Quantum radiation from a 5-dimensional rotating black hole

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We study a massless scalar field propagating in the background of a five-dimensional rotating black hole. We show that in the Myers-Perry metric describing such a black hole the massless field equation allows the separation of variables. The obtained angular equation is a generalization of the equation for spheroidal functions. The radial equation is similar to the radial Teukolsky equation for the 4-dimensional Kerr metric. We use these results to quantize the massless scalar field in the space-time of the 5-dimensional rotating black hole and to derive expressions for energy and angular momentum fluxes from such a black hole.

I. INTRODUCTION

Solutions of Einstein's equations describing black holes in space-times with more than usual $(3+1)$ dimensions have been known in literature for 40 yr [1]. Until recently, they were practically only of academic interest. The situation has changed with the appearance of so-called brane world models [2] in which large extra spatial dimensions can exist (much larger than the apparent Planck length scale $\sim 10^{-33}$ cm). In this framework, the fundamental quantum gravity energy scale could be in the 1 TeV range, while the characteristic length scale (compactification radius of extra dimensions) can be as large as 0.1 mm. This allows the existence of higher dimensional mini-black holes which can be described within the classical theory of gravity. Being gravitational solitons, black holes in the brane world can "live" out of the brane in the bulk space. If the gravitational radius of such a black hole is much smaller than the distance to the brane and the characteristic length defined by the bulk curvature and/or the size of extra dimensions, the influence of the external conditions on the properties of the space-time near the horizon is small. Under these conditions in order to describe the black hole one can use solutions of vacuum Einstein equations. This also can be true for small black holes attached to the brane provided the brane is "soft"; that is, its tension is small.

From a phenomenological point of view, the most exciting possibility is that such mini-black holes can be produced in particle collisions in near future accelerator and cosmic ray experiments [3]. In order to predict the correct experimental signature of these events, one has to know the basic properties of solutions of these higher dimensional black holes. In general case, the impact parameter in particle collision will be nonzero. Therefore, majority of black holes produced in such way would be rotating.

Higher dimensional black holes were studied before in different contexts. For example, supersymmetric rotating black holes were studied in [4,5]. The solution analyzed there is not a vacuum solution of Einstein's equations and requires some special choice of parameters in order to accommodate the supersymmetry. In [6,7], rotating black holes were studied in the context of string theory. Scalar field graybody factors were calculated for a very general case of a boosted vacuum solution of Einstein's equations.

The metric for higher dimensional rotating black holes was first derived by Myers and Perry [8]. In the present paper, we study the massless scalar field equation in the background of a five-dimensional rotating black hole described by the Myers-Perry metric. We show that this equation allows the separation of variables. The obtained angular equation is a generalization of the equation for spheroidal functions. The radial equation is similar to the radial Teukolsky equation for the 4-dimensional Kerr metric.

In the case of $3+1$ dimensions, there is only one parameter of rotation and there is an axis of rotation which stays invariant under rotation. The rotation group in $4+1$ dimensions, SO(4), has two Casimir operators. Rather than an axis of rotation, there exist planes of rotation which stay invariant under the rotation. This implies that, in general, there are two parameters of rotation corresponding to two independent planes of rotation. In a special case, one can set one of the parameters to zero and consider rotations only in one plane. We find interesting that in another special case, when both parameters of rotation are nonzero and equal in magnitude, the angular equation reduces to a case of a nonrotating black hole. For this special case, the space-time has two additional Killing vectors. It is interesting that the spatial 3-dimensional slices of this space-time are homogeneous and belong to the Bianchi-type VIII class.

We analyze the structure and asymptotics of solutions of the radial equation which determine the black hole graybody factors. We use these results to quantize the massless scalar field in the space-time of the 5-dimensional rotating black hole and to derive expressions for energy and angular momentum fluxes from such a black hole.

II. MYERS-PERRY METRIC

A. Generic case

Following Myers and Perry [8], we write the metric of a 5-dimensional rotating black hole in the form

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$$ds^2 = -dt^2 + (r^2 + a_1^2)(d\mu_1^2 + \mu_1^2 d\phi_1^2)$$

$$+ (r^2 + a_2^2)(d\mu_2^2 + \mu_2^2 d\phi_2^2) + \frac{\Pi \mathcal{F}}{\Pi - r_0^2 r^2} dr^2$$

$$+ \frac{r_0^2 r^2}{\Pi \mathcal{F}} (dt + a_1 \mu_1^2 d\phi_1 + a_2 \mu_2^2 d\phi_2)^2, \tag{2.1}$$

$$\mathcal{F} = 1 - \frac{a_1^2 \mu_1^2}{r^2 + a_1^2} - \frac{a_2^2 \mu_2^2}{r^2 + a_2^2}, \tag{2.2}$$

$$\Pi = (r^2 + a_1^2)(r^2 + a_2^2). \tag{2.3}$$

Here r_0 is length parameter connected with the black hole mass M:

$$M = \frac{3 r_0^2}{8 \sqrt{\pi} G}, \tag{2.4}$$

where G is the $(4+1)$-dimensional gravitational coupling constant. Besides r_0 the metric (2.1) contains two rotation parameters a_1 and a_2. The variables μ_1 and μ_2 are not independent. They obey a constraint

$$\mu_1^2 + \mu_2^2 = 1. \tag{2.5}$$

Instead of keeping the symmetric form of the metric (2.1), we prefer to solve the constraint (2.5) explicitly. We use the following parametrization:

$$\mu_1 = \sin \theta, \quad \mu_2 = \cos \theta. \tag{2.6}$$

Let us also introduce the following notation:

$$a = a_1, \quad b = a_2, \quad \phi = \phi_1, \quad \psi = \phi_2, \tag{2.7}$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \tag{2.8}$$

$$\Delta = (r^2 + a^2)(r^2 + b^2) - r_0^2 r^2. \tag{2.9}$$

Then, the metric (2.1) takes the form

$$ds^2 = d\gamma^2 + \frac{r^2 \rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \tag{2.10}$$

$$d\gamma^2 \equiv \gamma_{AB} dx^A dx^B$$
$$= -dt^2 + (r^2 + a^2)\sin^2 \theta \, d\phi^2 + (r^2 + b^2)\cos^2 \theta \, d\psi^2$$
$$+ \frac{r_0^2}{\rho^2}[dt + a \sin^2 \theta \, d\phi + b \cos^2 \theta \, d\psi]^2. \tag{2.11}$$

Here $A, B = 0, 3, 4$ and $x^0 = t$, $x^3 = \phi$, $x^4 = \psi$. Angles ϕ and ψ take values from the interval $[0, 2\pi]$, while angle θ takes values $[0, \pi/2]$.

The metric (2.10) is invariant under the following transformation:

$$a \leftrightarrow b, \quad \theta \leftrightarrow \frac{\pi}{2} - \theta, \quad \phi \leftrightarrow \psi. \tag{2.12}$$

It possesses three Killing vectors, ∂_t, ∂_ϕ, and ∂_ψ. For this metric,

$$\sqrt{-g} = \sin \theta \cos \theta \, r \, \rho^2. \tag{2.13}$$

The black hole horizon is located at $r = r_+$, where

$$r_\pm^2 = \frac{1}{2}[r_0^2 - a^2 - b^2 \pm \sqrt{(r_0^2 - a^2 - b^2)^2 - 4 a^2 b^2}]. \tag{2.14}$$

The angular velocities Ω_a and Ω_b and the surface gravity κ are

$$\Omega_a = \frac{a}{r_+^2 + a^2}, \quad \Omega_b = \frac{b}{r_+^2 + b^2}, \tag{2.15}$$

$$\kappa = \frac{\partial_r \Pi - 2 r_0^2 r}{2 r_0^2 r^2}\bigg|_{r = r_+}. \tag{2.16}$$

B. Degenerate case

As we already mentioned in a general case the Myers-Perry metric has three Killing vectors. The space-time becomes more symmetric when $a = b$. To demonstrate this let us consider first the geometry of the section $t = $ const. It has the form

$$ds_4^2 = \frac{r^2 (r^2 + a^2)}{(r^2 + a^2)^2 - r_0^2 r^2} dr^2 + ds_3^2, \tag{2.17}$$

$$ds_3^2 = \alpha (d\theta^2 + \sin^2 \theta \, d\phi^2 + \cos^2 \theta \, d\psi^2)$$
$$+ \beta(\sin^2 \theta \, d\phi + \cos^2 \theta \, d\psi)^2, \tag{2.18}$$

where

$$\alpha = r^2 + a^2, \quad \beta = \frac{r_0^2 a^2}{r^2 + a^2}. \tag{2.19}$$

By introducing new coordinates

$$\Phi = \psi - \phi, \quad \Psi = \psi + \phi, \quad \vartheta = 2\theta, \tag{2.20}$$

where Ψ takes values from the interval $[0, 4\pi]$, Φ from $[-2\pi, 2\pi]$, and ϑ from $[0, \pi]$, one can rewrite the metric (2.18) as follows:

$$ds_3^2 = A (d\vartheta^2 + \sin^2 \vartheta \, d\Phi^2) + B (\cos \vartheta \, d\Phi + d\Psi)^2. \tag{2.21}$$

Here,

$$A = \frac{\alpha}{4}, \quad B = \frac{\alpha + \beta}{4}. \tag{2.22}$$

This is a canonical form of the Bianchi-type VIII metric when there exists a four-parameter group of isometries acting transitively in the 3-dimensional space [10].[1] The corresponding Killing vectors are

$$K_1 = \partial_\Phi, \quad K_2 = \partial_\Psi,$$

$$K_3 = \cos\Phi\, \partial_\vartheta - \cot\vartheta\,\sin\Phi\,\partial_\Phi + \frac{\sin\Phi}{\sin\vartheta}\,\partial_\Psi, \qquad (2.23)$$

$$K_4 = -\sin\Phi\,\partial_\vartheta - \cot\vartheta\,\cos\Phi\,\partial_\Phi + \frac{\cos\Phi}{\sin\vartheta}\,\partial_\Psi.$$

It is easy to check that these vectors (together with ∂_t) are also Killing vectors for the 5-dimensional space-time metric (2.10) when $a = b$.

Killing vectors can be viewed as generators of the symmetry group of the manifold. We can obtain a more familiar form of these Killing vectors. First note that

$$K_3 K_4 - K_4 K_3 = K_1, \qquad (2.24)$$

where multiplications denote successive applications of an operator. Then, the quadratic combination $K_3^2 + K_4^2 + K_1^2$ yields

$$K_3^2 + K_4^2 + K_1^2 = \frac{\partial^2}{\partial\vartheta^2} + \cot\vartheta\,\frac{\partial}{\partial\vartheta}$$

$$+ \frac{1}{\sin^2\vartheta}\left(\frac{\partial^2}{\partial\Phi^2} - 2\cos\vartheta\,\frac{\partial^2}{\partial\Phi\partial\Psi} + \frac{\partial^2}{\partial\Psi^2}\right). \qquad (2.25)$$

We can identify

$$J_\Phi \equiv -iK_1, \quad J_\Psi \equiv -iK_2, \quad J^2 \equiv -(K_3^2 + K_4^2 + K_1^2), \qquad (2.26)$$

where J_Φ, J_Ψ, and J^2 are familiar angular momentum operators, while Φ, Ψ, and ϑ are Euler angles for the rotation group $O(3)$.

The scalar curvature R for the metric (2.18),

$$R = 2\frac{3(r^2 + a^2)^2 - a^2 r_0^2}{(r^2 + a^2)^3}, \qquad (2.27)$$

is constant at fixed r and has a positive sign as long as $r > a\sqrt{(\sqrt{3}/3)(r_0/a) - 1}$.

Another interesting observation is the following. Instead of metric on slices $t = \text{const}$, one can consider the 4-dimensional foliation of the space-time by the Killing trajectories of the field $\xi_t = \partial_t$. The metric on this foliations is determined as [11]

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[1]In [9] rotating black hole solutions to four-dimensional Einstein-Maxwell-dilaton-axion gravity were analyzed. It was shown that they are closely related to the 5-dimensional rotating Myers-Perry black hole with $a = b$.

$$g^{(4)}_{\mu\nu} = g_{\mu\nu} - \frac{\xi_\mu \xi_\nu}{\xi^2}. \qquad (2.28)$$

For the Myers-Perry metric (2.1) this metric is

$$dS^2 \equiv g^{(4)}_{\mu\nu}dx^\mu\,dx^\nu = \frac{r^2(r^2 + a^2)}{(r^2 + a^2)^2 - r_0^2\,r^2}dr^2 + dS_3^2, \qquad (2.29)$$

where dS_3^2 has the form (2.18), with

$$\alpha = r^2 + a^2, \quad \beta = \frac{r_0^2\,a^2}{r^2 + a^2 - r_0^2}. \qquad (2.30)$$

Thus this metric is also a metric of homogeneous 3-dimensional space with the four-parameter group of motion.

The scalar curvature for the metric dS_3^2

$$R = 2\frac{3(r^2 + a^2)^2 - 3r_0^2 r^2 - 4r_0^2 a^2}{[(r^2 + a^2)^2(r^2 + a^2 - r_0^2)]} \qquad (2.31)$$

is also constant for $r = \text{const}$ and it is positive for

$$r > \tfrac{1}{6}\sqrt{18r_0^2 - 36a^2 + 6\sqrt{(9r_0^4 + 12r_0^2 a^2)}}. \qquad (2.32)$$

C. Flat space-time limit

When $r_0 = 0$ the metric takes the form

$$ds^2 = -dt^2 + (r^2 + a^2)\sin^2\theta\,d\phi^2 + (r^2 + b^2)\cos^2\theta\,d\psi^2$$

$$+ \frac{r^2\rho^2\,dr^2}{(r^2 + a^2)(r^2 + b^2)} + \rho^2\,d\theta^2. \qquad (2.33)$$

Such a space-time is flat and the metric can be rewritten as

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2 + dW^2$$

$$= -dT^2 + dR^2 + R^2\,d\Omega_3^2, \qquad (2.34)$$

where $T = t$ and

$$X = \sqrt{r^2 + a^2}\,\sin\theta\cos\tilde\phi, \quad Y = \sqrt{r^2 + a^2}\,\sin\theta\sin\tilde\phi,$$

$$Z = \sqrt{r^2 + b^2}\,\cos\theta\cos\tilde\psi, \quad W = \sqrt{r^2 + b^2}\,\cos\theta\sin\tilde\psi. \qquad (2.35)$$

Here,

$$\tilde\phi = \phi - \tan^{-1}(a/r), \quad \tilde\psi = \psi - \tan^{-1}(b/r), \qquad (2.36)$$

$$R^2 \equiv X^2 + Y^2 + Z^2 + W^2 = \rho^2, \qquad (2.37)$$

and $d\Omega_3^2$ is the line element on a unit three-sphere S^3. At far distances R and r differs only by terms of the order of a^2/r and b^2/r. Also normals to $R = \text{const}$ and $r = \text{const}$ coincide with the same accuracy.

A two-dimensional plane $Z = W = 0$ (X-Y-plane Π_{XY}) is a plane of rotation, while Π_{ZW} where $X = Y = 0$ is a two-plane orthogonal to Π_{XY}.

III. SCALAR FIELD EQUATION: SEPARATION OF VARIABLES

Let us consider a scalar massless field φ with the action

$$W[\varphi] = -\frac{1}{2} \int d^5 x \sqrt{-g} [(\nabla \varphi)^2 + \xi R \varphi^2]. \quad (3.1)$$

It obeys the following equation:

$$\Box \varphi - \xi R \varphi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \, g^{\mu\nu} \partial_\nu \varphi) - \xi R \varphi = 0. \quad (3.2)$$

The space-time (2.10) is Ricci flat so that $R = 0$. Let us denote $\Box^* = \rho^2 \Box$. Then the field equation

$$\Box^* \varphi = 0, \quad (3.3)$$

can be identically written in the form

$$H^{AB} \varphi_{,AB} + \frac{1}{r} \partial_r \left[\frac{\Delta}{r} \partial_r \varphi \right] + \frac{1}{\sin\theta \cos\theta} \partial_\theta [\sin\theta \cos\theta \partial_\theta \varphi] = 0, \quad (3.4)$$

The radial equation reads

$$\frac{\Delta}{r} \frac{d}{dr} \left[\frac{\Delta}{r} \frac{dR}{dr} \right] + WR = 0, \quad (3.11)$$

$$W = \Delta \left(-\lambda + \omega^2 (r^2 + a^2 + b^2) \right.$$
$$+ \frac{m^2(a^2 - b^2)}{r^2 + a^2} + \frac{k^2(b^2 - a^2)}{r^2 + b^2} \right)$$
$$+ r_0^2 (r^2 + a^2)(r^2 + b^2) \left(w - \frac{ma}{r^2 + a^2} - \frac{kb}{r^2 + b^2} \right)^2.$$
$$(3.12)$$

We used the freedom $\lambda \to \lambda + \text{const}$ in the choice of the separation constant λ in order to get the angular and radial equations in the most symmetric way. For this choice these equations are invariant under the transformation

$$a \leftrightarrow b, \quad \theta \leftrightarrow \frac{\pi}{2} - \theta, \quad m \leftrightarrow k. \quad (3.13)$$

where $H^{AB} = \rho^2 g^{AB}$. We used GRTENSOR program to calculate $g^{\mu\nu}$ and $\sqrt{-g}$. The components of H^{AB} are

$$H^{tt} = (a^2 - b^2)\sin^2\theta - \frac{(r^2 + a^2)[\Delta + r_0^2(r^2 + b^2)]}{\Delta}, \quad (3.5)$$

$$H^{\phi\phi} = \frac{1}{\sin^2\theta} - \frac{(a^2 - b^2)(r^2 + b^2) + b^2 r_0^2}{\Delta}, \quad (3.6)$$

$$H^{\psi\psi} = \frac{1}{\cos^2\theta} + \frac{(a^2 - b^2)(r^2 + a^2) - a^2 r_0^2}{\Delta}, \quad (3.7)$$

$$H^{t\phi} = \frac{a r_0^2 (r^2 + b^2)}{\Delta}, \quad H^{t\psi} = \frac{b r_0^2 (r^2 + a^2)}{\Delta},$$

$$H^{\phi\psi} = -\frac{a b r_0^2}{\Delta}. \quad (3.8)$$

The Eq. (3.4) allows the separation of variables. Namely, its solution can be decomposed into modes of the form

$$\varphi \sim e^{-i\omega t} e^{im\phi} e^{ik\psi} R(r) \Theta(\theta). \quad (3.9)$$

The angular function Θ obeys the equation

$$\frac{d}{d\theta} \left(\sin\theta \cos\theta \frac{d\Theta}{d\theta} \right) + \left[\lambda - \omega^2(a^2 \sin^2\theta + b^2 \cos^2\theta) - \frac{m^2}{\sin^2\theta} - \frac{k^2}{\cos^2\theta} \right] \sin\theta \cos\theta \, \Theta = 0. \quad (3.10)$$

It should be emphasized that the separability of the field equation (3.3) is directly connected with the existence of the Killing tensor for the metric (2.10),(2.11). The explicit form of this Killing tensor and discussion of its properties can be found in [12].

IV. HYPERSPHEROIDAL FUNCTIONS

A. Generic case

The angular equation can be rewritten for $S(\theta) = \sqrt{\cos\theta} \, \Theta(\theta)$ as follows:

$$\frac{d^2 S}{d\theta^2} + \frac{\cos\theta}{\sin\theta} \frac{dS}{d\theta} + U_\lambda S = 0, \quad (4.1)$$

$$U_\lambda = \tilde{\lambda} - \alpha \sin^2\theta - \frac{m^2}{\sin^2\theta} - \frac{k^2 - 1/4}{\cos^2\theta}, \quad (4.2)$$

where $\alpha = (a^2 - b^2)\omega^2$ and $\tilde{\lambda} = \lambda - \omega^2 b^2 + \frac{3}{4}$.

We discuss now properties of solutions of this equation. We are looking for solutions which are regular at singular points of the equation, $\theta = 0$ and $\theta = \pi/2$. This condition singles out special discrete values of λ which we enumerate by an integer number ℓ. Finding eigenvalues $\lambda_\ell(m, k | \alpha)$ and

eigenfunctions $S_\ell^{m,k}(\theta|\alpha)$ is a well defined problem. Standard arguments show that the eigenfunctions with different $\lambda_\ell(m,k|\alpha)$ are orthogonal to one another with the proper chosen measure. Thus we write

$$\int_0^{\pi/2} d\theta \sin\theta\, S_\ell^{m,k}(\theta|\alpha)\, S_{\ell'}^{m,k}(\theta|\alpha) = \delta_{\ell,\ell'}. \qquad (4.3)$$

In what follows, we shall use the following normalized set of functions:

$$\mathcal{Y}_{\ell m k}(\theta,\phi,\psi|\alpha) = \frac{e^{im\phi+ik\psi}}{2\pi}\frac{S_\ell^{m,k}(\theta|\alpha)}{\sqrt{\cos\theta}}. \qquad (4.4)$$

We shall also use the compact notation for the index $A = \{\ell,m,k\}$. This set of functions possess the following normalization conditions

$$\int d\gamma\, \mathcal{Y}_A \bar{\mathcal{Y}}_{A'} = \delta_{AA'}, \qquad (4.5)$$

where

$$\int d\gamma(\cdots) = \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$\times \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi(\cdots),$$

$$\delta_{AA'} = \delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}. \qquad (4.6)$$

With $x=\cos(\theta)$, the Eq. (4.1) can be rewritten as follows:

$$\frac{d}{dx}\left[(1-x^2)\frac{dS}{dx}\right] + U_\lambda S = 0, \qquad (4.7)$$

$$U_\lambda = \tilde{\lambda} + \alpha x^2 - \frac{m^2}{1-x^2} - \frac{k^2-1/4}{x^2}. \qquad (4.8)$$

This equation besides ∞ has three singular points $x=0,\pm1$ at which solutions have the following asymptotic behavior:

$$S(x)\sim x^{\pm k} \quad\text{at } x=0, \quad S(x)\sim(1\mp x)^{\pm m/2} \quad\text{at } x=\pm1. \qquad (4.9)$$

For a regular solution, we have

$$S(x) = x^{|k|}(1-x^2)^{|m|/2}F(x), \qquad (4.10)$$

where F is a solution of the equation

$$x^2(1-x^2)F'' + (-2x^3|m|-2x^3|k|$$

$$+2x|k|-2x^3)F'\left(a^2\omega^2 x^4 + [\lambda-(|m|+|k|)^2\right.$$

$$\left. -(|m|+|k|)]x^2 + \frac{1}{4}-|k|\right)F = 0. \qquad (4.11)$$

The regularity condition means that we are looking for a solutions $F(x)$ which are finite at $x=0$ and at $x=1$. This gives us the two-point boundary value problem.

B. Degenerate case

Consider now the degenerate case when $a=b$. A solution S of Eq. (4.1) is of the form

$$S_\ell^{|m|,|k|}(\theta) = \sin^{|m|}\theta \sqrt{\cos\theta}\left[C_1\cos^{|k|}\theta s_\ell^{|m|,|k|}(|k|,\theta)\right.$$

$$\left. + C_2\cos^{-|k|}\theta s_\ell^{|m|,|k|}(-|k|,\theta)\right], \qquad (4.12)$$

$$s_\ell^{|m|,|k|}(|k|,\theta) = F(\alpha_+,\alpha_-;1+|k|;\cos^2\theta), \qquad (4.13)$$

$$\alpha_\pm = \frac{|m|+1+|k|}{2} \pm \frac{1}{2}\sqrt{1+\lambda}. \qquad (4.14)$$

Here $F(a,b;c;z)$ is a hypergeometric function. The coefficients of this function in our case posses the property $a+b-c=|m|=$(positive) integer number. The regularity of S at $\theta=\pi/2$ requires $C_2=0$, while its regularity at $\theta=0$ implies

$$\frac{|m|+1+|k|}{2} - \frac{1}{2}\sqrt{1+\lambda} = -\ell, \qquad (4.15)$$

where ℓ is an integer number. Thus we have

$$\lambda_\ell(|m|,|k|) = (2\ell+|m|+|k|+1)^2 - 1. \qquad (4.16)$$

The corresponding eigenfunctions are the Jacobi polynomials $P_\ell^{|m|,|k|}(1-2\cos^2\theta)$. Thus

$$\mathcal{Y}_{\ell m k}(\theta,\phi,\psi|\alpha=0)$$

$$= \frac{e^{im\phi+ik\psi}}{2\pi}(1-z)^{|m|/2}(1+z)^{|k|/2}B_\ell^{|m|,|k|}P_\ell^{|m|,|k|}(z), \qquad (4.17)$$

where $z=\cos(2\theta)=\cos\vartheta$ and

$$B_\ell^{|m|,|k|} = 2^{-(|m|+|k|-1)/2}\sqrt{2\ell+|m|+|k|+1}$$

$$\times \sqrt{\frac{\Gamma(\ell+1)\Gamma(\ell+|m|+|k|+1)}{\Gamma(\ell+|m|+1)\Gamma(\ell+|k|+1)}}. \qquad (4.18)$$

Note that for $a=b$ the hyperspheroidal harmonics (4.4) are eigenfunctions of the operators J_Φ, J_Ψ, and J^2 determined by Eq. (2.26). Using this fact one can demonstrate that the set (4.4) of hyperspheroidal harmonics is complete (see, e.g., [13,14]). For this set, we have $m=0,\pm1,\pm2,\ldots$, $k=0,\pm1,\pm2,\ldots$, $l=0,1,2,\ldots$.

V. RADIAL EQUATION

It is convenient to rewrite the radial equation (3.11) as follows. Let us define

$$R = \left[\frac{r}{(r^2+a^2)(r^2+b^2)}\right]^{1/2}\mathcal{R}. \qquad (5.1)$$

Then, one has

$$\frac{d^2\mathcal{R}}{dr_*^2} + V\mathcal{R} = 0, \tag{5.2}$$

where r_* is a tortoise coordinate with the property

$$\frac{dr_*}{dr} = \frac{(r^2+a^2)(r^2+b^2)}{\Delta}. \tag{5.3}$$

The effective potential V is[2]

$$V = \frac{r^2\Delta\left[-\lambda + \omega^2(r^2+a^2+b^2) + \dfrac{m^2(a^2-b^2)}{r^2+a^2} + \dfrac{k^2(b^2-a^2)}{r^2+b^2}\right]}{(r^2+a^2)^2(r^2+b^2)^2} + \frac{Z\Delta^2}{4r^2(r^2+a^2)^4(r^2+b^2)^4}$$

$$+ \frac{r_0^2 r^2\left(\omega - \dfrac{ma}{r^2+a^2} - \dfrac{kb}{r^2+b^2}\right)^2}{(r^2+a^2)(r^2+b^2)}, \tag{5.4}$$

where

$$Z = 21r^8 + 14r^6(a^2+b^2) + 5r^4(a^2-b^2)^2$$
$$- 10r^2a^2b^2(a^2+b^2) - 3a^4b^4.$$

The radial equation is evidently invariant under the transformation

$$a \leftrightarrow b, \quad m \leftrightarrow k. \tag{5.5}$$

At the horizon, i.e., $r=r_+$, the effective potential V takes the form

$$V_{hor} = (\omega - m\Omega_a - k\Omega_b)^2, \tag{5.6}$$

with Ω_a and Ω_b defined by the Eq. (2.15). The asymptotic value of the potential V at infinity is

$$V_{inf} = \omega^2. \tag{5.7}$$

We define two sets of solutions $\mathcal{R}_A^{in}(r|\omega)$ and $\mathcal{R}_A^{up}(r|\omega)$ (with $A=\{\ell,m,k\}$) by the boundary conditions

$$\mathcal{R}_A^{in}(r|\omega) \sim \begin{cases} t_A^{in}(\omega)e^{-i\varpi r_*} & \text{as } r \to r_+ \\ e^{-i\omega r_*} + r_A^{in}(\omega)e^{+i\omega r_*} & \text{as } r \to \infty \end{cases} \tag{5.8}$$

and

$$\mathcal{R}_A^{up}(r|\omega) \sim \begin{cases} r_A^{up}(\omega)e^{-i\varpi r_*} + e^{+i\varpi r_*} & \text{as } r \to r_+ \\ t_A^{up}(\omega)e^{+i\omega r_*} & \text{as } r \to \infty. \end{cases} \tag{5.9}$$

Here,

$$\varpi = \omega - m\Omega_a - k\Omega_b. \tag{5.10}$$

Since the eigenvalues $\lambda_\ell(mk|\alpha)$ are real the functions complex conjugated to $\mathcal{R}_J^{in}$ and $\mathcal{R}_J^{up}$ are also solutions. Using the constancy of the Wronskian for various combinations of solutions of the radial equation one gets

$$1 - |r_A^{in}(\omega)|^2 = \frac{\varpi}{\omega}|t_A^{in}(\omega)|^2, \quad 1 - |r_A^{up}(\omega)|^2 = \frac{\omega}{\varpi}|t_A^{up}(\omega)|^2, \tag{5.11}$$

$$\omega\,\bar{t}_A^{up}(\omega)r_J^{in}(\omega) = -\varpi\,\bar{r}_A^{up}(\omega)\,t_A^{in}(\omega), \quad \omega\,t_A^{up}(\omega) = \varpi\,t_A^{in}(\omega). \tag{5.12}$$

It should be emphasized that, as it was the case in $3+1$ dimensions, for certain values of ω it is possible to have the reflection coefficients greater than one. This implies the existence of the so-called *super-radiance effect*. The condition for super-radiance is

$$0 < \omega < m\Omega_a + k\Omega_b. \tag{5.13}$$

In this case, it follows from Eq. (5.11) that the reflection coefficients $r_A^{in,up}(\omega)$ become greater than one. This is the consequence of the fact that the scalar field modes obeying (5.13) are amplified by the rotating black hole.

VI. ENERGY AND ANGULAR MOMENTUM FLUXES

Quantization of massless fields in the 4-dimensional space-time of a rotating black hole was discussed in [15–18] (see also [19]). Main aspects of the quantization including the choice of the state remain practically the same in the 5-dimensional case. For this reason, we shall not repeat here all the formal elements of the standard procedure but simply introduce the required notations and present the final results. We shall follow the paper [18], where necessary details can be found.

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[2]The more general setup adopted for the string theory framework was studied in [6,7].

In order to quantize the field one uses the complete set of orthonormal complex solutions of the field equation (3.2). The scalar product for these solutions is defined as follows:

$$\langle \varphi_1, \varphi_2 \rangle = \frac{i}{2} \int_{\Sigma} \sqrt{-g}\,(\bar{u}_{2,\mu} u_1 - \bar{u}_{1,\mu} u_2)\, d\Sigma^{\mu}, \quad (6.1)$$

where Σ is any complete Cauchy surface. Since the scalar product (6.1) does not depend on the choice of Σ, it is convenient in our case to choose $\Sigma = \mathcal{J}^- \cup H^-$, where $\mathcal{J}^-$ is the past null infinity and H^- is the past horizon. (For details, see, e.g., [19].)

Using solutions of the radial and angular equations, we define the following normalized solutions of the field equation (3.2) [in coordinates $(t, r, \theta, \phi, \psi)$]:

$$u_{\Lambda}^{\mathrm{in}}(x) = \left[\frac{r}{4\pi\omega(r^2 + a^2)(r^2 + b^2)} \right]^{1/2}$$
$$\times e^{-i\omega t} \mathcal{R}_{\ell m k}^{\mathrm{in}}(r|\omega) \mathcal{Y}_{\ell m k}(\theta, \phi, \psi | a\omega), \quad (6.2)$$

$$u_{\Lambda}^{\mathrm{up}}(x) = \left[\frac{r}{4\pi|\varpi|(r^2 + a^2)(r^2 + b^2)} \right]^{1/2}$$
$$\times e^{-i\omega t} \mathcal{R}_{\ell m k}^{\mathrm{up}}(r|\omega) \mathcal{Y}_{\ell m k}(\theta, \phi, \psi | a\omega). \quad (6.3)$$

Here $\Lambda = \{\omega \ell m k\}$. The solutions $u_{\Lambda}^{\mathrm{in}}$ are defined for $\omega > 0$. The solutions $u_{\Lambda}^{\mathrm{up}}$ are naturally defined for $\varpi > 0$. For superradiant modes, when condition (5.13) is satisfied, one uses solutions $u_{-\omega \ell - m - k}^{\mathrm{up}}$.

The unrenormalized expectation value of the stress-energy tensor in the state of the Unruh vacuum is (see [18])

$$\langle U | \hat{T}_{\mu\nu} | U \rangle = \sum_J \left[\int_0^{\infty} d\varpi \coth\left(\frac{\pi\varpi}{\kappa} \right) T_{\mu\nu}[u_{\Lambda}^{\mathrm{up}}, \bar{u}_{\Lambda}^{\mathrm{up}}] \right.$$
$$\left. + \int_0^{\infty} d\omega\, T_{\mu\nu}[u_{\Lambda}^{\mathrm{in}}, \bar{u}_{\Lambda}^{\mathrm{in}}] \right], \quad (6.4)$$

where κ is the surface gravity and

$$T_{\mu\nu}[u, \bar{u}] = \left(\frac{1}{2} - \xi \right)(u_{;\mu} \bar{u}_{;\nu} + u_{;\nu} \bar{u}_{;\mu}) - \xi(u_{;\mu\nu} \bar{u} + u \bar{u}_{;\mu\nu})$$
$$+ \left(-\frac{1}{2} + 2\xi \right) g_{\mu\nu} u_{;\lambda} \bar{u}^{;\lambda}. \quad (6.5)$$

Here ξ is the parameter of nonminimal coupling. For the conformal field in 5-dimensional space-time $\xi = 3/16$ and $T_{\mu}^{\nu}[u, \bar{u}] = 0$.

Let us define the following expressions for the energy and angular momentum density fluxes at infinity,

$$\varepsilon = -\langle U | \hat{T}_{\mu\nu} | U \rangle \xi_t^{\mu} n^{\nu}, \quad (6.6)$$

$$j = -\langle U | \hat{T}_{\mu\nu} | U \rangle \xi_{\phi}^{\mu} n^{\nu}. \quad (6.7)$$

Here n^{μ} is a unit vector in r direction at infinity and we denoted by ξ_t^{μ} and ξ_{ϕ}^{μ} the Killing vectors which generate

translation in time t and rotation in an ϕ direction. Substituting the asymptotics of functions $u_{\Lambda}^{\mathrm{in}}$ and $u_{\Lambda}^{\mathrm{up}}$ into Eq. (6.4) one gets for the renormalized value of fluxes (see [18])

$$\varepsilon(\theta) \sim \frac{1}{8\pi^3 r^3} \sum_{\ell, m, k} \int_0^{\infty} \frac{\omega^2\, d\omega}{\varpi(e^{2\pi\varpi/\kappa} - 1)} |t_{\ell m k}^{\mathrm{up}}(\omega)|^2$$
$$\times \frac{|S_{\ell}^{m,k}(\theta | \alpha)|^2}{\cos\theta}, \quad (6.8)$$

$$j(\theta) \sim \frac{1}{8\pi^3 r^3} \sum_{\ell, m, k} \int_0^{\infty} \frac{m\omega\, d\omega}{\varpi\,(e^{2\pi\varpi/\kappa} - 1)} |t_{\ell m k}^{\mathrm{up}}(\omega)|^2$$
$$\times \frac{|S_{\ell}^{m,k}(\theta | \alpha)|^2}{\cos\theta}. \quad (6.9)$$

As we already mentioned, in the case of $a = b \neq 0$ the angular harmonics $S_{\ell}^{m,k}$ are the same as the angular harmonics in the absence of rotation. However, the radial equations (and therefore the gray-body factors) are different for these two cases. Because of that, the energy and angular momentum fluxes differ from those derived in the nonrotating case.

In the degenerate case $a = b \neq 0$, we have

$$\varepsilon(\theta) \sim \frac{1}{8\pi^3 r^3} \sum_{\ell, m, k} \int_0^{\infty} \frac{\omega^2\, d\omega}{\varpi(e^{2\pi\varpi/\kappa} - 1)} |t_{\ell m k}^{\mathrm{up}}(\omega)|^2 (1 - z)^{|m|}$$
$$\times (1 + z)^{|k|} (B_{\ell}^{|m|, |k|})^2 [P_{\ell}^{|m|, |k|}(z)]^2, \quad (6.10)$$

$$j(\theta) \sim \frac{1}{8\pi^3 r^3} \sum_{\ell, m, k} \int_0^{\infty} \frac{m\omega\, d\omega}{\varpi(e^{2\pi\varpi/\kappa} - 1)} |t_{\ell m k}^{\mathrm{up}}(\omega)|^2 (1 - z)^{|m|}$$
$$\times (1 + z)^{|k|} (B_{\ell}^{|m|, |k|})^2 [P_{\ell}^{|m|, |k|}(z)]^2, \quad (6.11)$$

where $B_{\ell}^{|m|, |k|}$ is defined in Eq. (4.18) and $z = \cos(2\theta)$.

By integrating over the angle variables (over three-sphere boundary at infinity), we obtain the expressions for the total energy and angular moment emission

$$\dot{E} = \frac{1}{2\pi} \sum_{\ell, m, k} \int_0^{\infty} \frac{\omega^2\, d\omega}{\varpi(e^{2\pi\varpi/\kappa} - 1)} |t_{\ell m k}^{\mathrm{up}}(\omega)|^2, \quad (6.12)$$

$$\dot{J} = \frac{1}{2\pi} \sum_{\ell, m, k} \int_0^{\infty} \frac{m\omega\, d\omega}{\varpi(e^{2\pi\varpi/\kappa} - 1)} |t_{\ell m k}^{\mathrm{up}}(\omega)|^2. \quad (6.13)$$

VII. DISCUSSION

The results obtained in this paper are important from several points of view. First, they allow one to better understand the dynamics of massless scalar fields propagating near a rotating higher dimensional black holes. The separation of variables which occurs in 5-dimensional scalar field wave equation indicates that separation which occurred in 4-dimensional case was not merely an accident, but it is rather a property which follows from the symmetries of a rotating black hole.

The results are readily applicable to the brane world mod-

els where phenomenologically valid rotating higher dimensional black holes can exist. Nontrivial structure of rotational group in four spatial dimensions (such as the existence of two parameters of rotation, the notion of invariant planes of rotations rather than axis of rotation, etc.) gives rise to rich phenomenology concerning radiation from such black holes. For example, one of the parameters of rotation could be zero, in which case the black hole would be spinning only in one plane. This case is of particular interest in brane world models where in the first approximation the black holes produced in collisions of standard model particles can spin only in the planes defined by the brane where all the standard model fields are confined. However, if we take a back reaction into account, after such a black hole emits higher dimensional graviton into the bulk, it gains the general angular momentum which cannot be described with a single parameter of rotation. Due to stochastic nature of the black hole evaporation, it is not likely that the black hole rotation will spin itself up significantly in any general direction. However, this makes the general form of rotation important to study, at least, in principle. It is interesting to mention that if both parameters of rotation are nonvanishing but equal, the angular equation coincides with that of a nonrotating case. For each of these particular cases, using expressions given in Eqs. (6.10) and (6.11), we can calculate the angular distributions as well as the total amount of energy and angular momentum emitted by a black hole.

We demonstrated that similarly to the 4-dimensional case, 5-dimensional rotating black hole allows the super-radiance effect. If m and k are azimuthal quantum numbers with respect to two rotation axes, then the condition of super-radiance is $\omega < m\Omega_a + k\Omega_b$, where Ω_a and Ω_b are angular velocities. One can expect that another feature of quantum radiation from 4-dimensional rotating black holes, namely, its strong spin dependence [20], is also present in the 5-dimensional case. If it occurs then the bulk radiation of gravitons might be the leading channel of the black hole decay. In order to check this conjecture, it is necessary to solve higher spin massless field equations in the higher dimensional space-time of a rotating black hole. This is an interesting challenge.

ACKNOWLEDGMENTS

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[1] F.R. Thangherlini, Nuovo Cimento **77**, 636 (1963).

[2] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *ibid.* **436**, 257 (1998); N. Kaloper, J. March-Russel, G. Starkman, and M. Trodden, Phys. Rev. Lett. **85**, 928 (2000); G. Starkman, D. Stojkovic, and M. Trodden, *ibid.* **87**, 231303 (2001); Phys. Rev. D **63**, 103511 (2001); L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[3] S. Dimopoulos and G. Landsberg, Phys. Rev. Lett. **87**, 161602 (2001); D.M. Eardley and S.B. Giddings, Phys. Rev. D **66**, 044011 (2002); S.B. Giddings and S. Thomas, *ibid.* **65**, 056010 (2002); J.L. Feng and A.D. Shapere, Phys. Rev. Lett. **88**, 021303 (2002); G. Landsberg, hep-ph/0211043; hep-ph/0205174; S.B. Giddings, hep-th/0205027; V. Frolov and D. Stojkovic, Phys. Rev. Lett. **89**, 151302 (2002); Phys. Rev. D **66**, 084002 (2002); M.B. Voloshin, Phys. Lett. B **524**, 376 (2002); S. Solodukhin, *ibid.* **533**, 153 (2002); R. Emparan, G. Horowitz, and R.C. Myers, Phys. Rev. Lett. **85**, 499 (2000); M. Bleicher, Phys. Lett. B **548**, 73 (2002); A. Ringwald, hep-ph/0212342; A. Ringwald and H. Tu, Phys. Lett. B **525**, 135 (2002); M. Kowalski, A. Ringwald, and H. Tu, *ibid.* **529**, 1 (2002); H. Tu, hep-ph/0205024; hep-ph/0211159; Y. Uehara, Prog. Theor. Phys. **107**, 621 (2002); Mod. Phys. Lett. A **17**, 1551 (2002); R. Casadio and B. Harms, Phys. Lett. B **487**, 209 (2000); Phys. Rev. D **64**, 024016 (2001); Int. J. Mod. Phys. A **17**, 4635 (2002); D. Stojkovic, Phys. Rev. D **67**, 045012 (2003); V. Cardoso and J.P.S. Lemos, following paper, *ibid.* **67**, 084005 (2003); Phys. Lett. B **538**, 1 (2002); L. Anchordoqui and H. Goldberg, Phys. Rev. D **65**, 047502 (2002); **67**, 064010 (2003); L.A. Anchordoqui, J.L. Feng, H. Goldberg, and A.D. Shapere, *ibid.* **66**, 103002 (2002); L.A. Anchordoqui, T. Paul, S. Reucroft, and J. Swain, hep-ph/0206072; L.A. Anchordoqui, J.L. Feng, H. Goldberg, and A.D. Shapere, Phys. Rev. D **65**, 124027 (2002); T.G. Rizzo, J. High Energy Phys. **02**, 011 (2002); P. Kanti and J. March-Russell, hep-ph/0212199; Phys. Rev. D **66**, 024023 (2002); K. Cheung, Phys. Rev. Lett. **88**, 221602 (2002); Phys. Rev. D **66**, 036007 (2002); A. Chamblin and G.C. Nayak, *ibid.* **66**, 091901(R) (2002); G.C. Nayak, hep-ph/0211395; T. Han, G.D. Kribs, and B. McElrath, Phys. Rev. Lett. **90**, 031601 (2003); R. Emparan, R.W. Gregory, and C. Santos, Phys. Rev. D **63**, 104022 (2001); M. Rogatko, *ibid.* **64**, 064014 (2001); D. Ida, K. Oda, and S.C. Park, *ibid.* **67**, 064025 (2003); D. Ida, Y. Uchida, and Y. Morisawa, gr-qc/0212035; R. Emparan, A. Fabbri, and N. Kaloper, J. High Energy Phys. **08**, 043 (2002); R. Emparan, J. Garcia-Bellido, and N. Kaloper, *ibid.* **01**, 079 (2003).

[4] G.W. Gibbons and C.A.R. Herdeiro, Class. Quantum Grav. **16**, 3619 (1999).

[5] C.A.R. Herdeiro, Nucl. Phys. **B582**, 363 (2000).

[6] M. Cvetic and F. Larsen, Phys. Rev. D **56**, 4994 (1997).

[7] M. Cvetic and D. Youm, Nucl. Phys. **B476**, 118 (1996).

[8] R.C. Myers and M.J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).

[9] G. Clement, D. Gal'tsov, and C. Leygnac, Phys. Rev. D **67**, 024012 (2003).

[10] A.Z. Petrov, *Einstein spaces* (Pergamon, New York, 1969).

[11] R. Geroch, J. Math. Phys. **12**, 918 (1971).

[12] V. Frolov and D. Stojkovic, gr-qc/0301016.

[13] D.A. Varshalovich, A.N. Moskalev, and V.K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).

[14] A.O. Barut and R. Raszka, *Theory of Group Representations and Applications* (World Scientific, Singapore, 1986).

[15] P. Candelas, P. Chrzanowski, and K.W. Howard, Phys. Rev. D **24**, 297 (1981).

[16] V.P. Frolov and K.S. Thorne, Phys. Rev. D **39**, 2125 (1989).

[17] A.L. Matacz, P.C.W. Davies, and A.C. Ottewill, Phys. Rev. D **47**, 1557 (1993).

[18] A.C. Ottewill and E. Winstanley, Phys. Rev. D **62**, 084018 (2000).

[19] V. Frolov and I. Novikov, *Black Hole Physics: Basic Concepts and New Developments* (Kluwer Academic, Dordrecht, 1998).

[20] D.N. Page, Phys. Rev. D **13**, 198 (1976); **14**, 3260 (1976).