Bohm and Einstein-Sasaki metrics, black holes, and cosmological event horizons

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We study physical applications of the Bohm metrics, which are infinite sequences of inhomogeneous Einstein metrics on spheres and products of spheres of dimension $5 \leqslant d \leqslant 9$. We prove that all the Bohm metrics on $S^3 \times S^2$ and $S^3 \times S^3$ have negative eigenvalue modes of the Lichnerowicz operator acting on transverse traceless symmetric tensors, and by numerical methods we establish that Bohm metrics on S^5 have negative eigenvalues too. General arguments suggest that all the Bohm metrics will have negative Lichnerowicz modes. These results imply that generalized higher-dimensional black-hole spacetimes, in which the Bohm metric replaces the usual round sphere metric, are classically unstable. We also show that the classical stability criterion for Freund-Rubin solutions, which are products of Einstein metrics with anti–de Sitter spacetimes, is the same in all dimensions as that for black-hole stability, and hence such solutions based on the Bohm metrics will also be unstable. We consider possible end points of the instabilities, and in particular we show that all Einstein-Sasaki manifolds give stable solutions. Next, we show how analytic continuation of Bohm metrics gives Lorentzian metrics that provide counterexamples to a strict form of the cosmic baldness conjecture, but they are nevertheless consistent with the intuition behind the cosmic no-hair conjectures. We indicate how these Lorentzian metrics may be created ‘‘from nothing’’ in a no-boundary setting. We argue that Lorentzian Bohm metrics are unstable to decay to de Sitter spacetime. Finally, we argue that noncompact versions of the Bohm metrics have infinitely many negative Lichnerowicz modes, and we conjecture a general relationship between Lichnerowicz eigenvalues and nonuniqueness of the Dirichlet problem for Einstein's equations.

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I. INTRODUCTION

The properties of higher-dimensional black holes, and more generally spacetimes with event horizons, have come to play an increasingly important role in physics. This is not only for the purely theoretical reason that they may throw light on some hitherto intractable problems of black holes in $3+1$ spacetime dimensions, but also because, if current ideas about large extra dimensions are correct, then such black holes may possibly be created by high energy collisions in accelerator experiments, and their behavior might be accessible to direct observation [1,2]. As well as having possible applications to laboratory scale physics, higher-dimensional black holes, and other types of horizons such as cosmological event horizons [3], may also have played an important role in the early universe.

Many of the properties of black holes and event horizons in higher dimensions are very similar to their counterparts in $3+1$ dimensions. For example, the analogue of the spherically symmetric Schwarzschild black hole exists in all dimensions, with the two-sphere of the four-dimensional solution replaced by a $(D-2)$ sphere in D dimensions. Likewise, there is an obvious higher-dimensional analogue of the usual four-dimensional de Sitter spacetime. Moreover, subject to the strict condition of asymptotic flatness, the former and their charged versions are unique [4,5]. However, if one drops the condition of strict asymptotic flatness, by allowing other compact Einstein metrics in place of the usual round sphere in the spatial sections at constant radius, then in higher dimensions there are many more possibilities for black hole solutions, even on manifolds with the same topology as the higher-dimensional Schwarzschild solution. This is because of the remarkable fact, discovered by Bohm [6], that for $5 \leqslant d \leqslant 9$, the sphere S^d carries infinitely many other inhomogeneous Einstein metrics, in addition to its usual round metric. One might wonder whether the resulting black-hole solutions in spacetime dimensions $7 \leqslant D \leqslant 11$ could arise during scattering processes. This depends upon their stability. In this paper, we find evidence, and in some cases proofs, that they are in fact unstable. To do so we use methods developed in Ref. [7], which showed that the stability depends on the non-negativity of the spectrum of the operator

$$\Delta_{\text{stab}} = \Delta_L + \frac{\Lambda}{d-1}\left(4 - \frac{(5-d)^2}{4}\right), \tag{1}$$

where Δ_L is the Lichnerowicz Laplacian on transverse traceless second rank symmetric tensor fields on the d-dimensional compact Einstein space, which satisfies $R_{ab} = \Lambda g_{ab}$. We obtain numerical results establishing the existence of negative Lichnerowicz modes in Bohm metrics on S^5, hence demonstrating the instability of seven-dimensional black holes constructed using these metrics. We present general arguments suggesting that all the other Bohm sphere metrics will have negative Lichnerowicz modes too.

Bohm also showed the existence of infinitely many inhomogeneous Einstein metrics on the products of spheres $S^{N_1} \times S^{N_2}$, for $4 \leqslant N_1 + N_2 \leqslant 9$, with $N_1 \geqslant 2$ and $N_2 \geqslant 2$. The lowest-dimensional such examples are on $S^3 \times S^2$. We have also investigated the associated topologically nontrivial black holes (including the homogeneous product metric), and we prove that these are unstable. It is also known that $S^3 \times S^2$ admits infinitely many other homogeneous Einstein metrics, the so called $T^{p,q} \equiv \mathrm{spin}(4)/U(1)$ spaces. Of these, only $T^{1,1}$ admits Killing spinors, and in fact only the black hole associated with $T^{1,1}$ is stable [7]. There are in addition some other *inhomogeneous* metrics on $S^3 \times S^2$ with Killing spinors; these are examples of Einstein-Sasaki metrics. We show here that the associated black holes using these metrics are stable. One might think therefore that an unstable black hole based on the usual product $S^3 \times S^2$ metric would evolve dynamically into one based on one of the $S^3 \times S^2$ Einstein-Sasaki spaces. Presumably however, in doing so the area of the event horizon must increase. Now at fixed "mass parameter" we show the areas of the horizons of the Einstein-Sasaki horizons with topology $S^3 \times S^2$ are less than those of the $S^3 \times S^2$ Bohm metrics. Thus in the evolution, the mass parameter would have to increase, which seems rather paradoxical. Another way to say this is that at fixed temperature (which is proportional to the inverse of the mass parameter) the entropy of the Einstein-Sasaki metrics is less than that of the Bohm metrics.

In addition to constructing black holes, one may use Bohm's metrics to obtain static inhomogeneous Lorentzian solutions of the Einstein equations with a positive cosmological constant, which are topologically the same as the static de Sitter metric. As with the de Sitter spacetime, they contain a cosmological event horizon [3]. As such they provide counterexamples in $\leqslant D \leqslant 9$ spacetime dimensions to the long standing and hitherto intractable cosmic baldness conjecture [8], which would be a generalization of Israel's uniqueness theorem to cover the de Sitter situation. The area of the cosmological event horizon in the Lorentzian Bohm metrics is smaller than that in de Sitter spacetime. We believe therefore that they are unstable, and that under a small perturbation they would evolve, at least within the event horizon of a given observer, to a static de Sitter–like state. If this is the case, then although evading the strict letter of the cosmic baldness conjecture they would respect the spirit of the weaker no-hair conjecture. The latter asserts that apart from unstable cases of measure zero, the generic solution should settle down to a de Sitter–like state within the horizon of any given observer. This is all that is needed to justify the usual intuition behind inflationary models of the early universe. Another possible application for these generalized de Sitter spacetimes would be as tunneling metrics.

The plan of this paper is as follows. In Sec. II we review the link established in Ref. [7] between black-hole stability and the spectrum of the Lichnerowicz Laplacian on the compact d-dimensional Einstein space M_d that forms the constant-radius spatial sections. We also show that this stability criterion is identical, for all dimensions of M_d, to that for the stability of Freund-Rubin type $\mathrm{AdS}_n \times M_d$ solutions of gravity coupled to a d form. We then discuss a lower bound on the spectrum of the Lichnerowicz operator, based on considerations of the Weyl curvature, that was considered in Refs. [9,10].

In Sec. III we give a detailed discussion of the Bohm metrics, and we exhibit negative modes of the Lichnerowicz operator in some of these backgrounds. In certain cases, including all the Bohm metrics on $S^3 \times S^2$ and on $S^3 \times S^3$, we obtain an analytic proof of the existence of negative modes. Intuition leads one to expect negative modes in all the Bohm metrics, and we back this up with some numerical results in certain examples where an analytic proof is lacking. There are also noncompact examples of Bohm metrics, which are Ricci flat. We give analytic proofs that the noncompact Bohm metrics on $\mathbb{R}^3 \times S^2$ and $\mathbb{R}^3 \times S^3$, recently considered by Kol [11], have negative modes of the Lichnerowicz Laplacian. Again, intuition leads one to expect negative modes for all the noncompact Bohm examples.

In Sec. IV we discuss Einstein-Sasaki metrics, which may be defined as odd-dimensional Einstein metrics ds^2 whose cone $d\hat{s}^2 = dr^2 + r^2 ds^2$ is Ricci flat and Kähler. They admit Killing spinors, and we use this fact to obtain a lower bound on the spectrum of the Lichnerowicz operator. In particular we use this to demonstrate that the associated black holes are always stable. Likewise, this establishes that Freund-Rubin compactifications using Einstein-Sasaki manifolds will always be stable.

Section V contains a description of the analytic continuation of the Bohm metrics to give Lorentzian spacetimes that are generalizations of de Sitter spacetime. These provide counterexamples to the Cosmic Baldness conjecture. In Sec. VI we try to relate the existence of negative modes for the Lichnerowicz Laplacian to the nonuniqueness of the Dirichlet problem for the Einstein equations. Section VII gives our main conclusions, and points to some other applications of Bohm metrics. For example, they can provide magnetic monopole solutions in Kaluza-Klein theory. An appendix gives further details about our numerical techniques, and includes some graphs illustrating the behavior of the metric functions in the Bohm solutions.

II. STABILITY AND THE LICHNEROWICZ LAPLACIAN

Our first aim will be to study the classical stability of two types of spacetime constructed using a general positive curvature Einstein metric M_d. The first of these comprises generalizations to higher dimensions of the four-dimensional Schwarzschild black hole, in which the spatial two-sphere at constant radius is generalized to a higher-dimensional Einstein space M_d. The second class of examples comprises Freund-Rubin type solutions $\mathrm{AdS}_n \times M_d$ to a theory of Einstein gravity coupled to a d-form field strength. As we shall show below, the classical stability criteria for both of these classes of spacetimes are expressible as the *same* criterion on the spectrum of the Lichnerowicz operator acting on transverse traceless symmetric two-index tensors on M_d. To set the stage for this discussion, we begin in Sec. II A with a general discussion of the Lichnerowicz operator, reviewing the manner in which it arises from a consideration of the second variation of the Einstein-Hilbert action. We then review the black-hole stability [7] and $\mathrm{AdS}_n \times M_d$ stability [12,13] criteria in Secs. II B and II C, and in Sec. II D we review an argument given in Refs. [9,10] which shows how a lower bound on the Lichnerowicz spectrum can be obtained by considering the eigenvalues of the Weyl tensor.

A. Stability criteria

We begin by considering the Einstein-Hilbert action in d dimensions

$$S = \int_M \sqrt{g} d^d x [R - (d-2)\Lambda], \tag{2}$$

whose Euler-Lagrange equations give the Einstein equation

$$R_{ab} = \Lambda g_{ab}. \tag{3}$$

Under the perturbation

$$g_{ab} \to g_{ab} + h_{ab}, \tag{4}$$

one finds that up to quadratic order in h_{ab}, the action S is given on-shell by $S = S_0 + S_1 + S_2 + \cdots$, with

$$S_0 = 2\Lambda \int_M \sqrt{g} d^d x, \quad S_1 = 0,$$

$$S_2 = \int_M \sqrt{g} d^d x \left[-\tfrac{1}{4} h^{ab} \Delta_2 h_{ab} + \tfrac{1}{4} h \Delta_0 h + \tfrac{1}{2} (\nabla_a h^{ab})^2\right], \tag{5}$$

where $h \equiv h^a_a$, $\Box \equiv \nabla^a \nabla_a$ and

$$\Delta_0 h \equiv -\Box h + \tfrac{1}{2}(d-2)\Lambda h, \quad \Delta_2 \equiv \Delta_L - 2\Lambda. \tag{6}$$

Here Δ_L is the Lichnerowicz Laplacian operator acting on symmetric rank two tensors:

$$\Delta_L h_{ab} \equiv -\Box h_{ab} - 2R_{acbd} h^{cd} + R_{ca} h^c_b + R_{cb} h^c_a. \tag{7}$$

If we consider a transverse traceless perturbation

$$\nabla^a h_{ab} = 0, \quad h^a_a = 0, \tag{8}$$

then the second variation of the action is simply given by

$$S_2 = -\frac{1}{4} \int_M \sqrt{g} d^d x h^{ab} \Delta_2 h_{ab}. \tag{9}$$

We will be using these formulas on a d-dimensional Einstein manifolds that appear as part of the full spacetime. In particular we will consider generalized vacuum black holes solutions, with total spacetime dimension $D = d+2$ [7], where the two extra dimensions are the time and radial directions. Also we will be considering spacetimes that are a direct product of $(D-d)$-dimensional anti–de Sitter with a d-dimensional compact Einstein manifold [13]. The anti–de Sitter spacetimes are supported by a gauge field of appropriate rank.

In a perturbative classical stability analysis one asks whether there are finite energy solutions to the linearized equations of motion

$$\hat{\Delta}_2 \hat{h}_{AB} = 0 \tag{10}$$

that grow exponentially in time. We use hats to denote the full D-dimensional spacetime tensors and operators and upper case indices run over D dimensions.

B. Generalized Schwarzschild-Tangherlini spacetimes

Generalized Schwarzschild-Tangherlini black holes have the form

$$d\hat{s}^2 = -\left[1 - \left(\frac{\ell}{r}\right)^{d-1}\right] dt^2 + \frac{dr^2}{\left[1 - \left(\frac{\ell}{r}\right)^{d-1}\right]} + r^2 ds_d^2, \tag{11}$$

where ℓ is a constant and ds_d^2 is the metric on a d dimensional compact Einstein manifold B with the curvature normalized to be that of S^d

$$R_{ab} = (d-1) g_{ab}. \tag{12}$$

The black hole solution has vanishing cosmological constant.

It was found in Ref. [7] that the dangerous mode for instability is a transverse tracefree eigenfunction of the Lichnerowicz Laplacian on the Einstein manifold B:

$$\hat{h}_{0a} = \hat{h}_{1a} = \hat{h}_{11} = \hat{h}_{00} = \hat{h}_{10} = 0,$$

$$\hat{h}_{ab} = h_{ab}(x) r^2 \phi(r) e^{i\omega t}, \tag{13}$$

where x are coordinates on B and

$$\Delta_L h_{ab} = \lambda h_{ab}. \tag{14}$$

Here Δ_L is the Lichnerowicz Laplacian on B. The stability of the spacetime was found to depend on the spectrum $\{\lambda\}$ of the Lichnerowicz Laplacian acting on transverse tracefree modes on B. Concretely, if the spectrum contains an eigenvalue that is too negative, the spacetime is unstable because $\omega^2 < 0$, giving an exponential growth in time in Eq. (13):

$$\lambda_{\min} < \lambda_c \equiv 4 - \frac{(5-d)^2}{4} \Leftrightarrow \text{instability}. \tag{15}$$

This result follows from considering the behavior of the radial dependence of the perturbation (13) $\phi(r)$. When the criterion (15) is satisfied, the solution for $\phi(r)$ that decays at infinity also oscillates as $\sin \ln r$ for small r. This allows it to be matched to a solution that is well behaved at the horizon $r = \ell$ [7]. Thus a finite energy mode exists in this case and the spacetime is unstable.

It is perhaps worth pointing out here that a form of Birkhoff theorem holds for the metrics we are considering. In other words if we had assumed a general time dependent metric of the form

$$ds^2 = -e^{-2\phi(r,t)}dt^2 + e^{2\psi(r,t)}dr^2 + Y^2(r,t)ds_d^2, \quad (16)$$

where ds_d^2 is a d-dimensional *time-independent* Einstein metric with scalar curvature $d(d-1)$, we would have found, on imposing the Einstein equations, that the metrics had to be static. In fact this result also holds if the metric is coupled to a two-form field strength (in the electric case) or a d-form field strength the magnetic case. It also holds if one includes a cosmological term. It means that when perturbing the static metric we must consider time dependent perturbations of the transverse or base Einstein metric ds_d^2. This is another way of seeing why we need information about the the spectrum of the Lichnerowicz operator on this space.

We shall not give a detailed proof of Birkhoff's theorem here, but merely indicate how to modify an existing treatment of Wiltshire [14] which assumes $SO(d+1)$ invariance, i.e., that ds_d^2 is the unit round metric on S^d. Wiltshire gives a proof which also covers the case when Gauss-Bonnet terms are present. The argument he presents will not go over to the case of a general Einstein metric, since it makes special use of properties of its Riemann tensor. Thus in what follows we ignore that term, which means we set $\tilde{\alpha}=0$ in his equations. It is an interesting question to ask whether Birkhoff's theorem remains true when one includes a Gauss-Bonnet term.

The discussion depends upon whether ∂Y is spacelike, timelike, or null. We assume the first case, and make a coordinate choice such that $Y=r$. The field equation $R^r_t=0$ [Eq. (6b) in his paper] then yields

$$\partial_t \psi = 0.$$

The equation $R^t_t + R^r_r = 0$ then gives

$$\partial_r \phi + \partial_r \psi = 0.$$

This means that $\phi+\psi=f(t)$, where $f(t)$ is an arbitrary differentiable function of t. By choice of the coordinate t, $f(t)$ may be taken to vanish. It follows that both ϕ and ψ are independent of time t, and the metric is therefore static. The remaining field equations show that it takes the Schwarzschild-Tangherlini form.

C. Anti–de Sitter product spacetimes

These are solutions to a system with a d-form field strength minimally coupled to gravity

$$d\hat{s}^2 = ds^2_{\mathrm{AdS}_{D-d}} + ds_d^2,$$

$$F_d = \left(\frac{2(D-2)(d-1)}{D-d-1} \right)^{1/2} \mathrm{vol}_B, \quad (17)$$

where we have taken the same normalization for the curvature of B as previously [Eq. (12)]. It was shown in Refs. [12,13] that the dangerous mode is a transverse tracefree mode on the manifold B multiplying a scalar on the AdS spacetime

$$\hat{h}_{0a} = \hat{h}_{1a} = \hat{h}_{11} = \hat{h}_{00} = \hat{h}_{10} = 0,$$

$$\hat{h}_{ab} = h_{ab}(x)\phi(y), \quad (18)$$

where y are the coordinates on the AdS. The mode $h_{ab}(x)$ on B is an eigentensor as in Eq. (14). From the AdS point of view, $\phi(y)$ is seen as a scalar field with mass given by [13]

$$m^2 = \lambda - 2(d-1). \quad (19)$$

Instability of massive scalars on AdS spacetime is expressed in terms of the Breitenlohner-Freedman bound [15,16]. In our units, this reads

$$m^2 \left(\frac{D-d-1}{d-1} \right)^2 < -\frac{(D-d-1)^2}{4} \Leftrightarrow \text{instability}. \quad (20)$$

Using the value of the mass in Eq. (19), the criterion (20) is just

$$\lambda_{\min} < \lambda_c \equiv 4 - \frac{(5-d)^2}{4} \Leftrightarrow \text{instability}. \quad (21)$$

This is immediately seen to be the same as the criterion found for the black-hole spacetimes (15). This is an intriguing match.

D. Lichnerowicz Laplacian and the Weyl tensor

In order to make estimates of the lowest eigenvalue of the Lichnerowicz Laplacian, it is convenient first to rewrite Eq. (7) in terms of the Weyl tensor. Since we are assuming that the metric is Einstein, with $R_{ab} = \Lambda g_{ab}$, this is given in d dimensions by

$$C_{abcd} = R_{abcd} - \frac{\Lambda}{d-1}(g_{ac}g_{bd} - g_{ad}g_{bc}). \quad (22)$$

Thus the Lichnerowicz Laplacian becomes

$$\Delta_L h_{ab} = -\Box h_{ab} - 2C_{acbd}h^{cd} + \frac{2d\Lambda}{d-1}h_{ab}. \quad (23)$$

A method for obtaining a lower bound on the smallest eigenvalue of Δ_L was introduced in Ref. [9]. One considers the integral of $3[\nabla_{(a}h_{bc)}]^2$, which, after performing an integration by parts and using the transversality and tracelessness of h_{ab}, gives

$$\int_M \sqrt{g} d^d x h^{ab} \Delta_L h_{ab} = \int_M \sqrt{g} d^d x \{3[\nabla_{(a} h_{bc)}]^2 - 4h^{ab} R_{abcd} h^{cd} + 4\Lambda H_{ab} h^{ab}\},$$
$$= \int_M \sqrt{g} d^d x \left\{ 3[\nabla_{(a} h_{bc)}]^2 - 4h^{ab} C_{abcd} h^{cd} + \frac{4d\Lambda}{d-1} h_{ab} h^{ab} \right\}$$
$$\geqslant \int_M \sqrt{g} d^d x \left(-4h^{ab} C_{abcd} h^{cd} + \frac{4d\Lambda}{d-1} h_{ab} h^{ab} \right). \quad (24)$$

Viewing C_{abcd} as a map acting on symmetric traceless tensors h_{ab}, we can define its eigenvalues κ by

$$C_{acbd} h^{cd} = \kappa h_{ab}. \quad (25)$$

For homogeneous spaces, we therefore have a simple inequality

$$\Delta_L \geqslant \frac{4d\Lambda}{d-1} - 4\kappa_{\max}, \quad (26)$$

where $\kappa_{\max}$ is the largest eigenvalue of the Weyl tensor. Equality is attained if the corresponding eigentensor h_{ab} is a Staeckel tensor, satisfying $\nabla_{(a} h_{bc)} = 0$. The bound (26) was derived for homogeneous Einstein seven-manifolds M_7 in Ref. [9]. It was also shown that in the case of $U(1)$ bundles over the Einstein-Kähler product six-manifolds $S^2 \times \mathbb{CP}^2$ and $S^2 \times S^2 \times S^2$, the eigentensor that maximises κ is in fact Staeckel, and thus the equality in Eq. (26) is attained [9,10].

For inhomogeneous spaces, such as the Bohm metrics that we shall be studying later in this paper, the bound (24) must be kept in its integrated form, and so we have

$$\int_M \sqrt{g} d^d x h^{ab} \Delta_L h_{ab} \geqslant \int_M \sqrt{g} d^d x \left(\frac{4d\Lambda}{d-1} - 4\kappa_{\max}(x) \right) h_{ab} h^{ab}, \quad (27)$$

where $\kappa_{\max}(x)$ represents the (position dependent) largest eigenvalue of the Weyl tensor. If we define $\kappa^0_{\max}$ to be the largest value attained by any of the eigenvalues of the Weyl tensor anywhere in M, $\kappa^0_{\max} \equiv \sup_x \kappa_{\max}(x)$, then we obtain the inequality

$$\Delta_L \geqslant \frac{4d\Lambda}{d-1} - 4\kappa^0_{\max}. \quad (28)$$

It is clear, however, that this bound is not likely to be very sharp, especially if $\kappa_{\max}$ depends strongly on position.

We can, nevertheless, extract a general feature of the spectrum of the Lichnerowicz operator from the above considerations, namely that in Einstein spaces of positive Ricci tensor, the minimum eigenvalue tends to be lowered by having a large positive eigenvalue of the Weyl tensor. Indeed, we can see from Eq. (28) that if the largest Weyl tensor eigenvalue is not sufficiently positive, then Δ_L could never be negative or zero. We shall see when we study the Bohm metrics in detail that in these cases the Weyl tensor can in fact have sufficiently large eigenvalues that the Lichnerowicz spectrum includes negative eigenvalues. By using a Rayleigh-Ritz variational method we shall be able to obtain upper bounds on the lowest eigenvalue of the Lichnerowicz Laplacian, allowing us in some cases to give an analytic proof of the existence of negative-eigenvalue modes.

III. THE BOHM EINSTEIN METRICS ON S^N AND $S^{N_1} \times S^{N_2}$

A. Description of the Bohm construction

Bohm's construction [6] gives rise to a countable infinity of Einstein metrics with positive Ricci tensor on the spheres S^N for $5 \leqslant N \leqslant 9$, and on the product topologies $S^{N_1} \times S^{N_2}$ for $5 \leqslant N_1 + N_2 \leqslant 9$ with $N_1 \geqslant 2$ and $N_2 \geqslant 2$. We can use these metrics in the black-hole and AdS$\times M_d$ spacetimes of the previous section.

The starting point for the construction is the following ansatz for metrics of cohomogeneity one:

$$ds^2 = d\rho^2 + a^2 d\Omega_p^2 + b^2 d\tilde{\Omega}_q^2, \quad (29)$$

where a and b are functions of the radial variable ρ, and $d\Omega_p^2$ and $d\tilde{\Omega}_q^2$ are the standard metrics on the unit spheres S^p and S^q. An elementary calculation shows that in the orthonormal frame $e^0 = dt$, $e^i = a\bar{e}^i$, $e^\alpha = b\bar{e}^\alpha$, where $\bar{e}^i \bar{e}^i = d\Omega_p^2$ and $\bar{e}^\alpha \bar{e}^\alpha = d\tilde{\Omega}_q^2$, the components of the Riemann tensor are given by

$$R_{0i0j} = -\frac{\ddot{a}}{a}\delta_{ij}, \quad R_{0\alpha0\beta} = -\frac{\ddot{b}}{b}\delta_{\alpha\beta}, \quad R_{i\alpha j\beta} = -\frac{\dot{a}\dot{b}}{ab}\delta_{ij}\delta_{\alpha\beta},$$
$$R_{ijk\ell} = \frac{1-\dot{a}^2}{a^2}(\delta_{ik}\delta_{k\ell} - \delta_{i\ell}\delta_{jk}),$$
$$R_{\alpha\beta\gamma\delta} = \frac{1-\dot{b}^2}{b^2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}). \quad (30)$$

Note that $\dot{a} = \partial a/\partial\rho$. The components of the Ricci tensor are given by

$$R_{00} = -\frac{p\ddot{a}}{a} - \frac{q\ddot{b}}{b},$$
$$R_{ij} = -\left[\frac{\ddot{a}}{a} + \frac{q\dot{a}\dot{b}}{ab} + \frac{(p-1)(\dot{a}^2-1)}{a^2}\right]\delta_{ij},$$
$$R_{\alpha\beta} = -\left[\frac{\ddot{b}}{b} + \frac{p\dot{a}\dot{b}}{ab} + \frac{(q-1)(\dot{b}^2-1)}{b^2}\right]\delta_{\alpha\beta}. \quad (31)$$

The Einstein equations $R_{ab}=\Lambda g_{ab}$ give rise to two second-order differential equations for a and b

$$\frac{\ddot{a}}{a}+\frac{q\dot{a}\dot{b}}{ab}+\frac{(p-1)(\dot{a}^2-1)}{a^2}=-\Lambda,$$

$$\frac{\ddot{b}}{b}+\frac{p\dot{a}\dot{b}}{ab}+\frac{(q-1)(\dot{b}^2-1)}{b^2}=-\Lambda, \tag{32}$$

together with the first-order constraint

$$\frac{p(p-1)(\dot{a}^2-1)}{a^2}+\frac{q(q-1)(\dot{b}^2-1)}{b^2}+\frac{2pq\dot{a}\dot{b}}{ab}+(p+q-1)\Lambda=0. \tag{33}$$

We shall adopt the conventional normalization, when considering Einstein metrics with positive Λ, of taking

$$\Lambda=p+q, \tag{34}$$

which is one less than the total dimension of the space.

When $p>1$ and $q>1$, which we shall be considering here, the general solution of the Einstein equations is not known explicitly. A well-known special solution is

$$a=\sin\rho, \quad b=\cos\rho, \tag{35}$$

in which case the metric (29) becomes just the standard round metric on S^{p+q+1}

$$ds^2=d\rho^2+\sin^2\rho d\Omega_p^2+\cos^2\rho d\tilde{\Omega}_q^2, \tag{36}$$

written as a foliation by $S^p\times S^q$. This can easily be recognized as the metric on the unit S^{p+q+1} by introducing coordinates x^A on $\mathbb{R}^{p+q+2}$, subject to the unit-radius constraint $x^Ax^A=1$, and then introducing orthogonal unit vectors m^A and n^A in $\mathbb{R}^{p+q+2}$, such that a general point on the unit S^{p+q+1} in $\mathbb{R}^{p+q+2}$ can be written as

$$x^A=m^A\sin\rho+n^A\cos\rho. \tag{37}$$

A second well-known special solution to the Einstein equations is

$$a=\sqrt{\frac{p}{\Lambda}}\sin\left(\sqrt{\frac{\Lambda}{p}}\rho\right), \quad b=\sqrt{\frac{q-1}{\Lambda}}, \tag{38}$$

with, using our conventional choice, $\Lambda=p+q$. This gives the standard homogeneous Einstein metric on $S^{p+1}\times S^q$,

$$ds^2=d\rho^2+\frac{p}{\Lambda}\sin^2\left(\sqrt{\frac{\Lambda}{p}}\rho\right)d\Omega_p^2+\frac{q-1}{\Lambda}d\tilde{\Omega}_q^2. \tag{39}$$

There is an analogous solution for $S^p\times S^{q+1}$ too. Since there is obviously always a discrete transformation under which the roles of the spheres S^p and S^q are interchanged, we shall not in general bother to mention the symmetry-related possibility.

It is shown in Ref. [6] that the Einstein equations (32) and (33) admit a countably infinite number of solutions giving rise to inequivalent metrics that extend smoothly onto manifold of topology S^{p+q+1}, and another countable infinity of solutions for which the metrics extend smoothly onto manifolds of topology $S^{p+1}\times S^q$. We shall denote these metrics by $\mathrm{Bohm}(p,q)_n$, where the integer n runs over $n=0,2,4,\ldots$, for the S^{p+q+1} sequence, and $n=1,3,5,\ldots$, for the $S^{p+1}\times S^q$ sequence. The standard unit metric (35) on S^{p+q+1} corresponds to $\mathrm{Bohm}(p,q)_0$, and the standard product Einstein metric (38) on $S^{p+1}\times S^q$ corresponds to $\mathrm{Bohm}(p,q)_1$.

The higher metrics $\mathrm{Bohm}(p,q)_n$ with $n\geqslant 2$ are all inhomogeneous. The radial coordinate ρ runs between endpoints which can be taken to be 0 and ρ_f, defined by the vanishing of one or other of the metric functions a and b. The metric extends onto the corresponding degenerate orbit because the associated metric function vanishes similar to ρ or $(\rho_f-\rho)$, so that one has a regular collapsing of p spheres or q spheres as in the origin of spherical polar coordinates. For all the S^{p+q+1} metrics $\mathrm{Bohm}(p,q)_{2m}$ one has

$$a(0)=0, \ \dot{a}(0)=1, \ b(0)=b_0, \ \dot{b}(0)=0;$$

$$a(\rho_f)=a_0, \ \dot{a}(\rho_f)=0, \ b(\rho_f)=0, \ \dot{b}(\rho_f)=-1. \tag{40}$$

On the other hand, for the $S^{p+1}\times S^q$ metrics $\mathrm{Bohm}(p,q)_{2m+1}$ one has

$$a(0)=0, \ \dot{a}(0)=1, \ b(0)=b_0, \ \dot{b}(0)=0;$$

$$a(\rho_f)=0, \ \dot{a}(\rho_f)=-1, \ b(\rho_f)=\tilde{b}_0, \ \dot{b}(\rho_f)=0. \tag{41}$$

The functions a and b are strictly positive for $0<\rho<\rho_f$, and the quantities a_0, b_0, and $\tilde{b}_0$ are certain constants.

Plots of the metric functions a and b for various Bohm metrics are presented in the Appendix. These have been obtained by performing a numerical integration of the Einstein equations (32). It can be seen that as the index n for $\mathrm{Bohm}(p,q)_n$ increases, the metrics rapidly become approximations to the "double-cone" Einstein metric

$$ds^2=d\rho^2+\frac{1}{(p+q-1)}\sin^2\rho[(p-1)d\Omega_p^2+(q-1)d\tilde{\Omega}_q^2], \tag{42}$$

for most of the range of the radial coordinate. The metric (42) itself is singular at the apexes $\rho=0$ and $\rho=\pi$, since near to each of these points one has a collapse of $S^p\times S^q$ surfaces. The actual $\mathrm{Bohm}(p,q)_n$ metrics with large n deviate from Eq. (42) just in the vicinity of the apexes, instead approaching the forms given in Eqs. (40) or (41). It is interesting to note that Eq. (42) is in fact the singular limit both of the regular S^{p+q+1} sequence $\mathrm{Bohm}(p,q)_{2m}$ and the regular $S^{p+1}\times S^q$ sequence $\mathrm{Bohm}(p,q)_{2m+1}$.

Note that in the case of the Bohm metrics $\mathrm{Bohm}(p,q)_{2m+1}$ with the topology $S^{p+1}\times S^q$, the fact that the metric function $b(\rho)$ never vanishes means that we can replace the associated round sphere S^q with its metric $d\tilde{\Omega}_q^2$ in Eq. (29) by *any* Einstein space Q_q of dimension q, whose (positive) Ricci curvature is normalized to $\tilde{R}_{\alpha\beta}=(q-1)\tilde{g}_{\alpha\beta}$, and we will again have a complete and nonsingular Bohmian metric in $(p+q+1)$ dimensions, now with the topology $S^{p+1}\times Q_q$. The Einstein space Q_q could itself be taken to be a Bohm metric such as $\mathrm{Bohm}(2,2)_n$, $\mathrm{Bohm}(2,3)_n$, or $\mathrm{Bohm}(3,2)_n$.

B. Estimates and bounds for Lichnerowicz in Bohm metrics

1. Eigenvalues of the Weyl tensor

We saw earlier, in Sec. II D, that positive eigenvalues of the Weyl tensor tend to drive the lowest mode of the Lichnerowicz more negative. Accordingly, we can gain insights into the bounds on the spectrum of the Lichnerowicz operator in the Bohm metrics by studying the Weyl tensor. From Eq. (30), and our choice of normalization where $R_{ab}=(p+q)g_{ab}$, we have

$$C_{0i0j}=x_1\delta_{ij},\quad C_{0\alpha0\beta}=x_2\delta_{\alpha\beta},\quad C_{i\alpha j\beta}=x_3\delta_{ij}\delta_{\alpha\beta},$$

$$C_{ijk\ell}=x_4(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk}),$$

$$C_{\alpha\beta\gamma\delta}=x_5(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}), \tag{43}$$

where

$$x_1=-1-\frac{\ddot a}{a},\quad x_2=-1-\frac{\ddot b}{b},\quad x_3=-1-\frac{\dot a\dot b}{ab},$$

$$x_4=\frac{1-\dot a^2-a^2}{a^2},\quad x_5=\frac{1-\dot b^2-b^2}{b^2}. \tag{44}$$

Note that these coefficients x_i are not all independent, and thus

$$px_1=-qx_2=\tfrac12 q(q-1)x_5-\tfrac12 p(p-1)x_4,$$

$$x_3=-\frac{p(p-1)x_4+q(q-1)x_5}{2pq}. \tag{45}$$

It is straightforward to see that with multiplicities $\mathbf{m}$, the traceless eigenvectors h_{AB} and eigenvalues κ of the Weyl tensor are given by

$$h_{0i}:\ \mathbf{m}=p,\ \kappa=-x_1,$$

$$h_{0\alpha}:\ \mathbf{m}=q,\ \kappa=-x_2,$$

$$\{h_{ij}|h_{ii}=0\}:\ \mathbf{m}=\tfrac12 p(p+1)-1,\ \kappa=-x_4,$$

$$\{h_{\alpha\beta}|h_{\alpha\alpha}=0\}:\ \mathbf{m}=\tfrac12 q(q+1)-1,\ \kappa=-x_5,$$

$$h_{i\alpha}:\ \mathbf{m}=pq,\ \kappa=-x_3, \tag{46}$$

together with two eigenvectors of the form

$$h_{00}=-pu-qv,\quad h_{ij}=u\,\delta_{ij},\quad h_{\alpha\beta}=v\,\delta_{\alpha\beta} \tag{47}$$

for which the eigenvalues are given by the roots of a quadratic equation

$$\kappa_\pm=\tfrac12(p-1)x_4+\tfrac12(q-1)x_5\pm\frac{\sqrt{pq[(p-1)x_4+(q-1)x_5]^2+(p+q+1)[p(p-1)x_4-q(q-1)x_5]^2}}{2\sqrt{pq}}. \tag{48}$$

The coefficients u and v are then given by

$$u=-2q\kappa_\pm-p(p-1)x_4+q(q-1)x_5,$$

$$v=2p\kappa_\pm-p(p-1)x_4+q(q-1)x_5. \tag{49}$$

In total, we have the expected $\frac12(p+q+1)(p+q+2)-1$ symmetric traceless eigenmodes in $(p+q+1)$ dimensions.

Using the output of the numerical integration of the Einstein equations for the Bohm metrics, we find that the eigenvalue of the Weyl-tensor that achieves the largest positive value is κ_+ given by Eq. (48). It is therefore in this sector that one can expect to find the lowest-lying eigenmodes of the Lichnerowicz operator. We can see from Eq. (47) that the associated eigenvector is of a type that may be thought of as a "ballooning mode" in time t. That is to say, if we considered the associated black hole spacetime with its associated time dependent perturbation in t (13), it is a mode where one of the spheres S^p or S^q tends to inflate at the expense of the other. This accords with one's intuition, which would suggest that the most likely instability for metrics with direct-product orbits would be ballooning modes of this general type.

Some examples of our numerical results for the maximum value of the largest eigenvalue of the Weyl tensor are as follows. For the $\mathrm{Bohm}(2,2)_n$ metrics on S^5 and $S^3\times S^2$ we find from Eq. (48) that κ_+ attains its maximum value at the endpoints of the radial coordinate range, and so

$$\kappa^0_{\max}=\frac{5(1-b_0^2)}{3b_0^2}. \tag{50}$$

In fact, as n increases, the function κ_+ peaks more and more strongly around the endpoints. For $n=0,\ldots,6$, we have approximately

$$\kappa^0_{\max}=\{0,5,24.26,118.45,579.76,3013.72,15106.9\}. \tag{51}$$

(The results for $n=0$ and $n=1$ are exact, since these the standard homogeneous metrics on S^5 and $S^3\times S^2$.) Using Eq. (28), we obtain the lower bound

$$\Delta_L \geqslant -\frac{20(1-4b_0^2)}{3b_0^2} \tag{52}$$

on the spectrum of the Lichnerowicz operator on $\mathrm{Bohm}(2,2)_n$. From our numerical results for the values of b_0 for the first few examples, we find

$$\begin{aligned}
&\mathrm{Bohm}(2,2)_0: \ \Delta_L \geqslant 20,\\
&\mathrm{Bohm}(2,2)_1: \ \Delta_L \geqslant 0,\\
&\mathrm{Bohm}(2,2)_2: \ \Delta_L \geqslant -77.04,\\
&\mathrm{Bohm}(2,2)_3: \ \Delta_L \geqslant -453.8,\\
&\mathrm{Bohm}(2,2)_4: \ \Delta_L \geqslant -2342,\\
&\mathrm{Bohm}(2,2)_5: \ \Delta_L \geqslant -11972,\\
&\mathrm{Bohm}(2,2)_6: \ \Delta_L \geqslant -60407.
\end{aligned} \tag{53}$$

The bounds for $n=0$ and $n=1$ are in fact exactly attained, corresponding to the the cases of the homogeneous S^5 and $S^3\times S^2$ metrics, respectively. The zero mode in the latter case is the ballooning mode on $S^3\times S^2$. As we mentioned previously, the lower bounds we obtain for the inhomogeneous Bohm metrics are not expected to be very sharp.

In general for the $\mathrm{Bohm}(p,q)_n$ metrics we find

$$\kappa^0_{\mathrm{max}} = \frac{(q-1)(p+q+1)(1-b_0^2)}{(p+1)b_0^2}, \tag{54}$$

and hence we have the lower bound

$$\Delta_L \geqslant -\frac{4(p+q+1)[q-1-(p+q)b_0^2]}{(p+1)b_0^2}, \tag{55}$$

where as usual we have normalized the scale so that $R_{ab} = (p+q)\ g_{ab}$.

2. Transverse tracefree ballooning modes

In order to study transverse tracefree perturbations, we consider a metric of the form

$$ds^2 = c^2 d\rho^2 + a^2 d\Omega_p^2 + b^2 d\tilde{\Omega}_q^2. \tag{56}$$

This is similar to Eq. (29), except that we have, for convenience, introduce the coordinate gauge function $c(t)$ in the metric. Substituting into the Einstein-Hilbert action

$$S = \int \sqrt{g} d^d x [R-(d-2)(d-1)] \tag{57}$$

(where $d=p+q+1$) and omitting a constant factor equal to the volume of the product metric on the unit $S^p\times S^q$, this gives

$$S = \int a^p b^q c \left[2pq\frac{\dot{a}\dot{b}}{abc^2} + p(p-1)\frac{\dot{a}^2}{a^2c^2} + q(q-1)\frac{\dot{b}^2}{b^2c^2} + \frac{p(p-1)}{a^2} + \frac{q(q-1)}{b^2} - (p+q)(p+q-1)\right] d\rho. \tag{58}$$

Ballooning modes in product metrics, in which one factor contracts and the other expands, are Lichnerowicz zero modes and are typically associated with instabilities [12,9,13,7]. It is reasonable to expect, therefore, as we argued in Sec. III B 1, that if instabilities were to arise in the Bohm metrics, they would be associated with modes of a similar type. We are therefore led to seek a generalization of ballooning modes to the warped product of spheres present in Eq. (56). The perturbation

$$a \to a\sqrt{1+u}, \quad b \to b\sqrt{1+v}, \quad c=1 \to \sqrt{1+\gamma} \tag{59}$$

is tracefree at the linearized level if $\gamma + pu + qv = 0$, and transverse if

$$\dot{\gamma} + \left(p\frac{\dot{a}}{a} + q\frac{\dot{b}}{b}\right)\gamma - p\frac{\dot{a}}{a}u - q\frac{\dot{b}}{b}v = 0. \tag{60}$$

These two conditions can be used in order to solve for u and v in terms of γ:

$$u = \frac{\dot{\gamma} + [(q+1)\dot{b}b^{-1} + p\dot{a}a^{-1}]\gamma}{p(\dot{a}a^{-1} - \dot{b}b^{-1})},$$

$$v = \frac{\dot{\gamma} + [(p+1)\dot{a}a^{-1} + q\dot{b}b^{-1}]\gamma}{q(\dot{b}b^{-1} - \dot{a}a^{-1})}. \tag{61}$$

One is free to choose the function γ, which completely determines the perturbation through Eq. (61). However, the $\mathrm{Bohm}(p,q)_n$ metric has n interior points at which $(\dot{a}a^{-1} - \dot{b}b^{-1})$ vanishes, and hence the expressions for u and v are singular for generic choices of γ. This problem can be solved by inverting Eq. (61) to give γ in terms of u,

$$\gamma = \frac{1}{a^p b^{q+1}} \int a^p b^{q+1} \left[\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right] u\, d\rho. \tag{62}$$

The remaining function v is given by the trace-free condition $pu + qv + \gamma = 0$. One is now free to choose a nonsingular function u to obtain a perturbation that will at worst be singular at the end points $\rho = 0$, $\rho = \rho_f$. These singularities can be avoided as described in the next subsection. One simple choice is

$$u = \frac{p-m}{pa^m b^{p+q+1-m}}, \quad v = \frac{-(p+1-m)}{qa^m b^{p+q+1-m}},$$

$$\gamma = \frac{1}{a^m b^{p+q+1-m}}. \tag{63}$$

We will use these expressions below in order to exhibit negative Lichnerowicz modes in some of the Bohm metrics.

3. Rayleigh-Ritz estimates for the lowest Lichnerowicz eigenvalue

In the eigenvalue problem $\Delta\phi=\lambda\phi$ for a self-adjoint operator Δ, one can obtain an upper bound on the lowest eigenvalue by the Rayleigh-Ritz method: namely,

$$\lambda_{\min}\leqslant\frac{\int_M \psi\Delta\psi d\rho}{\int_M \psi^2 d\rho}, \tag{64}$$

with equality being achieved if the trial function ψ is actually the eigenfunction corresponding to the lowest eigenvalue. In this section, we shall apply this method to obtain an upper bound on the lowest eigenvalue of the Lichnerowicz operator on transverse traceless (TT) modes in the Bohm metrics, and in particular, we shall find that there is a negative-eigenvalue mode in some of cases we examine.

The easiest cases to consider are the Bohm metrics $\text{Bohm}(p,q)_{2m+1}$ on the product topologies $S^{p+1}\times S^q$. In these metrics, the function b in Eq. (29) is nowhere vanishing, and so we can take our trial function to be given by Eq. (63) with $m=0$:

$$u=\frac{\epsilon}{b^{p+q+1}},\quad v=\frac{-(p+1)\epsilon}{q b^{p+q+1}},\quad \gamma=\frac{\epsilon}{b^{p+q+1}}. \tag{65}$$

Here we have introduced ϵ as a small constant order parameter. As we shall see below, this trial function allows us to prove that certain of the Bohm metrics on products of spheres have negative eigenvalue modes of the Lichnerowicz operator.

C. Negative Lichnerowicz eigenvalues in Bohm metrics

1. Analytic results for negative modes for Bohm metrics on $S^3\times S^2$

It turns out that the easiest cases to study are the Bohm metrics $\text{Bohm}(2,2)_{2m+1}$, whose topology is $S^3\times S^2$. We are able to obtain completely analytic and explicit results that prove the existence of negative modes of the Lichnerowicz operator for all these examples (for $m\geqslant 1$), and so we shall present the details for these metrics here. For these examples, we take $\gamma=\epsilon/b^5$ as our trial function, as suggested by Eq. (65), implying that we have $u=\epsilon/b^5$ and $v=-3\epsilon/(2b^5)$. From the expansion of S given in Eq. (58), with the perturbation (59), we can easily extract the terms quadratic in ϵ, and so by comparing with Eqs. (6) and (9) we obtain a Rayleigh-Ritz bound for the lowest eigenvalue $\lambda_{\min}$ of the Lichnerowicz operator

$$\lambda_{\min}\leqslant\frac{\int_0^{t_f} P d\rho}{\int_0^{t_f} Q d\rho}, \tag{66}$$

where

$$P=-\frac{3a^2}{b^{10}}+\frac{7}{b^8}-\frac{30a^2}{b^8}+\frac{7\dot a^2}{b^8}-\frac{112 a\dot a\dot b}{b^9}-\frac{91a^2\dot b^2}{2b^8},$$

$$Q=\frac{15a^2}{2b^8}, \tag{67}$$

and $\rho_f=2\rho_c$ is the upper limit of the range of the radial coordinate ρ. Note that in fact it suffices to evaluate the integrals only up to the midpoint $\rho=\rho_c$ in these cases, since the metric functions are symmetric about $\rho=\rho_c$ here.

Using the constraint (33), with $p=q=2$, we can eliminate the term involving $\dot a^2$ in P. We also note that upon use of the second-order equations (32), we can prove the identities

$$\frac{d}{d\rho}\left(\frac{a\dot a}{b^8}\right)=-\frac{10 a\dot a\dot b}{b^9}+\frac{1-4a^2}{b^8},$$

$$\frac{d}{d\rho}\left(\frac{a^2\dot b}{b^9}\right)=-\frac{10a^2\dot b^2}{b^{10}}+\frac{a^2(1-4b^2)}{b^{10}}. \tag{68}$$

Since $a\dot a/b^8$ and $a^2\dot b/b^9$ vanish at both endpoints of the full integration range, we can use these in order to perform integrations by parts in the evaluation of $\int P$. Specifically, we use the former to remove the term in P involving $\dot a\dot b$, and then using the latter, we find that

$$\int_0^{\rho_f} P d\rho=-\frac{25}{2}\int_0^{\rho_f}\frac{a^2\dot b^2}{b^{10}}d\rho, \tag{69}$$

which is manifestly nonpositive. We therefore have the Rayleigh-Ritz bound

$$\lambda_{\min}\leqslant-\frac{5}{3}\,\frac{\int_0^{\rho_f} a^2 b^{-10}\dot b^2 d\rho}{\int_0^{\rho_f} a^2 b^{-8} d\rho} \tag{70}$$

for the lowest eigenvalue of the Lichnerowicz operator. (Recall that we are working in units where $R_{ab}=4g_{ab}$.) This proves that the Einstein metrics $\text{Bohm}(2,2)_{2m+1}$ have a negative eigenvalue for the Lichnerowicz operator on transverse traceless symmetric two-index tensors, for $m\geqslant 1$. (The case $m=0$ is the standard product Einstein metric on $S^3\times S^2$, with $b=\frac{1}{2}$. In this case the numerator gives zero, and in fact we exactly saturate the upper bound, finding the known lowest eigenvalue $\Delta_L=0$ on the product metric.)

Using the numerical results described in the Appendix, we find the following upper bounds on the lowest Lichnerowicz eigenvalue for the Bohm(2,2)$_3$ and Bohm(2,2)$_5$ Einstein metrics on $S^3\times S^2$:

$$\text{Bohm}(2,2)_3:\ \ \lambda_{\min}\leqslant -7.2766,$$

$$\text{Bohm}(2,2)_5:\ \ \lambda_{\min}\leqslant -198.008. \tag{71}$$

(Recall that we are normalizing the metrics so that $R_{ab}=4g_{ab}$.) As one goes to higher examples Bohm(2,2)$_{2m+1}$ with increasing m, one finds that the upper bound on the lowest Lichnerowicz eigenvalue becomes increasingly negative, tending to $-\infty$ in the limit as $m\to\infty$. Note that our upper bounds (71) are considerably larger than the rather crude lower bounds (53) that we obtained by considering the eigenvalues of the Weyl tensor.

2. Analytic results for negative modes for Bohm metrics on $S^3\times S^3$

For general values of p and q, the analogous expression for the integrands Y in Eq. (69) and Q in Eq. (67) that appear in the numerator and denominator of the Rayleigh-Ritz functional (66) turn out to be

$$P=q^{-1}(p+q+1)^2(p^2-2p-3+pq-q)a^p b^{-2p-q-4}\dot{b}^2,$$

$$Q=q^{-1}(p+q+1)(p+1)a^p b^{-2p-q-2}, \tag{72}$$

if we take the trial function $\gamma=b^{-p-q-1}$. In general, this gives us a rather weak positive upper bound on the lowest Lichnerowicz eigenvalue for the Bohm$(p,q)_{2m+1}$ metrics on $S^{p+1}\times S^q$. In fact only for $p=q=2$, which we discussed above, and $p=2$, $q=3$, does one get a nonpositive bound from this choice of trial function. Interestingly, for $p=2$, $q=3$ the numerator integrand P in Eq. (72) vanishes identically, and so we obtain the bound

$$\lambda_{\min}\leqslant 0 \tag{73}$$

in this case. It is straightforward to see that for $m\geqslant 1$ the trial function $\gamma=1/b^6$ does not give an eigenfunction, and hence the inequality in Eq. (73) is not saturated. Thus we have an analytic proof that for the Bohm(2,3)$_{2m+1}$ Einstein metrics on $S^3\times S^3$ with $m\geqslant 1$, the lowest eigenvalue of the Lichnerowicz operator on TT symmetric tensors is strictly negative.

For all other cases aside from Bohm(2,2)$_{2m+1}$ and Bohm(2,3)$_{2m+1}$, the trial function $\gamma=b^{-p-q-1}$ does not give a negative upper bound on $\lambda_{\min}$. We believe that this is a consequence of a nonoptimal choice of trial function, since the qualitative arguments would suggest the existence of negative Lichnerowicz modes for all the Bohm metrics.

3. Numerical results for negative modes for Bohm(2,2)$_{2m}$ metrics

The analytic methods that allowed us to prove the existence of negative modes of the Lichnerowicz operator in the $S^3\times S^2$ and $S^3\times S^3$ Bohm metrics do not directly extend to any of the Bohm metrics on S^{p+q+1}. The reason for this is that whilst our trial function $\gamma=1/b^{p+q+1}$ is regular everywhere in the metrics on $S^{p+1}\times S^q$, it diverges at the right-hand end point of the range of the radial coordinate ρ in the metrics on S^{p+q+1}. A natural modification to the trial function to take account of this is to interpolate smoothly between $\gamma=1/b^{p+q+1}$ on the left-hand side and $\gamma=1/a^{p+q+1}$ on the right-hand side of the range of ρ. We have carried out this procedure numerically in the case of examples of the Bohm(2,2)$_{2m}$ metrics on S^5, and we find that indeed there are negative modes of the Lichnerowicz operator, in accordance with the qualitative arguments. Specifically, we find approximately

$$\text{Bohm}(2,2)_2:\ \ \lambda_{\min}\leqslant -0.7937,$$

$$\text{Bohm}(2,2)_4:\ \ \lambda_{\min}\leqslant -38.86,$$

$$\text{Bohm}(2,2)_6:\ \ \lambda_{\min}\leqslant -1040.6. \tag{74}$$

These upper bounds are again all considerably larger than the corresponding lower bounds in Eq. (53) that we obtained from the eigenvalues of the Weyl tensor.

D. Noncompact Bohm metrics

A class of complete and nonsingular noncompact metrics was also constructed by Bohm [17]. These include examples where the metric ansatz is again taken to be Eq. (29), but now the metric is required to be Ricci flat. These metrics have been considered recently in Ref. [11] in studies of the possibility of topology change. It was shown in Ref. [17] that regular metrics exist in which $a(\rho)$ and $b(\rho)$ satisfy the boundary conditions

$$a(0)=0,\ \ \dot{a}(0)=1,\ \ b(0)=b_0,\ \ \dot{b}(0)=0. \tag{75}$$

Unlike the previous compact examples, here regularity imposes no constraint on the allowed values for the constant b_0, and in fact the value of b_0 now merely sets the overall scale of the metric. Note that b is everywhere nonvanishing, and so the there is an S^q bolt at $\rho=0$. The metrics are asymptotically conical, approaching cones over the standard product Einstein metric on $S^p\times S^q$.

A representative example is presented in the Appendix, for the case of $p=2$, $q=2$. The Rayleigh-Ritz method that we described earlier for finding an upper bound on the smallest Lichnerowicz eigenvalue can be applied in these noncompact Bohm metrics too. In fact the trial function $\gamma=b^{-p-q-1}$ can be considered here too, since it remains finite everywhere and it falls off rapidly at large ρ. We find that the numerator and denominator integrands are then again given by Eq. (72), and so again we obtain a negative upper bound on the lowest Lichnerowicz eigenvalue for the cases $p=q=2$, and $p=2$, $q=3$.

Evaluating the integrands numerically for the case $p=q=2$, we find that

$$\lambda_{\min}\leqslant -\frac{0.110433}{b_0^2}. \tag{76}$$

For $p=2$, $q=3$, we find, as for $\mathrm{Bohm}(2,3)_{2m+1}$, that the bound is $\lambda_{\min}\leqslant 0$, and we can again argue that since the trial function $\gamma=1/b^6$ does not give an eigenfunction, we must have $\lambda_{\min}<0$. For all other p and q, our choice of trial function does not give a negative upper bound on the lowest eigenvalue of the Lichnerowicz operator. Again, we believe that this is because the trial function is nonoptimal in these other cases, since general arguments suggest that the noncompact Bohm metrics should all have negative eigenvalue modes of the Lichnerowicz operator.

IV. EINSTEIN-SASAKI MANIFOLDS

A. Introduction and definition

In this section we remind the reader that as well as the infinite sequence of cohomogeneity one Bohm metrics that have featured in our discussion, the manifold $S^3\times S^2$ admits many other Einstein metrics. For example, it has been known for some time that there are infinitely-many homogeneous but nonsupersymmetric $T^{p,q}$ spaces, corresponding to $U(1)$ bundles over $S^2\times S^2$ in which the $U(1)$ fibers wind p times over one S^2, and q times over the other. These all have the topology $S^3\times S^2$ and they all admit an Einstein metric. Only $T^{1,1}$ admits Killing spinors.

There are, by contrast, also many inequivalent supersymmetric examples of $S^3\times S^2$ Einstein metrics, which do admit Killing spinors. They can thus be used in the AdS conformal field theory (CFT) correspondence, replacing S^5 in the D3 brane metric and its near-horizon limit.

An Einstein-Sasaki metric may be defined as a $(2m+1)$-dimensional Einstein metric such that the cone over it is a Calabi-Yau metric

$$ds^2_{\text{Calabi-Yau}}=dR^2+R^2 ds^2_{\text{Einstein-Sasaki}}, \tag{77}$$

or in other words, the cone is a Ricci-flat Kähler metric. The Killing spinors in the Einstein-Sasaki metric come by direct projection from the covariantly-constant spinors of the Calabi-Yau metric. If one uses the complex structure J of the Calabi-Yau metric to act on the Euler vector of the cone $R(\partial/\partial R)$, one gets a Killing vector on the Einstein-Sasaki manifold with constant magnitude, and thus we may write locally

$$ds^2_{\text{Einstein-Sasaki}}=(d\psi+A)^2+ds^2_{\text{Einstein-Kähler}}, \tag{78}$$

where $J(\partial/\partial R)=\partial/\partial\psi$ and $ds^2_{\text{Einstein-Kähler}}$ is locally Einstein-Kähler with positive scalar curvature. Globally the $U(1)$ action generated by $\partial/\partial\psi$ may be free (in which case one speaks of a regular Sasaki structure) and the base is a smooth Einstein-Kähler manifold, or it may have fixed points in which case the Einstein-Kähler base has orbifold singularities. The total space however will still be smooth. We give a more detailed discussion of the relation between the Einstein-Kähler and Einstein-Sasaki spaces below.

Taking $\mathbb{CP}^2$ or $\mathbb{CP}^1\times\mathbb{CP}^1$ as the Einstein-Kähler base metric gives the standard homogeneous Sasaki metrics on S^5 or $T^{1,1}$, respectively. If the fibration is regular the only remaining possible base metrics for five-dimensional Einstein-Sasaki metrics are inhomogeneous metrics on del Pezzo surfaces, i.e., $\mathbb{CP}^2$ blown up at k points, with $3\leqslant k\leqslant 8$, giving Einstein-Sasaki metrics on the connected sum of k copies of $S^3\times S^2$.[1]

Recently Boyer, Galicki *et al.* [18–21] have constructed many inhomogeneous Einstein-Sasaki $(2n-1)$ metrics on the links $L_f=C_f\cap S^{2n+1}$ of weighted homogeneous polynomials f on $\mathbb{C}^{n+1}$. The notation is as follows: $C_f\in\mathbb{C}^{n+1}$ is the zero set $f(z)=0$ of the polynomial, and S^{2n+1} is the standard sphere. One readily sees that the Hopf fibration desends to L_f, and this gives the fibration associated to the Sasaki structure. Note that this description is purely topological. The metric is obtained indirectly by means of an existence proof. The present state of the art is that there are at least 14 inequivalent Einstein-Sasaki structures on $S^3\times S^2$ [21]. Of these, only $T^{1,1}$ is homogeneous, and so if used in the AdS/CFT correspondence the other 13 examples would give supersymmetric vacua with no R symmetry.

The volume of $T^{1,1}$ is well known to be $16\pi^3/27$. According to Ref. [22], the volume of L_f is given, in five dimensions, by

$$\frac{\pi^3}{27w}(|w|-d)^3, \tag{79}$$

where d is the degree of f, $\mathbf{w}=(w_0,w_1,w_2,w_3)$ are the weights, $|\mathbf{w}|=w_0+w_1+w_2+w_3$, and $w=w_0w_1w_2w_3$. The two inhomogeneous Einstein-Sasaki metrics on $S^3\times S^2$ constructed in Ref. [18] are both of degree 256 and have weights $\mathbf{w}=(11,49,69,128)$ and $\mathbf{w}=(13,35,81,128)$. They therefore have volumes $\pi^3/27\times 4760448$ and $\pi^3/27\times 4717440$, respectively. These may be compared with the volume of the product metric on $S^3\times S^2$, which is $\pi^3/\sqrt{2}$, and of the limiting singular double cone Bohm metric, which is $2\pi^3/3$.

B. Einstein-Sasaki manifolds as $U(1)$ bundles over Einstein-Kähler manifolds

In this subsection, we present what is essentially a review of how Einstein-Sasaki manifolds can be constructed as $U(1)$ bundles over Einstein-Kähler manifolds, focusing in particular on the construction of the Killing spinors. The construction can be applied to obtain Einstein-Sasaki manifolds in any odd dimension, and so we shall give the construction for this general case.

Suppose we have an Einstein-Kähler metric g_{ab} on a manifold M_n of (even) dimension $n=2m$. By the standard formulas of Kaluza-Klein reduction, the $(n+1)$-dimensional metric

$$d\hat{s}^2=(d\psi+A)^2+ds^2 \tag{80}$$

has Ricci tensor $\hat{R}_{AB}$ whose frame components are given by

[1] It is worth remarking that *any* five-dimensional closed simply connected spin manifold with no torsion in the second homology group is diffeomorphic to a connected sum of copies of $S^3\times S^2$.

TABLE I. The progression from Einstein-Kähler to Einstein-Sasaki to Ricci-flat Kähler cone.

$2m$-dimensional Einstein-Kähler	$(2m+1)$-dimensional Einstein-Sasaki	$(2m+2)$-dimensional Calabi-Yau cone
$\Lambda>0$	$\Lambda>0$	$R_{ab}=0$
$(D_a-ieA_a)\varepsilon=0$	$D_a\eta=\pm im\Gamma_a\eta$	$D_a\eta=0$

$$\hat{R}_{ab}=R_{ab}-\frac{1}{2}F^c_aF_{bc}\,,\quad \hat{R}_{00}=\frac{1}{4}F_{ab}F^{ab},\quad \hat{R}_{0a}=\frac{1}{2}\nabla^bF_{ab}\,, \tag{81}$$

where $F=dA$ and $\hat{e}^0=d\psi+A$, $\hat{e}^a=e^a$. Taking $F_{ab}=\mu J_{ab}$, where J_{ab} is the Kähler form on M_n, we therefore have

$$\hat{R}_{ab}=\left(\Lambda-\frac{1}{2}\mu^2\right)\hat{g}_{ab}\,,\quad \hat{R}_{00}=\frac{1}{4}n\mu^2,\quad \hat{R}_{0a}=0, \tag{82}$$

where $R_{ab}=\Lambda g_{ab}$ in M_n, and so the $(n+1)$-dimensional metric $d\hat{s}^2$ will be Einstein, $\hat{R}_{AB}=\hat{\Lambda}\hat{g}_{AB}$, and

$$\hat{\Lambda}=n,\quad \Lambda=n+2, \tag{83}$$

provided that we take $\mu=2$.

The covariant exterior derivative on spinors, $\hat{D}\equiv d+1/4\hat{\omega}_{AB}\Gamma^{AB}$, is easily seen to be given by

$$\hat{D}=D-\tfrac{1}{4}AJ_{ab}\Gamma^{ab}+\tfrac{1}{2}J_{ab}e^b\Gamma^{0a}-\tfrac{1}{4}J_{ab}d\psi\Gamma^{ab}, \tag{84}$$

where $D=d+\frac{1}{4}\omega_{ab}\Gamma^{ab}$ is the covariant exterior derivative on spinors in the base space M_n. Note that since n is necessarily even, the spinors in the total space $\hat{M}$ have the same dimension as those in base space M_n, and so we do not need to make any tensor-product decomposition of the Dirac matrices. The equation for Killing spinors in the $(n+1)$-dimensional bundle space $\hat{M}$, in the normalization $\hat{R}_{AB}=n\hat{g}_{AB}$ that we established above, is simply $\hat{D}_A\eta=\frac{1}{2}i\sigma\Gamma_A\eta$, where $\sigma=\pm1$. From Eq. (84), this gives the equations

$$D_a\eta-A_a\frac{\partial\eta}{\partial\psi}=\tfrac{1}{2}J_{ab}\Gamma^{0b}\eta+\tfrac{1}{2}i\sigma\Gamma_a\eta,$$

$$\frac{\partial\eta}{\partial\psi}=\tfrac{1}{4}J_{ab}\Gamma^{ab}\eta+\tfrac{1}{2}i\sigma\Gamma_0\eta. \tag{85}$$

As is well known, the Einstein-Kähler space M_n admits a gauge-covariantly constant spinor ε, satisfying

$$\mathcal{D}_a\varepsilon-ieA_a\varepsilon=0, \tag{86}$$

where as above we have $dA=F=2J$, and e is the electric charge carried by ε. This can be determined by examining the integrability condition $[\mathcal{D}_a,\mathcal{D}_b]\varepsilon=\frac{1}{4}R_{abcd}\Gamma^{cd}\varepsilon-2ieJ_{ab}\varepsilon$. Multiplying by Γ^{ab}, this gives $in(n+2)\varepsilon=4eJ_{ab}\Gamma^{ab}\varepsilon$. It is a straightforward exercise to calculate the eigenvalues of the matrix $J_{ab}\Gamma^{ab}$, and to show, in particular, that in general it has only two singlet eigenvalues, which are $\pm in$. It is these singlets that are associated with the gauge-covariant constant spinor ε (and its charge conjugate), and so we can deduce that

$$e=\tfrac{1}{4}(n+2). \tag{87}$$

One can also then easily show that $\Gamma^0\varepsilon=\sigma\varepsilon$, where $\sigma=\pm1$. From the second equation in Eq. (85) we therefore deduce that if we take $\eta=f(\psi)\varepsilon$ we shall have

$$f=e^{(1/4)(n+2)i\psi}, \tag{88}$$

and then the first equation in Eq. (85) confirms that indeed ε satisfies

$$D_a\varepsilon-\tfrac{1}{4}i(n+2)A_a\varepsilon=0. \tag{89}$$

In other words, we have proved that if ε is the gauge-covariantly constant spinor in the Einstein-Kähler manifold M_n, then $\eta=e^{(1/4)(n+2)i\psi}\varepsilon$ is a Killing spinor in the $U(1)$ bundle over M_n, which is therefore an Einstein-Sasaki manifold $\hat{M}$. The conjugate spinor satisfies the Killing-spinor equation with the opposite sign on the right-hand side. Lifted up further using Eq. (77), one obtains the conjugate pair of covariantly constant spinors in the Ricci-flat Kähler cone over the Einstein-Sasaki manifold. The situation is summarized in Table I.

C. Lichnerowicz bound for Einstein-Sasaki spaces

In any Einstein space M that admits Killing spinors, we can prove that the bound (21) that governs the stability of AdS$\times M$ solutions, and also the stability of Schwarzschild-Tangherlini black holes, is always satisfied. In other words, we can prove that an Einstein space in d dimensions with cosmological constant Λ has a Lichnerowicz spectrum such that

$$\Delta_L\geqslant\frac{\Lambda}{d-1}\left(4-\frac{(5-d)^2}{4}\right). \tag{90}$$

To prove this we shall first, for convenience, make our conventional choice of normalization $\Lambda=d-1$. A Killing spinor therefore satisfies $D_a\eta=\frac{1}{2}i\Gamma_a\eta$. Suppose that h_{ab} is a transverse traceless mode of the Lichnerowicz operator on M:

$$\Delta_Lh_{ab}=\lambda h_{ab}\,,\;\;\nabla^ah_{ab}=0,\;\;h^a_a=0. \tag{91}$$

We now define two vector spinors

$$\phi_a\equiv h_{ab}\Gamma^b\eta,\;\;\chi_a\equiv(\nabla_bh_{ac})\Gamma^{bc}\eta. \tag{92}$$

The assumed properties of h_{ab} can easily be seen to imply that

$$D^a\phi_a=0,\quad \Gamma^a\phi_a=0,\quad D^a\chi_a=0,\quad \Gamma^a\chi_a=0. \tag{93}$$

We now calculate the action of the Rarita-Schwinger operator on the vector-spinors, finding after some algebra that

$$i\Gamma^b D_b \phi_a = \chi_a - \frac{i}{2}(d-2)\phi_a,$$

$$i\Gamma^b D_b \chi_a = -(\lambda - d)\phi_a + \frac{i}{2}(d-4)\chi_a. \tag{94}$$

Thus by taking an appropriate linear combination of the two vector-spinors, we can form an eigenfunction $\psi_a = \phi_a + k\chi_a$ of the Rarita-Schwinger operator on transverse gamma-traceless spin $\frac{3}{2}$ modes

$$i\Gamma^b D_b \psi_a = \mu \psi_a. \tag{95}$$

It follows immediately from Eq. (94) that we shall have an eigenfunction if

$$\mu = \tfrac{1}{2}(d-2) - ik(\lambda - d), \quad k\mu = i - \tfrac{1}{2}k(d-4). \tag{96}$$

These equations determine the constant of proportionality to be $k = i/[\mu + \frac{1}{2}(d-4)]$, and hence that the Rarita-Schwinger eigenvalue μ satisfies

$$4\mu^2 - 4\mu - d^2 + 10d - 8 = 4\lambda. \tag{97}$$

Reorganizing this we obtain

$$\lambda = \tfrac{1}{4}(2\mu - 1)^2 + 4 - \tfrac{1}{4}(d-5)^2. \tag{98}$$

From the reality of the Rarita-Schwinger eigenvalue μ, we therefore deduce that

$$\lambda \geqslant 4 - \tfrac{1}{4}(d-5)^2. \tag{99}$$

Restoring the cosmological constant, we therefore obtain the claimed inequality (90), which must hold for any Einstein space of positive Ricci tensor that admits Killing spinors. In particular, this encompasses the case of all Einstein-Sasaki manifolds, in all odd dimensions.

It is worth remarking that the above proof is a generalisation of an argument that was used in Ref. [12] in the case of seven-dimensional Einstein-Sasaki manifolds. It was argued there that such a manifold M_7 could be used in order to obtain a supersymmetric solution $AdS_4 \times M_7$ of eleven-dimensional supergravity. Now it is known that eigenfunctions of the Lichnerowicz operator in the internal space give rise to scalar fields in the AdS spacetime. The supersymmetry of the background implies that these scalars must be members of supermultiplets, including fermions. Since the Kaluza-Klein reduction must necessarily give rise to *real* masses for the fermions, it follows that the masses of the bosons (making due allowance for the need to define mass carefully in AdS) must be real also. This translates into the statement [12] that the $(\text{mass})^2$ of the scalars must respect the Breitenlohner-Freedman [15] bound for stability, and hence it follows that the spectrum of the Lichnerowicz operator must be bounded from below by the stability limit, as given in Eq. (21), for the case $d=7$. The same argument was used recently for $AdS_5 \times M_5$ compactifications in Ref. [13]. Of course our general proof above can be seen to be essentially an extension of the supersymmetry argument of Ref. [12], since in fact the crucial ingredient was not really supersymmetry *per se*, but rather, the fact that the mass spectrum of scalar fields can be related to the mass spectrum of spin $\frac{1}{2}$ fields. Since the Lichnerowicz Laplacian is the mass operator for scalar fields, and the Rarita-Schwinger operator is the mass operator for spin $\frac{1}{2}$ fields, our demonstration above that the eigenfunctions of the two operators are related when there are Killing spinors can be seen to reduce to the supersymmetry argument in those special dimensions where supersymmetric AdS vacua can be found. Our argument above is much more general, however, since it dispenses with the excess baggage of supersymmetry, and the need to interpret mass in AdS backgrounds.

In Refs. [9,10] the Lichnerowicz bounds were investigated for seven-dimensional Einstein metrics on the spaces $M(m,n)$ and $Q(k,\ell,m)$ which are $U(1)$ bundles over $\mathbb{CP}^2 \times S^2$ and $S^2 \times S^2 \times S^2$, respectively, with the integers specifying the winding numbers of the $U(1)$ fibers over the base components. It is was found that for the Einstein-Sasaki examples, namely, $M(3,2)$ and $Q(1,1,1)$, the bound $\Delta_L \geqslant \frac{1}{2}\Lambda$ in Eq. (90) is strictly exceeded. This has the interesting consequence that for a range of ratios $m:n$ or $k:\ell:m$ around the Einstein-Sasaki values, the stability bound is still satisfied despite the absence of supersymmetry [9,10]. By contrast, it was shown recently in an analogous five-dimensional calculation for the $T^{p,q}$ spaces with Einstein metrics that the bound $\Delta_L \geqslant \Lambda$ in Eq. (90) is exactly saturated by the Einstein-Sasaki case $T^{1,1}$, and that all the nonsupersymmetric $p \neq q$ spaces have a Lichnerowicz mode lying strictly below the bound [13].

It is worth remarking that, in view of the equivalence of the criteria for black hole stability and AdS stability described in Secs. II B and II C, we have the immediate consequence that Einstein-Sasaki manifolds will always give stable Schwarzschild-Tangherlini black holes.

A further consequence is that any Einstein metric whose Lichnerowicz spectrum does not respect the lower bound (90) cannot admit Killing spinors, and so it cannot give rise to supersymmetric backgrounds in any supergravity compactification. Examples include not only the case of product metrics, for which it has long been known that there exists a Lichnerowicz zero mode [12], but also cases such as the Bohm metrics whose negative Lichnerowicz modes we have demonstrated in this paper.

V. LORENTZIAN BOHM METRICS, REAL TUNNELING GEOMETRIES, AND COUNTEREXAMPLES TO THE COSMIC BALDNESS CONJECTURE

In this section we discuss metrics obtained by analytic continuation of the Bohm metrics. These metrics, which provide generalizations of de Sitter spacetime as locally static solutions with cosmological horizons, have a number of applications. In particular, they provide counterexamples to a certain form of the Cosmic Baldness conjecture. Furthermore, the Riemannian Bohm metrics have a totally geodesic hypersurface. This allows them to be viewed as real tunneling geometries for the creation of the Lorentzian Bohm metrics "from nothing." We first review the geometry by discussing the case of the round $S^5 = \text{Bohm}(2,2)_0$.

A. Round S^5 and dS_5

The round metric on S^5 may be written as

$$ds^2 = d\rho^2 + \sin^2\rho(d\theta^2 + \sin^2\theta d\phi^2) + \cos^2\rho(d\theta'^2 + \sin^2\theta'^2 d\phi'^2). \quad (100)$$

This provides an isometric embedding into $\mathbb{E}^6$, with Cartesian coordinates denoted by the variables $(X_1, X_2, X_3, X_1', X_2', X_3')$, via

$$(X_1, X_2, X_3) = \sin\rho(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad (101)$$

and

$$(X_1', X_2', X_3') = \cos\rho(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta'). \quad (102)$$

Thus, as required, one has

$$X_1^2 + X_2^2 + X_3^2 + X_1'^2 + X_2'^2 + X_3'^2 = 1. \quad (103)$$

Note also, for later use, that

$$X_1^2 + X_2^2 + X_3^2 = \sin^2\rho, \quad X_1'^2 + X_2'^2 + X_3'^2 = \cos^2\rho. \quad (104)$$

The range of ρ is seen to be $\rho \in [0, \pi/2]$.

To get the locally static Lorentzian de Sitter solution dS_5, one can set the angle $\phi = \mathrm{i}t$ with t real. This means that $X_2 = \mathrm{i}T$ with T real, and the embedding is into (but not onto) the quadric

$$X_1^2 - T^2 + X_3^2 + X_1'^2 + X_2'^2 + X_3'^2 = 1. \quad (105)$$

Equations (104) now become

$$X_1^2 - T^2 + X_3^2 = \sin^2\rho, \quad X_1'^2 + X_2'^2 + X_3'^2 = \cos^2\rho. \quad (106)$$

It is clear that there can be points on dS_5 for which $X_1^2 - T^2 + X_3^2$ is negative, and therefore for which $X_1'^2 + X_2'^2 + X_3'^2$ exceeds unity. It follows that we need to use a different parameterization. We set $\sin^2\rho = 1 - b^2$ and get the metric

$$ds^2 = \frac{db^2}{1-b^2} + (1-b^2)(d\theta^2 - \sin^2\theta dt^2) + b^2(d\theta'^2 + \sin^2\theta' d\phi'^2). \quad (107)$$

In this metric $b \in [0, \infty)$. However, $b = 1$ is a coordinate singularity, and for $b > 1$ the orbits of ∂_t are spacelike. We use Eq. (107) in the region $0 \leqslant b < 1$, where ∂_t is timelike. There are Killing horizons of ∂_t at $\theta = 0$ and $\theta = \pi$. The metric on the horizon is

$$ds^2 = \frac{db^2}{1-b^2} + b^2(d\theta'^2 + \sin^2\theta' d\phi'^2), \quad (108)$$

which is the standard metric on S^3. It is important to note, that in order to get all of the S^3, we need both of the copies of this metric with $0 \leqslant b < 1$. These arise from the two values $\theta = 0$ *and* $\theta = \pi$, each of which covers half of the horizon. This is best seen from the embedding of Eqs. (101) and (102). The cosmological event horizon is not unique, since one may act with the $SO(5,1)$ isometry group. This fact is connected with the observer dependence of the associated Hawking thermal radiation [3]. The coordinate singularity at $b = 1$ arises because the two-dimensional orbit of $SO(2,1)$ on the dS_2 factor changes from being timelike to spacelike as it crosses the surface $b = 1$.

It is interesting to observe that with the metric on S^5 written as in (100), the map $\phi \to -\phi$ is an isometry, which fixes pointwise a separating totally-geodesic hypersurface Σ, given by $\phi = 0$ *and* $\phi = \pi$ (because $-\pi \sim \pi$ here). In terms of the embedding described earlier, Σ is the hypersurface $X_2 = 0$. The $\mathbb{Z}_2$ isometry implies the vanishing of the second fundamental form on Σ, i.e., $K \equiv \frac{1}{2}\mathcal{L}_n g = 0$, and hence the totally geodesic property. Here n is the normal to Σ. The metric on Σ is

$$ds^2 = \frac{db^2}{1-b^2} + (1-b^2)d\theta^2 + b^2(d\theta'^2 + \sin^2\theta' d\phi'^2), \quad (109)$$

which is in fact just the round metric on S^4. We get all of S^4 because we have two copies of this metric, each with $\theta \in [0, \pi]$, corresponding to $\phi = 0$ and $\phi = \pi$. Again, this is seen most immediately in terms of the embedding Eqs. (101) and (102), where $X_2 = 0$ manifestly defines an S^4. Thus we have a real tunneling geometry in the sense of Ref. [23]. That is, we have a compact gravitational instanton with totally geodesic boundary, such as one might use to approximate a proposed wave function for the universe. The Riemannian metric may be grafted onto the Lorentzian dS_5 metric (107) at $t = 0$, where it is clear that Eq. (107) also has the same totally geodesic hypersurface with metric (109) defined as the fixed point set of $t \to -t$.

B. Lorentzian Bohm metrics

The setup of the previous Sec. V A generalizes straightforwardly to Bohm metrics. An isometric embedding is no longer possible, because spheres are the only positive curvature Einstein metrics that may be embedded isometrically into Euclidean space of one dimension higher. However, the topological statements go through. The analytic continuation, $\phi \to it$, of the five-dimensional Bohm metrics gives

$$ds^2 = d\rho^2 + a^2(\rho)(d\theta^2 - \sin^2\theta dt^2) + b^2(\rho)(d\theta'^2 + \sin^2\theta' d\phi'^2). \quad (110)$$

Note that we could have analytically continued the second sphere instead. In some cases this gives two inequivalent Lorentzian metrics; we shall discuss making this interchange $a \leftrightarrow b$ below. We will distinguish between these two analytic continuations using subscripts. Thus Σ_1 and Σ_2 correspond to the totally geodesic hypersurfaces in each case. Again, we have Killing horizons at $\theta = 0$ and $\theta = \pi$. The range of ρ depends on the specifics of the Bohm metric.

Consider first the cases where the corresponding Riemannian Bohm metric has topology S^5. The topology of the horizon is S^3, as it was for the round S^5 of the previous Sec.

TABLE II. Volumes of various Bohm metrics.

Bohm	Topology	b_0	ρ_{fin}	$V=16\pi^2 \times\int a^2b^2d\rho$	$\mathrm{Vol}(\Sigma_2)=8\pi^2 \times\int a^2bd\rho$	$\mathrm{Vol}(\Sigma_1)=8\pi^2 \times\int ab^2d\rho$
$(2,2)_0$	S^5	1	1.57079	31.006	26.320	26.320
$(2,2)_2$	S^5	0.253554	2.68470	20.814	20.302	20.302
$(2,2)_4$	S^5	0.053054	3.04979	20.672	20.2605	20.2605
$(2,2)_1$	$S^3\times S^2$	0.5	2.22143	21.924	21.924	19.739
$(2,2)_3$	$S^3\times S^2$	0.117794	2.93537	20.684	20.189	20.335
$(2,2)_5$	$S^3\times S^2$	0.023571	3.10092	20.6709	20.267	20.254
$(2,2)_\infty$	bi-cone on $S^2\times S^2$	0	3.14159	20.6708	20.2603	20.2603

V A. This follows from the fact that the metric functions $a(\rho)$ and $b(\rho)$ behave near the two end points $\rho=0$ and $\rho=\rho_f$ in the same way as $\sin\rho$ and $\cos\rho$, respectively, behave near the end points $\rho=0$ and $\rho=\pi/2$ of the round S^5 metric. As we shall see shortly, the area A of this cosmological event horizon is always less than in the round case. Interchanging a and b in the S^5 Bohm examples gives the same Lorentzian metric.

When the Riemannian Bohm metric has topology $S^3\times S^2$, an exchange of functions $a\leftrightarrow b$ in Eq. (110) will change the topology of the Lorentzian manifold, and in particular the topology of the event horizon. This is because in these cases $a(\rho)$ goes to zero at both endpoints, while $b(\rho)$ never goes to zero. For topological purposes, we may think of a as behaving as $\sin\rho$ with end points $\rho=0$, $\rho=\pi$ and b behaving as a constant function [just as in the "trivial" Bohm metric Bohm$(2,2)_1$, which is simply the product Einstein metric on $S^3\times S^2$]. Thus we have two possibilities. The metric (110) has a horizon with topology $S^1\times S^2$. If we exchange a and b, the horizon will have topology S^3. These topologies are seen in the same way as in the previous Sec. V A, and as always we should take care to include the two values $\theta=0$ and $\theta=\pi$. The area of the horizons are

$$A_1=8\pi\int a^2d\rho \tag{111}$$

and

$$A_2=8\pi\int b^2d\rho, \tag{112}$$

respectively. Note that there is an extra factor of 2 because there are contributions from both $\theta=0$ and $\theta=\pi$. The arguments of the following sections suggest that these two quantities should be equal, and less than the horizon area for the de Sitter spacetime dS_5. These are nontrivial conditions on the functions a and b. The nonuniqueness of these cosmological horizons is reduced compared to the de Sitter case, because the relevant isometry group is now only $SO(2,1)$.

The $\mathbb{Z}_2$ isometry of the previous Sec. V A is also present in the Bohm metrics, and therefore we recover a totally-geodesic submanifold. Thus one might consider using Bohm metrics in tunneling calculations for the creation of a Lorentzian Bohm universe. In that application the number of negative modes of Δ_2 should be an odd number, so as to get an imaginary part for the free energy when one evaluates the functional integral. Usually one expects just one negative mode, and the contribution of instantaneous with more than one is often ignored.

Another interesting question is what is the volume of Σ, the totally geodesic boundary. For tunneling geometries constructed from hyperbolic tunneling manifolds [24],[2] $\mathrm{Vol}(\Sigma)$ is a measure of the complexity of Σ and it is possible to bound the volume of the tunneling geometry in terms of the volume of the boundary such that larger complexity, i.e., larger $\mathrm{Vol}(\Sigma)$, means larger volume [25]. In the case of positive scalar curvature, and with boundaries Σ of simple topology (S^4 or $S^2\times S^2$ in our case), the notion of complexity is not relevant. However, it is still interesting to know how the volume of the manifold is related to the volume of the totally geodesic boundary. Specifically, the volume of the five-dimensional manifold is

$$V=16\pi^2\int a^2b^2d\rho, \tag{113}$$

while

$$\mathrm{Vol}(\Sigma_2)=8\pi^2\int a^2bd\rho,$$

$$\mathrm{Vol}(\Sigma_1)=8\pi^2\int ab^2d\rho. \tag{114}$$

The two different values for Σ correspond to interchanging a and b in the metric. In the S^5 cases these will be the same, but for the $S^3\times S^2$ cases they will be different. Some examples are illustrated in Table II. The table also collects information about the corresponding values of b_0 and ρ_{fin}.

In this table the results for the double cone are found analytically, and the Bohm$(2,2)_0$ metric (the round five-sphere) may also be calculated analytically as a check on the numerics. The volumes decrease from the round sphere to the double cone, as expected from Bishop's theorem [26]. The volumes of the Σ's decrease with the volume in the S^5 cases, but not in the $S^3\times S^2$ cases.

[2]Note that Δ_2 has no negative modes in this case, and so one might worry about the tunneling interpretation.

TABLE III. Volumes of some higher dimensional Bohm metrics.

Bohm	Topology	b_0	ρ_{fin}	$V=4\pi^4 \int a^3b^3d\rho$
$(3,3)_0$	S^7	1	1.5707	32.470
$(3,3)_2$	S^7	0.305521	2.6933	24.499
$(3,3)_1$	$S^4\times S^3$	0.577351	2.2207	24.995
$(3,3)_3$	$S^4\times S^3$	0.14291	2.9322	24.482
$(3,3)_\infty$	bi-cone on $S^3\times S^3$	0	3.1416	24.4816

Similar calculations may of course be done with higher-dimensional Bohm metrics, as illustrated in Table III. The generalization of Σ to higher dimensions is slightly more involved, and will not be discussed here.

There is another analytic continuation of the Bohm metrics to a Lorentzian metric that is possible in the cases $B_{2m+1}(p,p)$. In these cases the metric functions $a(\rho)$, $b(\rho)$ are symmetric about the midpoint $\rho=\rho_c$. Thus $\rho=\rho_c$ defines a totally geodesic hypersurface, with topology $S^p \times S^p$, stabilized by a reflection. It follows that setting $\rho-\rho_c=it$ gives a Lorentzian expanding universe with spatial cross sections $S^p\times S^p$. We anticipate that the scale factors of each of the spheres will be expanding. A special case of this situation is $B_1(p,p)$, which corresponds to analytically continuing $S^3\times S^2$ to $dS^3\times S^2$. Note that in this case the S^2 factor does not expand, but this will not be the case for the general Bohm metrics where neither of the functions $a(\rho)$, $b(\rho)$ are constant.

C. Cosmological event horizons

In the Riemannian metrics, the circle action on S^5 generated by ∂_ϕ rotates the X_1-X_2 plane. The action has an S^3's worth of fixed points for which $X_1=0$ and $X_2=0$, corresponding to $\theta=0$ and $\theta=\pi$. Because the reversal of ϕ is also an isometry, we have in fact an $O(2)$ action, which allows the analytic continuation to a locally static, i.e., time-reversal invariant, metric with a hypersurface-orthogonal locally timelike Killing vector field with a Killing horizon. These are the cosmological horizons of the previous subsections. In such cases the Lorentzian metric may be written locally as

$$ds^2=-U^2dt^2+g_{ij}dx^idx^j, \tag{115}$$

where g_{ij} is the metric on the orbit space Q of ∂_t, and the gravitational field equations imply in particular that

$$\nabla^2U=-\Lambda U. \tag{116}$$

The quantity U vanishes on the horizon, and its normal derivative $\partial U/\partial n$ on the horizon $U=0$ is a constant, which is called the surface gravity κ. The period in imaginary time, i.e., the real period of $\tau\equiv it$, is $2\pi/\kappa$. If V is the volume of the corresponding Riemannian manifold, and A is the area of the event horizon in the Lorentzian manifold, one has

$$V=\frac{2\pi}{\kappa}\int_Q U\sqrt{g}d^{n-1}x. \tag{117}$$

The boundary ∂Q is the event horizon, where the orbits degenerate. For example, when the Bohm metric has topology S^5 the boundary Q is a four-ball, B^4, with boundary the event horizon S^3. The metric on Q is in fact given by Eq. (109). However, the crucial difference is that we take only $\phi=0$ to intersect all the orbits of ∂_ϕ once, whilst for Σ we needed to take both $\phi=0$ and $\phi=\pi$. The S^4 that we had before corresponded to gluing two copies of B^4s across their boundary S^3.

Integration of ∇^2U gives

$$\Lambda\int_Q U\sqrt{g}d^{n-1}x=\kappa A, \tag{118}$$

whence

$$V\Lambda=2\pi A. \tag{119}$$

This argument shows that when there are two possible inequivalent analytic continuations, such as for the Bohm metrics on $S^3\times S^2$, the horizon areas should be the same, $A_1=A_2$. This is illustrated in Table IV, which also illustrates the relationship (119), showing that it works for the various topologies. The values of V are repeated from Table III.

Note that κ cancels in Eq. (119), as it must since it depends on the normalization of the length of the Killing field, which is arbitrary. This relation between area and volume is quite universal and holds for any Einstein metric admitting an $O(2)$ action. It allows us to relate the on-shell action to the area of the horizon, and hence to show that formally at least, the entropy S is given by

$$S=\tfrac{1}{4}A, \tag{120}$$

just as in four dimensions. This general argument was first given in four dimensions in Ref. [27]. Now, a theorem of

TABLE IV. Horizon areas and volumes of Bohm metrics.

Topology	b_0	A_1	$A_2=A_1$?	V	$4V=2\pi A$?
S^5	1	19.74	Yes	31.006	Yes (=124.02)
S^5	0.253554	13.25	Yes	20.814	Yes (=83.25)
S^5	0.053054	13.160	Yes	20.672	Yes (=82.688)
$S^2\times S^3$	0.5	13.96	Yes	21.925	Yes (=87.69)
$S^2\times S^3$	0.117794	13.168	Yes	20.684	Yes (=82.74)
$S^2\times S^3$	0.023571	13.15954	Yes	20.6709	Yes (=82.684)
bi-cone on $S^2\times S^2$	0	13.15948	Yes	20.6708	Yes (=82.683)

Bishop [26] tells us that for fixed Λ, the volume V never exceeds its value for the round sphere, with equality only in the case of roundness. It follows that this area or entropy is always less than the area or entropy of the corresponding horizon in dS_5.

Since dynamically one expects the area to increase, and thermodynamically one expects the entropy to increase, there seem to be some physical grounds for believing that the static Lorentzian Bohm metrics are dynamically unstable. Indeed, one might conjecture that if they are perturbed slightly at some initial time, then they will evolve to an asymptotically de Sitter state, and that this evolution will be such that the area of the cosmological horizon increases monotonically from a value near, but smaller than, its value for the initial Bohm metric, to its value for the de Sitter spacetime. It should not be impossible to investigate this conjecture numerically.

D. Consequences for cosmic baldness

It has been conjectured for some time [8] (in four dimensions) that there should exist only one regular static solution of the Einstein equations with a cosmological constant that has only a single cosmological horizon (so that $\partial Q = S^{n-2}$). In fact one usually thinks of Q as being topologically an $(n-1)$ ball, as described above. This is a much stronger statement than that locally, within the event horizon of every observer or most observers, the metric will settle down to the static de Sitter form. In fact for generic initial data one cannot hope that the metric will settle down globally to the de Sitter state, as was originally made clear in [8] [see also Ref. [28] for a detailed discussion using the exact Lorentzian Taub-NUT (Newman-Unti-Tanburino) metrics].

It is now clear that the Bohm metrics provide infinitely many counterexamples to the cosmic baldness conjecture in dimensions $5 \leqslant n \leqslant 9$. The situation in four dimensions remains unclear. It is still possible to believe an even stronger conjecture, the truth of which would imply the cosmic baldness conjecture, namely, that there is only one Einstein metric on S^4. At present all that is known is that if there is another Einstein metric on S^4, then its volume must be less than that of the round metric by a factor of 3 [29], and that the magnitude of the Weyl tensor must exceed a certain threshold [30]. This is interesting in the light of the fact that it is the magnitude of the Weyl tensor which appears to play a role in controlling the spectrum of the Lichnerowicz operator. It may perhaps suggest that any counterexample will have a negative mode of Δ_2.

Curiously, there are some proofs of a form of the cosmic no-hair conjecture in the literature [31,32], but these proofs require a smooth structure at future spacelike infinity I^+. It seems likely that in our examples, the future timelike infinity will not be of the sort envisaged in those proofs. It would be interesting to investigate this point further, but this would seem to require analytic formulas for $a(\rho)$ and $b(\rho)$. As mentioned in the previous subsection it seems likely that these static metrics will be dynamically unstable, and they may well evolve into an asymptotically de Sitter–like state. If this is true then the main physical spirit of the no-hair conjecture will hold, even though the letter of the baldness conjecture is broken.

VI. NEGATIVE MODES AND NONUNIQUENESS OF THE DIRICHLET PROBLEM

The existence of negative Lichnerowicz modes has been a major theme in this work. This section contains speculative comments on a generic connection between negative Lichnerowicz modes and the nonuniqueness of solutions to the Dirichlet problem. In particular, we argue that there will be infinitely many negative Lichnerowicz modes on the *noncompact* Ricci-flat Bohm metrics. The existence of L^2 negative modes for the Lichnerowicz operator for noncompact Ricci flat manifolds such as the Riemannian Schwarzschild solution first came to light when considering the negative specific heat of black holes. Since then a considerable literature has grown up, analyzing that and related cases.

In the noncompact Ricci flat case, it seems that the general picture is as follows. One has a class of metrics on a manifold M_n depending on some parameters, in the simplest case just one overall scaling parameter μ, such as the mass in the Schwarzschild case. One asks whether this metric can fill in a given boundary Σ_{n-1} that has a given metric h_{ij}. That is, one tries to solve the Dirichlet problem for the Einstein equations. In the four-dimensional Riemannian-Schwarzschild case, for example, the boundary is taken to be $S^2 \times S^1$ with the product metric. This is specified by the radius R of the two-sphere and the period β of the circle. Physically, we are putting a black hole in a spherical box of radius R, and fixing the temperature on the boundary of the box to equal $T = \beta^{-1}$. If the metric and the boundary data are to agree then we must have

$$8\pi\mu\sqrt{1-\frac{2\pi}{R}}=\beta. \tag{121}$$

The number of solutions of this equation for μ depends on the ratio β/R that specifies, up to a scale, the boundary metric h_{ij}. One finds that if the ratio is small there are two solutions for μ. The Einstein action I of the two solutions differs. The action for the smaller value of μ is the smaller. We shall refer to these two solutions as branches. If the ratio is large there are no solutions for μ. At the critical value the two solutions for μ coincide, and give $\mu = \frac{1}{3}R$.

Now consider the operator Δ_2 (which equals Δ_L in this Ricci-flat situation), subject to Dirichlet boundary conditions, which gives the Hessian of the action I. For a large box (in relation to the scale set by the temperature $T = \beta^{-1}$), Δ_2 has a single negative mode for the branch with the smaller value of μ, and a positive but no negative mode for the branch with the larger value. As one reaches the critical value, the two branches coincide and so do the two eigenvalues. At the critical point there is an eigenmode of Δ_L with zero eigenvalue. In other words, at the critical point there is a marginally stable mode. Because the specific heat is given essentially by the Hessian of the action (i.e., the free energy) considered as a function of the boundary data, it changes sign at this value.

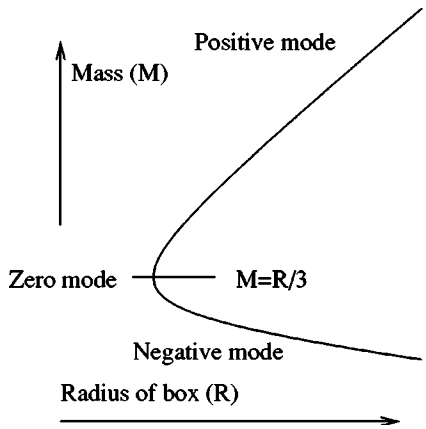

FIG. 1. Relationship between Lichnerowicz modes and masses for the Schwarzschild solution in a finite cavity.

Next, consider what happens as the radius of the box increases to infinity, so that the ratio β/R goes to zero while keeping β a constant, and take the branch on which μ is smaller. In the limit, one finds that $\mu\rightarrow\beta/(8\pi)$. On the other branch, one finds that $\mu\rightarrow\infty$. Thus, in the limit $R\rightarrow\infty$ one gets a noncompact Ricci-flat Einstein manifold, namely the standard Schwarzschild solution, and by following the mode which first appears as a zero-mode at the critical value, one gets an L^2 negative mode for Δ_L on that manifold. This process is illustrated at fixed temperature in Fig. 1.

The picture described above has been vindicated by detailed numerical calculations in this and related cases [33–37]. For example, Hawking and Page [38] studied black holes in anti–de Sitter spacetime. The classical solution is the Kottler or Schwarzschild anti–de Sitter solution, an Einstein metric with negative scalar curvature 4Λ. The role of the radius R is now played by the cosmological constant Λ, and the manifolds considered are always noncompact, but the general picture is similar.

The arguments given above are heuristic rather than being completely rigorous, but they suggest the following generalization. One considers a one-parameter family of Dirichlet problems for the Einstein equations. As the parameter varies one finds a discrete nonuniqueness, with more and more branches appearing, generically in pairs, and as each new branch appears a zero mode of Δ_2 occurs, which then splits into a pair of modes, one with positive eigenvalue and one with negative eigenvalue. In the limit that one gets a noncompact manifold, one should have found, on the correct branch, as many L^2 negative modes as the number of critical values one has passed.

An obvious example on which to try this argument is the noncompact metric of Bohm on $\mathbb{R}^3\times S^2$ [17], recently considered by Kol [11] and discussed above. In fact it exhibits a feature not seen previously, which is that even within the restricted framework of cohomogeneity one metrics, the Dirichlet problem may have infinitely many solutions.

These metrics are determined by a single (scale) parameter, which may be taken to be the radius b_0 of the two-sphere bolt. Now the geometry of a boundary at some fixed value of the radius R is given by the ratio $a(R)/b(R)$. Thus the possible filling solutions are determined by the intersections of the curve of a/b with a straight line at 45° through the origin in the (a,b) plane. Clearly, if we let a/b tend to one there are more and more intersections, which appear at critical points in pairs when the straight line touches the slightly wiggly but almost straight a/b curve. It seems reasonable to suppose that an additional zero mode of Δ_L appears at this point, and then as the slope of the straight line gets closer to unity, a pair of eigenvalues, one positive and one negative, branches off. If this intuition is correct, and if the branches are connected at $a/b=1$ say, then one expects that the noncompact Bohm metrics on $\mathbb{R}^3\times S^3$ should have infinitely many L^2 Lichnerowicz negative modes. This process is illustrated in Fig. 2.

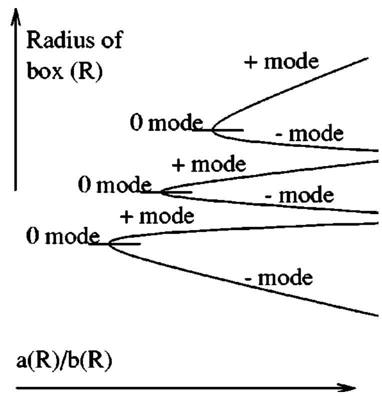

FIG. 2. The noncompact Bohm metric in a box with a fixed value of a/b at the boundary. Allowed values of R are shown along with the corresponding positive, negative, and zero modes from branching. It is expected that the branches will join at $a(R)/b(R)=1$.

Some evidence for this viewpoint comes from examining $SO(p)\times SO(q)$-invariant transverse trace-free perturbations of the singular Ricci-flat cone on $S^p\times S^q$ obtained in the limit $b_0\rightarrow 0$. That is to say, the cone is

$$ds^2=dr^2+\frac{r^2}{p+q-1}[(p-1)d\Omega_p^2+(q-1)d\tilde{\Omega}_q^2], \tag{122}$$

and the perturbation is

$$h_{\alpha\beta}=r^2\phi(r)\begin{pmatrix}\frac{1}{p}g_p & \\ & -\frac{1}{q}\tilde{g}_q\end{pmatrix}_{\alpha\beta}, \tag{123}$$

where g_p and $\tilde{g}_q$ are the round metrics on S^p and S^q, respectively. This is a zero mode on $S^p\times S^q$. If we want a mode on the cone with Lichnerowicz eigenvalue $-\lambda$, the equation for ϕ is [7]

$$\frac{d^2\phi}{dr^2}+\frac{d}{r}\frac{d\phi}{dr}+\frac{2d-2}{r^2}\phi=\lambda\phi. \tag{124}$$

The solutions to this equation for $\lambda\neq 0$ are, writing $d=p+q$,

$$\phi(r)=r^{(1-d)/2}[AI_\mu(\lambda^{1/2}r)+BK_\mu(\lambda^{1/2}r)],$$

$$\mu=\frac{1}{2}\sqrt{(d-1)(d-9)}, \tag{125}$$

where A and B are constants, and I_μ and K_μ are the modified Bessel functions. When $\lambda=0$ we get

$$\phi(r)=r^{(1-d)/2\pm i\sqrt{(d-1)(9-d)}/2}. \tag{126}$$

This expression was written down in Ref. [11] (his d is shifted by one). It is also the behavior of the Bessel function solutions in Eq. (125) as $r\rightarrow 0$. As $r\rightarrow\infty$, the Bessel function solutions go as $r^{-d/2}e^{\pm r}$. The K_μ Bessel function is the better behaved.

We are interested in the Bohm cases $4<d<9$ that coincide with oscillatory behavior in Eqs. (125) and (126). For these dimensions, the zero-mode solutions are not normalizable as $r\rightarrow\infty$ or as $r\rightarrow 0$, although they are bounded as $r\rightarrow\infty$. The negative mode solution with K_μ is normalizable at infinity.

Thus the K_μ solutions decay at infinity, and as we move in towards the origin, start oscillating at $r\sim\lambda^{-1/2}$. In the singular cone limit, the modes are not normalisable at the origin. However, suppose we are in a rounded-off cone. For $r\gg b_0$ the metric is essentially that of the singular cone, and we may use our solutions (125). If further we have $\lambda^{-1/2}\gg b_0$, logarithmic oscillations will set in, within this asymptotic regime. We should then expect to be able to match this solution to a solution in the inner regions that is well behaved at the origin (see Ref. [7]), for a certain discrete set of values for λ. This will give us a spectrum of negative Lichnerowicz modes. It would seem that there will be an infinity of such modes, accumulating at zero.

VII. CONCLUSIONS AND DISCUSSION

The principal focus of this paper has been to study applications of the countable infinities of inhomogeneous Einstein metrics on certain spheres and products of spheres, which were discovered recently by Bohm [6]. These occur for the topologies S^{p+q+1} and $S^{p+1}\times S^q$, for $5\leqslant p+q\leqslant 9$ and $p\geqslant 2$, $q\geqslant 2$. They may be used in place of the usual round-sphere Einstein metrics in a variety of constructions including black holes and Freund-Rubin solutions, and after a Wick rotation to a Lorentzian section, they may be interpreted as spacetime metrics in their own right.

The stability of generalized Schwarzschild-Tangherlini black holes, where the d-dimensional constant-radius spatial sections M_d in the $(d+2)$-dimensional spacetime are taken to be positive Ricci curvature Einstein spaces, was studied recently in Ref. [7]. It was shown that a solution will be classically stable if the spectrum of eigenvalues of the Lichnerowicz operator on transverse traceless symmetric two-index tensors in M_d is bounded below by a value corresponding to $\Delta_{\rm stab}\geqslant 0$ in Eq. (1). One of our results in this paper has been to show that this stability criterion is identical to one obtained in Refs. [12,13] for the stability of Freund-Rubin solutions $AdS_n\times M_d$ of gravity coupled to a d-form (or n-form) field strength. Thus it becomes of considerable interest to try to obtain bounds on the spectrum of the Lichnerowicz operator on Einstein spaces M_d.

The Bohm metrics are sequences of cohomogeneity one Einstein metrics of the form (29), which more and more nearly approach a "double-cone" form (42) as one progresses along the sequence. One therefore intuitively expects that the "ballooning" instabilities associated with direct products of the sphere metrics forming the principal orbits will give rise to negative-eigenvalue modes of the Lichnerowicz operator. Such modes would *a fortiori* violate the stability criterion described above, implying that Schwarzschild-Tangherlini black holes or $AdS_n\times M_d$ solutions constructed using the Bohm metrics would be unstable. In certain cases, including all the Bohm metrics on $S^3\times S^2$ and $S^3\times S^3$, we constructed analytic proofs that indeed show the existence of negative modes of the Lichnerowicz operator. Numerical calculations for other examples, namely Bohm metrics on S^5, have confirmed that these too have negative-eigenvalue modes of the Lichnerowicz operator. We believe that in fact all the Bohm metrics have negative Lichnerowicz modes.

One can perform analytic coordinate continuations in the Bohm metrics in order to obtain spacetimes with positive cosmological constant that generalize de Sitter spacetime. If one does this for the Bohm metrics that are themselves topologically spheres, then the resulting spacetimes have the same topology and global structure as de Sitter spacetime itself. These metrics provide infinitely many counterexamples, in dimensions $5\leqslant n\leqslant 9$, to the cosmic baldness conjecture, which asserted the uniqueness of regular static solutions of the Einstein equations with a single cosmological horizon. However, although the Bohmian analogues of de Sitter spacetime are regular, we have argued that they are unstable and that they are likely to decay into a de Sitter–like state. This would mean that the no-hair conjecture would remain inviolate.

In order to explore possible endpoints for the decay of spacetimes constructed using Bohm metrics, we were also led to consider other geometries for Einstein spaces M_d that would satisfy the criteria for stability. In particular, we considered compact Einstein spaces of positive Ricci curvature that admit Killing spinors. We showed that in all such spaces, in any dimension, one can derive a lower bound on the spectrum of the Lichnerowicz operator which implies that the stability criterion $\Delta_{\rm stab}\geqslant 0$ is satisfied. These examples include all the Einstein-Sasaki spaces, which may be defined as odd-dimensional Einstein spaces whose cones give Ricci-flat Kähler spaces in one higher dimension. It is straightforward to see that the covariantly constant spinors on the Ricci-flat Kähler cone project down as Killing spinors on the Einstein-Sasaki base. A by-product of our results is that it demonstrates that the Bohm metrics, for which Δ_L (and hence *a fortiori* $\Delta_{\rm stab}$) can be negative, cannot admit Killing spinors.

There also exist Ricci-flat Bohm metrics, with noncompact topology. The structure of these metrics at short distance looks very similar to that near one of the two endpoints of the compact metrics. However, lacking the cosmological term that causes the metric functions to turn over and recollapse in the compact examples, the noncompact spaces asymptotically approach the cone over the an $S^p\times S^q$ direct-product base. The noncompact spaces have the topology $\mathbb{R}^{p+1}\times S^q$. The only parameter in the noncompact metrics is the overall scale. We constructed an analytic proof that the noncompact Bohm metrics for $p=q=2$ and $p=2$, $q=3$ have negative Lichnerowicz modes, and we presented a general argument that indicates that all the noncompact Bohm metrics will have infinitely many L^2 normalizable negative-eigenvalue Lichnerowicz modes.

Another possible use of the Bohm metrics is to construct four-dimensional gravitating monopoles and black holes by dimensional reduction, as studied in Ref. [39]. This is possible because many of the Bohm metrics have S^3 factors. Recalling that S^3 is isomorphic to $SU(2)$, one can quotient by a $U(1)$ action on S^3 to end up with $SU(2)/[U(1)\times\mathbb{Z}_2]\simeq S^2$. Thus the resulting lower-dimensional space will have an S^2 factor, i.e., it will look similar to a monopole or black hole, and it will come with a $U(1)$ gauge field. Explicitly, one can write the metric on S^3 using Euler angles as

$$ds^2=d\theta^2+\sin^2\theta d\psi^2+(d\phi+\cos\theta d\psi)^2, \qquad (127)$$

where $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$, and $0\leqslant\psi\leqslant 4\pi$. The quotient by the ∂_ϕ isometry, and the $\mathbb{Z}_2$ quotient $\psi\sim\psi+2\pi$, leaves us with the standard metric on S^2, namely, $d\theta^2+\sin^2\theta d\psi^2$, and a charge-two Dirac monopole $A=\cos\theta d\psi$. One can also quotient by the whole $SU(2)$, and in this case because there is no fibration over the S^3 the $SU(2)$ gauge fields obtained will be trivial.

Thus, for example, take the seven-dimensional noncompact Bohm metric over two copies of S^3, supplement the metric by an eighth timelike direction $-dt^2$, and dimensionally reduce on $SU(2)\times U(1)$. One will obtain a gravitating $U(1)$ monopole with four scalar fields. Another possibility would be to take the generalized black hole in eight dimensions over a compact Bohm metric in six dimensions with topology $S^3\times S^3$. Again quotient by $SU(2)\times U(1)$, where the $SU(2)$ is acting on the round S^3 in the Bohm metric. We will obtain a $U(1)$ magnetically charged black hole in four dimensions. Because the S^3 that the $U(1)$ was acting on is not round, the black hole will not be spherically symmetric.

ACKNOWLEDGMENTS

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APPENDIX: NUMERICAL SOLUTIONS FOR BOHM METRICS

Numerical techniques. The Einstein equations (32) and (33) cannot be solved explicitly when $p>1$ and $q>1$. It was shown in Ref. [6] that countable infinities of smooth solutions satisfying the S^{p+q+1} or $S^{p+1}\times S^q$ boundary conditions (40) or (41) exist. It is quite a straightforward matter to obtain these solutions by numerical methods, since it turns out that the second-order equations (32) are quite stable.

We have constructed numerical solutions by first obtaining Taylor-series expansions for the metric functions a and b near to $\rho=0$, imposing the $\rho=0$ boundary conditions given in Eq. (40) [or, equivalently, in Eq. (41)]. To the first couple of orders, these give

$$a=\rho-\frac{q(q-1)+b_0^2(p+q)(p-q+1)}{6b_0^2p(p+1)}\rho^3+O(\rho^5),$$

$$b=b_0+\frac{q-1-b_0^2(p+q)}{2b_0(p+1)}\rho^2+O(\rho^4). \qquad (A1)$$

Note that a is an odd function of ρ, while b is an even function.

Using a Taylor expansion of the form (A1) (which we actually evaluated up to order ρ^9), we then set initial data just outside the singular point, for a very small positive value of ρ. These data are then evolved forward in ρ numerically, using the second-order equations (32). The exercise then becomes a "shooting problem," in which one seeks to adjust the one free initial parameter b_0 so as to achieve a smooth termination of the evolved data at a point $\rho=\rho_f$ where a and b satisfy one or other of the $\rho=\rho_f$ boundary conditions given in Eqs. (40) or (41).

In cases where $p=q$, the numerical analysis is simpler, since the regular solutions are all symmetric under reflection about the midpoint $\rho=\rho_c=\rho_f/2$. Thus one can avoid the need to handle the integrations in the region near $\rho=\rho_f$ where one or other metric function is tending to zero. Instead, the shooting problem reduces to finding a b_0 for which either $\dot{a}=\dot{b}=0$ at some point $\rho=\rho_c$ [for the Bohm$(p,p)_{2m}=S^{2p+1}$ metrics] or else for which $a=b$ and $\dot{a}=-\dot{b}$ [for the Bohm$(p,p)_{2m+1}=S^{p+1}\times S^p$ metrics].

It is known from the results in Ref. [6] that there is a countable infinity of values b_0 for which a regular termination at some ρ_f occurs. The largest b_0 yielding a regular solution is $b_0=1$, leading to the standard unit S^{p+q+1} metric (35), which is called Bohm$(p,q)_0$. The next value is $b_0=\sqrt{(q-1)/(p+q)}$, giving the direct-product Einstein metric (38) on $S^{p+1,q}$ that we call Bohm$(p,q)_1$. There is then a monotonically decreasing sequence of b_0 values, giving the Bohm$(p,q)_n$ sequence of Einstein metrics, alternating between terminating ρ_0 boundary conditions given by Eqs. (40) and (41). The limit point of the sequence is $b_0=0$, giving the double-cone singular metric (42).

Plots for the five-dimensional Einstein metrics Bohm$(2,2)_n$ on S^5 and $S^3\times S^2$ are given in Figs. 3–11 for $0\leqslant n\leqslant 6$, with $b_0=(1,\frac{1}{2},0.253554255,0.117794,0.053054,0.023571,0.010503)$. We also give a plot for the case of

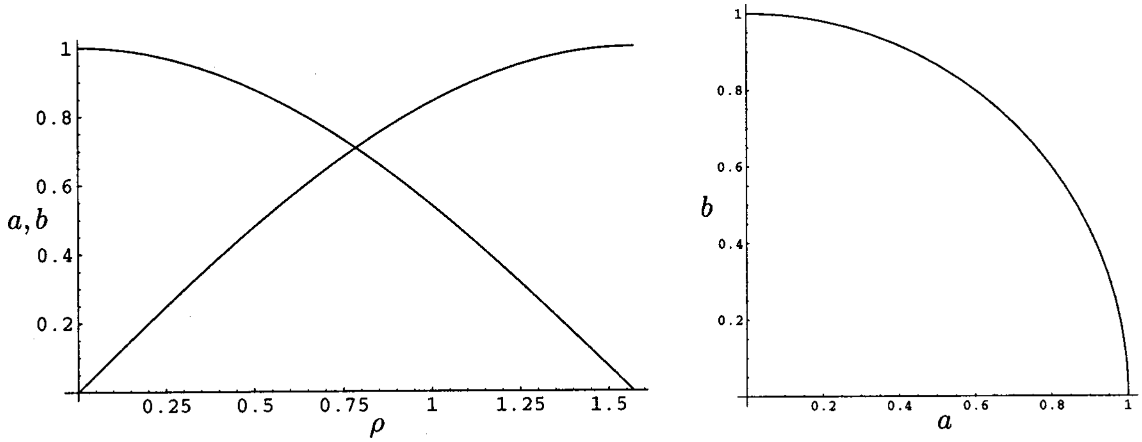

FIG. 3. The Bohm$(2,2)_0$ (standard) Einstein metric on S^5. The left-hand figure shows the metric coefficients a and b as functions of the radial variable ρ. The function a vanishes at $\rho=0$ and $b=b_0=1$ there. The crossover occurs at $\rho=\rho_c=\frac{1}{4}\pi$. The right-hand figure is a parametric plot of b vs a.

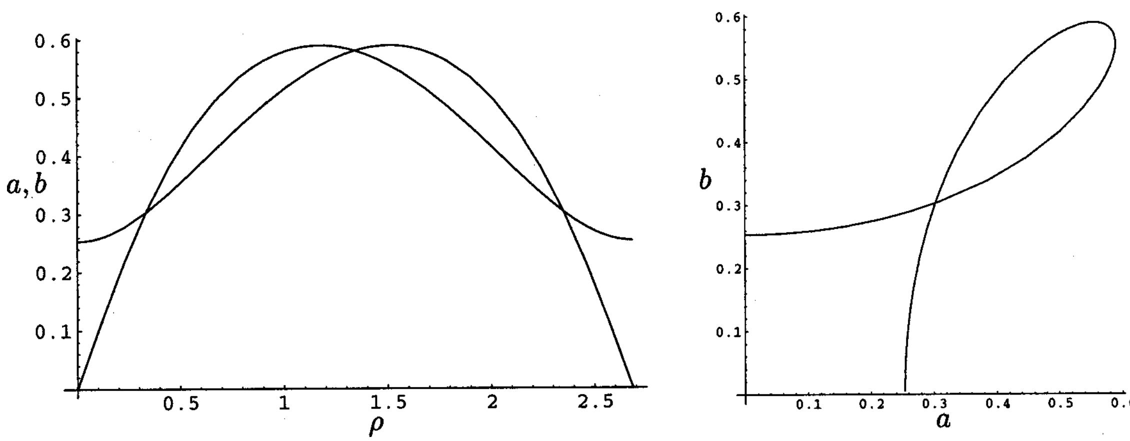

FIG. 4. The Bohm$(2,2)_2$ Einstein metric on S^5. The left-hand figure shows the metric coefficients a and b as functions of the radial variable ρ. At $\rho=0$ the function a vanishes and $b=b_0\approx0.253554255$. The midpoint is at $\rho_c\approx1.34235319$. The right-hand figure is a parametric plot of b vs a.

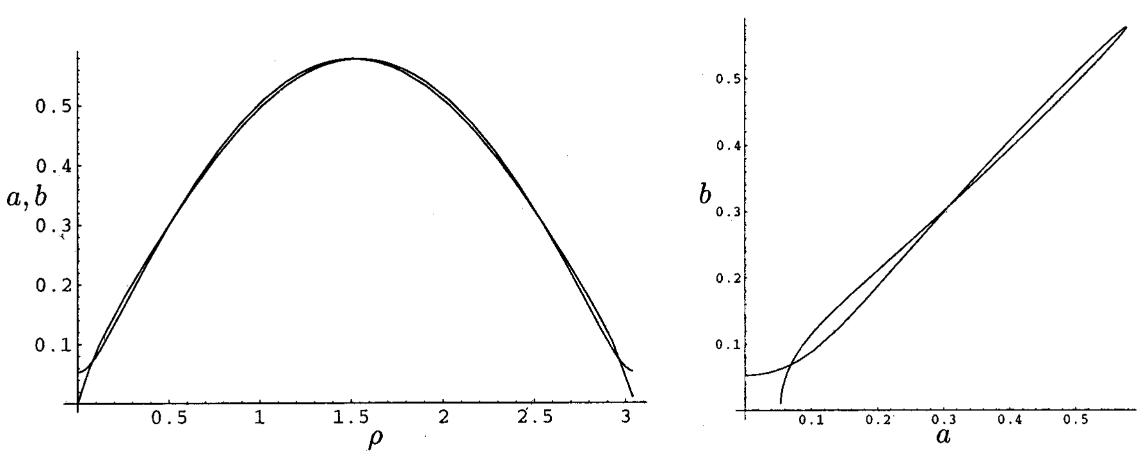

FIG. 5. The Bohm$(2,2)_4$ Einstein metric on S^5. The function b starts at $b_0\approx0.053054$, and the midpoint is at $\rho_c\approx1.524951$.

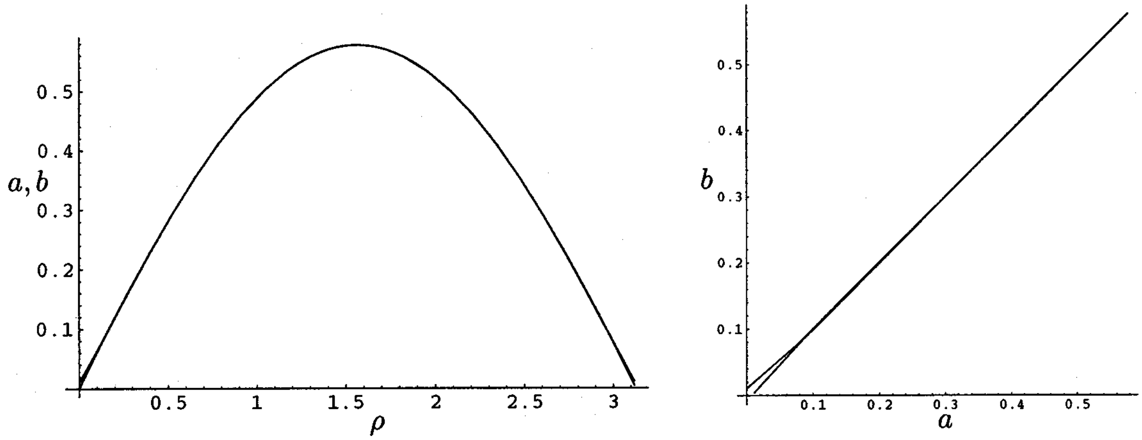

FIG. 6. The Bohm$(2,2)_6$ Einstein metric on S^5. The function b starts at $b_0 \approx 0.010503$ and the midpoint is at $\rho_c \approx 1.56174$.

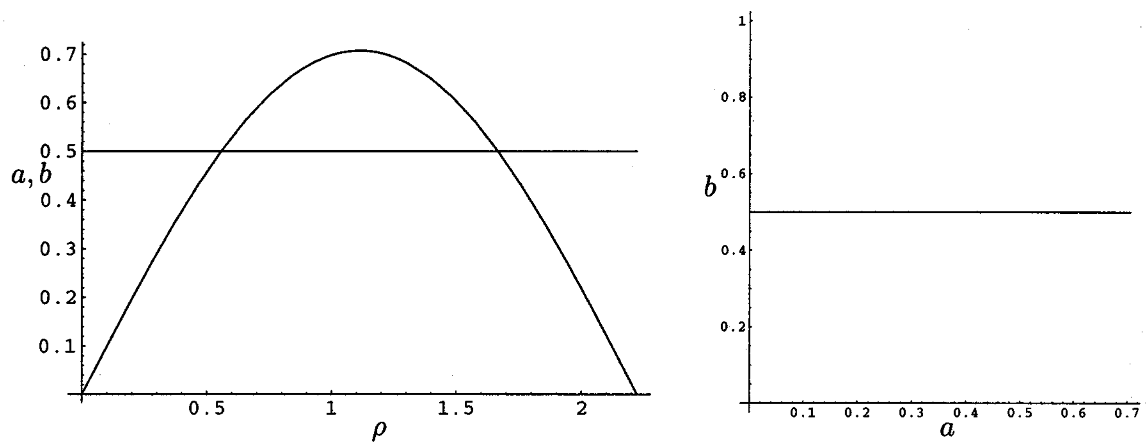

FIG. 7. The Bohm$(2,2)_1$ (standard) Einstein metric on $S^3 \times S^2$. The left-hand figure shows the metric coefficients a and b as functions of the radial variable ρ. At $\rho = 0$ the function a vanishes and $b_0 = \frac{1}{2}$. The midpoint is at $\rho_0 = (1/2\sqrt{2})\pi$. The right-hand figure is a parametric plot of b vs a.

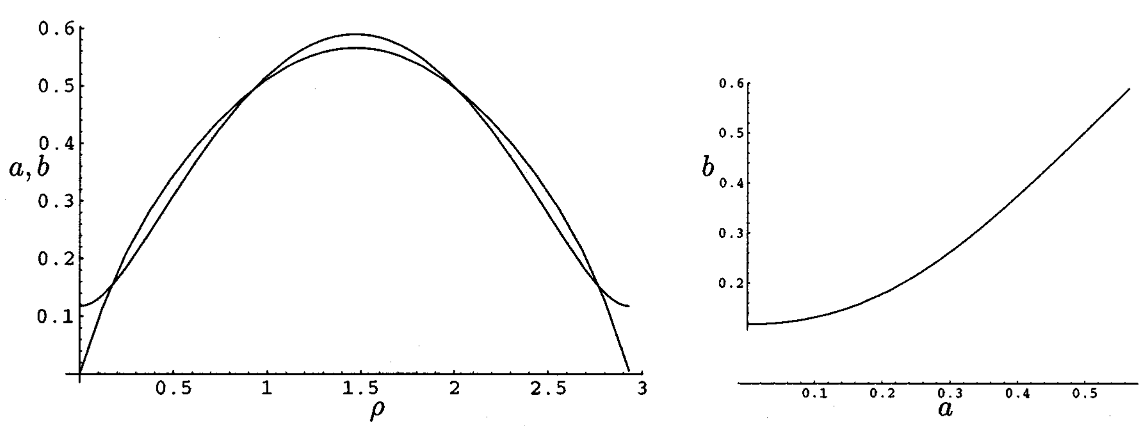

FIG. 8. The Bohm$(2,2)_3$ Einstein metric on $S^3 \times S^2$. The left-hand figure shows the metric coefficients a and b as functions of the radial variable ρ. At $t = 0$ the function a vanishes and $b_0 \approx 0.117794$. The midpoint is at $\rho_c \approx 1.46768843$. The right-hand figure is a parametric plot of b vs a.

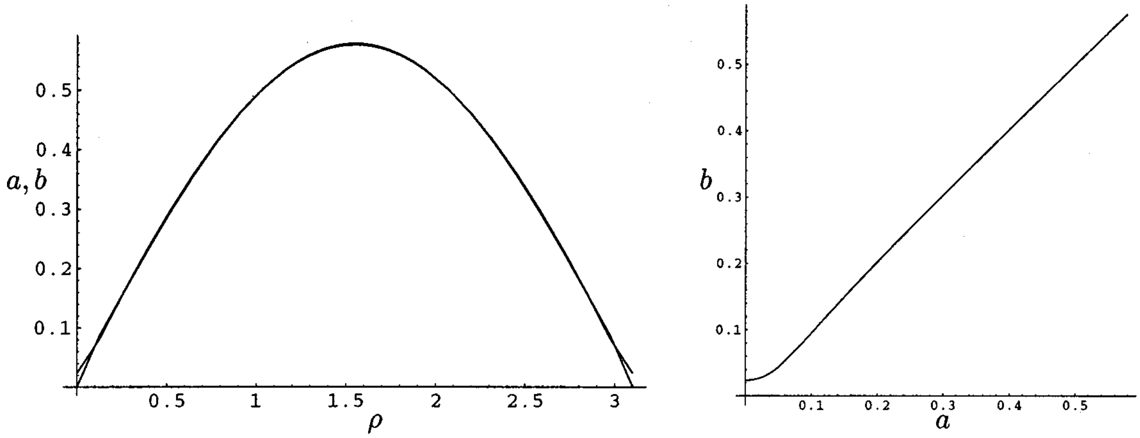

FIG. 9. The Bohm$(2,2)_5$ Einstein metric on $S^3\times S^2$. The left-hand figure shows the metric coefficients a and b as functions of the radial variable ρ. At $\rho=0$ the function a vanishes and $b_0\approx 0.023571$. The midpoint is at $t_c\approx 1.550472593$. The right-hand figure is a parametric plot of b vs a.

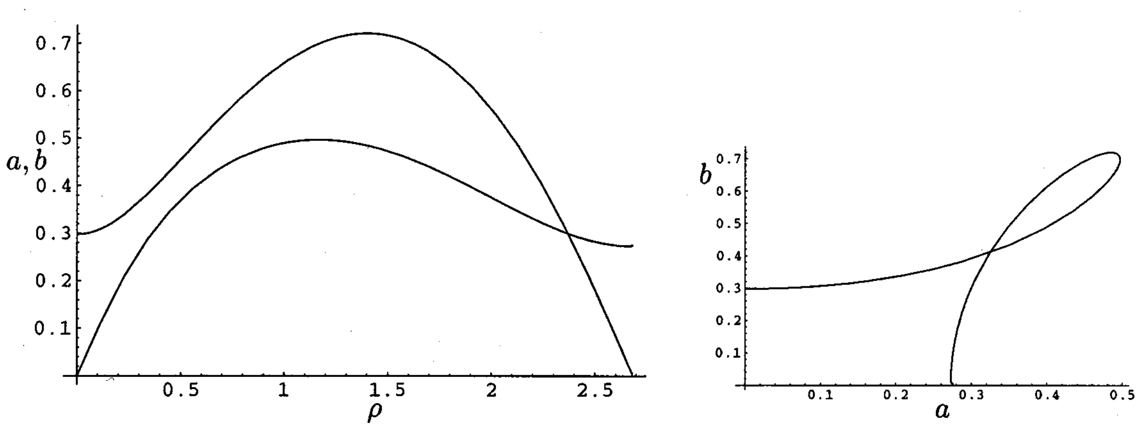

FIG. 10. The Bohm$(2,2)_2$ Einstein metric on S^6. The left-hand figure shows the metric coefficients a and b as functions of the radial variable ρ. At $\rho=0$ the function a vanishes and $b_0\approx 0.297647$. The end point is at $\rho_f\approx 2.68296$. The right-hand figure is a parametric plot of b vs a.

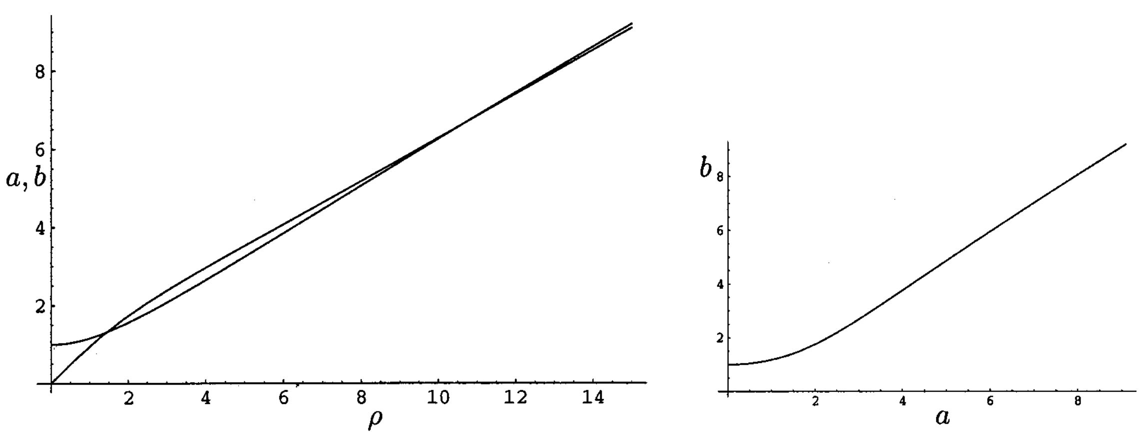

FIG. 11. The noncompact Ricci-flat Bohm metric on $\mathbb{R}^3\times S^2$. The left-hand figure shows the metric coefficients a and b as functions of the radial variable ρ. At $\rho=0$ the function a vanishes and b_0 is taken to be 1. The right-hand figure is a parametric plot of b vs a.

Bohm$(2,3)_2$, which is topologically S^6, to illustrate an example where there is no symmetry between a and b. This has $b_0 \approx 0.297647$, and the end point is at $\rho_f \approx 2.68296$ (there is no natural significance to the midpoint of the radial coordinate range in the $p \neq q$ examples).

A few isolated examples for other values of p and q are as follows. The Bohm$(3,3)_2$ metric on S^7 has $b_0 \approx 0.3055210896$, with the midpoint occurring at $\rho = \rho_c \approx 1.34689859293$. The Bohm$(3,3)_3$ metric on $S^4 \times S^3$ has $b_0 \approx 0.14291337$ and $\rho_c \approx 1.4691901856$. The Bohm$(4,4)_2$ metric on S^9 has $b_0 \approx 0.2851829$ and $\rho_c \approx 1.376730624$, while the Bohm$(4,4)_3$ metric on $S^5 \times S^4$ has $b_0 \approx 0.09135$ and $\rho_c \approx 1.5099148$.

We can also treat the analysis of the noncompact Bohm metrics described in Sec. III D in a similar fashion. Since these are Ricci-flat solutions of the Einstein equations, the terms involving the cosmological constant will be absent in Eqs. (32) and (33), but otherwise all the formulas are analogous. The short-distance Taylor expansions (A1) now become

$$a = \rho - \frac{q(q-1)}{6b_0^2 p(p+1)}\rho^3 + O(\rho^5),$$

$$b = b_0 + \frac{q-1}{2b_0(p+1)}\rho^2 + O(\rho^4). \tag{A2}$$

Using this, taken to order ρ^9, to set initial data just outside the S^q bolt at $\rho = 0$, we again performed numerical integrations. The plots for the functions a and b in the representative example $p = q = 2$ are given in Figs. 3–11.

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