Low energy effective theory for a two branes system: Covariant curvature formulation

Tetsuya Shiromizu
Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
and Advanced Research Institute for Science and Engineering, Waseda University, Tokyo 169-8555, Japan

Kazuya Koyama
Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan
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We derive the low energy effective theory for a two branes system solving the bulk geometry formally in the covariant curvature formalism developed by Shiromizu, Maeda, and Sasaki. As expected, the effective theory looks like an Einstein-scalar system. Using this theory we can discuss the cosmology and nonlinear gravity at low energy scales.

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I. INTRODUCTION

The recent progress in superstring theory provides us with a new picture of the universe, the so-called brane world, where our universe is like a domain wall in higher dimensional spacetime. The matter is confined on the three-branes. The simplest model was proposed by Randall and Sundrum [1,2]. In their model the bulk spacetime is a five-dimensional anti–de Sitter spacetime and the brane is a four-dimensional Minkowski spacetime. Their first model (RS1) gives us a geometrical solution to the gauge hierarchy problem. The RS1 model consists of two branes, a positive and negative tension brane. For the gauge hierarchy problem, it is supposed that the visible brane where we are is the negative tension one. The linealized theory has been carefully investigated in [3,4] (see Refs. [5–7] for the cosmological cases. See Ref. [8] for other issues.). As a result, it turns out that the gravity on the brane looks like scalar-tensor gravity. However, we have no successful analysis on the nonlinear aspects of the gravity except for the second order perturbation [9]. Recently there has been impressive progress [10–12] on this issue. In Ref. [11], the effective equation is derived at the low energy scale.

In this paper, we rederive the effective theory for a two branes systems which was obtained by Kanno and Soda [11] in the metric based approach. On the other hand, our approach is based on the covariant curvature formalism [13,14]. As seen later, our derivation is much simpler and more straightforward than the metric based approach. The covariant curvature formalism gives us a gravitational equation on the branes. For RS2 models, which consist of a single brane, this approach is a powerful tool to look at the full view of the brane world. Indeed, it was easy to see that Newton's gravity is recovered at low energy. The approach, however, is not regarded as so useful for RS1 models. We shall show this is not true. This is the main purpose of this paper. For simplicity, we will not address the stabilization issue of two branes (radion stabilization problems) [15–17].

The rest of this paper is organized as follows. In Sec. II, we summarize the covariant curvature formalism. In Sec. III, we formally solve the bulk perturbatively up to the 1st order. The infinitesimal and dimensionless parameter is the ratio of the bulk to brane curvature radii. After that we derive the effective theory at the 1st order. The equation includes the nonlinear part of the induced gravity on the branes. In the Sec. IV, we summarize the present work.

II. COVARIANT CURVATURE FORMALISM

We employ the following metric form:[1]

$$ds^2 = e^{2\phi(y,x)}dy^2 + q_{\mu\nu}(y,x)dx^\mu dx^\nu. \tag{1}$$

In the above it is supposed that the positive and negative tension branes are located at $y=0$ and $y=y_0$, respectively. The proper distance between two branes is given by $d_0(x) = \int_0^{y_0} dy e^{\phi(y,x)}$. $q_{\mu\nu}(y,x)$ is the induced metric of y $=$constant hypersurfaces.

We follow the geometrical procedure (covariant curvature formalism) developed in Refs. [13,14]. For simplicity, we do not include the bulk fields except for the bulk consmological constant. From the Gauss-Codacci equations, first of all, we have two key equations

$${}^{(4)}G^\mu_\nu = \frac{3}{\ell^2}\delta^\mu_\nu + KK^\mu_\nu - K^\mu_\alpha K^\alpha_\nu - \frac{1}{2}\delta^\mu_\nu(K^2 - K^\alpha_\beta K^\beta_\alpha) - E^\mu_\nu \tag{2}$$

and

$$D_\mu K^\mu_\nu - D_\mu K = 0, \tag{3}$$

where D_μ is the covariant derivative with respect to $q_{\mu\nu}$. ℓ is the bulk curvature radius. ${}^{(4)}G_{\mu\nu}$ is the four-dimensional Einstein tensor with respect to $q_{\mu\nu}$, $K_{\mu\nu}$ is the extrinsic curvature of $y=$constant hypersurfaces defined by

$$K_{\mu\nu} = \frac{1}{2}\pounds_n q_{\mu\nu} = \nabla_\mu n_\nu + n_\mu D_\nu \phi, \tag{4}$$

[1]In Ref. [11] it is assumed that $\phi(y,x)$ does not depend on y. However, as seen later, the assumption can be removed.

where $n=e^{-\phi}\partial_y$. Here note that $a^\mu=n^\nu\nabla_\nu n^\mu=-D^\mu\phi(y,x)$. $E_{\mu\nu}$ is a part of the projected Weyl tensor defined by

$$\begin{aligned}E^\mu_\nu&={}^{(5)}C_{\mu\alpha\nu\beta}n^\alpha n^\beta\\&=-D^\mu D_\nu\phi-D^\mu\phi D_\nu\phi-\pounds_n K^\mu_\nu-K^\mu_\alpha K^\alpha_\nu+\frac{1}{\ell^2}\delta^\mu_\nu,\end{aligned}\tag{5}$$

where ${}^{(5)}C_{\mu\nu\alpha\beta}$ is the five-dimensional Weyl tensor.

The junction conditions on the branes are

$$[K^\mu_\nu-\delta^\mu_\nu K]_{y=0}=-\frac{\kappa^2}{2}(-\sigma_1\delta^\mu_\nu+T_1{}^\mu{}_\nu)\tag{6}$$

and

$$[K^\mu_\nu-\delta^\mu_\nu K]_{y=y_0}=\frac{\kappa^2}{2}(-\sigma_2\delta^\mu_\nu+T_2{}^\mu{}_\nu).\tag{7}$$

$T_1{}^\mu{}_\nu$ and $T_2{}^\mu{}_\nu$ are the energy-momentum tensor localized on the positive and negative brane. σ_1 and σ_2 are the brane tensions. If one substitutes the above conditions to Eq. (2), we might be able to derive the Einstein equation on the brane. Indeed, this was a successful procedure for a single brane [13]. This is because $E_{\mu\nu}$ comes from just Kaluza-Klein modes and vanishes at low energy [3,13]. For a two branes system, on the other hand, we have to carefully evaluate $E_{\mu\nu}$ due to the existence of the radion fields. Otherwise, we have a wrong prediction on the gravity on the branes. So we need the evolutional equation for $E_{\mu\nu}$ in the bulk. Even for the low energy scale, we learned from the linealized theory [3] that $E_{\mu\nu}$ is not negligible.

To evaluate $E_{\mu\nu}$ in the bulk, we derive its evolutional equation. The result is

$$\begin{aligned}\pounds_n E_{\alpha\beta}&=D^\mu B_{\mu(\alpha\beta)}+K^{\mu\nu(4)}C_{\mu\alpha\nu\beta}+4K^\mu_{(\alpha}E_{\beta)\mu}-\frac{3}{2}KE_{\alpha\beta}\\&\quad-\frac{1}{2}q_{\alpha\beta}K^{\mu\nu}E_{\mu\nu}+2D^\mu\phi B_{\mu(\alpha\beta)}+2\tilde K^\mu_\alpha\tilde K_{\mu\nu}\tilde K^\nu_\beta\\&\quad-\frac{7}{6}\tilde K_{\mu\nu}\tilde K^{\mu\nu}\tilde K_{\alpha\beta}-\frac{1}{2}q_{\alpha\beta}\tilde K_{\mu\nu}\tilde K^\mu_\rho\tilde K^{\rho\nu},\end{aligned}\tag{8}$$

where $B_{\mu\nu\alpha}=q^\rho_\mu q^\sigma_\nu{}^{(5)}C_{\rho\sigma\alpha\beta}n^\beta$ and $\tilde K_{\mu\nu}=K_{\mu\nu}-\frac{1}{4}q_{\mu\nu}K$. Since the right-hand side contains $B_{\mu\nu\alpha}$, $K_{\mu\nu}$, and ${}^{(4)}C_{\mu\nu\alpha\beta}$, we also need their evolutional equations. After a long calculation we arrive at

$$\begin{aligned}&\pounds_n^{(4)}R_{\mu\nu\alpha\beta}+2^{(4)}R_{\mu\nu\rho[\alpha}K^\rho_{\beta]}+2D_{[\mu}B_{|\alpha\beta|\nu]}+2(D_\mu D_{[\alpha}\phi\\&\quad+D_\mu\phi D_{[\alpha}\phi)K_{\beta]\nu}-2(D_\nu D_{[\alpha}\phi-D_\nu\phi D_{[\alpha}\phi)K_{\beta]\mu}\\&\quad-2B_{\alpha\beta[\mu}D_{\nu]}\phi-2B_{\mu\nu[\alpha}D_{\beta]}\phi=0,\end{aligned}\tag{9}$$

$$\begin{aligned}&\pounds_n B_{\mu\nu\alpha}+2D_{[\mu}E_{\nu]\alpha}+2D_{[\mu}\phi E_{\nu]\alpha}-B_{\mu\nu\beta}K^\beta_\alpha+2B_{\alpha\beta[\mu}K^\beta_{\nu]}\\&\quad+({}^{(4)}R_{\mu\nu\alpha\beta}-K_{\mu\alpha}K_{\nu\beta}+K_{\mu\beta}K_{\nu\alpha})D^\beta\phi=0,\end{aligned}\tag{10}$$

and

$$e^{-\phi}\partial_y K^\mu_\nu=-D^\mu D_\nu\phi-D^\mu\phi D_\nu\phi-K^\mu_\alpha K^\alpha_\nu+\frac{1}{\ell^2}\delta^\mu_\nu-E^\mu_\nu.\tag{11}$$

Equation (11) is just the rearrangement of Eq. (5). The derivation is basically the same as that in Ref. [13].

The junction condition directly implies the boundary condition on the branes for $K_{\mu\nu}$ and $B_{\mu\nu\alpha}$ because of

$$B_{\mu\nu\alpha}=2D_{[\mu}K_{\nu]\alpha}.\tag{12}$$

III. DERIVATION OF LOW ENERGY EFFECTIVE THEORY

We are now ready to derive the low energy theory for a two branes system. To do so, as stressed in the previous section, we must know $E_{\mu\nu}$ and solve the equation for $E_{\mu\nu}$ in the bulk. By low energy we mean that the typical scale of the curvature scale L on the brane is much larger than the bulk curvature scale ℓ, that is, $L\gg\ell$. The dimensionless parameter is $\epsilon=(\ell/L)^2$ which is tacitly entered into the equations below. We expand K^μ_ν and E^μ_ν as

$$K^\mu_\nu={}^{(0)}K^\mu_\nu+{}^{(1)}K^\mu_\nu+\cdots\tag{13}$$

and

$$E^\mu_\nu={}^{(1)}E^\mu_\nu+\cdots.\tag{14}$$

A. 0th order

At the 0th order, the evolutional equation which we have to solve is only one for $K_{\mu\nu}$:

$$e^{-\phi}\partial_y^{(0)}K^\mu_\nu=\frac{1}{\ell^2}\delta^\mu_\nu-{}^{(0)}K^{\mu(0)}_\alpha K^\alpha_\nu.\tag{15}$$

And $K_{\mu\nu}$ satisfies the constraint

$$D^{(0)}_\mu K^\mu_\nu-D_\nu{}^{(0)}K=0.\tag{16}$$

It is easy to see that the solution is

$${}^{(0)}K^\mu_\nu=-\frac{1}{\ell}\delta^\mu_\nu.\tag{17}$$

From the definition

$$\frac{1}{2}e^{-\phi}\partial_y^{(0)}q_{\mu\nu}=-\frac{1}{\ell}{}^{(0)}q_{\mu\nu},\tag{18}$$

the metric at the 0th order becomes

$${}^{(0)}q_{\mu\nu}(y,x)=e^{-2d(y,x)/\ell}h_{\mu\nu}(x),\tag{19}$$

where $h_{\mu\nu}(x)$ is a tensor field depending only on the coordinate x on the brane and $d(y,x)=\int_0^y e^{\phi(y',x)}dy'$.

At this order the junction condition is

$$[{}^{(0)}K^\mu_\nu - \delta^\mu_\nu {}^{(0)}K]_{y=0} = \frac{\kappa^2}{2}\sigma_1 \delta^\mu_\nu \tag{20}$$

and

$$[{}^{(0)}K^\mu_\nu - \delta^\mu_\nu {}^{(0)}K]_{y=y_0} = -\frac{\kappa^2}{2}\sigma_2 \delta^\mu_\nu . \tag{21}$$

Thus the junction condition implies the relation between the bulk curvature radius ℓ and the brane tension $\sigma_{1,2}$ as

$$\frac{1}{\ell} = \frac{1}{6}\kappa^2\sigma_1 = -\frac{1}{6}\kappa^2\sigma_2 . \tag{22}$$

This is just the fine tuning of Randall-Sundrum models [1,2].

B. 1st order

At the 1st order, the Riemann tensor does appear in the basic equations. So we can expect that the Einstein field equation will be able to be described at this order. Indeed, the Gauss equation of Eq. (2) becomes

$$ {}^{(4)}G^\mu_\nu = -\frac{2}{\ell}({}^{(1)}K^\mu_\nu - \delta^\mu_\nu {}^{(1)}K) - {}^{(1)}E^\mu_\nu . \tag{23}$$

The first term in the right-hand side will be easily written in terms of the energy-momentum tensor on the branes using the junction condition at this order. So the unknown tensor is ${}^{(1)}E^\mu_\nu$.

The evolutional equations which we must solve are

$$e^{-\phi}\partial_y {}^{(1)}E_{\mu\nu} = \frac{2}{\ell}{}^{(1)}E_{\mu\nu} \tag{24}$$

and

$$e^{-\phi}\partial_y {}^{(1)}K^\mu_\nu = -(D^\mu D_\nu \phi + D^\mu\phi D_\nu\phi) + \frac{2}{\ell}{}^{(1)}K^\mu_\nu - {}^{(1)}E^\mu_\nu . \tag{25}$$

Equation (24) is easily solved as

$$ {}^{(1)}E_{\mu\nu} = e^{2d(y,x)/\ell} e_{\mu\nu}(x) \tag{26}$$

or

$$ {}^{(1)}E^\mu_\nu = e^{4d(y,x)/\ell}\hat{e}^\mu_\nu(x), \tag{27}$$

where $\hat{e}^\mu_\nu(x) = h^{\mu\alpha}e_{\alpha\nu}(x)$.

Substituting the expression of Eq. (27) into Eq. (25), we can obtain the solution for ${}^{(1)}K^\mu_\nu$ easily:

$$ {}^{(1)}K^\mu_\nu(y,x) = e^{2d/\ell}{}^{(1)}K^\mu_\nu(0,x) - \frac{\ell}{2}(1-e^{-2d/\ell}){}^{(1)}E^\mu_\nu(y,x) - \left[D^\mu D_\nu d - \frac{1}{\ell}\left(D^\mu d D_\nu d - \frac{1}{2}\delta^\mu_\nu (Dd)^2\right)\right]. \tag{28}$$

For comparison, note that $e_{\mu\nu}$ cannot be determined by the junction condition in the RS2 model [12,13]. To do so, we need the boundary condition near the Cauchy horizon.

C. Low energy effective theory for two brane systems

At the 1st order the junction condition on the positive and negative tension brane becomes

$$ {}^{(1)}K^\mu_\nu(0,x) - \delta^\mu_\nu {}^{(1)}K(0,x) = -\frac{\kappa^2}{2}T_1{}^\mu{}_\nu \tag{29}$$

and

$$ {}^{(1)}K^\mu_\nu(y_0,x) - \delta^\mu_\nu {}^{(1)}K(y_0,x) = \frac{\kappa^2}{2}T_2{}^\mu{}_\nu . \tag{30}$$

Using the above Eq. (30), the Gauss equation on the negative tension brane becomes

$$\begin{aligned} {}^{(4)}G^\mu_\nu &= -\frac{2}{\ell}[{}^{(1)}K^\mu_\nu(y_0,x) - \delta^\mu_\nu {}^{(1)}K(y_0,x)] - {}^{(1)}E^\mu_\nu(y_0,x) \\ &= -\frac{\kappa^2}{\ell}T_2{}^\mu{}_\nu - {}^{(1)}E^\mu_\nu(y_0,x). \end{aligned} \tag{31}$$

Using the expression of Eq. (28), the junction condition on the negative tension brane is written as

$$\begin{aligned} &{}^{(1)}K^\mu_\nu(y_0,x) - \delta^\mu_\nu {}^{(1)}K(y_0,x) \\ &= -\frac{\kappa^2}{2}e^{2d_0/\ell}T_1{}^\mu{}_\nu - (D^\mu D_\nu d_0 - D^\mu D_\nu d_0) \\ &\quad + \frac{1}{\ell}\left(D^\mu d_0 D_\nu d_0 + \frac{1}{2}\delta^\mu_\nu (Dd_0)^2\right) \\ &\quad - \frac{\ell}{2}(1-e^{-2d_0/\ell}){}^{(1)}E^\mu_\nu(d_0,x) \\ &= \frac{\kappa^2}{2}T_2{}^\mu{}_\nu . \end{aligned} \tag{32}$$

On the right-hand side of the first line, we used the solution of Eq. (28). The second line is just the junction condition. From Eq. (32), then, the tensor $e_{\mu\nu}$(and then ${}^{(1)}E_{\mu\nu}$) is completely fixed as

$$\begin{aligned} &\frac{\ell}{2}(1-e^{-2d_0/\ell})e^{4d_0/\ell}\hat{e}^\mu_\nu(x) \\ &= -\frac{\kappa^2}{2}(e^{2d_0/\ell}T_1{}^\mu{}_\nu + T_2{}^\mu{}_\nu) - (D^\mu D_\nu d_0 - \delta^\mu_\nu D^2 d_0) \\ &\quad + \frac{1}{\ell}\left(D^\mu d_0 D_\nu d_0 + \frac{1}{2}\delta^\mu_\nu (Dd_0)^2\right). \end{aligned} \tag{33}$$

The trace of Eq. (33) gives the equation for d_0 because e^μ_ν is traceless. In general the radion field is massive. Substituting

Eq. (33) into the Gauss equation at the 1st order, we can obtain the effective equation on the branes. On the negative tension brane, we have

$$
{}^{(4)}G^{\mu}_{\nu}=\frac{\kappa^2}{\ell}\frac{1}{\Phi}T^{\mu}_{2\ \nu}+\frac{\kappa^2}{\ell}\frac{(1+\Phi)^2}{\Phi}T^{\mu}_{1\ \nu}+\frac{1}{\Phi}(D^{\mu}D_{\nu}\Phi - \delta^{\mu}_{\nu}D^2\Phi)+\frac{\omega(\Phi)}{\Phi^2}\left(D^{\mu}\Phi D_{\nu}\Phi-\frac{1}{2}\delta^{\mu}_{\nu}(D\Phi)^2\right), \tag{34}
$$

where $\Phi=e^{2d_0/\ell}-1$ and $\omega(\Phi)=-\frac{3}{2}\Phi/(1+\Phi)$. As should be so, this is exactly the same result obtained by Kanno and Soda [11]. We also derive the effective equation on the positive tension brane easily:

$$
{}^{(4)}G^{\mu}_{\nu}=\frac{\kappa^2}{\ell}\frac{1}{\Psi}T^{\mu}_{1\ \nu}+\frac{\kappa^2}{\ell}\frac{(1-\Psi)^2}{\Psi}T^{\mu}_{2\ \nu}+\frac{1}{\Psi}(\hat{D}^{\mu}\hat{D}_{\nu}\Psi - \delta^{\mu}_{\nu}\hat{D}^2\Psi)+\frac{\omega(\Psi)}{\Psi^2}\left(\hat{D}^{\mu}\Psi\hat{D}_{\nu}\Psi-\frac{1}{2}\delta^{\mu}_{\nu}(\hat{D}\Psi)^2\right), \tag{35}
$$

where $\Psi=1-e^{-2d_0/\ell}$, $\omega(\Psi)=\frac{3}{2}\Psi/(1-\Psi)$, and $\hat{D}$ is the covariant derivative with respect to the induced metric $h_{\mu\nu}$ on the positive tension brane.

IV. SUMMARY

In this paper, we derived the gravitational equation on the branes for a two branes system at the low energy using the covariant curvature formalism [13] and low energy expansion scheme [11,12]. The theory obtained here is presumably applicable to the cosmology and nonlinear gravity at low energy scales. What we have done here is the evaluation of $E_{\mu\nu}$ at low energy. In Ref. [13], we thought that the antigravity appears on the negative tension brane supposing $E_{\mu\nu}$ is negligible. However, this is not correct and $E_{\mu\nu}$ is not negligible even at low energy.

Here we should comment on the difference between the study in Ref. [11] and the present one. In Ref. [11] $E_{\mu\nu}$ appears as the "constant of integration." Thus it is difficult to proceed with the discussion while keeping the physical meaning. This is just because of the metric based approach. On the other hand, $E_{\mu\nu}$ explicitly enters into the basic equations in the covariant curvature formalism and its physical meaning is manifest. Yet, the evolutional equation for $E_{\mu\nu}$ is simple at the low energy limit.

For simplicity, we focused on the two branes system without the radion stabilization. If one is serious about the gauge hierarchy problem, we must assume that we are living on the negative tension brane. In this case, the gravity on the negative tension brane is scalar-tensor type and the scalar coupling is not permitted from the experimental point of view [3]. So we should reconstruct our formalism in a two branes system with the radion stabilization mechanism. This issue is left for future study.

We are also interested in the higher order effects. We can expect that the effective theory with higher order corrections is higher-derivative type due to the nonlocal feature of the brane world [11,12,18].

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