Field theoretic approach to structure formation in an anisotropic medium

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(Received 17 October 2002; published 25 April 2003)

Considering a real scalar field distribution which is assumed to be locally anisotropic and coupled to a Bianchi type-I background spacetime, the energy density and pressure associated with the anisotropic matter field distribution are evaluated. The vanishing of the expectation values of the nondiagonal components of $T_{\mu\nu}$ allows us to treat the scalar field in complete analogy with the distribution of fluid. The primeval density perturbations produced by the vacuum fluctuations of the scalar field are considered and the Jeans criterion for structure formation is obtained. The metric and matter field perturbations are considered and it is found that for the present anisotropic case the perturbations of the pressure in the radial and tangential directions are different. The Jeans instability is discussed and the Jeans wave number for the present case is evaluated. It is found that for the anisotropic case the Jeans length depends on the velocity of the fluctuations in the radial and transverse directions and thus on the direction of propagation of the perturbations.

DOI: 10.1103/PhysRevD.67.084029 PACS number(s): 04.62.+v, 98.80.Bp, 98.80.Cq

I. INTRODUCTION

Superdense matter may be anisotropic, at least in certain density ranges. It is of considerable interest to determine the extent to which local anisotropy can alter the structure of massive objects [1].

Gravity is the dominant force that governs the large scale dynamics of the universe. The standard theory of cosmological structure formation [2–4] is based on the idea of gravitational instability [5–7], according to which small initial irregularities in the distribution of matter get amplified by the attractive nature of gravity. Small fluctuations in the density result in gravitational instability and gravitational instability causes the growth of perturbations in an expanding universe [8,9]. The structure we observe in the universe today is the end result of the gravitational amplification of small primeval perturbations [10].

Jeans considered the problem of the formation of galaxies in the universe as a process involving the interplay between gravitational attraction and the pressure force acting on a mass of nonrelativistic fluid [8]. Jeans explained that, starting from a homogeneous and isotropic fluid, small fluctuations in the density and velocity could evolve with time. He showed that density fluctuations can grow in time if the stabilizing effect of pressure is much smaller than the tendency of the self-gravity of a density fluctuation to induce collapse. When the pressure inside the perturbed fluid is greater than the self-gravity, the perturbation will propagate like an acoustic wave. There exists a critical wave number, the Jeans wave number k_J, above which disturbances cannot grow but only oscillate like acoustic waves.

Chan, Herrera, and Santos [11] studied the role played by local pressure anisotropy in the onset of instabilities and it has been shown that small anisotropies may drastically change the stability of the system. Herrera and Santos [12] studied the Jeans instability criterion for an interstellar gas which can produce anisotropies during its evolution.

The quantization of linear cosmological fluctuations [13,14] is not more complicated than the quantization of matter fields in an external background: it is a straightforward application of canonical quantization [15].

In the present work we consider a distribution of matter which is assumed to be locally anisotropic. The anisotropic fluid distribution will have different values for the transverse and radial components of pressure. The density perturbations in such a distribution are studied. We are interested in studying the type of effects that can result from anisotropy. In Sec. II we consider a real scalar field ϕ gravitationally coupled to an anisotropic Bianchi type-I background spacetime. The energy-momentum tensor expectation values are calculated and the energy density and pressure associated with the anisotropic matter field distribution are evaluated. The application of classical Jeans theory to the scalar field is conditioned by the vanishing of the expectation values of the nondiagonal components of the energy-momentum tensor. Vanishing of the expectation values of the nondiagonal components of $T_{\mu\nu}$ allows one to treat the scalar field in complete analogy to the distribution of fluid. The primeval density perturbations produced by the vacuum fluctuations of the scalar field are considered and the Jeans criterion for structure formation is obtained in Sec. III. The metric and matter field perturbations are considered in Sec. IV and it is found that for the present anisotropic case the perturbations of pressure in the radial and tangential directions are different. The Jeans instability is discussed and the Jeans wave number for the present case is evaluated in the next section. The discussions and conclusions are presented in Sec. VI.

II. ENERGY DENSITY AND PRESSURE ASSOCIATED WITH THE ANISOTROPIC MATTER FIELD DISTRIBUTION

Consider a real scalar matter field distribution ϕ which is assumed to be locally anisotropic, gravitationally coupled to a background spacetime and described by the Lagrangian density

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$$\mathcal{L}=\sqrt{-g}\left\{\frac{1}{2}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-(m^2+\xi R)\phi^2]\right\}. \quad (1)$$

The stress-energy tensor $T_{\mu\nu}$ is

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}L, \quad (2)$$

where $L=(-g)^{-1/2}\mathcal{L}$. In the gravitationally coupled case,

$$T_{\mu\nu}=(1-2\xi)\partial_\mu\phi\partial_\nu\phi+(2\xi-\tfrac{1}{2})g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi -2\xi\phi\nabla_\mu\nabla_\nu\phi+2\xi g_{\mu\nu}\phi\Box\phi-\xi G_{\mu\nu}\phi^2+\frac{m^2}{2}g_{\mu\nu}\phi^2. \quad (3)$$

Let the scalar field ϕ be gravitationally coupled to a (3 +1)-dimensional Bianchi type-I background spacetime which is spatially homogeneous with small anisotropy and has the line element

$$ds^2=dt^2-\sum_{i=1}^{3}a_i^2(t)(dx^i)^2. \quad (4)$$

The conformal time transformation is defined as $\partial t =C^{1/2}\partial\eta$ where $C=(a_1a_2a_3)^{2/3}$. Taking the expectation values of the components of $T_{\mu\nu}$ it is found that the nondiagonal terms vanish [16]. This allows one to treat the scalar field in complete analogy to the distribution of fluid. The similarity of the gravitational instabilities of a free scalar field and dustlike matter was pointed out by Turner [17]. For a perfect fluid we have the relation [2,3]

$$\langle T^{\mu\nu}\rangle=(\rho-p)u^\mu u^\nu-pg^{\mu\nu}, \quad (5)$$

where $u^\mu=(1,0,0,0)$, the comoving velocity. The energy density and pressure associated to the matter field distribution are

$$\rho=\langle T_0^0\rangle \quad \text{and} \quad p=-\langle T_i^i\rangle. \quad (6)$$

For anisotropic matter the energy-momentum tensor is $T_\nu^\mu=\mathrm{diag}(\rho,-p_r,-p_\theta,-p_\phi)$. Let us consider the case with $p_\theta=p_\phi$, which we denote by $p_\perp$, called the tangential pressure, and p_r the radial pressure. So we can write

$$p_r=-\langle T_r^r\rangle \quad \text{and} \quad p_\perp=-\langle T_\theta^\theta\rangle=-\langle T_\phi^\phi\rangle. \quad (7)$$

With the Bianchi type-I background spacetime and from Einstein's field equation we obtain [16] the set of equations

$$8\pi\rho=-\left[2\frac{\dot R_1\dot R_2}{R_1R_2}+\frac{\dot R_2^2}{R_2^2}\right]-\frac{3k}{R_1^2}+\frac{1}{r^2}\left[\frac{1}{R_1^2}-\frac{1}{R_2^2}\right],$$

$$8\pi p_r=-\left[2\frac{\ddot R_2}{R_2}+\frac{\dot R_2^2}{R_2^2}+\frac{k}{R_1^2}\right]+\frac{1}{r^2}\left[\frac{1}{R_1^2}-\frac{1}{R_2^2}\right],$$

$$8\pi p_\perp=-\left[\frac{\ddot R_1}{R_1}+\frac{\ddot R_2}{R_2}+\frac{\dot R_1\dot R_2}{R_1R_2}+\frac{k}{R_1^2}\right]. \quad (8)$$

III. PRIMEVAL DENSITY PERTURBATIONS AND THE JEANS CRITERION

Some residual fluctuations will always be in the scalar field. The primordial quantum fluctuations of the scalar field $\delta\phi(x,t)$ get amplified and evolve to become classical seed perturbations [3,4,18,19]. Such a primeval perturbation is regarded as a viable candidate for the origin of large scale structure. Take the matter in the universe to be a scalar field minimally coupled to gravity and described by the Lagrangian density in Eq. (1). In the conformal time coordinates the field equation is

$$\ddot\phi+2\tilde H\dot\phi-\nabla^2\phi+a^2V'(\phi)=0 \quad (9)$$

where $V'=\partial V/\partial\phi$ and $\tilde H=\dot a/a$ using conformal time.

We can split the field into unperturbed and perturbed parts as

$$\phi=\phi_0+\delta\phi. \quad (10)$$

Perturbing Eq. (9) and linearizing, we get the equation of motion for the scalar field perturbations as

$$(\ddot{\delta\phi})+2\tilde H(\dot{\delta\phi})-v_s^2\nabla^2(\delta\phi)+m^2a^2(\delta\phi)=0 \quad (11)$$

where we have $\nabla^2=-k^2/a^2$ and $m^2=V''$. v_s is the speed of propagation of the perturbation [13,19,20]. $v_s^2=1$ for scalar field matter and $v_s^2=\delta p/\delta\rho$ for ideal gas matter [19,20]. Neglecting the last term [18–20], for cosmological perturbations we can rewrite the above equation as

$$(\ddot{\delta\phi})+2\tilde H(\dot{\delta\phi})+\frac{v_s^2k^2}{a^2}(\delta\phi)=0. \quad (12)$$

The perturbations $\delta\phi(x,t)$ in the scalar field leads to perturbations in the energy density $\delta\rho(x,t)$. We consider that the perturbations $\delta\phi(x,t)$ are the only reason for the energy density perturbations $\delta\rho(x,t)$. So using Eq. (12) we can write

$$\ddot\delta+2\tilde H\dot\delta+\frac{v_s^2k^2}{a^2}\delta=0, \quad (13)$$

where $\delta=\delta\rho/\rho$ is the density contrast parameter. Since the scalar field is coupled to the gravitational background field, we have to consider the effects of the gravitational field potential also. Including the gravitational field potential, the above equation becomes

$$\ddot\delta+2\tilde H\dot\delta=\left(4\pi G\rho-\frac{v_s^2k^2}{a^2}\right)\delta. \quad (14)$$

The primeval density perturbations satisfy the adiabatic condition [18] for density contrasts. For cosmological perturbations v_s is the speed of sound.

The above equation tells us how or whether gravitational instability leads to the growth of condensation in the expanding universe. The right hand side of the equation shows the competing effects of gravity and the pressure gradient force. At very long wavelength $k \to 0$, the equation reduces to the zero pressure case. At very short wavelength, large k, the pressure term dominates and δ tends to oscillate as a sound wave. The pressure and gravity terms balance when the wavelength is equal to the Jeans length given by

$$\lambda_J = v_s(\pi/G\rho)^{1/2}, \tag{15}$$

which corresponds to the classical Jeans criterion [2–8] for the formation of structure in the universe.

IV. METRIC AND MATTER FIELD PERTURBATIONS

To model the universe more realistically the perturbation in the background metric also is to be included. The perturbation $\delta\phi(x,t)$ leads to perturbations in the energy density $\delta\rho(x,t)$ and hence in the metric of spacetime. In this case, it is convenient to split the metric into two parts, the first being the background metric and the other describing how the background spacetime deviates from the idealized background model:

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu}. \tag{16}$$

The metric perturbations are of three distinct types: scalar, vector, and tensor perturbations. Neither vector nor tensor perturbations exhibit instabilities. Vector perturbations decay kinematically in an expanding universe, whereas tensor perturbations lead to gravitational waves which do not couple to the energy density and pressure inhomogeneities. However, scalar perturbations may lead to growing inhomogeneities [13,18], which, in turn have an important effect on the dynamics of matter.

The energy-momemtum tensor can also be decomposed into background and perturbed parts:

$$T^\mu_\nu = {}^{(0)}T^\mu_\nu + \delta T^\mu_\nu, \tag{17}$$

where δT^μ_ν is linear in matter and the metric perturbations $\delta\phi$ and $\delta g_{\alpha\beta}$. Substituting Eqs. (10) and (16) into Eq. (3) we obtain the background energy-momentum tensor for the minimally coupled case in conformal time as

$${}^{(0)}T^\eta_\eta = \frac{1}{2C}\dot\phi_0^2 + \frac{1}{2}\sum_{i=1}^{3} \frac{1}{a_i^2}(\partial_i\phi_0)^2 + \frac{m^2}{2}\phi_0^2 \tag{18}$$

and

$${}^{(0)}T^i_i = -(\partial_i\phi_0)^2 - \frac{1}{2C}\dot\phi_0^2 + \frac{1}{2}\left[\sum_{i=1}^{3} \frac{1}{a_i^2}(\partial_i\phi_0)^2\right] + \frac{m^2}{2}\phi_0^2, \tag{19}$$

and the first-order perurbation as

$$\delta T^\eta_\eta = \frac{1}{C}\dot\phi_0(\dot{\delta\phi}) + \frac{1}{2}\delta g^{\eta\eta}\dot\phi_0^2 - \frac{1}{2C^2}\delta g_{\eta\eta}\dot\phi_0^2 - \frac{1}{2}\delta g^{ii}(\partial_i\phi_0)^2 + \frac{1}{2C}\frac{1}{a_i^2}\delta g_{\eta\eta}(\partial_i\phi_0)^2 + \frac{1}{a_i^2}(\partial_i\phi_0)(\partial_i\delta\phi) + m^2\phi_0\delta\phi \tag{20}$$

and

$$\delta T^i_i = -\frac{1}{a_i^2}(\partial_i\phi_0)(\partial_i\delta\phi) - \frac{1}{C}\dot\phi_0(\dot{\delta\phi}) - \frac{1}{2}\delta g^{00}\dot\phi_0^2 + \frac{1}{2C}\frac{1}{a_i^2}\delta g_{ii}\dot\phi_0^2 - \frac{1}{2}\delta g^{ii}(\partial_i\phi_0)^2 - \frac{1}{2}\frac{1}{a_i^4}\delta g_{ii}(\partial_i\phi_0)^2 + m^2\phi_0\delta\phi. \tag{21}$$

From Eq. (21) it is clear that the perturbation in pressure is different in different directions and $\delta p_r \neq \delta p_\perp$. This implies that the velocity of perturbations will be different in the radial and transverse directions.

V. JEANS INSTABILITY AND JEANS WAVE NUMBER

Any region with very slightly higher density will gravitationally attract matter from surrounding regions and thereby increase in density. Correspondingly, any regions of density lower than average will have matter removed by the gravitational attraction of neighboring regions. So long as the pressure forces are negligible an overdense region is expected to accrete material from its surroundings by the gravitational attraction and thus becoming even more dense. The denser it becomes, the more it will accrete, resulting in an instability which can ultimately cause the collapse of a fluctuation to a gravitationally bound object. Jeans calculations were done in the context of a static background fluid [5]. The theory of instabilities of an expanding universe was given by Lifshitz in 1946 [21]. He also concluded that there exists a critical wave number k_J above which disturbances cannot grow but can only oscillate like sound waves.

From the definition of sound velocity of adiabatic perturbations we get the expression for velocity perturbations in the radial direction,

$$v_{s_r}^2 = \frac{\delta p_r}{\delta\rho}, \tag{22}$$

and in the tranverse direction,

$$v_{s_\perp}^2 = \frac{\delta p_\perp}{\delta\rho}, \tag{23}$$

where δp_r and $\delta p_\perp$ denote the first-order fluctuation amplitudes of the corresponding quantities.

Let us take the scale factor as

$$a_i(t)=t^{2/3}[1+b_i(x_i)]. \tag{24}$$

The term $b_i(x_i)$ causes anisotropy in the scale factors, which we take as time independent. The above form of the scale factor gives a direction independent Hubble constant, as is expected in the actual case. When $b_i(x_i)=0$ the above form leads to the isotropic case. The Hubble constant and deceleration parameter are

$$H=\frac{\dot{a}}{a}=\frac{2}{3t} \quad \text{and} \quad q=-\frac{\ddot{a}}{a}\frac{1}{H^2}=\frac{1}{2}. \tag{25}$$

For $k=0$, the spatially flat case [2],

$$\rho=\frac{1}{6\pi G t^2}. \tag{26}$$

The equation that tells us how or whether gravitational instability leads to the growth of condensation in the expanding universe is given by Eq. (14) and for the present anisotropic case we can rewrite it as

$$\ddot{\delta}+2H\dot{\delta}+(v_{s_i}^2K_i^2-4\pi G\rho)\delta=0, \tag{27}$$

where K_i is the physical wave number [3,4] of perturbations in the ith direction.

For a general specific heat ratio γ, the pressure varies as ρ^γ and the speed of sound is

$$v_{s_r}=\left(\frac{\gamma p_r}{\rho}\right)^{1/2}=\left(\frac{\gamma\rho^\gamma}{\rho}\right)^{1/2}\propto\rho^{(\gamma-1)/2} \tag{28}$$

and

$$v_{s_\perp}=\left(\frac{\gamma p_\perp}{\rho}\right)^{1/2}=\left(\frac{\gamma\rho^\gamma}{\rho}\right)^{1/2}\propto\rho^{(\gamma-1)/2},$$

and Eq. (26) implies that

$$t^{-2}\propto\rho. \tag{29}$$

From Eqs. (28) and (29) we can find that

$$v_s\propto t^{1-\gamma} \tag{30}$$

and Eq. (27) now takes the form

$$\ddot{\delta}+\frac{4}{3t}\dot{\delta}+\left(\frac{\Lambda^2}{t^{2(\gamma-1)}}-\frac{2}{3t^2}\right)\delta=0 \tag{31}$$

where

$$\Lambda^2=t^{2(\gamma-1)}v_{s_i}^2K_i^2. \tag{32}$$

The solutions of Eq. (31) are

$$\delta_\pm\propto t^{-1/6}J_{\mp 5/2}(3\Lambda t^{1/3}) \tag{33}$$

where $J_n(x)$ are Bessel functions of the first kind. The Bessel function $J_n(x)$ oscillates for $x\gg 1$. For $x<1$, both growing and damped modes are present and the growing modes dominate over the decaying modes.

The critical condition $x=1$ gives

$$t^{1/3}\approx\Lambda. \tag{34}$$

Equations (32) and (34) together imply that

$$t^{-2}\sim 6\pi G\rho\sim v_{s_i}^2K_i^2, \tag{35}$$

which corresponds to the classical Jeans criterion. Thus we get the classical Jeans length for the perturbations in the ith direction as

$$\lambda_{J_i}=2\pi/K_{J_i},$$

where

$$K_{J_i}^2\sim\frac{6\pi G\rho}{v_{s_i}^2}. \tag{36}$$

Equation (36) shows that the Jeans length depends on the velocity of fluctuations in the radial and tranverse directions and thus on the direction of wave propagation.

VI. DISCUSSION AND CONCLUSIONS

Chan *et al.* [11] showed that small anisotropies may, in principle, drastically change the stability of the system. Herrera and Santos [12] showed that for systems with anisotropic pressures instabilities may develop for masses of several orders of magnitude smaller than the corresponding Jeans mass for an ideal locally isotropic gas. Khlopov and co-workers [7] discussed the gravitational instability of a free scalar field and a self-interacting scalar field. The Jeans wave number is obtained from the solution of the dispersion relation of the perturbations of a scalar field. Jetzer and Scialom considered linear scalar mode perturbations and they obtained an expression for the Jeans wave number starting from the general relativistic wave equations and solving the dispersion relation [22].

In the present work a scalar field approach to the Jeans mass calculation is discussed. We discuss the possibility of using the Jeans instability mechanism to form self-gravitating configurations from an anisotropic field distribution. We consider the distribution of a matter field which is assumed to be locally anisotropic and is coupled to an anisotropic background space-time. The energy density and pressure associated with the anisotropic matter field distribution are evaluated. The vanishing of the expectation values of the nondiagonal components of $T_{\mu\nu}$ allows one to treat the scalar field in complete analogy to the fluid distribution. The primeval density perturbations produced by the vacuum fluctuations of the scalar field are considered and the Jeans criterion for structure formation is obtained. Considering the metric and matter field perturbations, it is found that for the present anisotropic case the perturbations of pressure in the radial and tangential directions are different. This implies that the velocity of perturbations will be different in the radial and tangential directions. The Jeans instability is discussed and the Jeans wave number for the present case is evaluated. It is found that the Jeans length depends on the velocity of fluctuations in the radial and transverse directions and thus on the direction of propagation of the fluctuations.

ACKNOWLEDGMENTS

The authors would like to thank Professor Wayne Hu for useful discussions. One of us (M.J.) would like to thank UGC, New Delhi for financial support. V.C.K. acknowledges IUCAA, Pune.

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