Quark number susceptibilities of hot QCD up to $g^6 \ln g$

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The pressure of hot QCD has recently been determined to the last perturbatively computable order $g^6 \ln g$ by Kajantie *et al.* using three-dimensional effective theories. A similar method is applied here to the pressure in the presence of small but non-vanishing quark chemical potentials, and the result is used to derive the quark number susceptibilities in the limit $\mu = 0$. The diagonal quark number susceptibility of QCD with n_f flavors of massless quarks is evaluated to order $g^6 \ln g$ and compared with recent lattice simulations. It is observed that the results qualitatively resemble the lattice ones, and that when combined with the fully perturbative but yet undetermined g^6 term they may well explain the behavior of the lattice data for a wide range of temperatures.

I. INTRODUCTION

The grand potential is the most fundamental function describing the equilibrium properties of a thermodynamic system. By itself it amounts to minus the pressure $p(T, \mu_1, \mu_2, \dots)$ times the volume and from its derivatives one immediately obtains such quantities as entropy, specific heats, number densities and susceptibilities. In quantum chromodynamics a lot of attention has been devoted to determining the pressure in the quark-qluon plasma phase and e.g. lattice simulations have been successfully applied to the problem. They are, however, only available for temperatures no more than a few times above the critical temperature T_c of the deconfinement phase transition and have only recently been extended to finite baryon densities.

So far the most powerful tool available for analytic calculations in QCD has been perturbation theory, the use of which is justified by the small value of the coupling constant g at high energy densities due to asymptotic freedom. At $\mu = 0$ the perturbative series of the pressure has recently been extended to order $g^6 \ln g$ [1], which marks the final step in a series of impressive computations starting from the determination of the order g^2 contribution [2] and leading through the orders g^3 [3], $g^4 \ln g$ [4], g^4 [5] and g^5 [6,7]. The next $\mathcal{O}(g^6)$ term in the series is already out of reach for analytic computations due to infrared problems [8].

At non-zero quark chemical potentials the pressure is at present known only to order $g^4 \ln g$ [4] at finite T and to order g^4 at $T = 0$ [9] reflecting in part the computational complications induced by a finite value of μ. From these results one may derive the quark number susceptibilities χ defined as second derivatives of the pressure with respect to the chemical potentials. They are both important and very interesting quantities, since they describe the effects of finite density being at the same time directly measurable on a lattice in the limit $\mu \to 0$ (see e.g. [10–12]). The perturbative results for χ obtained from [4] in this limit are, however, unable to produce even the qualitative behavior of the lattice data. Recent HTL computations (see e.g. [13]) have improved the situation somewhat.

In the present paper the pressure of quark-gluon plasma with massless quarks is computed to order $g^6 \ln g$ in the limit of small but non-vanishing chemical potentials $0 \leq \mu/T \ll 1$. The calculation is a generalization of the recent $\mu = 0$ paper [1] and applies the same method, matching of effective three-dimensional theories to four-dimensional QCD [14], in separating the contributions of different momentum scales to p_{QCD}. The result is used in deriving the quark number susceptibilities at $\mu = 0$ to order $g^6 \ln g$, and the diagonal susceptibility is subsequently analyzed and compared with results of lattice simulations. One observes that the results of the present computation follow the same trend as the lattice ones, but that the effects of the yet undetermined contributions may still affect their behavior considerably.

The non-trivial part of deriving the susceptibilities is the evaluation of the fermionic three-loop diagrams of full QCD that contribute to the pressure. To do that one applies here the results of yet unpublished work [15], in which the necessary diagrams are computed for arbitrary T and μ. In this paper the results are, however, only quoted as expanded to second order with respect to μ/T, which is sufficient for the present purposes. Due to the use of the effective theory approach one may here perform the calculations without any resummations applying dimensional regularization in the $\overline{\text{MS}}$ scheme is to regulate both ultraviolet and infrared divergences. All fields are considered massless, while the chemical potentials of the quark flavors are regarded as being independent and non-zero and the temperature higher than T_c.

The general setup of the paper is presented in Sec. II, while the results for the pressure are assembled in Sec. III. Section IV is then devoted to explaining the computation of the matching coefficients needed in deriving the pressure, and the diagonal quark number susceptibility at $\mu = 0$ is discussed in Sec. V. Conclusions are finally drawn in Sec. VI and the values of the matching coefficients listed in the Appendix.

II. SETUP AND NOTATION

In Euclidean metric quantum chromodynamics is defined by the Lagrangian density

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$$\mathcal{L}_{\text{QCD}} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi} \slashed{D} \psi, \qquad (2.1)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \qquad (2.2)$$

$$D_\mu = \partial_\mu - i g A_\mu = \partial_\mu - i g A^a_\mu T^a, \qquad (2.3)$$

and where the massless quark fields have been combined into a multi-component spinor ψ. The $N_c \times N_c$ matrices T^a are here the generators of the fundamental representation of $SU(N_c)$, and the relevant group theory factors read

$$C_A \delta_{cd} \equiv f^{abc} f^{abd} = N_c \delta_{cd}, \qquad (2.4)$$

$$C_F \delta_{ij} \equiv (T^a T^a)_{ij} = \frac{N_c^2 - 1}{2 N_c} \delta_{ij}, \qquad (2.5)$$

$$T_F \delta_{ab} \equiv \text{Tr} \, T^a T^b = \frac{n_f}{2} \delta_{ab} \qquad (2.6)$$

and

$$D \delta_{cd} \equiv d^{abc} d^{abd} = \frac{N_c^2 - 4}{N_c} \delta_{cd}. \qquad (2.7)$$

The dimensions of the representations are $d_A \equiv \delta_{aa} = N_c^2 - 1$ for the adjoint one and $d_F \equiv \delta_{ii} = d_A T_F / C_F = N_c n_f$ for the fermionic one.

The partition function is defined as a path integral over all fields of the functional

$$\exp\left\{ -\left(S_{\text{QCD}} - \sum_f \mu_f N_f \right) \right\} = \exp\left\{ - \int_0^\beta d\tau \right.$$
$$\left. \times \int d^d x (\mathcal{L}_{\text{QCD}} - \psi^\dagger \boldsymbol{\mu} \psi) \right\}, \qquad (2.8)$$

where $\beta = 1/T$ and $\boldsymbol{\mu}$ is a diagonal $n_f \times n_f$-matrix in flavor space representing the different chemical potentials of the quark flavors. In Eq. (2.8) the space dimensionality is $d = 3 - 2\epsilon$ signifying the use of dimensional regularization. The partition function gives the pressure through the relation $p = T/V \ln Z$, where the infinite-volume limit is assumed.

At high temperatures and small chemical potentials $\mu \ll T$ the pressure of QCD can be separated into three parts $p_{\text{QCD}} = p_E + p_M + p_G$ corresponding respectively to the different momentum scales $2\pi T$, gT and $g^2 T$ contributing to it [14,7]. By definition

$$p_E(T,\mu) \equiv p_{\text{QCD}}(T,\mu) - \frac{T}{V} \ln \int \mathcal{D} A^a_i \, \mathcal{D} A^a_0 \exp\{-S_E\}, \qquad (2.9)$$

where S_E is the action of a three-dimensional effective theory with the Lagrangian density [16]

$$\mathcal{L}_E = \frac{1}{2} \, \text{Tr} \, F^2_{ij} + \text{Tr}[D_i, A_0]^2 + m_E^2 \text{Tr} A_0^2 + \frac{i g^3}{3\pi^2} \sum_f \mu_f \, \text{Tr} A_0^3$$
$$+ \delta \mathcal{L}_E, \qquad (2.10)$$

obtained from QCD by dimensional reduction [17]. In Eq. (2.10) m_E and the coupling constant g_E appearing in F_{ij} are parameters to be determined in full QCD, and terms that contribute to the pressure starting at order g^6 or higher have been assembled in $\delta \mathcal{L}_E$.

The functions $p_M(T,\mu)$ and $p_G(T)$ are similarly defined by

$$p_M(T,\mu) \equiv p_{\text{QCD}}(T,\mu) - p_E(T,\mu)$$
$$- \frac{T}{V} \ln \int \mathcal{D} A^a_i \exp\{-S_M\}$$
$$\equiv p_{\text{QCD}}(T,\mu) - p_E(T,\mu) - p_G(T), \qquad (2.11)$$

$$\mathcal{L}_M = \frac{1}{2} \, \text{Tr} \, F^2_{ij} + \delta \mathcal{L}_M \qquad (2.12)$$

with yet another coupling constant g_M appearing in the definition of F_{ij}. At leading order the different parts contribute to the pressure as $p_E \sim g^0$, $p_M \sim g^3$ and $p_G \sim g^6 \ln g$. As is indicated above by writing $p_G(T)$, it will be seen that the dependence of p_G on μ is of higher order than what is considered here.

The contributions of the different momentum scales to the $\mu = 0$ pressure have been analyzed in detail in [1]. Since the effects of finite chemical potentials with few exceptions manifest themselves merely as changes in the matching coefficients defined there, the treatment of the pressure in the present paper will be restricted to quoting the results for the p_N's from [1] and discussing the effects of finite μ on the coefficients.

The momentum integration measure and the shorthands for sum-integrals used from here onwards are

$$\int_p \equiv \int \frac{d^d p}{(2\pi)^d} = \Lambda^{-2\epsilon} \left(\frac{e^\gamma \bar{\Lambda}^2}{4\pi} \right)^\epsilon \int \frac{d^d p}{(2\pi)^d}, \qquad (2.13)$$

$$\sum\!\!\!\!\!\!\int_{P/\{P\}} \equiv T \sum_{p_0 / \{p_0\}} \int_p, \qquad (2.14)$$

where $\bar{\Lambda}$ is the $\overline{\text{MS}}$ scale and $p_0 \equiv 2 n \pi T$ stands for bosonic and $\{p_0\} \equiv (2n+1)\pi T - i\mu$ for fermionic Matsubara frequencies. In the following sections the chemical potentials usually appear in the dimensionless combination

$$\bar{\mu} \equiv \frac{\mu}{2\pi T}. \qquad (2.15)$$

III. THE PRESSURE AT $0 \leqslant \mu \ll T$

Collecting results from [1] and [16] one may, in analogy with [1], write the different parameters required in computing p_{QCD} to order $g^6 \ln g$ in the form

$$\frac{p_G(T)}{T\Lambda^{-2\epsilon}} = d_A C_A^3 \frac{g_M^6}{(4\pi)^4} \ln \frac{\bar{\Lambda}}{2m_M} [8\alpha_G + \mathcal{O}(\epsilon)], \qquad (3.1)$$

$$\frac{p_M(T,\mu)}{T\Lambda^{-2\epsilon}} = \frac{1}{(4\pi)} d_A m_E^3 \left[\frac{1}{3} + \mathcal{O}(\epsilon) \right]$$

$$+ \frac{1}{(4\pi)^2} d_A C_A g_E^2 m_E^2 \left[-\frac{1}{4\epsilon} - \frac{3}{4} - \ln \frac{\bar{\Lambda}}{2m_E} \right.$$

$$+ \mathcal{O}(\epsilon) \bigg] + \frac{1}{(4\pi)^3} d_A C_A^2 g_E^4 m_E \left[-\frac{89}{24} - \frac{1}{6}\pi^2 \right.$$

$$+ \frac{11}{6} \ln 2 + \mathcal{O}(\epsilon) \bigg]$$

$$+ \frac{1}{(4\pi)^4} d_A C_A^3 g_E^6 \ln \frac{\bar{\Lambda}}{2m_E} [8\alpha_{M1} + \mathcal{O}(\epsilon)]$$

$$+ \frac{1}{(4\pi)^4} d_A D T_F^2 g_E^6 \ln \frac{\bar{\Lambda}}{2m_E} [8\alpha_{M2} + \mathcal{O}(\epsilon)], \qquad (3.2)$$

$$m_M = C_A g_M^2, \qquad (3.3)$$

$$g_M^2 = g_E^2 [1 + \mathcal{O}(g_E^2/m_E)], \qquad (3.4)$$

$$\frac{p_E(T,\mu)}{T\Lambda^{-2\epsilon}} = T^3 \left[\alpha_{E1} + g^2(\alpha_{E2} + \mathcal{O}(\epsilon)) \right.$$

$$+ \frac{g^4}{(4\pi)^2} (\alpha_{E3} + \mathcal{O}(\epsilon)) \bigg], \qquad (3.5)$$

$$m_E^2 = T^2 \left[g^2(\alpha_{E4} + \alpha_{E5}\epsilon + \mathcal{O}(\epsilon^2)) \right.$$

$$+ \frac{g^4}{(4\pi)^2} (\alpha_{E6} + \mathcal{O}(\epsilon)) \bigg], \qquad (3.6)$$

$$g_E^2 = T \left[g^2 + \frac{g^4}{(4\pi)^2} (\alpha_{E7} + \mathcal{O}(\epsilon)) \right], \qquad (3.7)$$

where the effects of finite μ can be explicitly seen only in the appearance of an additional term proportional to α_{M2} in Eq. (3.2). Here g^2 is the renormalized coupling of full QCD, and the values of the matching coefficients α can be immediately obtained from the results of [1,2,13] and [16] with the exception of α_{E3} and α_{E5}. While the latter is found almost trivially through a one-loop computation, determining the first one requires the evaluation of all three-loop diagrams of full QCD that contribute to the pressure. This procedure is explained in Sec. IV.

Adding together Eqs. (3.1), (3.2) and (3.5) one has obtained a compact expression for the perturbative expansion of the pressure up to $\mathcal{O}(g^6 \ln g)$

$$\frac{p_{\text{QCD}}(T,\mu)}{T^4\Lambda^{-2\epsilon}} = \frac{p_E(T,\mu) + p_M(T,\mu) + p_G(T)}{T^4\Lambda^{-2\epsilon}}$$

$$= g^0\{\alpha_{E1}\} + g^2\{\alpha_{E2}\} + \frac{g^3}{(4\pi)} \left\{ \frac{d_A}{3} \alpha_{E4}^{3/2} \right\} + \frac{g^4}{(4\pi)^2} \left\{ \alpha_{E3} - d_A C_A \left[\alpha_{E4}\left(\frac{1}{4\epsilon} + \frac{3}{4} + \ln \frac{\bar{\Lambda}}{2gT\alpha_{E4}^{1/2}} \right) + \frac{1}{4} \alpha_{E5} \right] \right\}$$

$$+ \frac{g^5}{(4\pi)^3} \left\{ d_A \alpha_{E4}^{1/2} \left[\frac{1}{2}\alpha_{E6} - C_A^2 \left(\frac{89}{24} + \frac{\pi^2}{6} - \frac{11}{6}\ln 2 \right) \right] \right\} + \frac{g^6}{(4\pi)^4} \{ d_A C_A (\alpha_{E6} + \alpha_{E4}\alpha_{E7}) \ln[g\alpha_{E4}^{1/2}]$$

$$- 8 d_A C_A^3 (\alpha_{M1} \ln[g\alpha_{E4}^{1/2}] + 2\alpha_G \ln[gC_A^{1/2}]) - 8 d_A D T_F^2 \alpha_{M2} \ln[g\alpha_{E4}^{1/2}] \} \qquad (3.8)$$

with the values of the coefficients α listed in the Appendix and the pole of α_{E3} exactly cancelling the $1/\epsilon$ term appearing in the order g^4 contribution. One should notice here that the numerical factors appearing inside the logarithms of the last $g^6 \ln g$ term can be unambiguously defined only after the full order g^6 contribution has been determined.

IV. THE COMPUTATION OF α_{E3}

In order to find the matching coefficient α_{E3} one needs to compute the diagrammatic expansion of the QCD pressure to three-loop order at finite T and μ, but without any resummations. Since the corresponding expansion has already been derived for $\mu=0$ [5], one may restrict the treatment here to the μ-specific part, i.e. to the diagrams containing fermionic lines. These diagrams are shown in Fig. 1 and contribute to the expansion as

$$p_{\text{ferm}} = p_{\text{ferm}}^0 + \frac{1}{2} I_a + \frac{1}{3} I_b + \frac{1}{4} I_c + \frac{1}{2} I_d + \frac{1}{4} I_e + \frac{1}{2} I_f + \frac{1}{4} I_g + \frac{1}{4} I_h, \qquad (4.1)$$
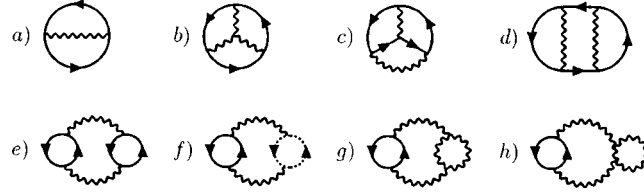
FIG. 1. The two- and three-loop fermionic diagrams contributing to the pressure. The solid, wiggly and dashed lines stand respectively for the quark, gluon and ghost fields.

where p_{ferm}^0 represents the pressure of non-interacting quarks (see e.g. [18]).

Applying the usual Euclidean space finite temperature Feynman rules to the individual diagrams, performing several shifts of integration momenta and taking advantage of the fact that the purely bosonic version of $\tilde{\mathcal{T}}$ vanishes at $\mathcal{O}(\epsilon^0)$ [5], the diagrams may be rewritten in terms of the sum-integrals

$$\mathcal{I}_n^m \equiv \sum\hspace{-1.2em}\int_P \frac{(p_0)^m}{(P^2)^n} \, , \tag{4.2}$$

$$\tilde{\mathcal{I}}_n^m \equiv \sum\hspace{-1.2em}\int_{\{P\}} \frac{(p_0)^m}{(P^2)^n} \, , \tag{4.3}$$

$$\tilde{\mathcal{T}} \equiv \sum\hspace{-1.2em}\int_{\{PQ\}} \frac{1}{P^2 Q^2 (P-Q)^2} \, , \tag{4.4}$$

$$\tilde{\mathcal{T}}' \equiv \sum\hspace{-1.2em}\int_{\{PQ\}} \frac{p_0}{P^2 Q^2 (P-Q)^4} \, , \tag{4.5}$$

$$\tilde{\mathcal{M}}_{m,n} \equiv \sum\hspace{-1.2em}\int_{\{PQR\}} \frac{1}{P^2 Q^2 (R^2)^m ((P-Q)^2)^n (P-R)^2 (Q-R)^2} \, , \tag{4.6}$$

$$\mathcal{N}_{m,n} \equiv \sum\hspace{-1.2em}\int_{\{PQ\}R} \frac{1}{P^2 Q^2 (R^2)^m ((P-Q)^2)^n (P-R)^2 (Q-R)^2} \, , \tag{4.7}$$

defined here in analogy with [7]. For example, for the diagram b one obtains in the Feynman gauge

$$I_b = -\sum\hspace{-1.2em}\int_{\{PQR\}} \mathrm{Tr}\left[\frac{1}{\not{P}} (g\gamma^\mu T_{ij}^a) \frac{1}{\not{Q}} (g\gamma^\nu T_{jk}^b) \frac{1}{\not{R}} (g\gamma^\rho T_{ki}^c) \right] i g f_{abc}$$

$$\times \frac{g_{\mu\rho}(2P_\nu - Q_\nu - R_\nu) + g_{\rho\nu}(2R_\mu - P_\mu - Q_\mu) + g_{\nu\mu}(2Q_\rho - R_\rho - P_\rho)}{(P-Q)^2 (Q-R)^2 (R-P)^2}$$

$$= \frac{3}{2} d_A C_A T_F g^4 \sum\hspace{-1.2em}\int_{\{PQR\}} \frac{P_\alpha Q_\beta R_\gamma (2P_\nu - Q_\nu - R_\nu)}{P^2 Q^2 R^2 (P-Q)^2 (Q-R)^2 (R-P)^2} \; \mathrm{Tr}[\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu \gamma^\gamma \gamma_\mu]$$

$$= 48(1-\epsilon) d_A C_A T_F g^4 \sum\hspace{-1.2em}\int_{\{PQR\}} \frac{P\cdot(P-Q) Q\cdot R}{P^2 Q^2 R^2 (P-Q)^2 (Q-R)^2 (R-P)^2}$$

$$= 12(1-\epsilon) d_A C_A T_F g^4 \left[(\mathcal{I}_1^0 - \tilde{\mathcal{I}}_1^0)\tilde{\mathcal{T}} + \frac{1}{2} \; \tilde{\mathcal{M}}_{0,0} \right], \tag{4.8}$$

and the other ones produce in the same gauge

$$I_a = -4(1-\epsilon) d_A T_F g^2 \tilde{\mathcal{I}}_1^0 (\tilde{\mathcal{I}}_1^0 - 2\mathcal{I}_1^0), \tag{4.9}$$

$$I_c = 4(1-\epsilon)d_A(2C_F-C_A)T_Fg^4[4(\mathcal{I}_1^0-2\tilde{\mathcal{I}}_1^0)\tilde{\mathcal{T}}+(2+\epsilon)\mathcal{N}_{0,0}-2\epsilon\tilde{\mathcal{M}}_{0,0}+2\mathcal{N}_{1,-1}], \tag{4.10}$$

$$I_d = -8(1-\epsilon)^2 d_A C_F T_F g^4[(\mathcal{I}_1^0-\tilde{\mathcal{I}}_1^0)^2\tilde{\mathcal{I}}_2^0-2\tilde{\mathcal{I}}_1^0\tilde{\mathcal{T}}+\tilde{\mathcal{M}}_{0,0}+\tilde{\mathcal{M}}_{1,-1}], \tag{4.11}$$

$$I_e = -8d_A T_F^2 g^4[4(1+\epsilon)(\tilde{\mathcal{I}}_1^0)^2\mathcal{I}_2^0-16\tilde{\mathcal{I}}_1^1\tilde{\mathcal{T}}'-(1-\epsilon)(\mathcal{N}_{0,0}-4\tilde{\mathcal{I}}_1^0\tilde{\mathcal{T}})-2\mathcal{N}_{1,-1}-2\mathcal{N}_{2,-2}], \tag{4.12}$$

$$I_f = -d_A C_A T_F g^4[8\mathcal{I}_1^0\tilde{\mathcal{I}}_1^0\mathcal{I}_2^0-2\mathcal{I}_1^0\tilde{\mathcal{T}}+\tilde{\mathcal{M}}_{0,0}-2\tilde{\mathcal{M}}_{-2,2}], \tag{4.13}$$

$$I_g = 4d_A C_A T_F g^4\left[4(6-5\epsilon)\mathcal{I}_1^0\tilde{\mathcal{I}}_1^0\mathcal{I}_2^0-(7-6\epsilon)\mathcal{I}_1^0\tilde{\mathcal{T}}-\left(\frac{3}{2}-2\epsilon\right)\tilde{\mathcal{M}}_{0,0}-(5-4\epsilon)\tilde{\mathcal{M}}_{-2,2}\right], \tag{4.14}$$

$$I_h = -8(3-2\epsilon)(1-\epsilon)d_A C_A T_F g^4[2\mathcal{I}_1^0\tilde{\mathcal{I}}_1^0\mathcal{I}_2^0-\mathcal{I}_1^0\tilde{\mathcal{T}}]. \tag{4.15}$$

Substituting Eqs. (4.8)–(4.15) into Eq. (4.1) and allowing explicitly flavor-dependent chemical potentials, one has obtained a representation for p_{ferm} in terms of the sum-integrals defined above

$$p_{\text{ferm}} = \frac{1}{n_f}\sum_f \{p_{\text{ferm}}^0-2(1-\epsilon)d_A T_F Z_g^2 g^2\tilde{\mathcal{I}}_1^0(\mathcal{I}_1^0-2\mathcal{I}_1^0)+d_A g^4(C_A T_F(1-\epsilon)[8(1+\epsilon)\mathcal{I}_1^0\tilde{\mathcal{I}}_1^0\mathcal{I}_2^0-4\epsilon\mathcal{I}_1^0\tilde{\mathcal{T}}+4\tilde{\mathcal{I}}_1^0\tilde{\mathcal{T}}+2\epsilon\tilde{\mathcal{M}}_{0,0}$$

$$-(2+\epsilon)\mathcal{N}_{0,0}-2\mathcal{N}_{1,-1}-4\tilde{\mathcal{M}}_{-2,2}]-2C_F T_F(1-\epsilon)[2(1-\epsilon)(\mathcal{I}_1^0-\tilde{\mathcal{I}}_1^0)^2\tilde{\mathcal{I}}_2^0-4\mathcal{I}_1^0\tilde{\mathcal{T}}+4(1+\epsilon)\tilde{\mathcal{I}}_1^0\tilde{\mathcal{T}}+2\tilde{\mathcal{M}}_{0,0}$$

$$-(2+\epsilon)\mathcal{N}_{0,0}+2(1-\epsilon)\tilde{\mathcal{M}}_{1,-1}-2\mathcal{N}_{1,-1}])\}-\frac{1}{n_f^2}\sum_{fg}d_A g^4 T_F^2\{8(1+\epsilon)\tilde{\mathcal{I}}_1^0[\mu_f]\,\tilde{\mathcal{I}}_1^0[\mu_g]\mathcal{I}_2^0$$

$$+4(1-\epsilon)(\tilde{\mathcal{I}}_1^0[\mu_f]\,\tilde{\mathcal{T}}[\mu_g]+\tilde{\mathcal{I}}_1^0[\mu_g]\,\tilde{\mathcal{T}}[\mu_f])-16(\tilde{\mathcal{I}}_1^1[\mu_f]\,\tilde{\mathcal{T}}'[\mu_g]+\tilde{\mathcal{I}}_1^1[\mu_g]\,\tilde{\mathcal{T}}'[\mu_f])-2(1-\epsilon)\mathcal{N}_{0,0}[\mu_f,\mu_g]$$

$$-4\mathcal{N}_{1,-1}[\mu_f,\mu_g]-4\mathcal{N}_{2,-2}[\mu_f,\mu_g]\}. \tag{4.16}$$

The renormalization coefficient of the gauge coupling, Z_g, is given by

$$Z_g^2 = 1 - \frac{11C_A-4T_F}{3}\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}, \tag{4.17}$$

and in the latter sum of Eq. (4.16) the $\mathcal{N}_{n,-n}$'s depend on both μ_f and μ_g through the respective fermionic momenta. The sum-integrals appearing in the result are computed[1] in [15] for arbitrary T and μ, and thus the coefficient α_{E3} is available by simply combining Eq. (4.16) with the bosonic part of the strict perturbation expansion of the pressure (see e.g. Eq. (31) of [7]). The outcome of the computation expanded to second order in $\bar{\mu}$ is displayed in Eq. (A6) of the Appendix.

V. QUARK NUMBER SUSCEPTIBILITIES AT $\mu = 0$

From the expression (3.8) one may at once extract the quark number susceptibilities defined by

$$\chi_{ij} \equiv \frac{\partial^2 p}{\partial\mu_i\partial\mu_j} = \chi\delta_{ij} + \tilde{\chi}(1-\delta_{ij}), \tag{5.1}$$

where the symmetry between the massless quark flavors at $\mu = 0$ has been exploited in the latter equality in dividing the result explicitly into a diagonal and an off-diagonal part. The off-diagonal susceptibility $\tilde{\chi}$ is already known to order $g^6\ln g$ [13], but the result obtained here for the diagonal one χ is new. Specializing to the physical case $N_c = 3$ one readily obtains

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[1]Recently there has appeared a new paper [19], in which one- and two-loop sum-integrals analogous to those considered in [15] have been independently calculated.
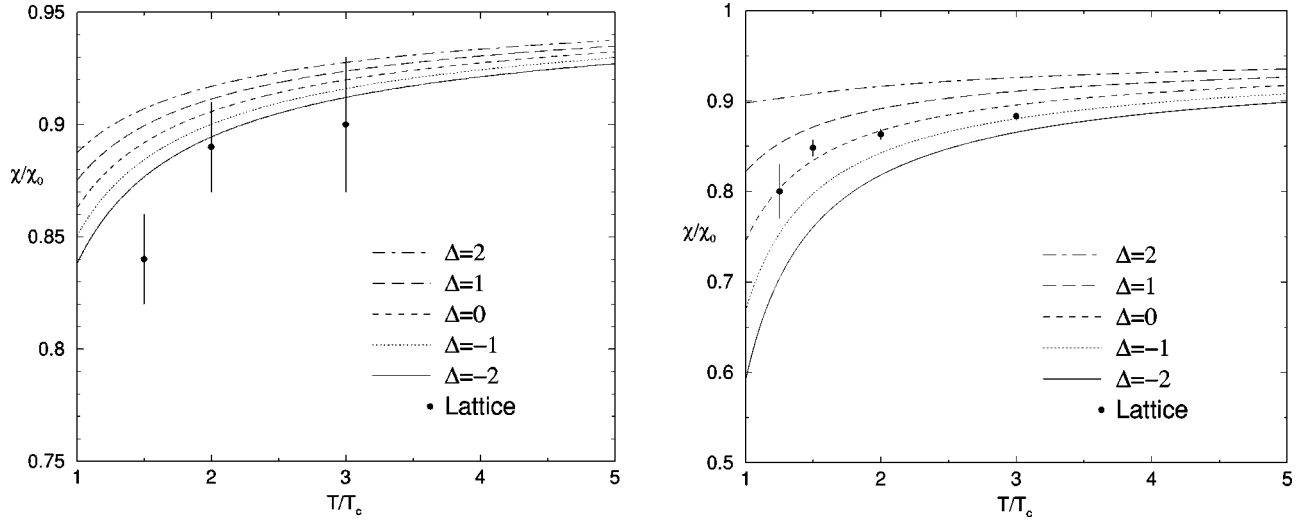
FIG. 2. χ/χ_0 plotted as a function of T/T_c for $n_f=0$ (left) and $n_f=2$ (right) with various values of C. The parameter Δ is defined as the ratio of the coefficients of the $g^6\ln g$ and g^6 terms in the expansion (5.2). The lattice data is from [10,11], and the values $T_c/\Lambda_{\overline{\rm MS}}|_{n_f=0}=1.15$ and $T_c/\Lambda_{\overline{\rm MS}}|_{n_f=2}=0.49$ [20] are used.

$$
\left.\frac{\chi}{\chi_0}\right|_{\mu=0} = 1-2\,\frac{g^2(\rho)}{4\pi^2}+8\,\sqrt{1+\frac{n_f}{6}}\left(\frac{g^2(\rho)}{4\pi^2}\right)^{3/2}+12\left(\frac{g^2(\rho)}{4\pi^2}\right)^2\ln\frac{g^2(\rho)}{4\pi^2}-\frac{1}{36}\left\{12(33-2n_f)\ln\frac{e^\gamma\rho}{4\pi T}-432\ln\left[1+\frac{n_f}{6}\right]\right.
$$

$$
\left.+133+26n_f+16(17+2n_f)\ln 2-432\gamma-432\frac{\zeta'(-1)}{\zeta(-1)}\right\}\left(\frac{g^2(\rho)}{4\pi^2}\right)^2+\frac{1}{12\sqrt{1+n_f/6}}\left\{4(6+n_f)(33-2n_f)\ln\frac{e^\gamma\rho}{4\pi T}\right.
$$

$$
\left.-669+30n_f+4n_f^2-36\pi^2+4(99-24n_f-4n_f^2)\ln 2+\frac{28}{9}\,(6+n_f)^2\zeta(3)\right\}\left(\frac{g^2}{4\pi^2}\right)^{5/2}+2\left\{2(33-2n_f)\ln\frac{e^\gamma\rho}{4\pi T}+\frac{59}{9}\right.
$$

$$
\left.+n_f-8n_f\ln 2+\frac{7}{3}\,(6+n_f)\zeta(3)\right\}\left(\frac{g^2}{4\pi^2}\right)^3\ln\frac{g^2}{4\pi^2}-\frac{33-2n_f}{108}\left\{6(33-2n_f)\ln\frac{e^\gamma\rho}{4\pi T}-432\ln\left[1+\frac{n_f}{6}\right]+88+26n_f\right.
$$

$$
\left.+16(17+2n_f)\ln 2-432\gamma-432\frac{\zeta'(-1)}{\zeta(-1)}-\frac{2889}{33-2n_f}\right\}\left(\frac{g^2}{4\pi^2}\right)^3\ln\frac{e^\gamma\rho}{4\pi T}+C(n_f)\left(\frac{g^2}{4\pi^2}\right)^3+\mathcal{O}(g^7),\tag{5.2}
$$

where $\chi_0=T^2$ is the free theory result. The two-loop running of g

$$
g^2(\Lambda)=g^2(\rho)\left[1+\frac{1}{6}\,(11C_A-4T_F)\ln\frac{\rho}{\Lambda}\frac{g^2(\rho)}{4\pi^2}+\frac{1}{12}\left(17C_A^2-10C_AT_F-6\frac{d_FC_F^2}{d_A}+\frac{1}{3}(11C_A-4T_F)^2\ln\frac{\rho}{\Lambda}\right)\ln\frac{\rho}{\Lambda}\left(\frac{g^2}{4\pi^2}\right)^2\right]\tag{5.3}
$$

has been used to determine the form of the second but last term, which cancels the scale dependence of the lower-order contributions. The symbol ρ is used in Eq. (5.2) in place of the scale of dimensional regularization to emphasize the fact that its value is arbitrary, though one expects it to be of order $2\pi T$.

The last term of Eq. (5.2), proportional to $C(n_f)$, represents the yet undetermined $\mathcal{O}(g^6)$ contribution to the diagonal susceptibility and can only be obtained through a massive computation involving the evaluation of all four-loop diagrams of full QCD contributing to the pressure. It is, however, worth noting that unlike in the case of the pressure [1] no lattice simulations will be required in this process due to the fact that at order g^6 the contribution of p_G to $p_{\rm QCD}$ is μ-independent. It is therefore obvious that $C(n_f)$ has no direct relation to the coefficient δ defined in [1] to represent the $\mathcal{O}(g^6)$ part of $p_{\rm QCD}$.

In Figs. 2 and 3 the result (5.2) is plotted as a function of T together with lattice data obtained from [10,11]. Figure 2 contains plots of the $n_f=0$ and $n_f=2$ cases for different $C(n_f)$ showing that for reasonable values of C the perturbative
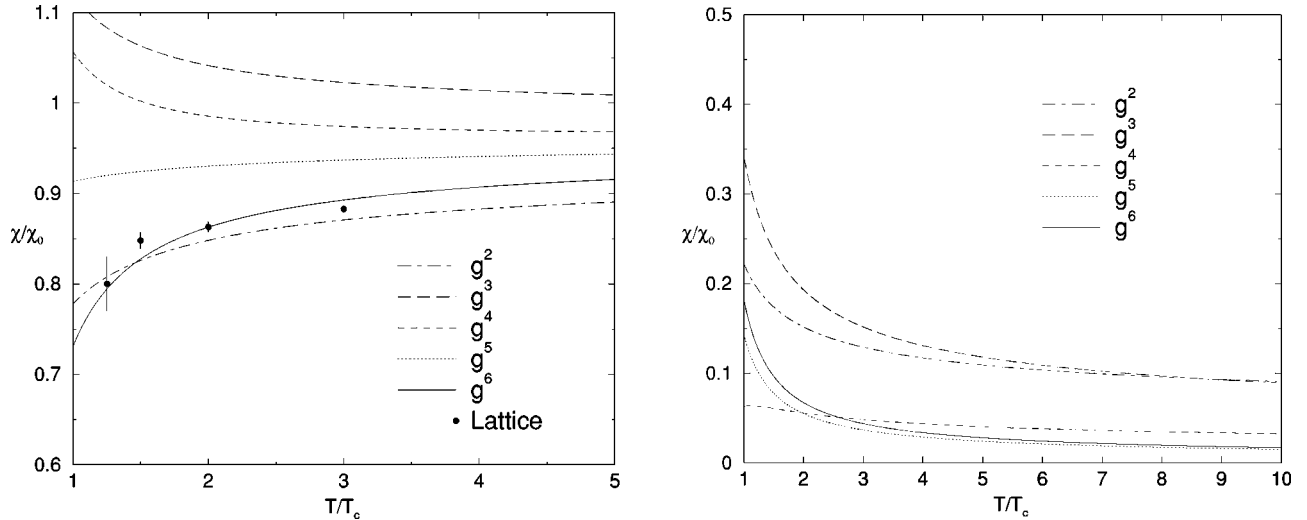
FIG. 3. $\chi/\chi_0|_{n_f=2}$ at different perturbative orders for $C(2)=-45$ or $\Delta\approx-0.18$ (left) and the absolute values of the individual terms of the series (right). The lattice results are from [11].

series can be made to approach the lattice results even at temperatures surprisingly close to T_c. This should, however, by no means be considered an argument suggesting that the region of applicability of the present results can be extended down to T_c. The success of these perturbative predictions at $T\lesssim3T_c$ is in large part merely a consequence of having a free parameter available for plotting, and the behavior of the result at these temperatures may be completely distorted by the eventual computation of $C(n_f)$. For reference, the susceptibility for $n_f=2$ and $C(2)=-45$ is nevertheless plotted in Fig. 3 to different perturbative orders alongside with the absolute values of the different terms of the series. For temperatures higher than a few times T_c the convergence properties of the series appear to be reasonably good. In the plots the value of the scale parameter ρ is set to $6.742T$ for $n_f=0$ and to $8.112T$ for $n_f=2$, for which the one-loop corrections to g_E^2 vanish [21].

A few words about the curious concept of susceptibility at zero quark flavors are probably in order. The quantity has been computed above as the formal $n_f\rightarrow0$ limit of the general result (5.2) and has been plotted in Fig. 2. The interest in this unphysical limit is due to the existence of recent lattice results from quenched QCD [10] that provide an interesting and powerful test for the validity of the perturbative expansion of the susceptibility. In particular, being able to compare the perturbation theory results and the lattice data for different numbers of fermion flavors makes it possible to qualitatively study the n_f dependence of the yet unknown coefficient $C(n_f)$.

VI. CONCLUSIONS

In this paper the diagonal quark number susceptibility of QCD at vanishing chemical potentials has been computed to order $g^6\ln g$ in perturbation theory. This is a three-order improvement to the previous result [4]. Since the next term in the series requires the determination of the perturbative part of the $\mathcal{O}(g^6)$ pressure at finite T and μ, and even the corre-

sponding $\mu=0$ computation seems to be out of reach for current computational techniques, the result obtained here will most likely not be subject to improvement in near future. Nevertheless, a need for further work aiming at the determination of the $\mathcal{O}(g^6)$ terms in the expansions of both the pressure and the susceptibilities is obviously present.

The result obtained here for the diagonal susceptibility resembles the lattice data available but, as demonstrated in the previous sections, may also be considerably modified by the yet undetermined contributions to χ revealing its lack of predictive power. This could actually have been anticipated due to similar problems encountered in connection with the perturbative expansion of the pressure. The problem can be viewed as a natural consequence of the relatively large value of the QCD coupling constant at the energy scale considered here and of the slow convergence of the perturbative expansion of the pressure in the 3D sector.

While the high temperature, small chemical potential region of the QCD phase diagram is without doubt of considerable special interest, it is also worthwhile to study how the results obtained there can be generalized to other parts of the (T,μ) plane. An especially interesting task waiting to be tackled is the building of a bridge between the order g^4 perturbative results for the pressure at $T\neq0$, $\mu=0$ [5] and $T=0$, $\mu\neq0$ [9]. In the $T\ll\mu$ region the true ground state of the theory may naturally be modified by the appearance of non-perturbative qq condensates [22].

APPENDIX: THE MATCHING COEFFICIENTS

The values of the matching coefficients α defined in Sec. III read

$$\alpha_G = \frac{43}{96} - \frac{157}{6144}\ \pi^2,$$

(A1)

$$\alpha_{M1} = \frac{43}{32} - \frac{491}{6144}\ \pi^2,$$

(A2)

$$\alpha_{M2} = -\frac{2}{3}\left(\frac{1}{n_f}\sum_f \bar{\mu}\right)^2,$$

(A3)

$$\alpha_{E1} = \frac{\pi^2}{45}\frac{1}{n_f}\sum_f \left\{ d_A + d_F\left(\frac{7}{4} + 30\bar{\mu}^2\right)\right\},$$

(A4)

$$\alpha_{E2} = -\frac{d_A}{144}\frac{1}{n_f}\sum_f \left\{ C_A + \frac{T_F}{2}\ (5 + 72\bar{\mu}^2)\right\},$$

(A5)

$$\alpha_{E3} = \frac{d_A}{144}\frac{1}{n_f}\sum_f \left\{ C_A^2\left[\frac{12}{\epsilon} + \frac{194}{3}\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{116}{5} + 4\gamma - \frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{220}{3}\frac{\zeta'(-1)}{\zeta(-1)}\ \right] + C_A T_F\left[12(1 + 12\bar{\mu}^2)\frac{1}{\epsilon}\right.\right.$$

$$+\left(\frac{169}{3} + 600\bar{\mu}^2\right)\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{1121}{60} - \frac{157}{5}\ln 2 + 8\gamma - 2(91 - 156\gamma - 176\ln 2)\bar{\mu}^2 - \frac{1}{3}\frac{\zeta'(-3)}{\zeta(-3)}$$

$$+\frac{2}{3}\ (73 + 432\bar{\mu}^2)\frac{\zeta'(-1)}{\zeta(-1)}\right] + C_F T_F\left[\frac{105}{4} - 24\ln 2 + 6(35 + 16\ln 2)\bar{\mu}^2\right] + T_F^2\left[\left(\frac{20}{3}\ + 96\bar{\mu}^2\right)\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{1}{3}\right.$$

$$\left.\left.-\frac{88}{5}\ln 2 + 4\gamma - 8(13 + 16\ln 2 - 12\gamma)\bar{\mu}^2 - \frac{8}{3}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{16}{3}\frac{\zeta'(-1)}{\zeta(-1)}\ \right]\right\},$$

(A6)

$$\alpha_{E4} = \frac{1}{3}\frac{1}{n_f}\sum_f \{C_A + T_F[1 + 12\bar{\mu}^2]\},$$

(A7)

$$\alpha_{E5} = \frac{1}{3}\frac{1}{n_f}\sum_f \left\{ 2C_A\left[\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{\zeta'(-1)}{\zeta(-1)}\ \right] + T_F\left[2(1 + 12\bar{\mu}^2)\ln\frac{\bar{\Lambda}}{4\pi T} + 1 - 2\ln 2 - 12(1 - 4\ln 2 - 2\gamma)\bar{\mu}^2\right.\right.$$

$$\left.\left.+2\frac{\zeta'(-1)}{\zeta(-1)}\ \right]\right\},$$

(A8)

$$\alpha_{E6} = \frac{1}{9}\frac{1}{n_f}\sum_f \left\{ C_A^2\left[22\ln\frac{e^\gamma\bar{\Lambda}}{4\pi T} + 5\right] + C_A T_F\left[2(7 + 132\bar{\mu}^2)\ln\frac{e^\gamma\bar{\Lambda}}{4\pi T}\ + 9 - 16\ln 2 + 4(33 + 14\ \zeta(3))\bar{\mu}^2\right]\right.$$

$$\left.-18 C_F T_F[1 + 12\bar{\mu}^2] - 4T_F^2(1 + 12\bar{\mu}^2)\left[2\ln\frac{e^\gamma\bar{\Lambda}}{4\pi T} - 1 + 4\ln 2 - 14\ \zeta(3)\bar{\mu}^2\right]\right\},$$

(A9)

$$\alpha_{E7} = \frac{1}{3}\frac{1}{n_f}\sum_f \left\{ C_A\left[22\ln\frac{e^\gamma\bar{\Lambda}}{4\pi T} + 1\right] - 4T_F\left[2\ln\frac{e^\gamma\bar{\Lambda}}{4\pi T} + 4\ln 2 - 14\zeta(3)\bar{\mu}^2\right]\right\}.$$

(A10)

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