2 + 1$) since the winding number, i.e. the (non-gauged) sigma model contribution of K^{i0}, does not satisfy Stokes theorem. The second term is however a boundary term, and we note that it vanishes for solutions for which $|\Phi|\to\infty$ as $r\to\infty$. In the O(3) model, these are solutions which do not break the U(1) symmetry asymptotically. Another possibility is solutions which do break the U(1) symmetry asymptotically, and thus do not cover all target space. This kind of solutions have non-integer winding number, but the topological charge has also a magnetic flux contribution, which may compensate for this fact.

This is exactly what happens for the self-dual cylindrically symmetric solutions of the O(3) GMH model, where we find that in all cases $T = n$. If the solution tends to a symmetry breaking vacuum, the boundary term of Eq. (7.5) does not vanish and evaluates to $n/(1 + \omega^2)$, where we define $\omega = w/\mu$. The winding number, which is the first term, is easily found to be $n\omega^2/(1 + \omega^2)$. They both add up to give $T = n$. In the case $\omega\to\infty$, where the solution does not break asymptotically the U(1) symmetry, the flux contribution vanishes but the first is an integer and we get $T = n$ again. The third possibility, $\omega = 0$ is however an exception: In this case, the winding number vanishes but we have a flux contribution to get $T = eA(\infty)$.

Another property, which we can note here, is the relation with the Euler number of the 2-surface generated by the self-dual GMH solutions. The Euler number is given by

$$\chi = -\frac{1}{4\pi}\int d^2x\sqrt{|\gamma|}R(\gamma)$$
(7.6)

while for (say) self-dual solutions ($\eta = 1$), we may write the topological charge as

$$T = -\frac{1}{2\Omega_T}\int d^2x\sqrt{|\gamma|}(\mathcal{E}_1(|\Phi|)\gamma^{ij}(D_i\Phi)^*(D_j\Phi)$$

$$+ 2\mathcal{E}_2(|\Phi|)B^2) - \frac{e}{\Omega_T}\mathcal{F}(v)\oint_{r\to\infty}A_i dx^i.$$
(7.7)

By the field equations (2.11) and (2.14), we see that the first term is just proportional to the Euler number, so we have a simple relation between the topological charge, the Euler number and the magnetic flux, which we denote here by Ψ:

$$T + \frac{\chi}{4G\Omega_T} - \frac{e\mathcal{F}(v)\Psi}{\Omega_T} = 0.$$
(7.8)

For the O(3) system, Eq. (7.8) yields the following relation:

$$2\pi T + \frac{\chi}{2G\mu^2} + \frac{e\Psi}{1 + \beta^2} = 0.$$
(7.9)

For asymptotically conic space, the Euler number is related to the deficit angle by $\chi = \delta\varphi/2\pi$, so Eq. (7.8) is easily verified by using Eq. (4.7).

Next we turn to the GCSH model, where things are quite similar. The topological current and charge are still given by Eqs. (7.1)–(7.5). However, since the self-duality condition is now modified by replacing A_i by $\bar{A}_i$, it is useful to write the topological charge also in a way that is ready for direct use of the modified self-duality. We therefore write the topological charge density as

$$K^0 = -\frac{\epsilon^{ij}}{2\Omega_T\sqrt{|g|}}\left\{i\mathcal{E}_1(|\Phi|)(\bar{D}_i\Phi)^*(\bar{D}_j\Phi)\right.$$

$$\left. + 2e\partial_i[\mathcal{F}(|\Phi|)A_0L_j]\right\} + \frac{e\mathcal{F}(|\Phi|)\bar{B}}{\Omega_T}.$$
(7.10)

Note however that the surface term vanishes only for quantized flux solutions $[c = \lambda = 0, A_0(w) = 0]$. If we proceed along similar lines as for the GMH case, we easily find the same relation between the topological charge, Euler number and the magnetic flux, i.e. Eq. (7.8). It simplifies again for the O(3) model to Eq. (7.9).

Finally, we use this relation in order to calculate the topological charge for the solutions with quantized as well as non-quantized flux, which is obtained directly from Eq. (7.5):

$$T = \begin{cases} eA(\infty), & w = 0, \\ n, & w = v, \\ n, & w \to \infty. \end{cases} \qquad (7.11)$$

A nice check of our results is that we can easily reproduce by substitution in Eq. (7.9) the expressions (5.39)–(5.41) for the deficit angles.

VIII. OUTLOOK

We have shown that the inclusion of dielectric functions gives the possibility of dramatic changes in the long range behavior of scalar, gauge and metric fields around a cosmic string or vortex. This was obtained even for very simple and symmetric models, like O(3) with the potential (3.13). We believe that the astrophysical and cosmological consequences deserve further studies.

Another continuation and generalization of our work is to consider theories with a more extended field content. One natural possibility is to gauge a U(1)×U(1) subgroup of the global symmetry group of a CP(2) non-linear sigma model noticing that CP(2) can be parametrized by two complex coordinates. One can therefore expect that in analogy with the way the O(3) model [with $S^2 \sim CP(1)$ as target space] generalizes the ordinary cosmic string, the gauged CP(2) model generalizes the superconducting cosmic string [43,44].

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