Causal perturbation theory in general FRW cosmologies: Energy-momentum conservation and matching conditions

G. Amery*
Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
and School of Mathematical and Statistical Sciences, University of Natal, Durban, 4041, Republic of South Africa

E. P. S. Shellard†
Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
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We describe energy-momentum conservation in relativistic perturbation theory in general Friedmann-Robertson-Walker (FRW) backgrounds with causal source terms, such as the presence of cosmic defect networks. A prescription for a linear energy-momentum pseudotensor in a curved FRW universe is provided, and it is decomposed using eigenfunctions of the Helmholtz equation. Conserved vector densities are constructed from the conformal geometry of these spacetimes and related to our pseudotensor, demonstrating the equivalence of these two approaches. We also relate these techniques to the role played by residual gauge freedom in establishing matching conditions at early phase transitions, which we can express in terms of components of our pseudotensor. This formalism is concise and geometrically sound on both sub- and superhorizon scales, thus extending existing work to a physically (and numerically) useful context.

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I. INTRODUCTION

Considerable challenges are presented by the study of the causal generation of perturbations seeding large-scale structure formation and anisotropies in the cosmic microwave background (CMB) [1]. Not only is the analytic treatment of the resulting inhomogeneous evolution equations extremely complicated, but their numerical implementation must also circumvent a number of subtle pitfalls before facing up to the severe dynamic range limitations of even supercomputer simulations. To date the only quantitative numerical studies with realistic causal sources, such as cosmic strings [2–4] or other global defect networks [5–7], have been performed in flat Friedmann-Robertson-Walker (FRW) ($K=0$) backgrounds. Despite positive indications about the large-scale structure power spectrum for models with a cosmological constant included [4], these defect networks in flat cosmologies appear to be unable to replicate the observed position of the first acoustic peak in the CMB angular power spectrum [3,6–9]—indeed the best results for defects are for $K\neq 0$ cosmologies [10].

This situation contrasts markedly with the standard inflationary paradigm in which reliable predictions about the CMB acoustic peaks are relatively straightforward to make and for which there appears to be remarkable accord with recent CMB experiments [11]. So the question arises as to the relevance and utility of complicated theoretical studies of causal perturbation generation when the simple primordial inflationary models appear to suffice. The first motivation is that the confrontation with observation remains indecisive, not only because of the significant experimental uncertainties—for example, even Microwave Anisotropy Probe (MAP) data will be insufficient to simultaneously constrain both the adiabatic and isocurvature inflationary modes, and cosmological parameters [12]—but also because good quantitative accuracy has not yet been achieved for the full range of cosmic defect theories. For example, even for flat universes, a subsidiary role for defect networks complementing the inflationary power spectrum cannot be excluded. Indeed, claims of improved fits in hybrid defect-inflation models [13,14] are not surprising given the extra degrees of freedom available.

There are a number of mechanisms by which defects can be produced at the end of inflation with the appropriate energy scale: Hybrid inflation typically ends through symmetry breaking which generates defects [15]. Phenomenological grand unified theory (GUT) models have been proposed which can produce superheavy strings after inflation [16]. ''Preheating'' as inflation ends is also capable of creating superheavy defects even for low energy inflation scales [17]. Given the foundational uncertainties that remain concerning inflation [18] and the lack of a widely accepted realistic phenomenology, it is only reasonable to continue to explore alternative paradigms such as late-time ''causal'' generation mechanisms—which are not exhausted by defect networks in any case, e.g. ''explosion'' [19], and other source models [7]. Moreover, in order to have confidence in cosmological parameter estimation, it will be necessary to constrain these alternative models, including the effects of vector and tensor modes, and $K\neq 0$ backgrounds. Here the combination of intrinsic curvature and defect sources is particularly interesting.

Cosmic defects would typically be expected to contribute

*Electronic address: g.amery@damtp.cam.ac.uk
†Electronic address: e.p.s.shellard@damtp.cam.ac.uk

to the nonGaussianity of CMB anisotropies on small scales—e.g. for cosmic strings, on arcminute scales [22]—with effects on larger scales being swamped by the central limit theorem. This nonGaussianity ultimately derives from the nonlinearity of the defect sources, which is in marked contrast with the linear quantum effects which imply that the simplest inflationary scenarios are typically Gaussian. Although there do exist inflationary models (such as nonvacuum initial state, and multifield inflation) that can yield non-Gaussian statistics (e.g. χ^2), the detection of a different distribution would imply that inflation was not the only significant source of perturbations in the early universe. The presence or absence of such distinct signatures may therefore provide observational tests with which to confront inflation and causal paradigms [20]. Another particularly exciting prospect is the detection of a CMB polarization signal for which the competing models give very different predictions and, indeed, some causal effects can be differentiated [21]. Of course, topological defects are strongly motivated in high energy physics and their discovery—from observations of such effects in the CMB, or from the study of high energy cosmic rays and gravitational lensing—would have profound implications for our understanding of the early universe.

Finally, we note that there is now a significant body of work about causal mechanisms for structure formation and this has raised a number of interesting issues within general relativistic perturbation theory. However, even with most work undertaken in a flat FRW background, the number of approaches to the problem almost equals the number of papers. A key aim of the present paper, then, is to demonstrate the equivalence of the most important of these approaches and to generalize this work to all FRW cosmologies, laying the foundations for quantitative studies in curved backgrounds in particular. We shall work in the synchronous gauge because of its ubiquity in numerical simulations and the physical transparency offered by this gauge choice.

As well as being of interest in its own right, energy-momentum conservation is both the physical constraint on the spurious (gauge) modes resulting from the residual freedom in the synchronous gauge, and a common technique used to ensure that numerical simulations are free from their effects. In the literature, treatments of the energy-momentum conservation of individual modes in the combined system of gravitational and matter fields have been variously phrased in terms of "compensation" [7,23], "integral constraints" [24,25], and the construction of "pseudotensors" to describe the energy and momentum densities and their conservation laws [5,23,24,26], as well as the use of matching conditions across a phase transition to set initial conditions [24]. The relationship between these notions and the initial conditions has been discussed to some extent in the case of a flat FRW background. For general FRW cosmologies, however, the situation is less clear and deeper conceptual issues have to be resolved. In Ref. [27] a definition of the pseudo-energy was motivated by a consideration of matching conditions at an instantaneous phase transition, which was related to "geometrically" obtained conservation laws in the superhorizon limit. However, to date, a systematic geometric definition (valid on all scales) of the complete pseudotensor for curved FRW cosmologies has not been given.

In Sec. II we shall provide a prescription for the construction of such a linear energy-momentum pseudotensor in the $K\neq 0$ FRW universe. The pseudotensor so obtained agrees in the flatspace limit ($K\rightarrow 0$) with the Landau-Lifshitz stress-energy pseudotensor $\tau_{\mu\nu}$ obtained in Ref. [23], and with the superhorizon pseudo-energy of Ref. [27]. We also discuss the philosophy underlying the notion of a pseudotensor and how its inherently global nature appears to be at odds with theories of local causal objects. In Sec. III we define energy and momentum with respect to a general FRW background manifold. This allows us to calculate conserved vector densities for the conformal geometry of these spacetimes, and to relate them to our pseudotensor, giving it a local geometrical meaning that is valid on all scales and demonstrating the equivalence of the two formalisms. In Sec. IV we apply the matching condition formalism [24,27] to a curved universe, and discuss how the residual gauge freedom in the synchronous gauge may be exploited to make the pseudo-energy continuous across the phase transition in which the defects (or other sources) appear. We also show that we may match the vector part of the pseudotensor across this transition. We conclude (Sec. V) with a discussion of the implications of this work.

II. A GENERALIZED ENERGY-MOMENTUM PSEUDOTENSOR

We wish to consider metric perturbations $h_{\mu\nu}$ about a general FRW spacetime

$$ds^2 = a^2(\gamma_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \tag{1}$$

where the comoving background line element in "conformal-polar" coordinates (τ,χ,ϕ,θ) is given by

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + \frac{1}{|K|}[d\chi^2 + \sin^2_K\chi(d\theta^2 + \sin^2\theta d\phi^2)], \tag{2}$$

with the function $\sin_K\chi$ depending on the spatial curvature K as

$$\sin_K\chi = \begin{cases} \sinh\chi, & K<0, \\ \chi, & K=0, \\ \sin\chi, & K>0. \end{cases} \tag{3}$$

Here, $a\equiv a(\tau)$ is the scalefactor, for which we can define the conformal Hubble factor $\mathcal{H}=\dot{a}/a$, with dots denoting derivatives with respect to conformal time τ. As emphasized earlier, we shall adopt the synchronous gauge defined by the choice

$$h^{0\mu} = 0, \tag{4}$$

where the trace is given by $h \equiv h^i{}_i$ (with the convention throughout that Greek indices run from 0 to 3 and Latin from 1 to 3).

The Einstein equations are given by $G_{\mu\nu}\equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$ (with $\kappa = 8\pi G$), and we will separate the energy-momentum tensor $T_{\mu\nu}$ into three parts:

$$T_{\mu\nu}=\overline{T}_{\mu\nu}+\delta T_{\mu\nu}+\Theta_{\mu\nu}. \quad (5)$$

The background tensor $\overline{T}_{\mu\nu}$ includes the dark energy of the universe (or cosmological constant), while the first order part $\delta T_{\mu\nu}$ incorporates the stress energy of the radiation fluid, baryonic matter, and cold dark matter. The final contribution $\Theta_{\mu\nu}$ represents the stress tensor of an evolving defect network or some other causal sources. This is assumed to be small (of order $\delta T_{\mu\nu}$) and "stiff," that is, its energy and momenta are conserved independently of the rest of the matter and radiation in the universe and to lowest order its evolution is unaffected by the metric perturbations $h_{\mu\nu}$. In principle, we should include also backreaction effects and (for example) the decay products from string loops, modeled as another fluid component. These are, however, relatively benign cosmologically, and are usually neglected in numerical studies. A scalar potential may similarly be included in the formalism—see, for example, Hu, Seljak, White, and Zaldarriaga [32]. Except in situations in which first order theory breaks down, the conservation laws discussed in this paper should still hold in these cases.

A. Conceptual discussion and pseudotensors in flat ($K=0$) FRW spacetimes

It is interesting also to consider the notion of the energy-momentum tensor of the geometry or gravitational field, which we shall denote as $t_{\mu\nu}$. If it were possible to define then we could reexpress the perturbed Einstein equations simply as a wave equation for $h_{\mu\nu}$ with a source term constructed from the "complete" energy-momentum tensor, that is, the sum $\tau_{\mu\nu}=T_{\mu\nu}+t_{\mu\nu}$. As we shall explain, the linearized Bianchi identities would imply that the sum $\tau_{\mu\nu}$ is (to linear order) locally conserved $\tau^{\mu\nu}{}_{,\nu}=0$, since it includes all the flux densities of matter and gravity (unlike the covariant conservation law $T^{\mu\nu}{}_{;\nu}=0$ which represents an exchange between matter and gravity). Such motivations for incorporating the geometry in a "complete" energy-momentum tensor $\tau_{\mu\nu}$ are discussed at considerable length in Ref. [28] using the example of metric perturbations about Minkowski space.

Einstein, as well as Landau and Lifshitz, have presented procedures whereby one may rewrite the Bianchi identities to obtain quantities that they call energy-momentum "pseudotensors." These have some of the above properties, and allow for the calculation of various conserved quantities [26]. Here both $t_{\mu\nu}$ and $\tau_{\mu\nu}$ are quadratic in the connection coefficients, so that they are "linear tensors," behaving like tensors under linear transformations.

For a Minkowski space, with $\gamma_{\mu\nu}=\eta_{\mu\nu}$, $a=1$ in Eq. (1), linearizing reveals this procedure to be essentially trivial because $t_{\mu\nu}$ vanishes to first order. However, for the flat space ($K=0$) expanding universe, the time dependence of the scalefactor a in Eq. (1) introduces additional terms at linear order. This has been used by Veeraraghavan and Stebbins [23] to define an energy-momentum pseudotensor in this case:

$$\tau_{00}=(\delta T_{00}+\Theta_{00})-\frac{\mathcal{H}\dot{h}}{\kappa},\qquad \tau_{0i}=\delta T_{0k}+\Theta_{0k},$$

$$\tau_{ij}=\delta T_{ij}+\Theta_{ij}-\frac{\mathcal{H}}{\kappa}(\dot{h}_{ij}-\dot{h}\,\delta_{ij}). \quad (6)$$

Here the components τ^{00}, τ^{0i}, and τ^{ij} defined in Eq. (6) can be identified as the pseudo-energy density $\mathcal{U}$, the pseudo-momentum density $\vec{\mathcal{S}}$, and the pseudo-stress tensor $\mathcal{P}_{ij}$, respectively. Using the stress-energy conservation equations (the Bianchi identities)

$$\tau^{\mu\nu}{}_{,\nu}=0, \quad (7)$$

various suitable choices of evolution variables have then been made: for example, [5,23]. This flat space result can be obtained from a straightforward manipulation of the field equations for $h_{\mu\nu}$ which involves moving any background-dependent terms to the right hand side [5]. However, for the generalization to curved spacetime backgrounds we need a more rigorous prescription for the energy-momentum pseudotensor, as well as the definition of its components in a coordinate system appropriate for practical applications—this is the subject of this section. We are also called upon to come to terms with the nonlocal nature of these objects.

The Landau-Lifshitz construction of $\tau_{\mu\nu}$ proceeds by appealing to the principle of equivalence, which allows one to choose a normal coordinate system so that the connection coefficients vanish in the neighborhood of a point. In a general spacetime, the interacting part of the geometry $t_{\mu\nu}$ cannot be made to vanish by this coordinate choice, although it then resides only in the second and higher order derivatives of the metric. Nevertheless it becomes significant over extended portions of the spacetime and so the energy-momentum of the geometry must be understood as global in nature [29]. This fact forbids the existence of a tensor density for the gravitational energy and momenta, so that the best that we can actually hope for in terms of local quantities is a "pseudotensorial"[1] object which, suitably integrated over a large region of spacetime, would lead to a quantity that is sufficiently gauge invariant for practical purposes.

However, in causal perturbation theory, we are particularly interested in a distribution of small perturbations each of which has associated energy and momentum. These objects (such as topological defects, and their associated perturbations) are not well modeled, even as a distribution, by quantities that have no meaning except over large portions of the spacetime, and one has a rather *ad hoc* balance between the requirement that one consider a sufficiently large volume, and the understanding that effect of the distribution of causal

[1]These objects are commonly known as "pseudotensors" for historical reasons, e.g. Einstein's antisymmetric construction. Here, the nomenclature refers to the fact that they require additional structure—such as a preferred coordinate system/background manifold—on the spacetime for their definition [30], rather than their transformation properties under reflections. They are not true tensors, but linear tensors.

objects should average to zero. This is also a major conceptual difficulty facing integral constraints for localized perturbations as discussed by Traschen *et al.* [25]. Fortunately, there exists a formalism [31] in which one can avoid these difficulties by defining energy and momentum with respect to a background manifold, so that one obtains conservation laws and conserved vector densities. We shall apply this formalism to the general FRW spacetime in Sec. III, thereby providing the results of this section with a local geometrical interpretation on all scales.

B. General FRW $(K\neq 0)$ spacetimes and curvilinear coordinates

Consider two spacetimes related via a conformal transformation—also known as a metric rescaling—of the metric tensor so that

$$\tilde{g}_{\mu\nu}=\Omega g_{\mu\nu}, \quad \tilde{g}^{\mu\nu}=\Omega^{-1}g^{\mu\nu}, \tag{8}$$

where Ω is a scalar function of the coordinates $\Omega(x^{\mu})$. A general FRW universe may be so rescaled to a stationary $(a=1)$ FRW universe. Since the nonzero intrinsic curvature of a general FRW spacetime manifests itself in the nonvanishing property of the background Einstein tensor (even in a stationary spacetime), we shall have to separate out the background from the perturbed parts. Moreover, since we wish to express perturbations in terms of the Helmholtz decomposition in polar coordinates, we shall write all spatial derivatives in terms of the covariant derivative with respect to γ_{ij}, rather than the partial derivatives as previously for the $K=0$ case in Cartesian coordinates.

Under Eq. (8) the Einstein tensor transforms as

$$\tilde{G}_{\mu\nu}=G_{\mu\nu}+t_{\mu\nu},$$
$$t_{\mu\nu}=-\psi_{\mu;\nu}+\frac{1}{2}\psi_{\mu}\psi_{\nu}+\frac{1}{4}g_{\mu\nu}\psi^{\sigma}\psi_{\sigma}+g_{\mu\nu}\psi^{\sigma}{}_{;\sigma}, \tag{9}$$

where $\psi_{\mu}\equiv(\ln\Omega)_{,\mu}$. Now let $g_{\mu\nu}=a^2(\gamma_{\mu\nu}+h_{\mu\nu})$ as in Eq. (1) with $\Omega=1/a^2$, so that $\tilde{g}_{\mu\nu}=\gamma_{\mu\nu}+h_{\mu\nu}$ is the metric for observers comoving with the expansion of the universe. If we raise the first index, we can make the identification $\psi_0=-2\mathcal{H}$, $\psi_i=0$. Hence, the components of a stress energy "pseudotensor" defined by

$$\tau^{\mu}{}_{\nu}\equiv\tilde{G}^{\mu}{}_{\nu}/\kappa, \tag{10}$$

may be written as

$$\kappa\tau^{0}{}_{0}=-3K+(a^2\delta G^{0}{}_{0}+\mathcal{H}\dot{h}),$$
$$\kappa\tau^{0}{}_{i}=a^2\delta G^{0}{}_{i},$$
$$\kappa\tau^{i}{}_{j}=-K\delta^{i}{}_{j}+(a^2\delta G^{i}{}_{j}-\mathcal{H}[\dot{h}^{i}{}_{j}-\dot{h}\delta^{i}{}_{j}]). \tag{11}$$

We note that, since metric rescalings (8) preserve its tensorial properties, the $\tau^{\mu}{}_{\nu}$ defined in Eq. (10) are true tensors in both the stationary and the expanding spacetimes. These may be further written as a sum of a background contribution from the unperturbed spacetime, and a perturbed part (unlike the $K=0$ case for which the background term vanishes). Thus, $\tau^{\mu}{}_{\nu}=\overline{\tau^{\mu}}{}_{\nu}+\delta\tau^{\mu}{}_{\nu}$, with the components given by

$$\kappa\overline{\tau^{0}}{}_{0}=-3K, \quad \kappa\delta\tau^{0}{}_{0}=a^2\delta G^{0}{}_{0}+\mathcal{H}\dot{h},$$
$$\kappa\overline{\tau^{0}}{}_{i}=0, \quad \kappa\delta\tau^{0}{}_{i}=a^2\delta G^{0}{}_{i},$$
$$\kappa\overline{\tau^{i}}{}_{j}=-K\delta^{i}_{j}, \quad \kappa\delta\tau^{i}{}_{j}=a^2\delta G^{i}{}_{j}-\mathcal{H}[\dot{h}^{i}{}_{j}-\dot{h}\delta^{i}{}_{j}]. \tag{12}$$

Since the $\tau^{\mu}{}_{\nu}$ are precisely the Einstein tensor (divided by κ) in the conformally related stationary spacetime $\tilde{g}_{\mu\nu}$, they must satisfy the Bianchi identities there. Hence, we know that

$$\tilde{D}_0\tau^{0}{}_{0}+\tilde{D}_j\tau^{j}{}_{0}=0, \quad \tilde{D}_0\tau^{0}{}_{i}+\tilde{D}_j\tau^{j}{}_{i}=0, \tag{13}$$

where $\tilde{D}_{\mu}$ denotes covariant differentiation with respect to the stationary 4-metric $\tilde{g}_{\mu\nu}$. Now, using the connections and Eq. (12), and working to first order, we may rewrite Eq. (13) as

$$\delta\tau^{0}{}_{0,0}+\delta\tau^{i}{}_{0|i}-\frac{K}{\kappa}\dot{h}=0,$$
$$\delta\tau^{0}{}_{i,0}+\delta\tau^{j}{}_{i|j}=0, \tag{14}$$

where the bar denotes the covariant derivative with respect to the 3-metric γ_{ij}, and the $-(K/\kappa)\dot{h}$ term is implicit in the covariant derivative $\tilde{D}_j\tau^{j}{}_{0}$.

This manner of rewriting the Einstein equations clearly reduces to that of [23]—see Eq. (6)—for $K=0$, where $\tau^{\mu}{}_{\nu}=\delta\tau^{\mu}{}_{\nu}$. However, Eqs. (14) obeyed by the $\delta\tau^{\mu}{}_{\nu}$ are more complicated than Eq. (7) because the nonzero intrinsic curvature manifests as a nonvanishing background Einstein tensor, which appears in the Bianchi identities for the full spacetime.

If we exploit the fact that the vector $\mathbf{P}_0$—Killing in $\tilde{\mathbf{g}}$—with components $\delta^{\mu}{}_{0}$ may be multiplied with itself to form the (reducible) Killing tensor $-\delta^{\mu}{}_{0}\delta^{0}{}_{\nu}$, then we may add $+K\dot{h}/\kappa$ times this tensor to $\delta\tau^{\mu}{}_{\nu}$ without disturbing the tensorial properties of the perturbed part of the $\tau^{\mu}{}_{\nu}$. This amounts to a redefinition of the 00-component only. Henceforth we shall consider $\delta\tau^{\mu}{}_{\nu}$ to be redefined in this fashion so that

$$\kappa\delta\tau^{0}{}_{0}\rightarrow\kappa\delta\tau^{0}{}_{0}-Kh=a^2\delta G^{0}{}_{0}+\mathcal{H}\dot{h}-Kh. \tag{15}$$

The new $\delta\tau^{\mu}{}_{\nu}$ will then satisfy the concise equations

$$\delta\tau^{0}{}_{0,0}+\delta\tau^{i}{}_{0|i}=0, \tag{16}$$

$$\delta\tau^{0}{}_{i,0}+\delta\tau^{j}{}_{i|j}=0. \tag{17}$$

We may justify this redefinition by noting that the 00-component so obtained is precisely the definition of energy (up to a factor $a^2\sqrt{\gamma}$) obtained from the conformal Killing vector $\mathbf{P}_0$ in the following section. Furthermore, since the volume element $\mathrm{d}V=\mathrm{d}\bar{V}+\mathrm{d}V_{\rm pert}$ where $\mathrm{d}\bar{V}=a^4\sqrt{\gamma}\mathrm{d}y\,\mathrm{d}\theta\,\mathrm{d}\phi$, $\mathrm{d}V_{\rm pert}=(h/2)\mathrm{d}\bar{V}$ and γ is the determinant of the spatial 3-metric, we may interpret the $-Kh$ term as representing the alteration to the flat space energy due to the effect of intrinsic curvature on the volume element.

Expressing the conservation properties of a general FRW cosmology in this fashion is particularly useful, as it produces equations phrased in terms of the spatial covariant derivative, which is precisely the language used to express the properties of the Helmholtz eigenfunctions $\mathbf{Q}^{(m)}$, commonly used to describe perturbations in such cosmologies.

C. The Helmholtz decomposed pseudotensor

For perturbations over a curved FRW background, we can no longer make use of standard Fourier expansions. Instead, it is usual to employ the Helmholtz decomposition using the linearly independent eigenfunctions of the Laplacian in polar coordinates (see Ref. [32]). We expand all perturbation quantities in terms of the eigenfunctions $\mathbf{Q}^{(m)}$, which are the scalar $(m=0)$, vector $(m=\pm1)$ and tensor $(m=\pm2)$ solutions to the Helmholtz equation

$$\nabla^2\mathbf{Q}^{(m)}\equiv\gamma^{ij}\mathbf{Q}^{(m)}_{|ij}=-k^2\mathbf{Q}^{(m)}, \tag{18}$$

where the generalized wave number q and its normalized equivalent β are related to k via $q^2=k^2+(|m|+1)K$, $\beta=q/\sqrt{|K|}$ and the eigentensor has $|m|$ suppressed indices (equal to the rank of the perturbation). The divergenceless and transverse-traceless conditions for the vector and tensor modes are expressed via $Q_i^{(\pm1)|i}=0$ and $\gamma^{ij}Q_{ij}^{(\pm2)}=Q_{ij}^{(\pm2)|i}=0$. Auxiliary vector and tensor modes may be constructed as follows:

$$Q_i^{(0)}=-k^{-1}Q^{(0)}_{|i},\quad Q_{ij}^{(0)}=k^{-2}Q^{(0)}_{|ij}+\frac{1}{3}\gamma_{ij}Q^{(0)},$$

$$Q_{ij}^{(\pm1)}=-(2k)^{-1}[Q^{(\pm1)}_{i|j}+Q^{(\pm1)}_{j|i}]. \tag{19}$$

The spectra for flat and open universes $(K\leqslant0)$ are continuous and complete for $\beta\geqslant0$. For the $K>0$ case, the spectrum is discrete because of the existence of periodic boundary conditions. For scalar perturbations, we then have $\beta=3,4,5,\ldots$ since the $\beta=1,2$ modes are pure gauge [33]. Using this decomposition the metric perturbation may be decomposed as

$$h_{ij}=2\int\mathrm{d}\mu(\beta)[h_L\gamma_{ij}Q^{(0)}+h_TQ_{ij}^{(0)}+h_V^{(1)}Q_{ij}^{(1)}+h_V^{(-1)}Q_{ij}^{(-1)}+h_G^{(2)}Q_{ij}^{(2)}+h_G^{(-2)}Q_{ij}^{(-2)}], \tag{20}$$

where h_L and h_T represent two "longitudinal" and "transverse" scalar degrees of freedom, $h_V^{\pm1}$ two vector modes and $h_G^{\pm2}$ two tensor modes. As well as the transform over the "radial" coordinate β, there is an implicit sum over indices lm which label the spherical harmonics encoding the angular dependence.

Decomposing the energy-momentum pseudotensor (15) in this fashion, we have

$$\delta\tau^0_0=\int\mathrm{d}\mu(\beta)\tau_SQ^{(0)},$$

$$\delta\tau^0_i=\int\mathrm{d}\mu(\beta)[\tau_{IV}Q_i^{(0)}+\tau_V^{(1)}Q_i^{(1)}+\tau_V^{(-1)}Q_i^{(-1)}],$$

$$\delta\tau^i_j=\int\mathrm{d}\mu(\beta)[2(\tau_L\gamma^i_jQ^{(0)}+\tau_TQ^{(0)i}{}_j)+\tau_{IT}^{(1)}Q^{(1)i}{}_j+\tau_{IT}^{(-1)}Q^{(-1)i}{}_j+\tau_G^{(2)}Q^{(2)i}{}_j+\tau_G^{(-2)}Q^{(-2)i}{}_j], \tag{21}$$

where the τ_{IV} and $\tau_{IT}^{\pm1}$ terms are the "induced-vector" and "induced-tensor" modes associated with $Q_i^{(0)}$ and $Q_{ij}^{(\pm1)}$ auxiliary modes. The quantities τ_S, $\tau_V^{(\pm1)}$, τ_L, τ_T and $\tau_G^{(\pm2)}$ are defined as

$$\kappa\tau_S=-2k^2\left[h_L+\left(\frac{1}{3}-\frac{K}{k^2}\right)h_T\right]=-\kappa a^2[\rho_f\delta_f+\rho_s]+6\mathcal{H}\dot{h}_L-6Kh_L,$$

$$\kappa\tau_V^{(\pm1)}=-\frac{1}{2}k\dot{h}_V^{(\pm1)}\left(1-\frac{2K}{k^2}\right)=\kappa a^2[(\rho_f+p_f)v_f^{(\pm1)}+v_s^{(\pm1)}],$$

$$\kappa\tau_L=\left[\left(K-\frac{k^2}{3}\right)h_L-\ddot{h}_L-\frac{k^2}{3}\left(\frac{1}{3}-\frac{K}{k^2}\right)h_T\right]=\frac{1}{2}\kappa a^2[\delta p_f^{(0)}+p_s^{(0)}]-\mathcal{H}\dot{h}_L,$$

$$\kappa\tau_T=\frac{1}{2}\left[\ddot{h}_T-\frac{k^2}{3}h_T-k^2h_L\right]=\frac{1}{2}a^2[p_f\Pi_f^{(0)}+\Pi_s^{(0)}]-\mathcal{H}\dot{h}_T,$$

$$\kappa\tau_G^{(\pm2)}=[(2K+k^2)h_G^{(\pm2)}+\ddot{h}_G^{(\pm2)}]=\kappa a^2[p_f\Pi_f^{(\pm2)}+\Pi_s^{(\pm2)}]-2\mathcal{H}\dot{h}_G^{(\pm2)}. \tag{22}$$

In the second equality for each of the above equations we have made use of decompositions similar to Eq. (21) for the fluid $\delta T^\mu{}_\nu$ (subscript f) and source $\Theta^\mu{}_\nu$ (subscript s) terms, so as to write the pseudotensor in terms of these variables [32]. Equations (16), (17) yield four equations for the remaining four variables:

$$\tau_{IV}=\frac{\dot{\tau}_S}{k},\quad k\dot{\tau}_{IV}=2k^2\left[\tau_L+2\left(\frac{K}{k^2}-\frac{1}{3}\right)\tau_T\right]=\ddot{\tau}_S,$$

$$\tau_{IT}^{(\pm1)}=-2k[k^2-2K]^{-1}\dot{\tau}_V^{(\pm1)}=\frac{\ddot{h}_V^{(\pm1)}}{\kappa},\tag{23}$$

where we have used the first equation to obtain the final equality in the second. We observe that we have six independent quantities: τ_S and one of τ_L, τ_T for the scalars, $\tau_V^{(\pm1)}$ for the vectors, and $\tau_G^{(\pm2)}$ for the tensors. (Note that τ_L is defined as the spatial trace: $6\tau_L Q^{(0)}=\tau^i{}_i$.)

Finally, we comment on the relation of our pseudotensor (22) to an alternative definition given by Uzan *et al.* in Ref. [27]. For perturbations over a curved ($K\neq 0$) FRW universe, there exist several possible (*ad hoc*) generalizations of the Landau-Lifshitz pseudotensor, depending upon the manner in which one removes the residual spatial gauge freedom present in the synchronous gauge (see later in Sec. IV). In Ref. [27] matching conditions were used (as an interesting aside) to define the $\tau_{0\mu}$ components of the pseudotensor as

$$\kappa\tau_{00}^{UDT}\equiv\sqrt{\gamma}[\kappa\delta T_{00}+\kappa\Theta_{00}+Kh^- -H\dot{h}],$$

$$\kappa\tau_{0k}^{UDT}\equiv\sqrt{\gamma}[\kappa\delta T_{0k}-2K\partial_k\dot{E}],\tag{24}$$

where $\partial_0\tau_{00}=\partial_k\tau_{0k}$, and h, E, h^- correspond to the formalism of this paper as: $h=6\int\mu(\beta)h_L Q^{(0)}$, $-\Delta E=\int\mu(\beta)h_T Q^{(0)}$, and $h^-=h-2\Delta E$, $\Delta=D^iD_i$.

Apart from providing a prescription for all the components of the energy-momentum pseudotensor (and in a more elegant decomposition), our definition (22) extends and improves upon that proposed in Ref. [27] on two counts. First, Eq. (24) was only given a geometrical interpretation on superhorizon scales. The Kh term in the (redefined) $\delta\tau^0{}_0$ component in Eq. (15) replaces a Kh^- term in their definition (24), where their variable $h^-=h-h^s$ is the sum $6(h_L+h_T/3)$. The two definitions agree in the superhorizon limit, in which case $h\sim h^-$, but our definition (22) and its physical interpretation are also valid on subhorizon scales.

Second, there are the limitations inherent in the manner in which the pseudotensor is defined in Ref. [27]: Unlike Eq. (12) the perturbed and background parts of the pseudotensor are not distinguished. Moreover, their quantity τ_{0i} is defined via a conservation equation, so that the pure divergenceless part $\tau_V^{(\pm1)}$, removed by the derivative in Eq. (16) is not specified. We shall show (in Sec. IV) that this part can be recovered as a vector quantity to be matched across the transition. Finally, the definition of $\tau_{00}=-a^2\tau^0{}_0$ in Eq. (24) and Ref. [27] differs by a factor $\sqrt{-g}=a^4\sqrt{\gamma}$ from our $\tau^0{}_0$, so that it is related (on superhorizon scales only) to the one conserved current $\hat{I}^\mu_{\mathbf{P}_0}$, whereas all components of our pseudotensor (12) can be related to the four conserved currents I^μ_ξ defined in the next section (Sec. III).

D. Relation to the superhorizon growing modes

The pseudo-energy $\delta\tau^0{}_0$ (or τ_S) obtained in this section may be simply related to the coefficient of the superhorizon growing modes for the cold dark matter (CDM) density perturbation δ_c in the radiation- and matter-dominated eras, as well as in the curvature-dominated epoch. Assuming adiabatic perturbations and ignoring the source terms in a two fluid radiation plus CDM model, it is well known that the CDM density perturbation obeys the equations:

$$\ddot{\delta}_c+\mathcal{H}\dot{\delta}_c-4[\mathcal{H}^2+K]\delta_c=0,\quad \Omega_r=1,\Omega_c=0,$$

$$\ddot{\delta}_c+\mathcal{H}\dot{\delta}_c-\frac{3}{2}[\mathcal{H}^2+K]\delta_c=0,\quad \Omega_c=1,\Omega_r=0.$$

In both the radiation and matter eras, there exists a superhorizon growing mode proportional to τ^2, while in the curvature-dominated regime, this becomes a constant term. If we let the coefficient of this mode be A, then we find that $\kappa\tau_S\approx-8A$ in the radiation era, $\kappa\tau_S\approx-20A$ in the matter era, and $\kappa\tau_S\approx 2KA$ in the curvature regime. Thus, our generalized pseudo-energy essentially tracks the growing mode of the density perturbation. This is a useful property for numerical simulations (as discussed for example in [5]), since we can replace $\dot{\delta}_c$ with τ_S, thus avoiding the possibility of spurious growing modes sourced by numerical errors. We shall further investigate the inclusion of the pseudotensor in numerical evolution schemes elsewhere [34].

III. FRW CONFORMAL GEOMETRY AND CONSERVED CURRENTS

The energy, momentum and their conservation laws for one spacetime may be defined with respect to another manifold in an inherently local manner [31]. In the context of perturbation theory, we already have a background, and it seems logical to employ this approach. However, this is not a very compact form of expressing the desired conservation laws which, unlike the pseudotensor of the previous section, are not phrased in terms of the spatial covariant derivative with respect to γ_{ij}, making it incompatible with the decomposition of perturbation quantities with respect to eigenfunctions of the Laplacian. Here we shall calculate the conserved vector densities for the conformal geometry of a general FRW spacetime, and relate these to our pseudotensor, giving it a geometrical meaning that is valid on all scales and demonstrating the equivalence of the two formalisms.

A. Conserved currents with respect to a FRW background

The longstanding problem of defining energy, momentum and angular momentum for general relativistic perturbations has been considered by Katz *et al.* [31]. They provide a general formalism by which one can define, for an arbitrary spacetime $(M,g_{\mu\nu})$ containing perturbations and any vector ξ, conserved vector densities $\hat{I}^\mu(\xi)$ with respect to a background $(\overline{M},\overline{g}_{\mu\nu})$ and a mapping between M and $\overline{M}$. Here, and hereafter, a caret shall denote multiplication by $\sqrt{-g}$

$=a^4\sqrt{\gamma}$ where $g=\det g_{\mu\nu}$ and $\gamma=\det\gamma_{\mu\nu}$. Although one may use any vector ξ, it is useful to choose ξ as the conformal Killing vectors of the background spacetime, so as to exploit its symmetry properties.

In general, the choice of a particular background is free. However, it makes sense to either choose simple backgrounds possessing maximal symmetry or to choose as a background one that is already commonly in use in cosmology such as an unperturbed FRW spacetime. Conceptually, one might desire a background possessing a maximal Killing geometry (spanned by 10 linearly independent Killing vectors), so as to immediately generate Noether conserved quantities and currents. Of the general FRW spacetimes, only de Sitter spacetime has this property. The implications of the de Sitter Killing geometry have been investigated by several authors [24] and it allows for a clear relation to Traschen's integral constraints [25]. We shall demonstrate that the choice of a FRW background is not only quite tractable (despite the complications introduced by the use of a conformal rather than pure Killing geometry), but also allows for a clear relation between the conserved vectors $\hat{I}^\mu$ and our $\delta\tau_{\mu\nu}$, which is valid on both sub- and superhorizon scales.

The details of the construction of the conserved vector densities $\hat{I}^\mu$ associated with a conformal Killing vector ξ shall be omitted. The general formalism is given in [31], and the details of the construction of a relation between the 00-component of a pseudo-energy momentum tensor and the conserved vector density associated with just the conformal Killing vector normal to a constant time hypersurface may be found in [27]. We shall simply quote those results required for the current analysis: for each conformal Killing vector ξ we may define a vector density $\hat{I}^\mu(\xi)=\sqrt{-g}I^\mu(\xi)$ by

$$\kappa I^\mu(\xi)=\delta G^\mu{}_\nu\xi^\nu+A^\mu{}_\nu\xi^\nu+\kappa\zeta^\mu \tag{25}$$

and

$$A^\mu{}_\nu\xi^\nu=\frac{1}{2}(\bar{R}^\mu{}_\nu\delta^\sigma{}_\rho-\bar{R}^\sigma{}_\rho\delta^\mu{}_\nu)h^\rho{}_\sigma\xi^\nu$$
$$=\frac{h}{a^2}[\dot{\mathcal{H}}-\mathcal{H}^2-K]\xi^0\delta^\mu{}_0 \tag{26}$$

$$8\kappa a^2\zeta^\mu=(h\bar{g}^{\mu\rho}-h^{\mu\rho})Z_{,\rho}-ZD_\rho(h\bar{g}^{\mu\rho}-h^{\mu\rho}) \tag{27}$$

where we have substituted for the background terms in Eq. (26), $Z=\bar{g}^{\mu\nu}Z_{\mu\nu}$, and $Z_{\mu\nu}=\mathcal{L}_\xi\bar{g}_{\mu\nu}=2\psi\bar{g}_{\mu\nu}$. Here, $\mathcal{L}$ denotes the Lie derivative, and ψ is the conformal factor for ξ with respect to $\bar{g}_{\mu\nu}$, so that $\zeta^\mu=0$ for ξ Killing. The vector density so constructed will satisfy:

$$\hat{I}^\mu_{\xi\ ,\mu}=0\Leftrightarrow I^\mu_{\xi\ ;\mu}=I^0_{\xi\ ,0}+I^k_{\xi\,|k}+4\mathcal{H}I^0_\xi=0 \tag{28}$$

where we have used the result $V^\mu{}_{;\mu}=(\sqrt{-g}V^\mu)_{,\mu}/\sqrt{-g}$ for an arbitrary vector $\mathbf{V}$ in the first equality of Eq. (28), and the FRW connection coefficients in the last. Here, and elsewhere, we have used the subscript ξ to denote that the conserved vector so labeled is generated by the vector ξ.

B. Relations between the $\delta\tau^\mu{}_\nu$ and the $\hat{I}^\mu$

Any conformally flat spacetime will admit a maximal conformal Lie algebra spanned by 15 linearly independent conformal Killing vectors. For the general FRW metric, these were obtained in Ref. [35] in the coordinates (τ,x,y,z). In principle, each of the 15 vectors will generate a conserved vector with four components, and one conservation equation, yielding at least 45 components. Given that the symmetric pseudotensor has only 10 linearly independent components, of which 4 are removed by the Bianchi equations (16) and (17), there is clearly a considerable redundancy in the information contained in the set of all the vector densities obtained using the FRW conformal geometry. Since we wish to relate these conserved currents to our pseudotensor, our choice of vectors is guided by the desire to keep ξ simple (so that $\delta G^\mu{}_\nu\xi^\nu$ may be simply related to the $\delta\tau_{\mu\nu}$), and for the vectors to pick out different components $\tau_{\mu\nu}$. We shall therefore be particularly concerned with: the conformal Killing vector $\mathbf{P}_0$ normal to constant time hypersurfaces with conformal factor $\psi_{P_0}=\mathcal{H}$; the angular Killing vectors $\mathbf{M}_{12}$ and $\mathbf{M}_{23}$; and the generalized isotropic conformal Killing vector $\mathbf{H}$ which has the conformal factor $\psi_H=\cos_K\chi[\mathcal{H}n(\tau)+n'(\tau)]$. In the coordinates (τ,χ,θ,ϕ), these vectors have components:

$$P_0^\mu=(1,0,0,0),\quad M_{12}^\mu=(0,0,0,1),$$
$$M_{23}^\mu=(0,0,-\sin\phi,-\cot\theta\cos\phi),$$
$$H^\mu=[\cos_K\chi\, n(\tau),\sin_K\chi\, n'(\tau),0,0]. \tag{29}$$

Here, $\sin_K\chi$ is defined in Eq. (3), while $\cos_K\chi=\{\cosh\chi,1,\cos\chi\}$; and $n(\tau)=\{\cosh\tau,\tau,\cos\tau\}$ for $K<0$, $K=0$ and $K>0$, respectively.

These conformal vectors reduce to Killing vectors under special conditions on the scale factor: for a flat $K=0$ FRW spacetime, the vector $\mathbf{P}_0$ is Killing if $a(t)=C$ where C is some constant so that we have the stationary Einstein spacetime; and $\mathbf{H}$ is Killing if $a(t)=C\exp(-t/C)$ so that we have a de Sitter background. In the case of the $K\neq0$ spacetimes, $\mathbf{P}_0$ is Killing if $a(t)=C$; and $\mathbf{H}$ is Killing if $a(t)=C/h(\tau)$, where $h(\tau)=\{\cos\tau,\cosh\tau\}$ for $K=\{-1,1\}$, respectively.

Using Eqs. (29) in (25) we obtain the following conserved vector densities which relate directly to our pseudotensor $\delta\tau^\mu{}_\nu$ given in Eq. (12):

$$\kappa\hat{I}^0_{\mathbf{P}_0}=a^2\sqrt{\gamma}\,\kappa\delta\tau^0{}_0,$$
$$\kappa\hat{I}^k_{\mathbf{P}_0}=a^2\sqrt{\gamma}[\kappa\delta\tau^k{}_0+\mathcal{H}(h^{kl}{}_{|l}-h^{|k})], \tag{30}$$

$$\kappa\hat{I}^0_{\mathbf{M}_{12}}=a^2\sqrt{\gamma}\,\kappa\delta\tau^0{}_3,$$
$$\kappa\hat{I}^k_{\mathbf{M}_{12}}=a^2\sqrt{\gamma}[\kappa\delta\tau^k{}_3+\mathcal{H}(\dot{h}^k{}_3-\delta^k{}_3\dot{h})], \tag{31}$$

$$\kappa \hat{I}^0_{\mathbf{M}_{23}} = a^2\sqrt{\gamma}[-\sin\phi\,\kappa\delta\tau^0{}_2 - \cot\theta\cos\phi\,\kappa\delta\tau^0{}_3],$$

$$\kappa \hat{I}^k_{\mathbf{M}_{23}} = a^2\sqrt{\gamma}[-\sin\phi\,\kappa\delta\tau^k{}_2 - \cot\theta\cos\phi\,\kappa\delta\tau^k{}_3 - \sin\phi\mathcal{H}(\dot{h}^k{}_2 - \delta^k{}_2\dot{h}) - \cot\theta\cos\phi(\dot{h}^k{}_3 - \delta^k{}_3\dot{h})], \tag{32}$$

valid for all FRW spacetimes, as well as

$$\kappa\hat{I}^0_{\mathbf{H}} = \begin{cases} a^2\sqrt{\gamma}[\kappa(\delta\tau^0{}_0 + Kh)\cosh\tau\cosh\chi + \kappa\delta\tau^0{}_1\sinh\tau\sinh\chi + \dot{h}\sinh\tau\cosh\chi], & K<0,\\ a^2\sqrt{\gamma}[\kappa\delta\tau^0{}_0\tau + \delta\tau^0{}_1 r + \dot{h}], & K=0,\\ a^2\sqrt{\gamma}[\kappa(\delta\tau^0{}_0 + Kh)\cos\tau\cos\chi - \kappa\delta\tau^0{}_1\sin\tau\sin\chi - \dot{h}\sin\tau\cos\chi], & K>0, \end{cases}$$

$$\kappa\hat{I}^k_{\mathbf{H}} = \begin{cases} a^2\sqrt{\gamma}[\kappa\delta\tau^k{}_0\cosh\chi\cosh\tau + \kappa\delta\tau^k{}_1\sinh\chi\sinh\tau + \mathcal{H}\sinh\chi\sinh\tau(\dot{h}^k{}_1 - \delta^k{}_1\dot{h}) \\ \quad + ((h\gamma^{k1} - h^{k1})\sinh\chi + (h^{kl}{}_{|l} - h^{|k})\cosh\chi)(\mathcal{H}\cosh\tau + \sinh\tau)], & K<0,\\ a^2\sqrt{\gamma}[\kappa\delta\tau^k{}_0\tau + \kappa\delta\tau^k{}_1 r + \mathcal{H}r(\dot{h}^k{}_1 - \delta^k{}_1\dot{h}) + (1+\mathcal{H}\tau)(h^{kj}{}_{|j} - h^{|k})], & K=0,\\ a^2\sqrt{\gamma}[\kappa\delta\tau^k{}_0\cos\chi\cos\tau - \kappa\delta\tau^k{}_1\sin\chi\sin\tau - \mathcal{H}\sin\chi\sin\tau(\dot{h}^k{}_1 - \delta^k{}_1\dot{h}) \\ \quad + (-(h\gamma^{k1} - h^{k1})\sin\chi + (h^{kl}{}_{|l} - h^{|k})\cos\chi)(\mathcal{H}\cos\tau - \sin\tau)], & K>0, \end{cases} \tag{33}$$

where we have used Eq. (15) in Eq. (25) for each of $\mathbf{P}_0$, $\mathbf{M}_{12}$, $\mathbf{M}_{23}$, and $\mathbf{H}$.

C. Alternative derivation of $\delta\tau^\mu{}_\nu$ from the $\hat{I}_\xi$'s

The constraint equations (16) and (17) satisfied by the energy-momentum pseudotensor are encoded in the vector density equations (28). For $\xi = \mathbf{P}_0$, Eq. (28) yields Eq. (16); for $\xi = \mathbf{M}_{12}$ it yields Eq. (17) with $i=3$; for $\xi = \mathbf{M}_{23}$ it yields Eq. (17) for $i=2,3$ in the following linear combination:

$$-\sin\phi(\delta\tau^0{}_{2,0} + \delta\tau^k{}_{2|k}) - \cot\theta\cos\phi(\delta\tau^0{}_{3,0} + \delta\tau^k{}_{3|k}) = 0,$$

while for $\xi = \mathbf{H}$ we obtain Eqs. (16) and (17) for $i=1$, in the combination

$$\cosh\tau\cosh\chi[\tau^0{}_{0,0} + \tau^k{}_{0|k}] + \sinh\tau\sinh\chi[\tau^0{}_{1,0} + \tau^k{}_{1|k}] = 0,$$

for the $K<0$ case, and similarly for $K>0$.

Note that since $\hat{I}^\mu{}_{,\mu} = 0 \Leftrightarrow I^\mu{}_{;\mu} = 0$, the identification of the components of the perturbed part of the pseudotensor as being proportional to the components I^μ leads one to expect a conservation law of the form given in Eqs. (16) and (17). The presence of terms in Eqs. (30)–(33) other than the $\delta\tau^\mu{}_\nu$ accounts for the difference between the general covariant derivative on the FRW spacetime, on the one hand, and the spatial covariant derivative and temporal partial derivative, on the other. Hence, we see that given the I^μ_ξ for the conformal geometry $\{\xi\}$ of the background FRW spacetime, we could construct the perturbed pseudotensor directly using Eqs. (30)–(33) and the final equation of Eq. (28): the two formalisms are equivalent. The results of this section also demonstrate that the use of a FRW spacetime as the background manifold has the effect of removing the background energy and momentum: there do not appear any contributions from the $\overline{\tau}^\mu{}_\nu$ in the vector densities $\hat{I}^\mu_\xi$.

Approaching the $\delta\tau^\mu{}_\nu$ from this point of view also lends weight to the (apparently) *ad hoc* inclusion of the $K\dot{h}/\kappa$ term into the perturbed pseudo-energy $\delta\tau^0{}_0$ as defined in Eq. (15) because it is this redefined quantity that appears in the conserved (energy) vector density associated with the conformal Killing vector $\mathbf{P}_0$. This is not surprising, as the isometry described by the Killing vector $\mathbf{P}_0$ in the spacetime $(M,\tilde{g})$ is not entirely lost as we go to the spacetime (M,g), where $\mathbf{P}_0$ is a conformal Killing vector. It is preserved in the evolution space $\mathcal{R}\times TM$—where $\mathcal{R}$ accounts for the affine parametrization of the geodesics, and TM is the tangent bundle—by the appearance of an irreducible Killing tensor $K^\mu{}_\nu = -a^2\delta^\mu{}_0\delta^0{}_\nu + a^2\delta^\mu{}_\nu$, related to the reducible Killing tensor $L^\mu{}_\nu = -\delta^\mu{}_0\delta^0{}_\nu + \delta^\mu{}_\nu$ in the $(M,\tilde{g})$ spacetime [36]. As this last tensor is reducible (a sum of products of the Killing vector $\mathbf{P}_0$ and the metric, with constant coefficients), it encodes the same information as the Killing vector itself. Thus, we may expect there to be an ''energy isometry'' associated with the tensor $\delta^\mu{}_0\delta^0{}_\nu$, which we used in Sec. II B.

The vector densities of this section provide a consistent definition of energy and momentum with respect to a FRW background and, as we have just shown, the identification of the quantities $\delta\tau^\mu{}_\nu$ (including the curvature term in the 00-component) leads naturally to a concise and algebraically useful conservation law, phrased as a differential equation.

IV. ENERGY-MOMENTUM PSEUDOTENSORS AND MATCHING CONDITIONS

We wish to consider the emergence of a topological defect network (or other causal sources) at some stage in cosmic history, that is, the time when defects ''switch on'' and are carved out of the background energy density during a phase transition. This process sets the initial conditions for all the perturbation variables prior to their sourced evolution, a state we must specify if we are to perform realistic numerical simulations. It is common to assume that any phase transition at which defects will appear will take less than one Hubble time, so it will be effectively ''instantaneous'' for all modes larger than the horizon at the time of the transition. Matching conditions have then been found to relate the resulting perturbation variables on superhorizon scales to their prior unperturbed state in a ''sourceless'' universe [24]. While this approach will apply in many physical situations, there are circumstances in which it may not, such as hybrid scenarios with mixed perturbation mechanisms or late-time phase transitions in which subhorizon modes might be important. Here, we have already defined a generalized energy-momentum pseudotensor applying to both sub- and superhorizon scales which should prove useful for this wider class of scenarios. We shall therefore ignore these difficulties so as to demonstrate consistency with existing analyses, and demonstrate, in an appropriate synchronous gauge, that its components can be used to specify the matching conditions valid for all length scales in a defect-forming transition.

A. Matching conditions on a constant energy density surface

If the phase transition appears instantaneous for a given mode, we need only to match the geometric and matter variables on the spacelike hypersurface surface Σ, described by the equation

$$\rho(x^{\mu})=\rho_0+\delta\rho=\text{const}, \tag{34}$$

where, up to a small perturbation, we have assumed homogeneity on either side of Σ [24]. Prior to the phase transition, the perfectly homogeneous and isotropic ''perturbation'' may always be absorbed into a redefinition of the (continuous) scale factor. In a simple model without surface layers [24] (i.e. ignoring the internal structure of the phase transition), the standard procedure used to match the geometric and matter variables is to insist that the induced 3-metric $\perp_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$ must be continuous over Σ. This task is simplified if, on either side of the phase transition, one uses the residual gauge freedom in the time coordinate $\tau \to \tilde{\tau}=\tau+T$, with T a nontrivial first order scalar function of the coordinates, to transform to a coordinate system in which Σ is defined by the equation $\tilde{\tau}=\text{const}$ ($\tilde{\rho}=\text{const}$), and $\widetilde{\delta\rho}=\delta\rho+\dot{\rho}_0 T=0$. Using the Friedman equations the appropriate transformation is therefore specified by

$$T=-\frac{\delta\rho}{\dot{\rho}_0}=\frac{\kappa a^2[\rho\delta+\rho^s]}{9\mathcal{H}(\mathcal{H}^2+K)(1+\omega)}, \tag{35}$$

which may be interpreted (at each point in 3-space) as moving the time-slicing forward/backward so that the surface Σ is a constant time hypersurface. Here $p=\omega\rho$ is the equation of state for the total fluid, but for the purposes of this paper, we may assume that we are in the radiation dominated epoch.

In setting up this gauge, no use is made of the residual scalar freedom, $x^k \to \tilde{x}^k=x^k+D^kL$, in the spatial coordinates. Here $D^kL=\partial^k L$ because L is another first order scalar function of the coordinates. Note that this new gauge cannot be comoving, as this would require that $T=0$, and we need this freedom to force the constant time and constant energy density surfaces to coincide.

B. Matching the scalar modes

In the gauge described above—denoted by a tilde—the metric is given by: $\tilde{g}_{\mu\nu}=a^2(\tilde{\tau})[\gamma_{\mu\nu}+\tilde{h}_{\mu\nu}]$ where we shall rewrite the spatial metric perturbation as

$$\tilde{h}_{ij}=2\tilde{\mathrm{h}}_L\gamma_{ij}+2\left(D_iD_j-\frac{1}{3}\gamma_{ij}\Delta\right)\tilde{\mathrm{h}}_T, \tag{36}$$

where $\mathrm{h}_L(\tau,x^k)=\int \mathrm{d}\mu(\beta)h_L(\tau,\beta)\mathbf{Q}^{(0)}(\tau,x^k,\beta)$ and similarly for $\mathrm{h}_T(\tau,x^k)$. These spatially dependent variables are used as the physical interpretation of the transformation is more transparent, and they facilitate comparisons to existing work [24,27]. We shall obtain results for the β dependent quantities later.

The gauge transformed quantities are given by

$$\tilde{h}_{00}=h_{00}+-2(\dot{T}+\mathcal{H}T), \quad \tilde{h}_{0i}=h_{0i}+\dot{L}_{|i}-T_{|i},$$

$$\tilde{\mathrm{h}}_L=\mathrm{h}_L+\mathcal{H}T+\frac{1}{3}\Delta L, \quad \tilde{\mathrm{h}}_T=\mathrm{h}_T+L. \tag{37}$$

Preservation of synchronicity ($\tilde{h}_{00}=0=\tilde{h}_{0i}$) thus provides the form of T and L:

$$T=\frac{f(x^k)}{a}, \quad L=g(x^k)+f(x^k)\int\frac{\mathrm{d}\tau}{a}. \tag{38}$$

where f, g are functions of the spatial coordinates only. As noted previously, f is completely determined by the process of establishing a time-slicing that also has constant energy density (at the phase transition). However, g is completely free, and may be chosen in such a manner as to simplify equations [27]. We shall demonstrate that this freedom may be more profitably used to specify gauges (for both $K=0$ and $K\neq 0$) in which the energy-momentum pseudotensor of Sec. II must be continuous across the phase transition.

The vector orthonormal to the constant time hypersurface is given by $n_{\mu}=(-a,0,0,0)$, so that the perturbed parts of the induced metric $\perp_{\mu\nu}$ and extrinsic curvature $K^{\mu}{}_{\nu}$ are

$$\delta\perp_{ij}=a^2\tilde{h}_{ij}, \quad \delta\perp_{\mu 0}=0,$$

$$\delta K^{\mu}{}_{0}=0=\delta K^{0}{}_{\mu}, \quad \delta K^{i}{}_{j}=-\frac{1}{2a}\tilde{\gamma}^{ik}\dot{\tilde{h}}_{kj}, \tag{39}$$

where we use $a(\tilde{\tau})\approx a(\tau)[1+\mathcal{H}T]$, obtained by Taylor expanding about τ. Assuming that the background is continuous across the phase transition, we need only match the perturbed parts; i.e. we insist that $[\delta\perp_{ij}]_{\pm}=0=[\delta K^{i}{}_{j}]_{\pm}$, where $[F]_{\pm}$ denotes the limit $\lim_{\epsilon\to 0^+}[F(\tau_{PT}+\epsilon)-F(\tau_{PT}-\epsilon)]$.

Substituting Eq. (36), transforming back to the original gauge and using Eq. (38) we find that

$$\left[\mathrm{h}_L+\frac{\mathcal{H}f}{a}+\frac{1}{3}\Delta g+\frac{1}{3}\Delta f\int\frac{\mathrm{d}\tau}{a}\right]_{\pm}=0, \quad (40)$$

$$\left[\dot{\mathrm{h}}_L+\frac{f}{a}\left(\frac{-3(\mathcal{H}^2+K)(1+\omega)}{2}+K\right)+\frac{1}{3}\frac{\Delta f}{a}\right]_{\pm}=0, \quad (41)$$

$$\left[\left(D_iD_j-\frac{1}{3}\gamma_{ij}\Delta\right)\left(\mathrm{h}_T+g+f\int\frac{\mathrm{d}\tau}{a}\right)\right]_{\pm}=0, \quad (42)$$

$$\left[\left(D_iD_j-\frac{1}{3}\gamma_{ij}\Delta\right)\left(\dot{\mathrm{h}}_T+\frac{f}{a}\right)\right]_{\pm}=0. \quad (43)$$

Taking the linear combination $-6\mathcal{H}\times$ (41) $+6K\times$ (40) we have

$$\left[\tau_S+2K\left(\Delta g+\Delta f\int\frac{\mathrm{d}\tau}{a}\right)-2\mathcal{H}\frac{\Delta f}{a}\right]_{\pm}=0, \quad (44)$$

where we have used Eqs. (35) and (38).

Decomposing with respect to the Helmholtz equation, and noting that the eigenfunctions separate and are time independent, we obtain

$$\left[h_L(\beta)+\frac{\mathcal{H}f(\beta)}{a}-\frac{k^2}{3}g(\beta)-\frac{k^2}{3}f(\beta)\int\frac{\mathrm{d}\tau}{a}\right]_{\pm}=0, \quad (45)$$

$$\left[\tau_S(\beta)+2K\left(-k^2g(\beta)-k^2f(\beta)\int\frac{\mathrm{d}\tau}{a}\right)+2\mathcal{H}k^2\frac{f}{a}\right]_{\pm}=0, \quad (46)$$

$$\left[h_T(\beta)+g(\beta)+f(\beta)\int\frac{\mathrm{d}\tau}{a}\right]_{\pm}=0, \quad (47)$$

$$\left[\dot{h}_T(\beta)+\frac{f(\beta)}{a}\right]_{\pm}=0, \quad (48)$$

where we have replaced Eq. (41) by Eq. (44).

There exists an entire class of objects related by gauge transformations to the "pseudo-energy" $\delta\tau^0{}_0$ corresponding to different choices for $g(x^k)$ in Eq. (44). Uzan *et al.* [27] make use of this freedom to specify

$$g=-h_T-f\int\frac{\mathrm{d}\tau}{a}$$

which eliminates the matching condition (42) and yields $[\tau_{00}^{UDT}]_{\pm}=0$, refer to Eq. (24). On superhorizon scales this reduces to a matching on our pseudo-energy: $[\tau_S]_{\pm}=0$. However, one is not using the gauge freedom to relate the matching condition to well-defined geometrical objects. It would be both more aesthetically appealing and more useful if one could employ this freedom to make $\delta\tau^0{}_0$ continuous across the transition. This is a subtle issue that shall be more fully explored elsewhere [37], where we discuss initial conditions and their consistency with causality. For now, we merely note that, for practical purposes in which we wish to describe the onset of defect induced perturbations carved out of the background (or inflationary) fluid, compensation between the fluid and the source densities implies that we can usually take f to be continuous across the transition. In the absence of primordial density perturbations, it will moreover initially vanish—see Eq. (35). In this physical context, we may then completely specify the gauge by choosing $[g]_{\pm}=0$ so that we obtain:

$$[h_L(\beta)]_{\pm}=0, \quad [\tau_S(\beta)]_{\pm}=0,$$
$$[h_T(\beta)]_{\pm}=0, \quad [\dot{h}_T(\beta)]_{\pm}=0. \quad (49)$$

C. Matching the vector and tensor modes

The residual gauge freedom in the vector modes may be expressed as invariance under the infinitesimal coordinate transformation $x^i\rightarrow\tilde{x}^i=x^i+L^i$, where $\mathbf{L}(\tau,x^k)$ is a divergenceless 3-vector: $D_iL^i=0$. Writing $\tilde{\mathrm{h}}_{ij}=2(\mathrm{h}^{(1)}_{V(i|j)}+\mathrm{h}^{(1)}_{V(i|j)})$ for the spatial metric perturbation, where $\mathrm{h}^{(\pm1)}_{V\,i}(\tau,x^k)=\int\mathrm{d}\mu(\beta)h_V^{(\pm1)}(\tau,\beta)Q_i^{(\pm1)}(\beta,x^k)$, the gauge transformed vector quantities are

$$\tilde{\mathrm{h}}_{0i}=\mathrm{h}_{0i}+\dot{L}_i, \quad \tilde{\mathrm{h}}_{V\,i}=\mathrm{h}_{V\,i}+L_i. \quad (50)$$

Preservation of synchronicity implies that $\dot{L}_i=0$ everywhere, so that $\mathbf{L}$ is a function of the spatial coordinates only. Proceeding as for the scalar perturbations we match the induced metric and extrinsic curvature.

After transforming back to the original gauge, and exploiting the fact that $\dot{L}_j=0$ everywhere so that $D^{(i}\dot{L}_{j)}=0$ is certainly true on the hypersurface, we obtain

$$[D_{(i}\mathrm{h}^{(1)}_{V\,j)}+D_{(i}\mathrm{h}^{(-1)}_{V\,j)}+D_{(i}L_{j)}]_{\pm}=0, \quad (51)$$

$$[D^{(i}\dot{\mathrm{h}}^{(1)}_{V\,j)}+D^{(i}\dot{\mathrm{h}}^{(-1)}_{V\,j)}]_{\pm}=0. \quad (52)$$

Helmholtz decomposing and assuming that β modes separate, Eq. (52) is equivalent to

$$[\dot{h}_V^{(\pm1)}(\beta)]_{\pm}=0, \quad (53)$$

as the $m=\pm1$ contributions are linearly independent. Hence, we find that

$$[\tau_V^{(\pm1)}(\beta)]_{\pm}=0. \quad (54)$$

This is precisely the divergenceless part of $\delta\tau^0{}_i$ which is not obtainable by integrating the conservation equation (23), unlike the induced vector mode $\tau_{IV}^{(0)}$ (constructed from scalars). Using Eq. (22) we see that the matching condition (54) implies a "compensation" between the source and fluid vorticities. We shall investigate this phenomenon further in the context of establishing consistent initial conditions in Ref. [37]. The remaining equation (51) may be written as $[h_V^{(\pm1)}(\beta)]_{\pm}=0$ by means of an appropriate specification $(L_i=0)$ of the residual gauge freedom in the vector mode.

For the gauge invariant tensors, we find that the (Helmholtz decomposed) tensor metric quantities are constrained to be continuous across the transition:

$$[h_G^{(\pm 2)}]_\pm = 0 = [\dot{h}_G^{(\pm 2)}]_\pm . \tag{55}$$

There is, however, no residual freedom in the tensor modes, so these matching conditions cannot completely constrain the continuity properties of the pure tensor contribution $\tau_G^{(\pm 2)}$. It may also be permissible to insist that $\tau_G^{(\pm 2)} = 0$ initially, although this is not mandated by our results.

V. DISCUSSION

In this paper we have considered definitions and conservation laws for quantities that may be used to define energy, momentum and the stresses, which are of relevance to setting the initial conditions for and/or constraining the evolution of numerical simulations. To this end, we have constructed an energy-momentum pseudotensor for FRW cosmologies with nonzero curvature and have generated conserved vector densities using the conformal geometry of a general FRW background manifold. We showed that these two formalisms are equivalent so that the pseudotensor components are geometrically well-defined objects on all scales. These results hold in the presence of a nonzero cosmological constant, as all the quantities discussed here are purely geometrical constructs, describing the symmetry properties of the FRW spacetime. This pseudotensor is likely to be a useful tool for detailed investigations of causal models in curved FRW universes, and also for hybrid models in which one has nontrivial contributions from both defects and inflation, with both phenomena coexisting. However, while the conservation laws encoded in the pseudotensor $\tau^\mu{}_\nu$ will remain valid in hybrid models, the matching analysis will have to be modified to incorporate preexisting perturbations (which we shall discuss elsewhere). We have phrased our results in terms of the commonly employed Helmholtz decomposition with respect to the eigenfunctions of the Laplacian.

We considered an instantaneous phase transition early in the universe as a first approximation to a model for the defects ''switching on,'' and employed a gauge in which constant energy and constant time surfaces coincide. Matching conditions then imply that there exists an entire class of objects which are continuous across the transition and are related by gauge transformation to our pseudotensor components. The notion of compensation together with a particular gauge specification removes this redundancy such that the τ_S (pseudo-energy) and $\tau_V^{(\pm 1)}$ (divergenceless vector) components of our generalized pseudotensor have this property. For a universe which was unperturbed (and hence homogeneous and isotropic) prior to the transition, we may then take $\tau_S = 0 = \tau_V^{(\pm 1)}$ as natural initial conditions. This result is true on all scales. In a subsequent paper [37], we shall establish with more rigor the effect of causality on the superhorizon behavior of the energy and momentum in general FRW cosmologies, as well as the implications for setting the initial conditions.

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