

Noncommutative inflation

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We show how a radiation dominated universe subject to space-time quantization may give rise to inflation as the radiation temperature exceeds the Planck temperature. We consider dispersion relations with a maximal momentum (i.e., a minimum Compton wavelength, or quantum of space), noting that some of these lead to a trans-Planckian branch where energy increases with *decreasing* momenta. This feature translates into negative radiation pressure and, in well-defined circumstances, into an inflationary equation of state. We thus realize the inflationary scenario without the aid of an inflaton field. As the radiation cools down below the Planck temperature, inflation gracefully exits into a standard big bang universe, dispensing with a period of reheating. Thermal fluctuations in the radiation bath will in this case generate curvature fluctuations on cosmological scales whose amplitude and spectrum can be tuned to agree with observations.

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I. INTRODUCTION

In spite of the success of the inflationary universe scenario [1] in solving some of the mysteries of standard cosmology and of providing a mechanism which explains the origin of density fluctuations on cosmological scales, a mechanism which to date has passed all of the observational challenges, we still do not have a convincing realization of inflation based on fundamental physics. Moreover, the usual realizations of inflation based on weakly coupled scalar matter fields (see e.g. [2,3] for comprehensive reviews) are plagued by important conceptual problems [4]. Thus, it is of great interest to explore possible realizations of inflation based on new fundamental physics. In particular, since inflation may occur at energy scales close to the Planck scale, it is of interest to consider the implications of the recent developments in our understanding of physics at the Planck scale for inflation.

Space-time noncommutativity is one of the key new ideas which follows from recent developments in string and matrix theory [5]. It is thus of great interest to explore the compatibility of noncommutative space-time structure with inflation (see [6,7] for ideas on how to solve some of the problems of standard cosmology without inflation in noncommutative geometry). Noncommutativity, and space-time quantization, in general lead to deformed dispersion relations (see e.g. [8]). It has been shown [9–11] (see also [12,13]) that this can have important consequences for the predictions of inflation. These authors demonstrated that the short distance cutoff given by modifying the usual commutation relations

$$[x, p] = i\hbar(1 + \beta p^2) \quad (1)$$

changes the perturbation spectrum due to quantum fluctua-

tions. Implicit in this approach is the necessity of an inflaton field generating a de Sitter phase.

In this Rapid Communication, we go one step further and identify dispersion relations for ordinary radiation which lead to inflation, without the need to introduce a new fundamental scalar field. Thermal fluctuations then replace quantum fluctuations as the seeds of cosmic structure, similar to what happens in the *warm inflation* scenario [14].

The dispersion relations derived from the noncommutative structure of space-time have the property that there is a maximum momentum (corresponding to a minimum Compton wavelength or quantum of space). Typically all trans-Planckian energies get mapped into this maximal momentum. However it is also possible to write down deformations for which trans-Planckian energies get mapped into all momenta smaller than this maximal momentum. In the latter case for a given momentum there are two energy levels, one sub-Planckian the other trans-Planckian. Along the trans-Planckian branch as one decreases the momentum of a particle its energy increases.

This unusual feature implies that as we expand a box with radiation thermally excited into the trans-Planckian branch, and thereby stretch the wavelength of all particles and decrease their momenta, their energies actually increase. Increased bulk energy as a result of expansion is the hallmark of negative pressure. We follow the thermodynamical calculation in detail, with a mixture of analytical (as developed in [7]) and numerical methods, to show that it is possible to generate an inflationary high energy equation of state for thermalized radiation subject to space-time noncommutativity. We identify a class of dispersion relations for which this occurs. We also find dispersion relations for which the equation of state corresponds to “phantom” matter [15].

II. MODIFIED DISPERSION RELATIONS, THERMODYNAMICS AND INFLATION

We start by recalling that noncommutativity leads to deformed dispersion relations, but whereas space-space noncommutativity must introduce anisotropic deformations, space-time noncommutativity preserves isotropy. Hence in the latter case, for massless particles, the dispersion relations may be written

$$E^2 - p^2 c^2 f^2 = 0 \quad (2)$$

where c is a constant reference speed, identified with the low-energy speed of light. We explore dispersion relations of the form

$$f = 1 + (\lambda E)^\alpha. \quad (3)$$

The case $\alpha=1$ was proposed in [16] and its implications considered in [7], and leads to a density dependent equation of state $w = p/\rho$ (p and ρ denoting pressure and energy density, respectively) which diverges like $\log(\lambda\rho)$. We also recall that for this model the color temperature (i.e., the peak of the thermal spectrum) saturates at $T_c \approx 1/\lambda$. In the case $\alpha \neq 1$, the high energy equation of state $w(\rho \rightarrow \infty)$ turns out to be a constant, and this leads to much simpler cosmological scenarios. Depending on α we obtain a realization of varying speed of light (VSL: [17,18]), inflation, or phantom matter [15]. We prove this feature by following the thermodynamical derivations described in [7].

As shown in [7] the deformed thermal spectrum is given by

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3} \frac{E^3}{e^{\beta E} - 1} \frac{1}{f^3} \left| 1 - \frac{f' E}{f} \right| \quad (4)$$

(note the modulus in the last factor, to be taken whenever the Jacobian of the transformation dE/dp is not positive definite). This leads to

$$\rho(E) = \frac{1}{\pi^2 \hbar^3 c^3} \frac{E^3}{e^{\beta E} - 1} \frac{|1 + (1 - \alpha)(\lambda E)^\alpha|}{(1 + (\lambda E)^\alpha)^4}. \quad (5)$$

We see that the peak of $\rho(E)$ scales like T for $\alpha < 2/3$, as illustrated in Fig. 1. For $\alpha > 2/3$ the peak saturates at $E = 1/\lambda$, but there is a wide tail up to $E = T$ for values in the range $2/3 < \alpha < 1$ (see Fig. 1). For $\alpha > 1$ the spectrum becomes double peaked, with peaks located at $\lambda E \sim 1$ for $\lambda T \gg 1$. The shape of the spectrum becomes temperature independent since $\rho(E)$ acquires the form of a temperature independent function of energy multiplied by T (see Fig. 1). Since the ambient speed of light is given by $c = dE/dp = c(E) = c(T_{peak})$ we see that only models with $\alpha < 1$ can be implemented as VSL models. For $\alpha > 1$ hotter radiation means more photons with the same maximal energy, and hence with the same speed. Only for $\alpha < 1$ does hotter radiation mean more energetic and faster photons, opening doors to VSL.

Next we examine whether or not denser radiation means hotter radiation. To answer this question we integrate $\rho(E)$

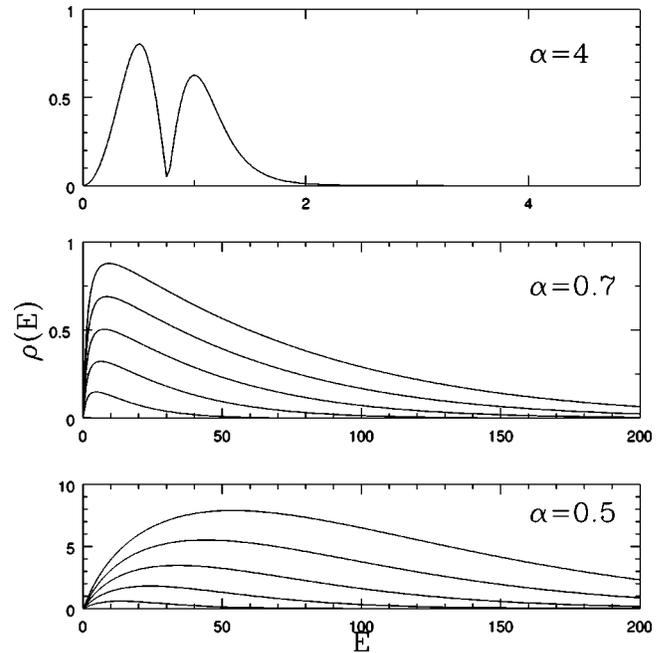


FIG. 1. The thermal spectrum for different temperatures for $\alpha = 0.5, 0.7, 4$. For $\alpha=0.5$, the peak of the spectrum scales like the temperature. For $\alpha=0.7$, although the peak does not vary much with the temperature, the tail of the spectrum extends to regions which scale with the temperature. For $\alpha=4$ we obtain a double peaked spectrum whose amplitude scales like T (we have divided all amplitudes by T for convenience), so that the shape does not depend on temperature.

to obtain a high-temperature Stephan-Boltzmann law relating ρ and T . We find a power-law of the form $\rho \propto T^\gamma$, with an asymptotic value for γ which varies from 4 (for $\alpha=0$) to 1 (for all $\alpha \geq 1$). The transition from low to high temperature behavior for different values of α is plotted in Fig. 2. The conclusion is that in all cases denser radiation corresponds to hotter radiation.

Finally the equation of state follows from (see [7])

$$p = \frac{1}{3} \int \frac{\rho(E) dE}{1 - \frac{f' E}{f}}. \quad (6)$$

Given that the denominator of the integrand is a constant at low and high energies, we may expect that the high energy equation of state is a constant approximated by

$$w(\rho \rightarrow \infty) \approx \frac{1}{3(1 - \alpha)}. \quad (7)$$

Of course, this formula assumes that the peak of $\rho(E)$ is located at super Planckian energies, where the denominator assumes its high energy constant value. This does not always happen (e.g. for $\alpha \geq 1$), so a numerical integration of Eq. (6) is necessary. We present the result in Fig. 3, where we also plot the approximation (7).

We see that in the regime where VSL may be realized ($\alpha < 1$) we do not have inflation, but for $\alpha > 1$ we have

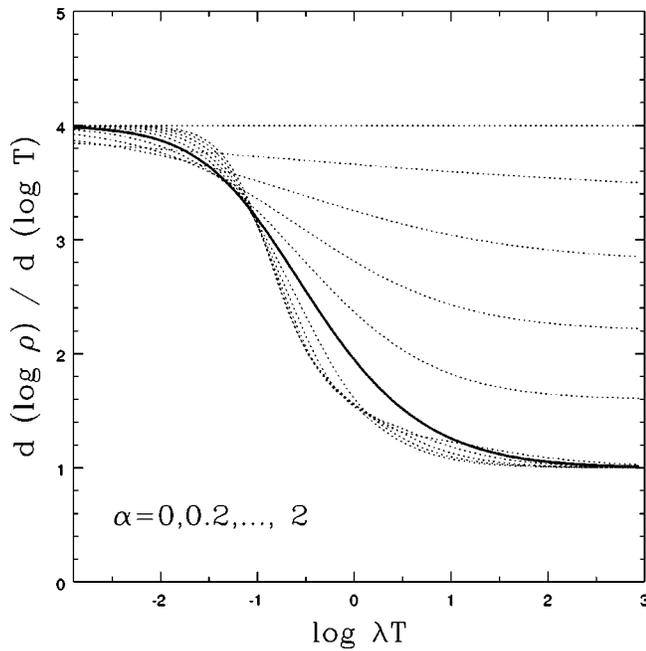


FIG. 2. The high-energy Stephan-Boltzmann law $\rho \propto T^\gamma$, for different values of α and different temperatures. The thick line denotes the case $\alpha=1$ studied in a previous work. We see that in all cases hotter radiation translates into denser radiation, although in general $1 < \gamma < 4$.

negative pressures. Numbers $1 < \alpha_1 < \alpha_2$ can be found such that for $1 < \alpha < \alpha_1$ noncommutative radiation at $T \gg 1/\lambda$ behaves like phantom matter ($w < -1$). For $\alpha_1 < \alpha < \alpha_2$ we have standard inflationary expansion, with $-1 < w < -1/3$, for temperatures $T \gg 1/\lambda$.

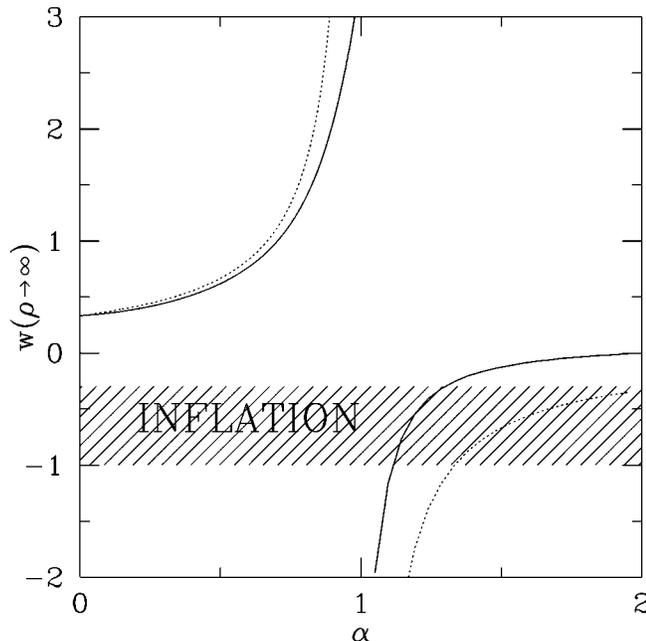


FIG. 3. The high energy equation of state for different values of α . We also plotted (dashed line) the approximation mentioned in the text. The shaded region delimits the values of α for which one may have inflationary expansion.

Hence when $\alpha_1 < \alpha < \alpha_2$, we have a scenario in which the Universe is always filled with radiation, but in the Planck epoch (or more precisely when $T \gg 1/\lambda$) radiation drives power-law inflation. As a result of expansion this inflationary radiation cools down, since $w < -1$ implies that ρ drops with expansion, and $\rho \propto T$ implies that T decreases too. When the radiation temperature drops below the Planck temperature its equation of state reverts to that of normal radiation. Then, the Universe enters a standard radiation dominated epoch. Our inflationary scenario does not have a graceful exit problem, and we have no need of a reheating period.¹

The critical case $\alpha = \alpha_1$, however, does not benefit from a graceful exit. It drives exponential inflation, but the radiation equation of state is that of a cosmological constant ($w = -1$). As a result ρ and T stay constant, and the Universe never exits the de Sitter phase to enter a radiation dominated phase.

The cases $1 < \alpha < \alpha_1$ are more complex and will be examined further elsewhere. For these models there is a critical ρ_c such that $w(\rho_c) = -1$; for $\rho < \rho_c$ we have $w > -1$ and for $\rho > \rho_c$ we have $w < -1$. If the Universe starts off trans-Planckian ($\rho > \rho_c$) and expanding, we have $a \propto (-t)^{2/3(1+w)}$, that is hyperinflation. However as the Universe expands it gets denser and hotter (since $w < -1$, and $\rho \propto T$), eventually reaching infinite density at $t=0$: perhaps this possibility may be used to realize the pre-big-bang scenario [19]. The only regular universe within this case has an infinite de Sitter past with $\rho = \rho_c$. However in this model de Sitter space is unstable. Any small nudge and it either plunges into an eternal Planck epoch with $\rho > \rho_c$ (with a possible pre-big-bang exit) or it decays into a standard radiation epoch.

To obtain a heuristic explanation for the origin of negative pressures for $\alpha > 1$, note that the pressure may be inferred from the change in the energy inside a box when its size is increased:

$$p = \sum_s n_s \left(\frac{-\partial E_s}{\partial V} \right) \quad (8)$$

where s labels states, n_s their occupation numbers, and $V = L^3$ the volume of a box of side L . The momenta are given by $\mathbf{p} = (2\pi\hbar/L)\mathbf{n}$, where \mathbf{n} is a triplet of quantum numbers indexing the state s . Hence as the volume increases the momenta of all states decreases, since their Compton wavelengths are stretched proportionally to L . Usually this translates into a decrease in the energy: hence the positive pressure of a gas. However for a high temperature gas living in noncommutative space with $\alpha > 1$ the energy of the dominant branch of the dispersion relation (the higher energy branch) is a decreasing function of the momentum. Hence a larger box is reflected in longer Compton wavelengths for

¹The viability of our scenario demands that the mode interactions in the expanding phase are sufficiently strong to maintain thermal equilibrium. In the absence of a fundamental theory which gives rise to our modified dispersion relation, a quantitative analysis of this issue is outside of the realm of this paper.

these states, and consequently smaller momenta, but now this implies a larger energy. Thus, by expanding a box of noncommutative radiation, the system gains energy, which corresponds to negative pressure.

What is the meaning of the parameters α and λ ? Using Eqs. (2) and (3) we can derive

$$p = \frac{E}{c(1+(\lambda E)^\alpha)} \quad (9)$$

from which we see that for $\alpha=1$ there is a maximum allowed momentum $p_{max}=1/(c\lambda)$. Its corresponding Compton wavelength is therefore the minimum length that can be physically probed, corresponding to the quantum of space. Hence λ is the parameter determining the size of the quantum of space. As $p \rightarrow p_{max}$, the energies $E(p)$ span all possible super Planckian energies all the way up to infinity, if $\alpha=1$. This changes dramatically if $\alpha>1$: then Eq. (9) also shows that there is a maximum momentum; however in this case super Planckian energies do not all get mapped into this momentum—rather we find that for all allowed momenta $p < p_{max}$ there are two energy levels, one sub-Planckian the other super-Planckian. The function $p(E)$ acquires two branches, along one of which p decreases with E . Finally the case $\alpha<1$ does not contain a sharp maximum momentum—merely a suppression of variation in momenta with energy for momenta above a given threshold.

III. COSMOLOGICAL FLUCTUATIONS

In our inflationary Universe scenario, it is thermal fluctuations which are responsible for generating the curvature fluctuations which develop into the observed perturbations on cosmological scales. A simple way to estimate the resulting spectrum (see [20] for a detailed analysis) is to assume fractional thermal density fluctuations of order unity on the thermal wavelength scale T^{-1} . Random superposition of these fluctuations leads to fractional mass fluctuations on Hubble radius scale H^{-1} measured at the time $t_i(k)$ that a particular wave number k crosses the Hubble scale during inflation

$$\frac{\delta M}{M}(t_i(k)) = A \left(\frac{H}{T} \right)^{3/2} \quad (10)$$

where A is a positive constant smaller than 1. It is convenient to express this result in terms of the physical scales λ and m_{pl} , and the number $N(k)$ of Hubble times between $t_i(k)$ and the end of inflation (which roughly occurs when $T=\lambda^{-1}$). Application of the Friedmann equations yields

$$\frac{\delta M}{M}(t_i(k)) = A(\lambda m_{pl})^{-3/2} e^{-3N/(2p)}, \quad (11)$$

where p is the power with which the scale factor $a(t)$ increases during the period of power law inflation.

In order to relate (10) with the fractional mass fluctuations when the scale re-enters the Hubble radius at time $t_f(k)$, we make use of the fact that fractional density fluctuations increase between $t_i(k)$ and $t_f(k)$ by a factor given by the ratio of $1+w$ at the respective times [21–24]. This factor is $2p$. In order to obtain a spectral slope consistent with the COBE data, the power p has to be sufficiently large. In this case, requiring that the amplitude of the fluctuations agree with the data requires λ^{-1} to be a couple of orders of magnitude smaller than m_{pl} , which from the point of view of string theory is not unreasonable.

IV. DISCUSSION AND CONCLUSIONS

In summary, noncommutative space-time geometry leads to modified dispersion relations. We have identified a class of dispersion relations which change the high-temperature equation of state of thermal relativistic matter into that of inflationary matter. In this scenario inflation does not require a different type of matter—standard radiation suitably heated up will behave like the proverbial inflaton field. As inflationary expansion proceeds, the radiation cools down until its equation of state reverts to that of ordinary radiation and consequently the Universe enters the standard hot-big-bang phase.

In the proposed scenario, thermal fluctuations in the radiation bath will generate the necessary density fluctuations to explain the structure of the universe. As discussed in detail in [20], the predicted spectrum is not exactly scale-invariant, as in power-law inflation. Given the improving limits on the deviation of the spectral slope from exact scale-invariance (see e.g. [25]), this more tightly constrains the parameter space of our scenario. Note, however, that the spectral amplitude can quite easily be tuned to agree with observations.

Note that the energy scale at which the modifications to the dispersion relation become important will most likely be comparable to the Planck scale of the underlying fundamental theory. Thus, inflation may well occur at energy scales higher than that of conventional grand unified models. Thus, there is the danger that in our scenario unwanted topological defects such as monopoles and domain walls will be produced. However, in the context of string theory motivated effective field theories at the unification scale, it is not required to have a unified gauge group (separate $U(1)$ factors are rather common). Thus, in this context one may well have models which do not yield any unwanted topological defects.

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