

# Negative energy densities in quantum field theory with a background potential

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We present a general procedure for calculating one-loop “Casimir” energy densities for a scalar field coupled to a fixed potential in renormalized quantum field theory. We implement direct subtraction of counterterms computed precisely in dimensional regularization with a definite renormalization scheme. Our procedure allows us to test quantum field theory energy conditions in the presence of background potentials spherically symmetric in some dimensions and independent of others. We explicitly calculate the energy density for several examples. For a square barrier, we find that the energy is negative and divergent outside the barrier, but there is a compensating divergent positive contribution near the barrier on the inside. We also carry out calculations with exactly solvable  $\text{sech}^2$  potentials, which arise in the study of solitons and domain walls.

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## I. INTRODUCTION

The weak energy condition of general relativity, the requirement that there be no negative energy densities, is sufficient to prevent the appearance of exotic features such as compactly generated closed timelike curves [1] and superluminal travel [2]. Quantum field theory appears to violate this condition, however. One example is the standard Casimir system of parallel plates, for which there is a negative energy density between the plates. However, this is an idealized system, which assumes a perfect conductor with an infinitely sharp and flat edge. A real material will have a rough surface at the atomic scale and will also appear transparent to very high energy modes. Since the Casimir energy is a sum over all energies, it will always include modes for which these effects are relevant, so this idealization could affect the value of the sum. In addition, since the boundary condition is imposed externally, there is no measure of the energy that would be required to maintain it.<sup>1</sup>

Although there are other ways to produce negative energy densities, for example, a superposition of states with zero and two photons, these cases are constrained by averaged energy conditions, which require the energy to be positive when averaged along an entire geodesic. They are also constrained by quantum inequalities [5], which limit the total negative energy that can exist when averaging over a certain period of time. Thus it is important to understand problems of the Casimir type if we want to know whether quantum field theory protects general relativity against negative energies.

In this paper, we reconsider the question of the energy density in such systems. To avoid the subtleties associated with the Casimir problem, we consider a quantum field in the presence of a background potential (i.e., a field with a mass that depends on position). In such an approach, one can choose a potential that depends only on one spatial dimension, and simulate the parallel plates in the Casimir problem. By generalizing the approach of Refs. [6–8] to local densities, we can precisely cancel the divergences in the calculation in a definite renormalization scheme.

Reference [9] considered similar problems for the special case of reflectionless potentials, such as the potentials for the supersymmetric kink and sine-Gordon models. Here we present a general approach suitable for numerical computation, in addition to analytic calculations in exactly solvable models [10]. These techniques are also useful for the study of Casimir forces and stresses [11].

## II. A SIMPLE MODEL

To illustrate our method, we will first consider a simple model. We take a real, massless scalar field in 2+1 dimensions in the background of a repulsive potential  $V$  that depends on one spatial dimension but not the other.

We start with the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} [\dot{\phi}^2 + (\nabla\phi)^2 + V\phi^2] \quad (1)$$

and expand the field  $\phi$  in terms of small oscillations, giving

$$\begin{aligned} \phi(x, y, t) = & \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2\pi\omega(p)}} \\ & \times \sum_{\chi=+,-} [\psi_k^\chi(x) e^{ipy} e^{-i\omega(p)t} a_{k,p}^\chi \\ & + \psi_k^\chi(x)^* e^{-ipy} e^{i\omega(p)t} a_{k,p}^{\chi\dagger}], \end{aligned} \quad (2)$$

where  $\omega(p) = \sqrt{k^2 + p^2}$ . The  $\psi_k^\chi(x)$  are normal mode wave

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<sup>1</sup>Recently, Helfer and Lang [3] showed that a frequency-independent dielectric would not be expected to give negative energy densities, but Sopova [4] showed that negative energy densities can be achieved in a Casimir system with Drude-model plates, as long as the spacing is very large compared to the plasma wavelength.

functions, which can be taken to be real. The sum is over the symmetric mode,  $\psi_k^+(x)$ , and the antisymmetric mode,  $\psi_k^-(x)$ . They satisfy

$$-\psi_k^{\chi''}(x) + V(x)\psi_k^\chi(x) = k^2\psi_k^\chi(x) \quad (3)$$

where prime denotes differentiation with respect to  $x$ , with the normalization

$$\int_{-\infty}^{\infty} dx \psi_k^+(x)\psi_k^-(x) = 0$$

and

$$\sum_{\chi=+,-} \int_{-\infty}^{\infty} dx \psi_k^\chi(x)\psi_{k'}^\chi(x) = 2\pi\delta(k-k'), \quad (4)$$

which gives  $\psi_k^{+(0)}(x) = \cos kx$  and  $\psi_k^{-(0)}(x) = \sin kx$  as the solutions in the free case,  $V(x) = 0$ .

The energy density is then

$$\begin{aligned} \langle \mathcal{H} \rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \int_0^{\infty} \frac{dk}{2\pi\omega(p)} \\ &\times \sum_{\chi=+,-} \left\{ \frac{1}{2} [\omega(p)^2 + p^2 + V(x)] \psi_k^\chi(x)^2 + \psi_k^{\chi'}(x)^2 \right\}. \end{aligned} \quad (5)$$

We write

$$\psi'(x)^2 = \frac{1}{2} \frac{d^2}{dx^2} \psi(x)^2 - \psi''(x)\psi(x) \quad (6)$$

and then use Eq. (3) to obtain

$$\begin{aligned} \langle \mathcal{H} \rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \int_0^{\infty} \frac{dk}{2\pi\omega(p)} \\ &\times \sum_{\chi=+,-} \left[ \omega(p)^2 \psi_k^\chi(x)^2 + \frac{1}{4} \frac{d^2}{dx^2} [\psi_k^\chi(x)^2] \right]. \end{aligned} \quad (7)$$

The integral is highly divergent, but by using dimensional regularization and introducing counterterms into the Hamiltonian, as discussed in Sec. III, we can render the integral finite. We then integrate out the transverse modes, giving

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} &= -\frac{\Gamma\left(-\frac{n+1}{2}\right)}{2(4\pi)^{(n+1)/2}} \int_0^{\infty} \frac{dk}{\pi} \\ &\times \sum_{\chi=+,-} \left\{ \omega^{n+1} \left[ \psi_k^\chi(x)^2 - \left(1 + \frac{V(x)}{2k^2}\right) \psi_k^{\chi(0)}(x)^2 \right] \right. \\ &\left. + \frac{n+1}{4} \omega^{n-1} \frac{d^2}{dx^2} [\psi_k^\chi(x)^2] \right\}, \end{aligned} \quad (8)$$

where  $n$  is the number of transverse dimensions, later to be set to 1. The Hamiltonian has been renormalized by subtracting a constant term for the vacuum energy and a term proportional to the potential, which is sufficient to render it finite in 2+1 dimensions.

We can relate the norms of the mode wave functions to the Green's function by (see, for example, Ref. [12])

$$\sum_{\chi=+,-} \psi_k^\chi(x)^2 = 2k \text{Im} G(x,x,k), \quad (9)$$

where  $G(x,x',k)$  is the Green's function, which satisfies

$$-G''(x,x',k) + V(x)G(x,x',k) - k^2G(x,x',k) = \delta(x-x') \quad (10)$$

and has only outgoing waves ( $\sim e^{ik|x|}$ ) at infinity.

The Green's function has the symmetry property  $G(x,x',k) = G(x,x',-k^*)^*$ , so for real  $k$  we can write

$$\sum_{\chi=+,-} \psi_k^\chi(x)^2 = \frac{k}{i} G(x,x,k) - \frac{k}{i} G(x,x,-k). \quad (11)$$

Thus we can compute the energy by extending the range of integration to  $-\infty$ , and using  $G^{(0)}(x,x,k) = i/(2k)$ ,

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} &= -\frac{\Gamma\left(-\frac{n+1}{2}\right)}{(4\pi)^{(n+3)/2}} \int_{-\infty}^{\infty} \omega^{n+1} \left[ \frac{2k}{i} G(x,x,k) - 1 - \frac{V(x)}{2k^2} \right. \\ &\left. + \frac{n+1}{2} \frac{k}{i\omega^2} \frac{d^2}{dx^2} G(x,x,k) \right] dk, \end{aligned} \quad (12)$$

where, since we are taking the massless limit,  $\omega = \sqrt{k^2}$ .

Next we would like to convert this expression into a contour integral by closing the contour at infinity in the upper half plane. The contour at infinity does not contribute, because for large, positive  $\text{Im} k$ ,

$$\frac{2k}{i} G(x,x,k) \rightarrow 1 + \frac{V(x)}{2k^2} + O(k^{-4}). \quad (13)$$

Singularities in the Green's function in the upper half plane correspond to normalizable eigenfunctions of the Hamiltonian, which represent bound states. Since the Hamiltonian is Hermitian, the bound states must have real energies, so the singularities must lie on the imaginary axis and have  $\text{Im} k < \mu$  where  $\mu$  is the mass. In this example, we have a repulsive potential and a massless particle, each of which is sufficient to ensure that there are no bound states at all. Thus the Green's function has no singularities for  $\text{Im} k \geq 0$ , and the only contribution to the integral comes from the branch cut along the positive imaginary axis coming from  $\omega^{n+1}$ . Integrating around the branch cut and using Eqs. (39) and (40) below gives

$$\langle \mathcal{H} \rangle_{\text{ren}} = - \frac{1}{2(4\pi)^{(n+1)/2} \Gamma\left(\frac{n+3}{2}\right)} \int_0^\infty \kappa^{n+1} \left[ 2\kappa G(x,x,k) - 1 + \frac{V(x)}{2\kappa^2} + \frac{n+1}{2\kappa} \frac{d^2}{dx^2} G(x,x,k) \right] d\kappa \quad (14)$$

and then setting  $n=1$  gives

$$\langle \mathcal{H} \rangle_{\text{ren}} = - \frac{1}{8\pi} \int_0^\infty d\kappa \left[ 2\kappa^3 G(x,x,i\kappa) - \kappa^2 + \frac{V(x)}{2} - \kappa \frac{d^2}{dx^2} G(x,x,i\kappa) \right]. \quad (15)$$

Once one has computed the Green's function, this integral is straightforward, though it may be necessary to resort to numerical techniques. We show the calculation for some example potentials in Sec. IV below.

### III. CALCULATIONAL METHOD

#### A. Model

We will now consider the more general case of a real scalar field of mass  $\mu$  in the background of a potential that is spherically symmetric in  $m$  nontrivial coordinates, which we label by  $x$ , and independent of the remaining  $n$  trivial coordinates, which we label by  $y$ . The energy density is

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} [\dot{\phi}^2 + (\nabla \phi)^2 + V(r) \phi^2 + \mu^2 \phi^2] \\ &= \frac{1}{2} \left( \dot{\phi}^2 + \frac{1}{2} \nabla^2(\phi^2) - \phi \nabla^2 \phi + V(r) \phi^2 + \mu^2 \phi^2 \right), \end{aligned} \quad (16)$$

where  $r=|x|$ . Decomposing the quantum field  $\phi$  in terms of modes gives

$$\begin{aligned} \phi(r, \Omega, t) &= \sum_{\ell, \ell_z} \sqrt{\frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}} \int \frac{d^n p}{(2\pi)^{n/2}} \frac{1}{\sqrt{2}} \\ &\times \left( \sum_j \frac{1}{\sqrt{\omega_j^\ell}} [\psi_j^\ell(r) * Y_{\ell\ell_z}^m(\Omega) * e^{-ipy} e^{i\omega_j^\ell(p)t} a_{j,p}^{\ell\ell_z \dagger} \right. \\ &+ \psi_j^\ell(r) Y_{\ell\ell_z}^m(\Omega) e^{ipy} e^{-i\omega_j^\ell(p)t} a_{j,p}^{\ell\ell_z}] \\ &+ \int_0^\infty \frac{dk}{\sqrt{\pi\omega(p)}} [\psi_k^\ell(r) * Y_{\ell\ell_z}^m(\Omega) * e^{-ipy} e^{i\omega(p)t} a_{k,p}^{\ell\ell_z \dagger} \\ &+ \psi_k^\ell(r) Y_{\ell\ell_z}^m(\Omega) e^{ipy} e^{-i\omega(p)t} a_{k,p}^{\ell\ell_z}] \Big), \end{aligned} \quad (17)$$

where  $\omega(p) = \sqrt{k^2 + p^2 + \mu^2}$ , the bound state energies are  $\omega_j^\ell(p) = \sqrt{p^2 + \mu^2 - \kappa_j^{\ell 2}}$ , and the sum over  $\ell$  gives the partial wave expansion in the  $m$  nontrivial dimensions.

The degeneracy factor  $D_\ell^m$  in each partial wave is given by the dimension of the space of symmetric tensors with  $\ell$  indices, each running from 1 to  $m$ , with all traces (contractions) removed [15]. By the symmetry of the indices, this dimension is given by the number of ways to make  $\ell$  indices out of 0 or more 1's, 0, or more 2's, and so on, which is the number of distinct ways to place  $m-1$  dividers into  $m+\ell-1$  slots. Removing all the traces requires subtracting the same quantity with  $\ell$  replaced by  $\ell-2$ . We thus obtain

$$\begin{aligned} D_\ell^m &= \frac{(m+\ell-1)!}{\ell!(m-1)!} - \frac{(m+\ell-3)!}{(\ell-2)!(m-1)!} \\ &= \frac{\Gamma(m+\ell-2)}{\Gamma(m-1)\Gamma(\ell+1)} (m+2\ell-2). \end{aligned} \quad (18)$$

The wave functions  $\psi_k^\ell(r)$  are the eigenstates of the time-independent radial Schrödinger equation

$$\begin{aligned} \left( -\frac{d^2}{dr^2} - \frac{m-1}{r} \frac{d}{dr} + \frac{\ell(\ell+m-2)}{r^2} + V(r) \right) \psi_k^\ell(r) \\ = k^2 \psi_k^\ell(r), \end{aligned} \quad (19)$$

which in general comprise both bound and scattering states. The wavefunctions and creation and annihilation operators are normalized as follows. For the spherical harmonics,

$$\int Y_{\ell\ell_z}^m(\Omega) * Y_{\ell'\ell'_z}^m(\Omega) d\Omega = \delta_{\ell\ell'} \delta_{\ell_z\ell'_z}, \quad (20)$$

for continuum states,

$$\begin{aligned} \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} \int_0^\infty r^{m-1} \psi_k^\ell(r) * \psi_{k'}^\ell(r) dr = \pi \delta(k-k'), \\ [a_{k,p}^{\ell\ell_z \dagger}, a_{k',p'}^{\ell'\ell'_z \dagger}] = [a_{k,p}^{\ell\ell_z}, a_{k',p'}^{\ell'\ell'_z}] = 0, \\ [a_{k,p}^{\ell\ell_z}, a_{k',p'}^{\ell'\ell'_z \dagger}] = \delta(k-k') \delta(p-p') \delta_{\ell\ell'} \delta_{\ell_z\ell'_z}, \end{aligned} \quad (21)$$

and for bound states,

$$\frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} \int_0^\infty r^{m-1} \psi_j^\ell(r) \psi_{j'}^{\ell'}(r) dr = \delta_{jj'} \delta_{\ell\ell'},$$

$$[a_{j,p}^{\ell\ell_z\ddagger}, a_{j',p'}^{\ell'\ell'_z\ddagger}] = [a_{j,p}^{\ell\ell_z}, a_{j',p'}^{\ell'\ell'_z}] = 0,$$

$$[a_{j,p}^{\ell\ell_z}, a_{j',p'}^{\ell'\ell'_z\ddagger}] = \delta_{jj'} \delta(p-p') \delta_{\ell\ell'} \delta_{\ell_z\ell'_z}. \quad (22)$$

Using these expressions, we obtain the vacuum expectation value of the Hamiltonian

$$\langle \mathcal{H} \rangle = \frac{1}{2} \sum_{\ell} D_{\ell}^m \int \frac{d^n p}{(2\pi)^n} \left[ \sum_j \omega_j^{\ell}(p) |\psi_j^{\ell}(r)|^2 \right. \\ \left. + \int_0^{\infty} \frac{dk}{\pi} \omega(p) |\psi_k^{\ell}(r)|^2 + \frac{1}{4} D_r^2 \left( \sum_j \frac{1}{\omega_j^{\ell}(p)} |\psi_j^{\ell}(r)|^2 \right) \right. \\ \left. + \int_0^{\infty} \frac{dk}{\pi} \frac{1}{\omega(p)} |\psi_k^{\ell}(r)|^2 \right], \quad (23)$$

where  $D_r^2 = d^2/dr^2 + [(m-1)/r](d/dr)$  is the radial Laplacian.

### B. Renormalization with one subtraction

For positive integer  $m$  and  $n$ , this quantity diverges, as we expect since we have not yet included the contribution of the counterterms. Therefore we will calculate the result using analytic continuation in  $m$  and  $n$  from values where it is convergent. After introducing counterterms also depending on  $m$  and  $n$ , we will then let the dimensions go to their physical values while holding the renormalization conditions fixed.

The first counterterm we will introduce renormalizes the cosmological constant. It is simply an overall constant in the Hamiltonian, and is fixed by the renormalization condition that the energy density of the trivial background  $V(r)=0$  is zero. The free wave functions are given by

$$\psi_k^{\ell(0)}(r) = \sqrt{\frac{\Gamma\left(\frac{m}{2}\right)}{\pi k} \frac{1}{2\pi^{m/2}} \frac{1}{r^{m/2-1}} J_{m/2+\ell-1}(kr)}, \quad (24)$$

which, by the Bessel function identity

$$\sum_{\ell=0}^{\infty} \frac{(2q+2\ell)\Gamma(2q+\ell)}{\Gamma(\ell+1)} J_{q+\ell}(z)^2 = \frac{\Gamma(2q+1)}{\Gamma(q+1)^2} \left(\frac{z}{2}\right)^{2q}, \quad (25)$$

satisfy the completeness relation

$$\sum_{\ell} D_{\ell}^m |\psi_k^{\ell(0)}(r)|^2 = \frac{1}{(4\pi)^{m/2-1}} \frac{k^{m-1}}{2\Gamma\left(\frac{m}{2}\right)}, \quad (26)$$

independent of  $r$ . Subtracting the energy in the trivial background we have

$$\langle \mathcal{H} \rangle - \langle \mathcal{H} \rangle_0 = \frac{1}{2} \sum_{\ell} D_{\ell}^m \int \frac{d^n p}{(2\pi)^n} \left[ \sum_j \omega_j^{\ell}(p) |\psi_j^{\ell}(r)|^2 \right. \\ \left. + \int_0^{\infty} \frac{dk}{\pi} \omega(p) [|\psi_k^{\ell}(r)|^2 - |\psi_k^{\ell(0)}(r)|^2] \right. \\ \left. + \frac{1}{4} D_r^2 \left( \sum_j \frac{1}{\omega_j^{\ell}(p)} |\psi_j^{\ell}(r)|^2 \right) \right. \\ \left. + \int_0^{\infty} \frac{dk}{\pi} \frac{1}{\omega(p)} |\psi_k^{\ell}(r)|^2 \right]. \quad (27)$$

By completeness, in each partial wave we have

$$\sum_j |\psi_j^{\ell}(r)|^2 + \int_0^{\infty} \frac{dk}{\pi} [|\psi_k^{\ell}(r)|^2 - |\psi_k^{\ell(0)}(r)|^2] = 0, \quad (28)$$

so we can implement a local version of the Levinson's theorem subtraction used in Ref. [6], giving

$$\langle \mathcal{H} \rangle - \langle \mathcal{H} \rangle_0 = \frac{1}{2} \sum_{\ell} D_{\ell}^m \int \frac{d^n p}{(2\pi)^n} \\ \times \left[ \sum_j (\omega_j^{\ell}(p) - \sqrt{p^2 + \mu^2}) |\psi_j^{\ell}(r)|^2 \right. \\ \left. + \int_0^{\infty} \frac{dk}{\pi} (\omega(p) - \sqrt{p^2 + \mu^2}) (|\psi_k^{\ell}(r)|^2 \right. \\ \left. - |\psi_k^{\ell(0)}(r)|^2) + \frac{1}{4} D_r^2 \left( \sum_j \frac{1}{\omega_j^{\ell}(p)} |\psi_j^{\ell}(r)|^2 \right) \right. \\ \left. + \int_0^{\infty} \frac{dk}{\pi} \frac{1}{\omega(p)} |\psi_k^{\ell}(r)|^2 \right]. \quad (29)$$

This subtraction is necessary to avoid the appearance of spurious infrared singularities in calculations in one space dimension. These singularities also appear in dimensions less than one, which we will need to consider as part of the dimensional regularization process.

Next we carry out the  $p$  integral, using

$$\int \frac{d^n p}{(2\pi)^n} (\sqrt{p^2 + q^2})^a \\ = \frac{2}{\Gamma\left(\frac{n}{2}\right) (4\pi)^{n/2}} \int_0^{\infty} p^{n-1} dp (\sqrt{p^2 + q^2})^a \\ = \frac{\Gamma\left(-\frac{n+a}{2}\right) q^{n+a}}{\Gamma\left(-\frac{a}{2}\right) (4\pi)^{n/2}}, \quad (30)$$

where we have done the integral by analytic continuation from values of  $a$  and  $n$  where it converges. We thus obtain

$$\begin{aligned} \langle \mathcal{H} \rangle - \langle \mathcal{H} \rangle_0 = & -\frac{\Gamma\left(-\frac{n+1}{2}\right)}{2(4\pi)^{(n+1)/2}} \\ & \times \sum_{\ell} D_{\ell}^m \left[ \sum_j [(\omega_j^{\ell})^{n+1} - \mu^{n+1}] |\psi_j^{\ell}(r)|^2 \right. \\ & + \int_0^{\infty} \frac{dk}{\pi} (\omega^{n+1} - \mu^{n+1}) \\ & \times [|\psi_k^{\ell}(r)|^2 - |\psi_k^{\ell(0)}(r)|^2] \\ & + \frac{n+1}{4} D_r^2 \left( \sum_j (\omega_j^{\ell})^{n-1} |\psi_j^{\ell}(r)|^2 \right. \\ & \left. \left. + \int_0^{\infty} \frac{dk}{\pi} \omega^{n-1} |\psi_k^{\ell}(r)|^2 \right) \right], \quad (31) \end{aligned}$$

where  $\omega = \sqrt{k^2 + \mu^2}$  for the scattering states and  $\omega_j^{\ell} = \sqrt{\mu^2 - \kappa_j^{\ell 2}}$  for the bound states with  $k = i\kappa_j^{\ell}$ .

Next, we must include the contribution of the counterterm proportional to  $V(r)$ , which is introduced to cancel the tadpole graph. In dimensional regularization, the contribution to the Hamiltonian from this counterterm is

$$\mathcal{H}_1 = \frac{\Gamma\left(\frac{1-n-m}{2}\right)}{2(4\pi)^{(m+n+1)/2}} \mu^{m+n-1} V(r), \quad (32)$$

so by using

$$\begin{aligned} & \int_0^{\infty} \frac{dk}{\pi} (\omega^{n+1} - \mu^{n+1}) k^{m-3} \\ & = \mu^{m+n-1} \frac{\Gamma\left(\frac{m-2}{2}\right) \Gamma\left(\frac{1-m-n}{2}\right)}{2\pi \Gamma\left(-\frac{n+1}{2}\right)} \quad (33) \end{aligned}$$

and Eq. (26) we have

$$\begin{aligned} \mathcal{H}_1 = & -\frac{\Gamma\left(-\frac{n+1}{2}\right)}{2(4\pi)^{(n+1)/2}} \sum_{\ell} D_{\ell}^m \int_0^{\infty} \frac{dk}{\pi} (\omega^{n+1} - \mu^{n+1}) \\ & \times (2-m) \frac{V(r)}{2k^2} |\psi_k^{\ell(0)}(r)|^2 \quad (34) \end{aligned}$$

so that

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} \equiv & \langle \mathcal{H} \rangle - \langle \mathcal{H} \rangle_0 - \langle \mathcal{H}_1 \rangle \\ = & -\frac{\Gamma\left(-\frac{n+1}{2}\right)}{2(4\pi)^{(n+1)/2}} \\ & \times \sum_{\ell} D_{\ell}^m \left[ \sum_j [(\omega_j^{\ell})^{n+1} - \mu^{n+1}] |\psi_j^{\ell}(r)|^2 \right. \\ & + \int_0^{\infty} \frac{dk}{\pi} (\omega^{n+1} - \mu^{n+1}) \\ & \times \left[ |\psi_k^{\ell}(r)|^2 - |\psi_k^{\ell(0)}(r)|^2 \left( 1 + (2-m) \frac{V(r)}{2k^2} \right) \right] \\ & + \frac{n+1}{4} D_r^2 \left( \sum_j (\omega_j^{\ell})^{n-1} |\psi_j^{\ell}(r)|^2 \right. \\ & \left. \left. + \int_0^{\infty} \frac{dk}{\pi} \omega^{n-1} |\psi_k^{\ell}(r)|^2 \right) \right]. \quad (35) \end{aligned}$$

We will use

$$|\psi_k^{\ell}(r)|^2 = 2k \text{Im} G_{\ell}(r, r, k), \quad (36)$$

where the Green's function is defined by

$$\begin{aligned} & -D_r^2 G_{\ell}(r, r', k) + \left( V(r) + \frac{\ell(\ell+m-1)}{r^2} - k^2 \right) G_{\ell}(r, r', k) \\ & = \delta^{(m)}(r-r') \quad (37) \end{aligned}$$

with the boundary conditions that it is regular at the origin and has only outgoing waves ( $\sim e^{ikr}$ ) at infinity. Using  $G(r, r, -k) = G(r, r, k)^*$ , we can rewrite Eq. (35) as

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} \equiv & \langle \mathcal{H} \rangle - \langle \mathcal{H} \rangle_0 - \langle \mathcal{H}_1 \rangle \\ = & -\frac{\Gamma\left(-\frac{n+1}{2}\right)}{2(4\pi)^{(n+1)/2}} \\ & \times \sum_{\ell} D_{\ell}^m \left[ \sum_j [(\omega_j^{\ell})^{n+1} - \mu^{n+1}] |\psi_j^{\ell}(r)|^2 \right. \\ & + \int_{-\infty}^{\infty} \frac{dk}{\pi} (\omega^{n+1} - \mu^{n+1}) \\ & \times \frac{k}{i} \left[ G_{\ell}(r, r, k) - G_{\ell}^{(0)}(r, r, k) \left( 1 + (2-m) \frac{V(r)}{2k^2} \right) \right] \\ & + \frac{n+1}{4} D_r^2 \left( \sum_j (\omega_j^{\ell})^{n-1} |\psi_j^{\ell}(r)|^2 \right. \\ & \left. \left. + \int_{-\infty}^{\infty} \frac{dk}{\pi} \omega^{n-1} \frac{k}{i} G_{\ell}(r, r, k) \right) \right]. \quad (38) \end{aligned}$$

These subtractions are sufficient to render the theory finite for  $m+n < 3$ . However, it appears that if we set  $n=1$ , the gamma function will cause Eq. (38) to diverge. In fact, as we will see in Appendix B, it does not, because the quantity in brackets vanishes. But here we will keep  $n$  general and instead close the contour of integration in the upper half  $k$  plane. For sufficiently small  $n$  the contour at infinity does not contribute. There is a pole for each bound state at  $k=iE_j$ , where  $E_j < \mu$  is the bound state energy, and the contributions from these poles exactly cancel the sum over bound states [16] in Eq. (38). Thus the final result is just given by the contribution from the branch cut along the imaginary axis from  $\mu$  to  $\infty$  resulting from  $\omega^{n+1}$ , which contributes

$$\Omega^{n+1}[i^{n+1} - (-i)^{n+1}] = 2i\Omega^{n+1} \sin \frac{(n+1)\pi}{2}, \quad (39)$$

where  $\Omega = \sqrt{\kappa^2 - \mu^2}$  and  $k = i\kappa$ . Then using the identity

$$\sin \pi z = -\frac{\pi}{\Gamma(z+1)\Gamma(-z)} \quad (40)$$

we have

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} &\equiv \langle \mathcal{H} \rangle - \langle \mathcal{H} \rangle_0 - \langle \mathcal{H}_1 \rangle \\ &= -\frac{1}{2(4\pi)^{(n+1)/2}\Gamma\left(\frac{n+3}{2}\right)} \sum_{\ell} D_{\ell}^m \int_{\mu}^{\infty} d\kappa \ 2\kappa \Omega^{n+1} \\ &\quad \times \left[ G_{\ell}(r, r, i\kappa) - G_{\ell}^{(0)}(r, r, i\kappa) \right. \\ &\quad \left. \times \left( 1 - (2-m) \frac{V(r)}{2\kappa^2} \right) - \frac{n+1}{4} \Omega^{n-1} D_r^2 G_{\ell}(r, r, i\kappa) \right]. \end{aligned} \quad (41)$$

We can now put in integer values of  $m$  and  $n$  without any divergence, as long as  $m+n < 3$ . Eq. (41) can also be efficiently evaluated numerically [11].

### C. Higher subtractions

When we have  $m+n=3$  space dimensions, we will need to introduce a second counterterm,  $\frac{1}{2}cV(x)^2$ . The first subtraction is particularly easy to define because there is a natural scheme, specified by the complete cancellation of the tadpole graph. Higher subtractions require a definition in terms of a renormalization scale, which can be chosen arbitrarily. In choosing this scale, we must be able to relate it to physical inputs, such as masses and coupling constants, in order to define a predictive theory.

To define the counterterm precisely, we consider the two-point function  $\Pi(p^2)$  in dimensional regularization. It diverges as we approach the physical dimension. The divergence is canceled by the contribution of the counterterm to

the two-point function, which is just  $c$ . We define the renormalization scale  $M$  by taking  $c = -\Pi(M^2)$ . With this definition, we have

$$\begin{aligned} c &= \frac{i}{2} \int_0^1 d\lambda \\ &\quad \times \int \frac{dE}{2\pi} \frac{d^{n+m}q}{(2\pi)^{n+m}} \frac{1}{(E^2 - q^2 - \mu^2 + M^2\lambda(1-\lambda) + i\epsilon)^2} \\ &= \frac{1}{2(4\pi)^{(n+m)/2}\Gamma\left(\frac{n+m}{2}\right)} \int_0^{\infty} \frac{q^{n+m-1}}{\omega(4\omega^2 - M^2)} dq, \end{aligned} \quad (42)$$

where  $q$  is the total momentum and we have integrated over the Feynman parameter  $\lambda$  and the loop energy  $E$ . Typically we will choose  $M^2 = \mu^2$ , except in the case of massless theories, where to avoid infrared singularities we will choose a spacelike renormalization point  $M^2 < 0$ .

This regulated expression is defined precisely as an analytic function of the dimension. Our goal is now to rewrite it in a way that allows us to incorporate it into our expression for the energy, Eq. (41), which is also given as an analytic function of the dimension. We express Eq. (26) in terms of Green's functions and analytically continue to express Eq. (42) as

$$\begin{aligned} c &= \frac{1}{2(4\pi)^{(n+1)/2}\Gamma\left(\frac{n+3}{2}\right)} \\ &\quad \times \int_{\mu}^{\infty} \Omega^{n+1} f(\kappa, M) \frac{1}{(4\pi)^{m/2-1}} \frac{\kappa^{m-1}}{2\Gamma\left(\frac{m}{2}\right)} d\kappa \\ &= \frac{1}{2(4\pi)^{(n+1)/2}\Gamma\left(\frac{n+3}{2}\right)} \\ &\quad \times \sum_{\ell} D_{\ell}^m \int_{\mu}^{\infty} \Omega^{n+1} f(\kappa, M) 2\kappa G_{\ell}^{(0)}(r, r, i\kappa) d\kappa, \end{aligned} \quad (43)$$

where  $f(\kappa, M)$  is given in terms of the hypergeometric function as

$$\begin{aligned} f(\kappa, M) &= \frac{2(m-4)(m-2)(2\kappa)^{2-m}}{(4\kappa^2 - M^2)^{3-m/2} \sin\left(\frac{m\pi}{2}\right)} \\ &\quad \times {}_2F_1\left(\frac{1}{2}, 3 - \frac{m}{2}, \frac{3}{2}, \frac{M^2}{M^2 - 4\kappa^2}\right) \end{aligned} \quad (44)$$

as we show in Appendix A.

Since we will eventually take the limit where  $m$  becomes an integer, we note that

$$f(\kappa, M) = \frac{12\kappa^2 - M^2}{2\kappa^2(4\kappa^2 - M^2)^2} \text{ for } m=1,$$

$$f(\kappa, M) = \frac{1}{\pi\kappa^2(4\kappa^2 - M^2)} \times \left( 1 + \frac{4\kappa^2 \arctan \frac{M}{\sqrt{4\kappa^2 - M^2}}}{M\sqrt{4\kappa^2 - M^2}} \right) \text{ for } m \rightarrow 2,$$

and

$$f(\kappa, M) = \frac{1}{2\kappa^2(4\kappa^2 - M^2)} \text{ for } m=3. \quad (45)$$

Equation (43) is now in a form where we can include it under the integral sign in Eq. (41) and obtain

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} &\equiv \langle \mathcal{H} \rangle - \langle \mathcal{H} \rangle_0 - \langle \mathcal{H}_1 \rangle - \langle \mathcal{H}_2 \rangle \\ &= - \frac{1}{2(4\pi)^{(n+1)/2} \Gamma\left(\frac{n+3}{2}\right)} \\ &\quad \times \sum_{\ell} D_{\ell}^m \int_{\mu}^{\infty} d\kappa \, 2\kappa \left\{ \Omega^{n+1} \left[ G_{\ell}(r, r, i\kappa) \right. \right. \\ &\quad \left. \left. - G_{\ell}^{(0)}(r, r, i\kappa) \left( 1 - \frac{V(r)}{2\kappa^2} (2-m) \right) \right. \right. \\ &\quad \left. \left. + V(r)^2 f(\kappa, M) \right] \right\} - \frac{n+1}{4} \Omega^{n-1} D_r^2 G_{\ell}(r, r, i\kappa). \end{aligned} \quad (46)$$

Before we can take the limit where  $m+n=3$ , however, there is one more potential divergence in Eq. (46). Our subtraction has cancelled the terms of order 1,  $V(r)/\kappa^2$  and  $V(r)/\kappa^4$  in the large- $\kappa$  expansion of the norm of the wave functions. But there could also be a term of order  $D_r^2 V(r)/\kappa^4$ , which will generate a divergence in this case. In the renormalization of the composite operator  $T_{\mu\nu}$ , we have a renormalization counterterm  $(c'/2)[(m+n-1)/[4(m+n)]](\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\partial_{\lambda}\partial^{\lambda})\phi^2$  [13]. Since we are considering just  $T_{00}$  here, this counterterm becomes  $(c'/2)[(m+n-1)/[4(m+n)]]\nabla^2\phi^2$ , exactly the form needed to cancel the remain-

ing divergence.<sup>2</sup> We fix this counterterm by subtracting the tadpole diagram with the composite operator carrying momentum  $p^2=M'^2$ . Aside from this change, it is analogous to the tadpole subtraction above, with  $\Omega^2 V(r)$  replaced by  $D_r^2 V(r)$ . The scale  $M'$  is then specified through the renormalization condition on the composite operator (and would typically be chosen equal to  $M$ ). As with  $M$ , a massless theory will require spacelike  $M'^2 < 0$ , while in a massive theory we may set  $M' = \mu$ . Thus we obtain the contribution

$$\begin{aligned} \langle \mathcal{H}_{2'} \rangle &= - \frac{(m+n-1)}{4(n+1)} \frac{1}{2(4\pi)^{(n+1)/2} \Gamma\left(\frac{n+1}{2}\right)} \\ &\quad \times \sum_{\ell} D_{\ell}^m \int_{\mu}^{\infty} d\kappa \, \Omega^{n-1} 2\kappa G_{\ell}^{(0)}(r, r, i\kappa) \\ &\quad \times \frac{D_r^2 V(r)}{\kappa^2 - M'^2} (2-m). \end{aligned} \quad (47)$$

This term is a total derivative, so it does not contribute to the total energy. We can split the contribution of this term between the bulk and derivative terms so that it renders them both separately finite at integer dimensions, giving

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} &= - \frac{1}{2(4\pi)^{(n+1)/2} \Gamma\left(\frac{n+3}{2}\right)} \sum_{\ell} D_{\ell}^m \int_{\mu}^{\infty} d\kappa \\ &\quad \times 2\kappa \left( \Omega^{n+1} \left\{ G_{\ell}(r, r, i\kappa) - G_{\ell}^{(0)}(r, r, i\kappa) \right. \right. \\ &\quad \times \left[ 1 - \frac{V(r)}{2\kappa^2} (2-m) + V(r)^2 f(\kappa, M) \right. \\ &\quad \left. \left. - \frac{D_r^2 V(r)}{8(\kappa^2 - M'^2)\Omega^2} (2-m)^2 \right] \right\} \\ &\quad \left. - \frac{n+1}{4} \Omega^{n-1} \left( D_r^2 G_{\ell}(r, r, i\kappa) + \frac{D_r^2 V(r)}{2(\kappa^2 - M'^2)} \right. \right. \\ &\quad \left. \left. \times (2-m) G_{\ell}^{(0)}(r, r, i\kappa) \right) \right). \end{aligned} \quad (48)$$

By Eq. (26), the sum over  $\ell$  of the free Green's function weighted by the degeneracy factor is independent of  $r$ , so we can pull the derivative outside in the last line, giving

<sup>2</sup>If we had chosen conformal instead of minimal coupling for the fields, which corresponds to adding the extra term  $\frac{1}{2}[(m+n-1)/[4(m+n)]](\partial_{\mu}\partial_{\nu} - g_{\mu\nu}\partial_{\lambda}\partial^{\lambda})\phi^2$  in the original Lagrangian, the divergent term would have cancelled automatically between the bulk term and the surface term and no renormalization would be necessary. However, conformally coupled theories have *classical* violations of the energy conditions [14].

$$\begin{aligned}
 \langle \mathcal{H} \rangle_{\text{ren}} = & - \frac{1}{2(4\pi)^{(n+1)/2} \Gamma\left(\frac{n+3}{2}\right)} \sum_{\ell} D_{\ell}^m \int_{\mu}^{\infty} d\kappa \, 2\kappa \left( \Omega^{n+1} \left\{ G_{\ell}(r,r,i\kappa) - G_{\ell}^{(0)}(r,r,i\kappa) \left[ 1 - \frac{V(r)}{2\kappa^2} (2-m) \right. \right. \right. \\
 & \left. \left. \left. + V(r)^2 f(\kappa, M) - \frac{D_r^2 V(r)}{8(\kappa^2 - M'^2) \Omega^2} (2-m)^2 \right] \right\} - \frac{n+1}{4} \Omega^{n-1} D_r^2 \left( G_{\ell}(r,r,i\kappa) + \frac{V(r)}{2(\kappa^2 - M'^2)} \right. \right. \\
 & \left. \left. \times (2-m) G_{\ell}^{(0)}(r,r,i\kappa) \right) \right). \tag{49}
 \end{aligned}$$

In this form, we see explicitly that the subtraction has cancelled the leading terms for large  $\kappa$ : In the surface term, inside the derivative we have implemented the same subtraction of the leading behavior of the Green's function as we found for the tadpole graph in the bulk term; in the bulk term, we have subtracted the leading term proportional to  $D_r^2 V(r)$  [which is derived explicitly for  $m=1$  in Eq. (B11)]. For the purposes of calculation, however, it is often easier to work with the combined expression

$$\begin{aligned}
 \langle \mathcal{H} \rangle_{\text{ren}} = & - \frac{1}{(4\pi)^{(n+1)/2} \Gamma\left(\frac{n+3}{2}\right)} \sum_{\ell} D_{\ell}^m \int_{\mu}^{\infty} \kappa \Omega^{n+1} \left[ \left( 1 - \frac{n+1}{4\Omega^2} D_r^2 \right) G_{\ell}(r,r,i\kappa) - G_{\ell}^{(0)}(r,r,i\kappa) \right. \\
 & \left. \times \left( 1 - \frac{(2-m)V(r)}{2\kappa^2} + V(r)^2 f(\kappa, M) + \frac{(2-m)D_r^2 V(r)}{8(\kappa^2 - M'^2) \Omega^2} (m+n-1) \right) \right] d\kappa. \tag{50}
 \end{aligned}$$

#### IV. EXAMPLES WITH ONE RELEVANT DIMENSION AND ONE IRRELEVANT DIMENSION

##### A. The general case

To illustrate our method, we would like to carry out some sample calculations in the case of  $m=1$  and  $n=1$ . Since we are in  $2+1$  dimensions, we only need one subtraction, proportional to  $V(x)$ . For  $m=1$ , the sum over partial waves reduces to a sum over the symmetric and antisymmetric channels. The free wave functions thus become

$$\psi_k^{+(0)}(x) = \cos kx, \quad \psi_k^{-(0)}(x) = \sin kx, \tag{51}$$

so that

$$|\psi_k^{+(0)}(x)|^2 + |\psi_k^{-(0)}(x)|^2 = 1. \tag{52}$$

We can sum over the two modes to get the overall Green's function

$$G(x,x,k) = G^+(x,x,k) + G^-(x,x,k), \tag{53}$$

with

$$G^{(0)}(x,x,k) = \frac{i}{2k}. \tag{54}$$

Then from Eq. (41) we have

$$\begin{aligned}
 \langle \mathcal{H} \rangle_{\text{ren}} = & - \frac{1}{2(4\pi)^{(n+1)/2} \Gamma\left(\frac{n+3}{2}\right)} \\
 & \times \int_{\mu}^{\infty} d\kappa \, \Omega^{n+1} \left[ 2\kappa G(r,r,i\kappa) - 1 + \frac{V(r)}{2\kappa^2} \right. \\
 & \left. - \frac{n+1}{2\Omega^2} \kappa D_r^2 G(r,r,i\kappa) \right] \tag{55}
 \end{aligned}$$

and for  $n=1$  we have

$$\begin{aligned}
 \langle \mathcal{H} \rangle_{\text{ren}} = & - \frac{1}{8\pi} \int_{\mu}^{\infty} d\kappa \, \Omega^2 \left[ 2\kappa G(r,r,i\kappa) - 1 + \frac{V(r)}{2\kappa^2} \right. \\
 & \left. - \frac{\kappa}{\Omega^2} \frac{d^2}{dx^2} G(r,r,i\kappa) \right], \tag{56}
 \end{aligned}$$

which reduces to Eq. (15) when  $\mu=0$ . For simplicity, we will consider the massless case for the remainder of this section.

##### B. Outside a potential with compact support

We next consider a potential that vanishes for all  $|x| > a$  and calculate the energy density in this region. In this case, the only counterterm is the vacuum energy, and we get



$$\langle \mathcal{H} \rangle_{\text{ren}} = -\frac{1}{8\pi} \int_0^\infty d\kappa \left[ \kappa^2 [2\kappa G(x, x, \kappa) - 1] - \kappa \frac{d^2}{dx^2} G(x, x, \kappa) \right]. \quad (57)$$

The Green's function for  $x, x' > a$  is

$$G(x, x', k) = \frac{i}{2k} [e^{-ikx} + r(k)e^{ikx}] e^{ikx'}, \quad (58)$$

where  $r(k)$  is the reflection amplitude. Thus

$$2\kappa G(x, x, i\kappa) = 1 + r(i\kappa)e^{-2\kappa x} \quad (59)$$

and

$$\langle \mathcal{H} \rangle_{\text{ren}} = \frac{1}{8\pi} \int_0^\infty d\kappa \kappa^2 r(i\kappa) e^{-2\kappa x}. \quad (60)$$

In the large  $x$  limit, only small  $\kappa$  contribute in the integral. As a result, the integral depends only on  $r(0) = -1$  (at  $k = 0$  we always have perfect reflection<sup>3</sup>), so we can approximate

$$\langle \mathcal{H} \rangle_{\text{ren}} \approx -\frac{1}{8\pi} \int_0^\infty d\kappa \kappa^2 e^{-2\kappa x} = -\frac{1}{32\pi x^3}. \quad (61)$$

### C. Square barrier

Next we consider a square barrier with  $V = V_0$  for  $|x| < a$  and  $V = 0$  otherwise. In this case, we can compute the normal mode wave functions in closed form, but must do a numerical integration at the end. Outside the barrier, the energy is given by Eq. (60) with

$$r = -\frac{V_0 e^{2\kappa a} \tanh 2\kappa' a}{2\kappa\kappa' + (\kappa^2 + \kappa'^2) \tanh 2\kappa' a} \quad (62)$$

and  $\kappa'^2 = \kappa^2 + V_0$ .

Thus, outside the barrier, the energy is

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} &= -\frac{V_0}{8\pi} \int_0^\infty d\kappa \frac{\kappa^2 e^{-2\kappa(x-a)} \tanh 2\kappa' a}{2\kappa\kappa' + (\kappa^2 + \kappa'^2) \tanh 2\kappa' a} \\ &= -\frac{V_0}{8\pi a} \int_0^\infty dq \frac{q^2 e^{-2q(y-1)} \tanh 2q'}{2qq' + (q^2 + q'^2) \tanh 2q'}, \end{aligned} \quad (63)$$

where we have defined the dimensionless quantities  $y = x/a$ ,  $q = \kappa a$ , and  $q' = \sqrt{q^2 + v} = \kappa' a$  where  $v = V_0 a^2$ . Note that the integrand cannot be less than 0, so the energy outside the barrier is always negative.

Far from the potential, specifically where  $y-1 \gg 1/\sqrt{v}$  and  $y-1 \gg 1/v$ , the contribution comes primarily from  $q \ll \sqrt{v}$ , and thus  $q' \approx \sqrt{v}$ . The integral is then

$$\int_0^\infty dq q^2 \frac{e^{-2q(y-1)}}{v} = \frac{1}{4(y-1)^3 v} \quad (64)$$

and

$$\langle \mathcal{H} \rangle_{\text{ren}} \approx -\frac{1}{32\pi(x-a)^3} \quad (65)$$

in agreement with Eq. (61).

Close to the potential, specifically when  $y-1 \ll 1/\sqrt{v}$  and  $y-1 \ll 1$ , the contribution comes mostly from  $q \gg \sqrt{v}$  and  $q \gg 1$ . Thus  $q' \approx q$  and  $\tanh 2q' \approx 1$ . The integral becomes

$$\frac{1}{8(y-1)} \quad (66)$$

and

$$\langle \mathcal{H} \rangle_{\text{ren}} \approx -\frac{V_0}{64\pi(x-a)}. \quad (67)$$

Inside the barrier, we have

$$G(x, x', k) = \frac{i}{k'} \frac{[k' \cos k'(x_{<} + a) - ik \sin k'(x_{<} + a)][k' \cos k'(x_{>} - a) + ik \sin k'(x_{>} - a)]}{2kk' \cos 2k'a - i(k^2 + k'^2) \sin 2k'a}, \quad (68)$$

where  $k' = \sqrt{k^2 - V_0}$ , so we can write

$$G(x, x, i\kappa) = \frac{1}{2\kappa'} \frac{(\kappa^2 + \kappa'^2) \cosh 2\kappa' a + 2\kappa\kappa' \sinh 2\kappa' a + V_0 \cosh 2\kappa' x}{2\kappa\kappa' \cosh 2\kappa' a + (\kappa^2 + \kappa'^2) \sinh 2\kappa' a}. \quad (69)$$

We can then split the energy into two parts,

$$\langle \mathcal{H} \rangle_{\text{ren}} = E_0 + E_1(x), \quad (70)$$

where  $E_1$  depends on position, but  $E_0$  does not.

<sup>3</sup>The only exceptions to this rule are potentials with a bound state precisely at threshold [12,17], which include reflectionless potentials.

The position-independent part is

$$\begin{aligned}
 E_0 &= -\frac{1}{8\pi} \int_0^\infty d\kappa \left\{ \frac{\kappa^3 (\kappa^2 + \kappa'^2) \cosh 2\kappa' a + 2\kappa\kappa' \sinh 2\kappa' a}{\kappa' 2\kappa\kappa' \cosh 2\kappa' a + (\kappa^2 + \kappa'^2) \sinh 2\kappa' a} - \kappa^2 + \frac{V_0}{2} \right\} \\
 &= -\frac{V_0^2}{8\pi} \int_0^\infty d\kappa \frac{1}{2\kappa'} \frac{2\kappa + \kappa' \tanh 2\kappa' a}{2\kappa\kappa' + (\kappa^2 + \kappa'^2) \tanh 2\kappa' a} \\
 &= -\frac{V_0^2 a}{8\pi} \int_0^\infty dq \frac{1}{2q'} \frac{2q + q' \tanh 2q'}{2qq' + (q^2 + q'^2) \tanh 2q'} \tag{71}
 \end{aligned}$$

and is always negative. In the limit where  $v \gg 1$ , we can approximate  $\tanh 2q' \approx 1$  to get

$$E_0 = -\frac{V_0^{3/2}}{12\pi}. \tag{72}$$

The position-dependent part is

$E_1(x)$

$$\begin{aligned}
 &= \frac{1}{8\pi} \int_0^\infty d\kappa \frac{\kappa}{\kappa'} \\
 &\quad \times \frac{(2\kappa'^2 - \kappa^2) V_0 \cosh 2\kappa' x}{2\kappa\kappa' \cosh 2\kappa' a + (\kappa^2 + \kappa'^2) \sinh 2\kappa' a} \\
 &= \frac{V_0}{8\pi a} \int_0^\infty \frac{q dq}{q'} \frac{(2q'^2 - q^2) \cosh 2q' y}{2qq' \cosh 2q' + (q^2 + q'^2) \sinh 2q'} \tag{73}
 \end{aligned}$$

and is always positive.

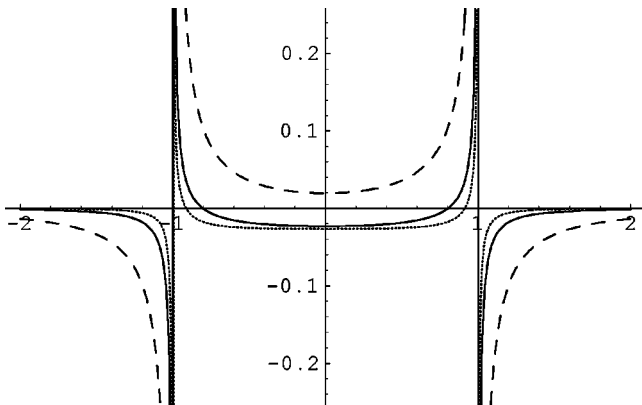


FIG. 1. Energy density in units of  $V_0^{3/2}$  for the square barrier of width 1 and heights 0.05 (dashed), 1 (solid), and 5 (dotted). As  $V_0$  increases, the positive energy becomes concentrated more and more near the edge of the barrier. In units of  $V_0^{3/2}$ , the outside energy decreases with  $V_0$ , but in absolute terms it approaches a fixed limit given by Eq. (65).

Note that the dominant term in the integrand in Eq. (73) is suppressed by  $e^{-2q'(1-y)} < e^{-2\sqrt{v}(1-y)}$ . Thus, far from the edge of the potential, where  $1-y \gg 1/\sqrt{v}$ ,  $E_1$  is negligible.

Close to the edge of the potential, with  $1-y \ll 1/\sqrt{v}$  and  $1-y \ll 1$ , the integral is  $1/[8(1-y)]$  and

$$E_1(x) \approx -\frac{V_0}{64\pi(a-x)} \tag{74}$$

which cancels, in a principal value sense, the divergence outside the barrier.

The sign of the energy density at the center of the barrier depends on the competition between the position-dependent and position-independent parts. For large  $v$ , the position-dependent part is suppressed in the center, and the energy density is negative. For small  $v$ , it is positive. The total energy density is shown for several values of  $v$  in Fig. 1.

It has long been known that the energy density near a perfectly reflecting boundary is zero if one uses the ‘‘conformally coupled’’ stress-energy tensor, but diverges if one uses the minimally coupled one, as we have done above. Kennedy, Critchley, and Dowker [18] argue that since the total energy is the same in the two cases, there must be a surface energy associated with the perfect conductor in the minimal case. Ford and Svaiter [19] found that the surface energy could be seen by allowing the boundary to fluctuate.

Here, we can see the situation by approximating a perfect conductor by a square barrier with  $a$  fixed and  $V_0 \rightarrow \infty$ . We can produce the conformal Hamiltonian density by including half the value of the total derivative term:

$$\mathcal{H}_{\text{conformal}} = \frac{1}{2} \dot{\phi}^2 + \frac{1}{8} \nabla^2(\phi^2) - \frac{1}{2} \phi \nabla^2 \phi + V(r) \phi^2. \tag{75}$$

This choice gives zero energy outside the barrier and removes the divergence of the energy density everywhere inside. With the minimally coupled Hamiltonian, the energy outside goes to Eq. (65) as  $V_0$  becomes large, while the positive energy inside clusters ever closer to the boundary, as shown in Fig. 1. Since the change to the total derivative term does not affect the total energy, we can see that the ‘‘surface energy’’ located just inside the boundary cancels the divergent negative energy outside.

## V. EXAMPLES WITH ONE RELEVANT DIMENSION AND TWO IRRELEVANT DIMENSIONS

### A. The general case

To carry out calculations in 3+1 dimensions, we now need subtractions proportional to  $V(x)$ ,  $V(x)^2$ , and  $V''(x)$ . We will use the renormalization scheme defined in Sec. III. For  $m=1$ , using Eq. (45) and evaluating Eq. (50) with  $m=1$  and  $n=2$  gives

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} = & -\frac{1}{12\pi^2} \int_{\mu}^{\infty} d\kappa \Omega^3 \left[ 2\kappa G(x,x,i\kappa) \right. \\ & - \frac{3\kappa}{2\Omega^2} \frac{d^2}{dx^2} G(x,x,i\kappa) - 1 + \frac{V(x)}{2\kappa^2} \\ & \left. - \frac{V(x)^2(12\kappa^2 - M^2)}{2\kappa^2(4\kappa^2 - M^2)^2} - \frac{V''(x)}{4(\kappa^2 - M'^2)\Omega^2} \right], \end{aligned} \quad (76)$$

where the Green's function has again been summed over the symmetric and antisymmetric channels. Again, we will restrict our attention to massless fields for simplicity.

### B. Outside a potential with compact support

The wave functions and Green's functions are just as in Sec. IV. Again, since the potential vanishes, the only counterterm is the vacuum energy. Thus Eq. (76) reduces to

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} = & -\frac{1}{12\pi^2} \int_0^{\infty} d\kappa \left[ \kappa^3 [2\kappa G(x,x,\kappa) - 1] \right. \\ & \left. - \frac{3}{2} \kappa^2 \frac{d^2}{dx^2} G(x,x,\kappa) \right] \end{aligned} \quad (77)$$

outside the potential, and so from Eq. (59),

$$\langle \mathcal{H} \rangle_{\text{ren}} = \frac{1}{6\pi^2} \int_0^{\infty} d\kappa \kappa^3 r(i\kappa) e^{-2\kappa x}. \quad (78)$$

In the large  $x$  limit, we can again take  $r(i\kappa) \approx r(0) = -1$ , to get

$$\langle \mathcal{H} \rangle_{\text{ren}} = -\frac{1}{6\pi^2} \int_0^{\infty} d\kappa \kappa^3 e^{-2\kappa x} = -\frac{1}{16\pi^2 x^4}, \quad (79)$$

a well-known result.

### C. Square barrier

Outside a square barrier with width  $a$  and height  $V_0$ , the reflection coefficient is given by Eq. (62), and the energy is

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} = & -\frac{V_0}{6\pi^2} \int_0^{\infty} d\kappa \frac{\kappa^3 e^{-2\kappa(x-a)} \tanh 2\kappa' a}{2\kappa\kappa' + (\kappa^2 + \kappa'^2) \tanh 2\kappa' a} \\ = & -\frac{V_0}{6\pi^2 a^2} \int_0^{\infty} dq \frac{q^3 e^{-2q(y-1)} \tanh 2q'}{2qq' + (q^2 + q'^2) \tanh 2q'}. \end{aligned} \quad (80)$$

Far from the potential, we approximate  $q' \approx \sqrt{v} \gg q$ . The integral is

$$\int_0^{\infty} dq q^3 \frac{e^{-2q(y-1)}}{v} = \frac{3}{8(y-1)^4 v} \quad (81)$$

and

$$\langle \mathcal{H} \rangle_{\text{ren}} \approx -\frac{1}{16\pi^2 (x-a)^4}, \quad (82)$$

in agreement with Eq. (79).

Close to the potential, we approximate  $q' \approx q$  and  $\tanh 2q' \approx 1$ . The integral becomes

$$\frac{1}{16(y-1)^2} \quad (83)$$

and

$$\langle \mathcal{H} \rangle_{\text{ren}} \approx -\frac{V_0}{96\pi^2 (x-a)^2}. \quad (84)$$

Inside the potential, we need the renormalized form,

$$\begin{aligned} \langle \mathcal{H} \rangle_{\text{ren}} = & -\frac{1}{12\pi^2} \int_0^{\infty} d\kappa \left[ \kappa^3 \left( 2\kappa G - 1 + \frac{V(x)}{2\kappa^2} \right. \right. \\ & \left. \left. - \frac{(12\kappa^2 - M^2)V(x)^2}{2\kappa^2(4\kappa^2 - M^2)^2} \right) - \frac{3}{2} \kappa^2 \frac{d^2}{dx^2} G \right], \end{aligned} \quad (85)$$

where  $M^2 < 0$  is the spacelike renormalization point.

For the square barrier, we get a position-independent part:

$$\begin{aligned} E_0 = & -\frac{1}{12\pi^2} \int_0^{\infty} d\kappa \\ & \times \left\{ \frac{\kappa^4 (\kappa^2 + \kappa'^2) \cosh 2\kappa' a + 2\kappa\kappa' \sinh 2\kappa' a}{\kappa' 2\kappa\kappa' \cosh 2\kappa' a + (\kappa^2 + \kappa'^2) \sinh 2\kappa' a} - \kappa^3 \right. \\ & \left. + \frac{\kappa V_0}{2} - \frac{\kappa(12\kappa^2 + \tilde{M}^2)V_0^2}{2(4\kappa^2 + \tilde{M}^2)^2} \right\} \\ = & -\frac{V_0^2}{12\pi^2} \int_0^{\infty} d\kappa \left\{ \frac{\kappa}{2\kappa'} \frac{2\kappa + \kappa' \tanh 2\kappa' a}{2\kappa\kappa' + (\kappa^2 + \kappa'^2) \tanh 2\kappa' a} \right. \end{aligned}$$

$$\begin{aligned}
 & -\frac{\kappa(12\kappa^2 + \tilde{M}^2)}{2(4\kappa^2 + \tilde{M}^2)^2} \Big\} \\
 = & -\frac{V_0^2}{12\pi^2} \int_0^\infty dq \left\{ \frac{q}{2q'} \frac{2q + q' \tanh 2q'}{2qq' + (q^2 + q'^2) \tanh 2q'} \right. \\
 & \left. - \frac{q(12q^2 + t^2)}{2(4q^2 + t^2)^2} \right\}, \quad (86)
 \end{aligned}$$

where  $\tilde{M}^2 = -M^2$  and  $t = \tilde{M}a$ .

We can isolate the dependence on the renormalization scale by using

$$\int_0^\infty dq \left\{ \frac{3}{8q'} - \frac{q(12q^2 + t^2)}{2(4q^2 + t^2)^2} \right\} = \frac{3}{16} \ln \frac{t^2}{v} + \frac{1}{8} \quad (87)$$

to obtain

$$\begin{aligned}
 E_0 = & -\frac{V_0^2}{12\pi^2} \left( \frac{3}{16} \ln \frac{\tilde{M}^2}{V_0} + \frac{1}{8} \right. \\
 & \left. + \int_0^\infty dq \left\{ \frac{q}{2q'} \frac{2q + q' \tanh 2q'}{2qq' + (q^2 + q'^2) \tanh 2q'} - \frac{3}{8q'} \right\} \right). \quad (88)
 \end{aligned}$$

In the limit where  $v \gg 1$ , we can approximate  $\tanh 2q' \approx 1$ , the integral gives  $-7/32$ , and we obtain

$$E_0 = \frac{V_0^2}{64\pi^2} \left[ \ln \frac{V_0}{\tilde{M}^2} + \frac{1}{2} \right], \quad (89)$$

consistent with the result obtained from the effective potential [20].

The position-dependent part is

$$\begin{aligned}
 E_1(x) = & \frac{1}{12\pi^2} \int_0^\infty d\kappa \\
 & \times \frac{\kappa^2}{\kappa'} \frac{(3\kappa'^2 - \kappa^2) V_0 \cosh 2\kappa' x}{2\kappa\kappa' \cosh 2\kappa' a + (\kappa^2 + \kappa'^2) \sinh 2\kappa' a} \\
 = & \frac{V_0}{12\pi^2 a^2} \int_0^\infty dq \\
 & \times \frac{q^2}{q'} \frac{(3q'^2 - q^2) \cosh 2q' y}{2qq' \cosh 2q' + (q^2 + q'^2) \sinh 2q'} \quad (90)
 \end{aligned}$$

and is always positive.

Far from the edge of the potential,  $E_1$  is negligible. Close to the edge, where we can approximate  $q' \approx q$  and  $\sinh 2q \approx \cosh 2q \approx e^{2q}/2$ , the integral becomes  $1/[8(1-y)^2]$ , and

$$E_1(x) \approx -\frac{V_0}{96\pi^2(a-x)^2}, \quad (91)$$

which cancels the divergence outside the barrier.

These results do not reflect any contribution from the  $V''(x)$  counterterm. In this case it vanishes for all  $|x| \neq a$ , since the potential is constant. Furthermore, the contribution to the total energy from this term is also zero, since it is a total derivative. If we imagine that the square barrier represents the limit in which a smooth potential gets steeper and steeper, we will find large equal and opposite contributions to the energy localized in the tiny region on both sides of the boundary. As long as we average over larger distance scales, this contribution will always cancel out, so it can be ignored in the square barrier limit.

#### D. The $\text{sech}^2$ potential in 3+1 dimensions

Finally, we consider the potential analyzed in 2+1 dimensions in Ref. [10],

$$V(x) = c^2 \text{sech}^2(x/a), \quad (92)$$

which arises frequently in soliton models. It is exactly solvable in terms of associated Legendre functions. For  $c^2 a^2 = -\ell(\ell+1)$  with integer  $\ell$  it becomes reflectionless. The Green's function at coincident points is

$$\begin{aligned}
 G(x, x, i\kappa) = & \frac{a}{2} \Gamma(1 + \kappa a + s) \Gamma(\kappa a - s) \\
 & \times P_s^{-\kappa a}[\tanh(x/a)] P_s^{-\kappa a}[-\tanh(x/a)], \quad (93)
 \end{aligned}$$

where  $P_s^\mu(x)$  is the associated Legendre function as defined in Ref. [21] for  $-1 < x < 1$ , and  $s = (\sqrt{1 - 4c^2 a^2} - 1)/2$ . Plugging this into Eq. (76), we have

$$\begin{aligned}
 \langle \mathcal{H} \rangle_{\text{ren}} = & -\frac{1}{12\pi^2} \int_\mu^\infty d\kappa \Omega^3 \left[ \frac{a}{2} \Gamma(1 + \kappa a + s) \Gamma(\kappa a - s) \right. \\
 & \times \left( 2\kappa - \frac{3\kappa}{2\Omega^2} \frac{d^2}{dx^2} \right) P_s^{-\kappa a}[\tanh(x/a)] \\
 & \times P_s^{-\kappa a}[-\tanh(x/a)] - 1 + \frac{c^2 \text{sech}^2(x/a)}{2\kappa^2} \\
 & - \frac{c^4 \text{sech}^4(x/a)(12\kappa^2 - M^2)}{2\kappa^2(4\kappa^2 - M^2)^2} \\
 & \left. + \frac{c^2 \text{sech}^2(x/a)(3\text{sech}^2(x/a) - 2)}{2a^2(\kappa^2 - M'^2)\Omega^2} \right], \quad (94)
 \end{aligned}$$

which can then be computed numerically. Figure 2 gives this energy density as a function of  $x$  for particular values of the parameters.

## VI. CONCLUSIONS

We have seen how to address the question of generation of negative energies through quantum fluctuations in the robust language of quantum field theory, where ambiguities

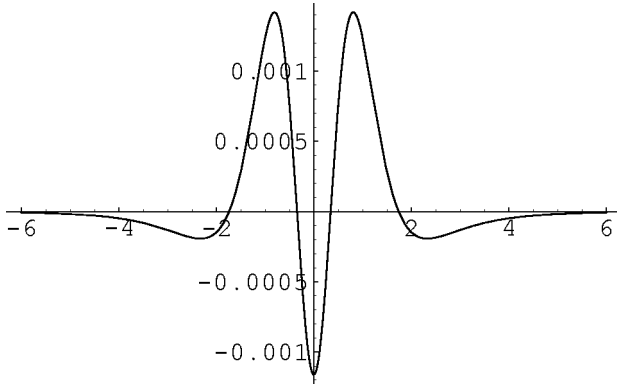


FIG. 2. Quantum energy density due to the potential of Eq. (92), for  $c = 1/a$ ,  $\mu = 0$ , and  $M^2 = M'^2 = -1/a^2$  in units where  $a = 1$ .

associated with idealized boundary conditions are absent. This approach implements standard renormalization procedures and is applicable to generic background potentials that are spherically symmetric in some dimensions and independent of the rest. Such potentials typically arise, for example, from topological defects or other extended objects. By using dimensional regularization, we have implemented a precise renormalization scheme, using only local subtractions for both the first- and second-order diagrams. We expect that this general formalism, together with fermion scattering theory in fractional dimensions developed in Ref. [8], will allow these results to be extended to fermions and gauge fields.

In the case of the square barrier, we have recovered the negative energy associated with perfect reflection at large distances from the barrier, and we have seen that the divergent negative energy outside the barrier is canceled by positive energy immediately inside. In a realistic example in which one includes the energy associated with the background potential, such cancellations might lead the averaged null energy condition to be obeyed even though the weak energy condition is violated [10]. Finally, we have calculated the energy density for a smooth background representing a domain wall in 3+1 dimensions.

*Note added in proof.* Reference [25] has calculated the surface tension for a bosonic  $\phi^4$  kink domain wall (and also its supersymmetric generalization) using an on-shell renormalization scheme, in space dimension one through four. In the language of the present paper, this calculation corresponds to the case of  $a = 2/\mu$  and  $c^2 = -3\mu^2/2$  in the potential of Sec. IV D, renormalized with  $M = \mu$ , setting  $m = 1$  and  $n$  to zero through three. The surface tension is obtained by then integrating this result over the one non-trivial dimension. (The choice of  $M'$  does not affect this calculation because the total derivative term integrates to zero.) Using the formulae in the present paper to carry out this calculation, we obtain results in agreement with the bosonic calculations in Ref. [25].

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#### APPENDIX A: CALCULATION OF $f$

By comparing the last line of Eq. (42) with the first equality in Eq. (43), we require

$$\begin{aligned} & \frac{1}{\Gamma\left(\frac{n+3}{2}\right)} \int_{\mu}^{\infty} \Omega^{n+1} \kappa^{m-1} \frac{\sqrt{\pi} f(\kappa, M)}{\Gamma\left(\frac{m}{2}\right)} d\kappa \\ &= \frac{1}{\Gamma\left(\frac{n+m}{2}\right)} \int_0^{\infty} \frac{q^{n+m-1}}{\omega(4\omega^2 - M^2)} dq, \end{aligned} \quad (\text{A1})$$

where  $\omega = \sqrt{q^2 + \mu^2}$  and  $\Omega = \sqrt{\kappa^2 - \mu^2}$ . Let us change variables on the left from  $\kappa$  to  $L = \Omega^2 = \kappa^2 - \mu^2$  and on the right from  $q$  to  $L = q^2$  to get

$$\begin{aligned} & \frac{1}{\Gamma\left(\frac{n+3}{2}\right)} \int_0^{\infty} L^{(n+1)/2} \kappa^{m-2} \frac{\sqrt{\pi} f(\kappa, M)}{\Gamma\left(\frac{m}{2}\right)} dL \\ &= \frac{1}{\Gamma\left(\frac{n+m}{2}\right)} \int_0^{\infty} \frac{L^{(n+m-2)/2}}{\omega(4\omega^2 - M^2)} dL \end{aligned} \quad (\text{A2})$$

with  $\kappa = \sqrt{L + \mu^2}$  on the left and  $\omega = \sqrt{L + \mu^2}$  on the right. We can write

$$\frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} dL L^{\alpha} h(L + \mu^2) = \frac{1}{\Gamma(1+\beta)} \int_0^{\infty} dL L^{\beta} j(L + \mu^2) \quad (\text{A3})$$

with

$$\alpha = (n+1)/2, \quad (\text{A4})$$

$$\beta = (n+m-2)/2, \quad (\text{A5})$$

$$h(x) = \frac{\sqrt{\pi} x^{m/2-1} f(\sqrt{x}, M)}{\Gamma\left(\frac{m}{2}\right)}, \quad (\text{A6})$$

$$j(x) = \frac{1}{\sqrt{x}(4x - M^2)}. \quad (\text{A7})$$

Denote the difference in the exponents as  $\delta = \alpha - \beta = (3 - m)/2$ . If  $\delta$  is a positive integer, the desired relationship is just integration by parts, and

$$h(x) = \left(-\frac{d}{dx}\right)^\delta j(x). \quad (\text{A8})$$

To extend this formula to non-integer  $\delta$ , we write  $j$  in terms of the hypergeometric function,

$$j(x) = \frac{1}{2\sqrt{\pi}} \left(\frac{4}{4x-M^2}\right)^d \Gamma(d) {}_2F_1\left(\frac{1}{2}, d; \frac{3}{2}; \frac{M^2}{M^2-4x}\right), \quad (\text{A9})$$

with  $d=3/2$ . The operator  $(-d/dx)$  just increments  $d$  in Eq. (A9), so we conjecture that the same relationship holds for all  $\delta$ , and thus that the desired  $h$  is given by Eq. (A9) with  $d=3/2+\delta=3-m/2$ . One can check that the conjecture is correct by explicitly performing the integrals in Eq. (A3), which both give

$$\frac{1}{2\sqrt{\pi}} \left(\mu^2 - \frac{M^2}{4}\right)^{\beta-1/2} \Gamma\left(\frac{1}{2}-\beta\right) \times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-\beta; \frac{3}{2}; \frac{M^2}{M^2-4\mu^2}\right). \quad (\text{A10})$$

Finally we find

$$\begin{aligned} f(\kappa, M) &= \frac{1}{2\pi} \Gamma\left(\frac{m}{2}\right) \Gamma\left(3-\frac{m}{2}\right) \left(\frac{4}{4\kappa^2-M^2}\right)^{3-m/2} \kappa^{2-m} \\ &\times {}_2F_1\left(\frac{1}{2}, 3-\frac{m}{2}; \frac{3}{2}; \frac{M^2}{M^2-4\kappa^2}\right) \\ &= \frac{2(m-4)(m-2)(2\kappa)^{2-m}}{(4\kappa^2-M^2)^{3-m/2} \sin\frac{\pi m}{2}} \\ &\times {}_2F_1\left(\frac{1}{2}, 3-\frac{m}{2}; \frac{3}{2}; \frac{M^2}{M^2-4\kappa^2}\right). \quad (\text{A11}) \end{aligned}$$

## APPENDIX B: LOCAL SUM RULES

### 1. General case

We used the analytic properties of the Green's function as a mathematical tool, enabling us to carry out calculations efficiently on the imaginary axis. In so doing, we avoided the apparent singularity in the gamma function coefficient of Eq. (35) for odd  $n$ . Nonetheless, this expression should be a valid result, finite for  $m+n < 3$ . As in the case of the total energy [6], the quantity in brackets must vanish for  $n=1$  in each partial wave individually. Furthermore, the combination of the first two terms in brackets vanishes separately from the total derivative term. These cancellations depend on a local analogue of the sum rules for the phase shift given in Refs. [22,23], which we demonstrate below.

When similar apparent divergences arise in the calculation of the total energy, they are canceled according to generalizations of Levinson's theorem [22,23]. For a system with

spherical symmetry, in each partial wave  $\ell$  these sum rules take the form

$$\sum_j (-\kappa_{\ell j}^2)^N + \int_0^\infty k^{2N} \frac{d}{dk} \left( \delta_\ell(k) - \sum_{s=1}^N \delta_\ell^{(s)}(k) \right) dk = 0, \quad (\text{B1})$$

where the bound states have  $k_{\ell j} = i\kappa_{\ell j}$ ,  $\delta_\ell(k)$  is the scattering phase shift, and  $\delta_\ell^{(s)}(k)$  is the scattering phase shift computed at order  $s$  in the Born approximation.<sup>4</sup> The  $N=0$  case gives Levinson's theorem. Like Levinson's theorem, these identities apply to general potentials in scattering theory and hold in each partial wave  $\ell$  individually. Also like Levinson's theorem, they are modified for the case of the symmetric channel in one dimension, as discussed in Ref. [23].

We have a relationship [11,12] between the phase shift, the change in the density of states, and the norm of the wave function

$$\begin{aligned} \frac{1}{\pi} \frac{d\delta_\ell(k)}{dk} &= \rho_\ell(k) - \rho_\ell^{(0)}(k) \\ &= \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} \frac{1}{\pi} \int_0^\infty dr r^{m-1} [|\psi_k^\ell(r)|^2 - |\psi_k^{\ell(0)}(r)|^2], \quad (\text{B2}) \end{aligned}$$

where the zero superscript indicates a quantity evaluated in the free case. This equation also holds order by order in the Born approximation. Using these relations we can rewrite Eq. (B1) as

$$\begin{aligned} \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} \int dr r^{m-1} \left[ \sum_j (-\kappa_{\ell j}^2)^N |\psi_j^\ell(r)|^2 \right. \\ \left. + \frac{1}{\pi} \int_0^\infty k^{2N} \left( |\psi_k^\ell(r)|^2 - \sum_{s=0}^N |\psi_k^{\ell(s)}(r)|^2 \right) dk \right] = 0, \quad (\text{B3}) \end{aligned}$$

where  $\psi_k^{\ell(s)}(r)$  is the Born approximation to the wave function computed at order  $s$  (the free wave function is the order zero term). The identities we need for the present application are simply the slightly stronger condition that Eq. (B3) holds for each  $r$  individually, rather than just as an integral. We can exploit the connection to the Green's function that was used in Refs. [22,23] to prove this result as well.

The case of  $N=0$  is particularly simple, because we know that

<sup>4</sup>In general, these identities continue to hold even if one subtracts  $N'$  orders in the Born approximation for any  $N' \geq N$ . However, there are some restrictions on such oversubtractions in the symmetric channel in one dimension [23].

$$\sum_j |\psi_j^\ell(r)|^2 + \frac{1}{\pi} \int_0^\infty [|\psi_k^\ell(r)|^2 - |\psi_k^{\ell(0)}(r)|^2] dk = 0 \quad (\text{B4})$$

by completeness; it is just the difference between the expectation value of a constant computed in the free and interacting bases. (After summing over the spectrum, each term is independent of  $r$ .) For higher  $N$ , we would like to show that

$$\begin{aligned} & \sum_j (-\kappa_{\ell j}^2)^N |\psi_j^\ell(r)|^2 \\ & + \frac{1}{\pi} \int_0^\infty k^{2N} \left( |\psi_k^\ell(r)|^2 - \sum_{s=0}^N |\psi_k^{\ell(s)}(r)|^2 \right) dk \end{aligned} \quad (\text{B5})$$

is zero.<sup>5</sup> We employ the relationship in Eq. (36) between the norm of the wave function and the Green's function to rewrite this expression as

$$\begin{aligned} & \sum_j (-\kappa_{\ell j}^2)^N |\psi_j^\ell(r)|^2 \\ & + \frac{1}{\pi} \int_{-\infty}^\infty k^{2N+1} \text{Im} \left( G_\ell(x, x, k) - \sum_{s=0}^N G_\ell^{(s)}(x, x, k) \right) dk, \end{aligned} \quad (\text{B7})$$

where we have extended the integral to the entire  $k$  axis by the symmetry of the integrand. To show this expression is zero, we would like to do the  $k$  integral as a contour, closed in the upper half plane. The singularities in the full Green's function correspond to bound states, and will exactly cancel the explicit contribution from the bound states [16]. The Born approximation has no singularities (since it does not see the bound states). Thus we are left with the contour at infinity. However, it does not contribute because we have subtracted enough Born approximations to ensure that the integrand falls like  $1/|k|^2$  at large  $|k|$  [24].

## 2. The symmetric channel

In one dimension, we have to consider the symmetric channel, which can have additional singularities at  $k=0$ . Such singularities, for example, lead to an extra  $1/2$  in Levinson's theorem [23], relating the phase shift at  $k=0$  to the number of bound states. We have

$$\delta_S(0) = \pi \left( n_S - \frac{1}{2} \right) \quad (\text{B8})$$

<sup>5</sup>As shown in Ref. [9], for reflectionless potentials in one dimension there is a stronger version of the first local sum rule,

$$\sum_j |\psi_j(r)|^2 \frac{2\kappa_j}{\kappa_j^2 + k^2} + |\psi_k(r)|^2 - |\psi_k^{(0)}(r)|^2 = 0, \quad (\text{B6})$$

which reduces to Eq. (B4) when integrated over  $k$ . It might be possible to find analogous results for the higher sum rules as well.

as opposed to the usual

$$\delta(0) = \pi n. \quad (\text{B9})$$

Analogously in our problem, Eq. (B4) must be modified to

$$\sum_j |\psi_j^S(x)|^2 - \frac{1}{2L} + \frac{1}{\pi} \int_0^\infty [|\psi_k^S(x)|^2 - |\psi_k^{S(0)}(x)|^2] dk = 0, \quad (\text{B10})$$

where  $L$  is the size of the system. Subtracting  $1/(2L)$  reflects the contribution from the state  $\psi(x) = \text{const}$  in the free spectrum. This state is ‘‘half-bound’’: While any potential will have a  $k=0$  state in the symmetric channel, in this case the wave function goes to a constant at infinity. Such states contribute to the spectrum with half the usual residue for a bound state, as the name indicates. (Generically a state with  $k=0$  will approach a line with nonzero slope, in which case no special treatment is necessary.) If a potential has a half-bound state, making the potential arbitrarily more attractive introduces a new bound state in the theory, and making it arbitrarily more repulsive eliminates the half-bound state.<sup>6</sup> There will be an analogous contribution to the energy density, so this term will cancel when we pass from Eq. (27) to Eq. (29) and the rest of the derivation of the energy density is unchanged.

For the other sum rules needed in our problem, however, we always multiply by enough powers of  $k$  to cancel any anomalous effects coming from states at  $k=0$ . We would have to be more careful if we do additional Born ‘‘oversubtractions,’’ in which case we could encounter additional terms analogous to those found in Ref. [23]. We can always avoid these problems as long as each ultraviolet Born subtraction is preceded by a corresponding infrared Levinson subtraction. For the first Born subtraction, the corresponding Levinson subtraction was done using Eq. (B4). Higher Levinson subtractions would use local analog of the higher sum rules in Ref. [6].

## 3. Local subtraction

For Casimir calculations it will be convenient to slightly modify the  $N=1$  sum rule. Our renormalization procedure subtracts not the full first Born approximation, but rather just a local part of it. However, this replacement does not affect the sum rule. For example, to apply the results of Sec. III B for  $m=n=1$ , we write

<sup>6</sup>A *reflectionless* potential will always have a half-bound state, because it must have  $\delta_S(k) = \delta_A(k)$  for all  $k$ . If this equality is to hold at  $k=0$ , to reconcile Eqs. (B8) and (B9) there must be a half-bound state, which contributes only a half to the number of bound states. The half-bound state in the free case (which is reflectionless) is just a consequence of this requirement.

$$\begin{aligned}
 & \sum_{x=+,-} |\psi_k(x)|^2 \\
 &= 1 + \int_x^\infty dy \frac{V(y)}{k} \sin 2k(y-x) + \dots \\
 &= 1 + \frac{V(x)}{2k^2} + \int_x^\infty dy \frac{V'(y)}{2k^2} \cos 2k(y-x) + \dots \\
 &= 1 + \frac{V(x)}{2k^2} - \int_x^\infty dy \frac{V''(y)}{4k^3} \sin 2k(y-x) + \dots \\
 &= 1 + \frac{V(x)}{2k^2} - \frac{V''(x)}{8k^4} - \int_x^\infty dy \frac{V'''(y)}{8k^4} \cos 2k(y-x) + \dots \\
 &= 1 + \frac{V(x)}{2k^2} - \frac{V''(x)}{8k^4} + \int_x^\infty dy \frac{V''''(y)}{16k^5} \sin 2k(y-x) + \dots
 \end{aligned} \tag{B11}$$

and subtract only the term directly proportional to  $V(x)$ , rather than all terms that are first order in the strength of the potential. However, the additional terms, proportional to the derivatives of  $V(x)$ , do not introduce any singularities in the integral and do not affect the contour at infinity because they fall like  $1/k^4$  or faster. Therefore, this modification does not affect the proof of the sum rule. This result allows us to apply the sum rule to Eq. (35).

#### 4. One irrelevant dimension

With the sum rules in hand, we can now extract a finite result from Eq. (35). Near  $n=1$  we have

$$\Gamma\left(-\frac{n+1}{2}\right) \approx \frac{2}{n-1} \quad \text{and} \quad a^{n-1} \approx 1 + \frac{n-1}{2} \log a^2 \tag{B12}$$

so that in the  $n \rightarrow 1$  limit we obtain

$$\begin{aligned}
 \langle \mathcal{H} \rangle_{\text{ren}} = & -\frac{1}{8\pi} \sum_\ell D_\ell^m \left\{ \sum_j (\omega_j^\ell)^2 \log(\omega_j^\ell)^2 |\psi_j^\ell(r)|^2 \right. \\
 & + \int_0^\infty \frac{dk}{\pi} \omega^2 \log \omega^2 \\
 & \times \left[ |\psi_k^\ell(r)|^2 - |\psi_k^{\ell(0)}(r)|^2 \left( 1 + (2-m) \frac{V(r)}{2k^2} \right) \right] \\
 & + \frac{1}{2} D_r^2 \left( \sum_j \log(\omega_j^\ell)^2 |\psi_j^\ell(r)|^2 \right. \\
 & \left. \left. + \int_0^\infty \frac{dk}{\pi} \log \omega^2 |\psi_k^\ell(r)|^2 \right) \right\}. \tag{B13}
 \end{aligned}$$

The local sum rule ensures that the scale of the logarithm does not affect the final result. In addition, the limit  $\mu \rightarrow 0$  is smooth (except when  $n=0$  and  $m=1$ , where we have the usual infrared divergences of one-dimensional field theory). If we extend the range of integration in Eq. (B13) as in Eq. (12), and then close the contour in the upper half plane, the branch cut associated with  $\log \omega^2$  will reproduce Eq. (15).

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