Power-law wave functions and generalized parton distributions for the pion

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We propose a model for generalized parton distributions of the pion based on the power-law *Ansatz* for the effective light-cone wave function.

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I. INTRODUCTION

Generalized parton distributions (GPDs) [1–3] are now the object of intensive theoretical studies, especially within the context of applications to deeply virtual [4,5] (for a detailed recent review see Ref. [6]) and large momentum transfer processes [7,8]. The main advantage of GPDs is their universality, allowing one to connect different hard processes, both exclusive and inclusive. The price for this is the complexity of GPDs: they are functions of three variables, e.g., skewed parton distributions (SPDs) $\mathcal{F}_\zeta(X,t)$ or $H(x,\xi;t)$ depend on the fraction X (or x) of the momentum carried by the active quark, the skewedness parameter ζ (or ξ), and the invariant momentum transfer t. For this reason, the most promising approach to disentangling GPDs from experimental data is to construct realistic models for GPDs and fix their parameters by fitting the data. The crucial point for the model building is that, in specific limits, GPDs reduce to more familiar functions describing the hadronic structure, such as usual parton densities, form factors, and distribution amplitudes. The ''reduction'' relations between GPDs and these functions have been used as a basis for building phenomenological models of GPDs [9]. Another fruitful idea used in the model building is to construct GPDs from the light-cone (LC) wave functions [7,8]. The most popular *Ansatz* [10] assumes a Gaussian dependence of the LC wave functions $\psi(x,k_\perp)$ on the transverse momentum $k_\perp$. A pragmatic reason behind this choice is the simplicity of Gaussian integrals allowing to obtain many results in analytic form. However, there are no *a priori* grounds to exclude wave functions with other types of transverse momentum dependence. In particular, the two-body (i.e., $\bar{q}q$) component of the pion wave function was calculated recently in a model [11] based on the one-gluon exchange approximation in the light-front framework. The wave function was found numerically, and it was observed that the fit is better if one uses a power-law form rather than a Gaussian [11,12]. Furthermore, the power-law wave functions were used some time ago in models for the nucleon form factors [13]. In the present paper, we show that a simple power-law *Ansatz* for the pion LC wave function allows one to obtain explicit analytic expressions for the form factor and generalized parton distributions. To make our presentation self-contained, in Sec. II we recall basic information about generalized parton distributions which is used in the following sections. In Sec. III, as a starting example, we consider the model with Gaussian dependence of the LC wave functions on the transverse momentum. In Sec. IV, we specify the explicit ''toy'' model expression for the effective pion wave function, which is then used in Sec. V to derive a parametric representation for the pion form factor. We show that the two parameters of this simple model, the constituent quark mass and the wave function width, can be easily adjusted to provide a curve close to existing experimental data. In Sec. VI, we analyze the asymptotic large-Q^2 behavior of the pion form factor. We consider both the massive $m\neq 0$ and massless $m=0$ cases. We show that, in the latter case, the pion form factor in our model with a power-law wave function $\psi(x,k_\perp) \sim 1/\{\sqrt{x(1-x)}(1+k_\perp^2/[\lambda^2 x(1-x)])^n\}$ has the same asymptotic behavior $F_\pi(Q^2)\sim 1/Q^2$ for any power n. Although this behavior is generated by the soft (Feynman) mechanism, it formally coincides with the quark counting law dictated by the hard one-gluon exchange mechanism. We show that the $1/Q^2$ behavior of the soft contribution is related to the fact that the parton distribution $f(x)$ in the massless case does not vanish for $x=1$. In the massive case, $f(x)\to 0$ as $x\to 1$ and the ultimate asymptotic behavior is $\ln(Q^2/\lambda^2)/Q^4$. However, for a wide range of accessible Q^2, the curve mimics the $1/Q^2$ behavior. In Sec. VII, we note that our parametric representation for the form factor has the form of the reduction relation connecting the pion form factor and the double distribution (DD) $F(x,y;t)$. The DD obtained in this way has correct spectral and symmetry properties. Moreover, it has the factorized structure proposed in Ref. [9]: it looks like a distribution amplitude with respect to the y variable and like a parton density with respect to the x variable. It also provides a nontrivial example of the interplay between x, y, and t dependence of DDs. With an explicit model for DDs at hand, one can calculate the relevant skewed distributions: the nonforward parton distribution $\mathcal{F}_\zeta(X;t)$ or Ji's off-forward parton distribution (OFPD) $H(x,\xi,t)$; see Sec. VIII. In the simple toy model that we use the pion is treated as an effectively two-body system, which is not very realistic: one may expect that the parton densities

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at small x are affected by many-body components. Indeed, the valence parton density obtained in our model differs rather strongly from the phenomenologically established form. In Sec. IX, we propose to fix this deficiency by adopting a model with a more realistic x profile at $t=0$, but preserving the analytic structure of the interplay between x,y, and t dependence generated by the power-law *Ansatz*. We show that by slightly changing the quark mass and the wave function width parameter it is still possible to get a good description of the pion form factor data. We present SPDs $\mathcal{F}_\zeta(X;t)$ obtained from the "realistic" DD. In particular, we show that in the "soft pion limit" $\zeta=1,t=0$, the isovector part of the "realistic" SPD has a shape close to the asymptotic form of the pion distribution amplitude. In Appendix A, to demonstrate that the variables x,y of the parametric representation for the form factor indeed have the meaning of the variables of double distributions, we give a covariant derivation of the toy model DD in a scalar model. In Appendix B, we discuss the structure of model SPDs in the impact parameter representation. In particular, we show how one can use superpositions of power-law DDs to build models for SPDs satisfying positivity bounds. In Appendix C, using again the toy scalar model, we briefly show how one can use our approach to build the models for two-pion distribution amplitudes that appear in the $\gamma^*\gamma\to\pi\pi$ reaction, which can be treated as the crossed-channel process to deeply virtual Compton scattering. Our conclusions are formulated in Sec. X.

Summarizing, in this paper we construct power-law models of the C-odd double distributions $F(x,y;t)$ for the pion and the relevant skewed parton distributions $\mathcal{F}_\zeta(X;t)$. By construction, the model GPDs satisfy such important constraints as reduction relations to usual parton densities and form factors, they have correct spectral and polynomiality properties, thus providing a model that can be used in phenomenological applications. For the simplified scalar case, we also build models that automatically satisfy the positivity constraints.

II. BASICS OF GENERALIZED PARTON DISTRIBUTIONS

Generalized parton distributions parametrize nonforward matrix elements of composite operators. To define the leading twist GPDs for the pion, we start with

$$\begin{aligned} i^n\langle P-r/2|\bar{\psi}_a\{\gamma_\mu \overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n}\}\psi_a|P+r/2\rangle &= 2\sum_{k=0}^{n}\frac{n!}{2^k k!(n-k)!}A_{nk}^{(a)}(t) \\ &\times\{P_\mu P_{\mu_1}\cdots P_{\mu_{n-k}}r_{\mu_{n-k+1}}\cdots r_{\mu_n}\} \\ &+\frac{1}{2^n}D_n^{(a)}(t)\{r_\mu r_{\mu_1}\cdots r_{\mu_n}\}, \end{aligned} \tag{2.1}$$

where $\overleftrightarrow{D}=(\overrightarrow{D}-\overleftarrow{D})/2$, $\{\cdots\}$ denotes the symmetric-traceless part of a tensor, a enumerates quark flavors, and the quark fields are taken at the origin. Compared to the more familiar case of forward matrix elements defining the usual parton densities, we have two four-vectors P and r, both of which can be used to build the tensor structure of the right hand side of Eq. (2.1). The index k specifies how many times the vector r appears in a particular term of the sum. Incorporating Hermiticity properties of the local operators and time-reversal invariance, one can show [14] that k is even. Now one can define double distributions $f(\beta,\alpha;t)$ as functions generating $A_{nk}(t)$ through its $\beta^{n-k}\alpha^k$ moments

$$\{1\pm(-1)^n\}A_{nk}^{(a)}(t)=\int_{-1}^{1}d\beta\int_{-1+|\beta|}^{1-|\beta|}\beta^{n-k}\alpha^k f_a^{\mp}(\beta,\alpha;t)\,d\alpha. \tag{2.2}$$

The spectral property $|\beta|+|\alpha|\leqslant 1$ can be proved for any relevant diagram of perturbation theory [1,5].

As usual, the Mellin moments define two functions: $f_a^-(\beta,\alpha;t)$ corresponds to even n and $f_a^+(\beta,\alpha;t)$ to odd n. They both are even functions of α. With respect to β, $f_a^-(\beta,\alpha;t)$ is even while $f_a^+(\beta,\alpha;t)$ is odd. For $\beta>0$, one can write $f_a^-(\beta,\alpha;t)$ as the difference $f_a(\beta,\alpha;t)-f_{\bar{a}}(\beta,\alpha;t)$ of quark and antiquark distributions [i.e., $f_a^-(\beta,\alpha;t)$ corresponds to a valence quark distribution: $f_a^-=f_a^{\rm val}$] and $f_a^+(\beta,\alpha;t)$ as their sum $f_a(\beta,\alpha;t)+f_{\bar{a}}(\beta,\alpha;t)$. The Polyakov-Weiss D term [15] is defined as the function $D_a(\alpha;t)$ whose α^n moments give the $D_n^{(a)}(t)$ coefficients. The latter are nonzero only for odd n; hence $D_a(\alpha;t)$ is an odd function of α.

We stress that this definition of double distributions is absolutely Lorentz invariant: it does not require reference to any particular frame. Moreover, the mutual orientation and relative size of the two momenta P and r are arbitrary. If, in some particular frame, the space part of the momentum P is oriented in the (longitudinal) x_3 direction, the four-momentum r may also have a nonzero longitudinal component, but it may be purely transverse as well, having nonzero components in the transverse x_1,x_2 plane only. The double distributions $f(\beta,\alpha;t)$ parametrizing the nonforward matrix element are Lorentz invariant objects and they are the same in all cases.

Usually, to extract the symmetric-traceless part of a tensor $O_{\mu\mu_1\cdots\mu_n}$, it is multiplied by $z^\mu z^{\mu_1}\cdots z^{\mu_n}$, where z^μ is a lightlike vector $z^2=0$. This trick corresponds to a projection of Eq. (2.1):

$$\begin{aligned} \langle P-r/2|\bar{\psi}_a\hat{z}(iz\overleftrightarrow{D})^n\psi_a|P+r/2\rangle &= 2(Pz)\sum_{k=0}^{n}\frac{n!}{2^k k!(n-k)!}A_{nk}^{(a)}(t)(Pz)^{n-k}(rz)^k \\ &+\frac{1}{2^n}D_n^{(a)}(t)(rz)^{n+1} \end{aligned} \tag{2.3}$$

(where $\hat{z}\equiv\gamma_\mu z^\mu$). The direction of z is arbitrary, but, to access all the coefficients $A_{nk}(t)$, one should have both $(Pz)\neq 0$ and $(rz)\neq 0$. In particular, if z has only the minus light-cone component, both P^+ and r^+ should be nonzero to make all the coefficients A_{nk} visible. Such a situation is characteristic for deeply virtual Compton scattering (DVCS) where the momentum transfer r must have a nonzero longitudinal component. To study DVCS, it is convenient to treat the ratio $\xi=r^+/2P^+$ as an independent variable and define off-forward parton distributions $H(\tilde{x},\xi;t)$ [2]. To this end, one introduces the functions

$$\mathcal{M}_n^{(a)}(\xi;t)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}A_{nk}^{(a)}(t)\xi^k+D_n^{(a)}(t)\xi^{n+1} \tag{2.4}$$

and declares $\mathcal{M}_n^{(a)}(\xi;t)$ to be the moments of $H_a(\tilde{x},\xi;t)$:

$$\{1\pm(-1)^n\}\mathcal{M}_n^{(a)}(\xi;t)=\int_{-1}^{1}\tilde{x}^n H_a^{\mp}(\tilde{x},\xi;t)d\tilde{x}. \tag{2.5}$$

These definitions provide a formal relation between $H(\tilde{x},\xi;t)$ and $f(\beta,\alpha;t)$:

$$H_a^{\pm}(\tilde{x},\xi;t)=\int_{-1}^{1}d\beta\int_{-1+|\beta|}^{1-|\beta|}f_a^{\pm}(\beta,\alpha;t)\,\delta(\tilde{x}-\beta-\xi\alpha)d\alpha +(1\pm 1)\mathrm{sgn}(\xi)D_a(\tilde{x}/\xi;t). \tag{2.6}$$

Combining Eqs. (2.2) and (2.3) gives the definition of DDs through the parametrization of nonforward matrix elements of nonlocal light cone operators

$$\begin{aligned}\langle P-r/2|\overline{\psi}_a(-z/2)\hat{z}\psi_a(z/2)|P+r/2\rangle|_{z^2=0} &=(Pz)\int_{-1}^{1}d\beta\int_{-1+|\beta|}^{1-|\beta|}e^{-i\beta(Pz)-i\alpha(rz)/2}(f_a^+(\beta,\alpha;t)\\ &\quad +f_a^-(\beta,\alpha;t))d\alpha+(rz)\int_{-1}^{1}e^{-i\alpha(rz)/2}D_a(\alpha;t)d\alpha.\end{aligned} \tag{2.7}$$

Using the symmetry of $f_a^{\pm}(\beta,\alpha;t)$ with respect to β and α, one can rewrite this representation in terms of quark and antiquark DDs taken for positive β only

$$\begin{aligned}\langle P-r/2|\overline{\psi}_a(-z/2)\hat{z}\psi_a(z/2)|P+r/2\rangle|_{z^2=0} &=2(Pz)\int_0^1 d\beta\int_{-1+\beta}^{1-\beta}(f_a(\beta,\alpha;t)e^{-i\beta(Pz)-i\alpha(rz)/2}\\ &\quad -f_{\bar{a}}(\beta,\alpha;t)e^{i\beta(Pz)+i\alpha(rz)/2})d\alpha+(rz)\\ &\quad \times\int_{-1}^{1}e^{-i\alpha(rz)/2}D_a(\alpha;t)d\alpha.\end{aligned} \tag{2.8}$$

In the forward limit, $r=0$, the left hand side of Eq. (2.8) coincides with the matrix element defining the usual parton densities $f_{a,\bar{a}}(x)$. This gives the reduction relations

$$\int_{-1+x}^{1-x}f_{a,\bar{a}}(x,\alpha;t=0)d\alpha=f_{a,\bar{a}}(x)$$

and

$$H_{a,\bar{a}}(x,\xi=0;t=0)=f_{a,\bar{a}}(x). \tag{2.9}$$

On the other hand, keeping $r\neq 0$ but taking $n=0$ in Eq. (2.1) one deals with the matrix element of the vector current which defines the a component $F_a(t)$ of the relevant form factor. The reduction relations connecting GPDs with form factors result from

$$F_a(t)=A_{00}^{(a)}(t)=\mathcal{M}_0^{(a)}(t) \tag{2.10}$$

and are given by the expressions

$$\int_0^1 d\beta\int_0^{1-\beta}(f_a(\beta,\alpha;t)-f_{\bar{a}}(\beta,\alpha;t))d\alpha=F_a(t),$$

$$\int_0^1(H_a(\tilde{x},\xi;t)-H_{\bar{a}}(\tilde{x},\xi;t))d\tilde{x}=F_a(t) \tag{2.11}$$

containing only the valence quark combinations, namely, $f_a^{\mathrm{val}}=f_a-f_{\bar{a}}$ and $H_a^{\mathrm{val}}=H_a-H_{\bar{a}}$.

The representation (2.8) has the structure of a plane wave decomposition, which provides the parton interpretation of DDs: the quarks carry the momentum $\beta P+(1+\alpha)r/2$ originating from both the average momentum P and the momentum transfer r. Another possibility (which is more convenient in applications involving light-cone wave functions) is to write the momenta of quarks as xp_1+yr, i.e., in terms of r and the original hadron momentum $p_1=P+r/2$. The new variables x,y are expressed through β,α by $x=\beta, y=(1+\alpha-\beta)/2$. The resulting DDs $F_{a,\bar{a}}(x,y;t)$ "live" on the triangle $0\leqslant x,y,x+y\leqslant 1$. Since $f(\beta,\alpha;t)$ are even functions of α, the DDs $F(x,y;t)$ are symmetric with respect to $y\to 1-x-y$ transformation ("Munich" symmetry [16]). For light-cone dominated processes, like DVCS, only the plus component $xp_1^+ + yr^+$ is essential. Defining the skewedness parameter $\zeta=r^+/p_1^+$, we introduce nonforward parton distributions [5]

$$\mathcal{F}_\zeta^{a,\bar{a}}(X,t)=\int_0^1 dx\int_0^{1-x}F^{a,\bar{a}}(x,y;t)\,\delta(X-x-\zeta y)dy. \tag{2.12}$$

These distributions are related to the usual parton densities by

$$\int_0^{1-x}F_{a,\bar{a}}(x,y;t=0)dy=f_{a,\bar{a}}(x),$$

$$\mathcal{F}^{a,\bar{a}}_{\zeta=0}(X;t=0)=f_{a,\bar{a}}(X) \tag{2.13}$$

and to form factors by

$$\int_0^1 dx\int_0^{1-x}F_a^{\mathrm{val}}(x,y;t)dy=F_a(t),$$

$$\int_0^1 \mathcal{F}_\zeta^{a,\mathrm{val}}(X,t)dX=F_a(t). \tag{2.14}$$

Note that the double distributions $F(x,y;t)$ are integrated over y in both of the above reduction relations. Thus, it makes sense to introduce intermediate functions

$$\mathcal{F}_{a,\bar{a}}(x,t)=\int_0^{1-x}F_{a,\bar{a}}(x,y;t)dy\equiv\mathcal{F}_{\zeta=0}(x;t). \tag{2.15}$$

They satisfy simpler reduction relations

$$\mathcal{F}_{a,\bar{a}}(x,t=0)=f_{a,\bar{a}}(x)\quad\text{and}\quad\int_0^1\mathcal{F}_a^{\mathrm{val}}(x,t)dx=F_a(t). \tag{2.16}$$

Thus, the functions $\mathcal{F}(x,t)$ are hybrids of the form factors $F(t)$ and the usual parton densities $f(x)$; that is why we call them *nonforward parton densities* (NDs) [7].

III. GAUSSIAN WAVE FUNCTION AND NONFORWARD PARTON DENSITIES

The concept of NDs can be easily illustrated within the framework of the light-cone formalism. Consider a two-body bound state whose lowest Fock component is described by a light-cone wave function $\psi(x,k_\perp)$. In a frame where the momentum transfer r is purely transverse $r=r_\perp$, one can write the two-body contribution to the form factor as [17]

$$F^{(2b)}(t)=\int_0^1 dx\int \psi^*(x,k_\perp+(1-x)r_\perp)\psi(x,k_\perp)d^2k_\perp$$

$$\equiv\int_0^1\mathcal{F}^{(2b)}(x,t)dx, \tag{3.1}$$

where $F^{(2b)}(x,t)$ is the two-body contribution into the nonforward parton density

$$\mathcal{F}^{(2b)}(x,t)=\int \psi^*(x,k_\perp+(1-x)r_\perp)\psi(x,k_\perp)d^2k_\perp . \tag{3.2}$$

Adding contributions from higher Fock components, one obtains the total ND $\mathcal{F}(x,t)$ whose integral over x gives the form factor $F(t)$ of the bound state. As discussed in the previous section, at zero momentum transfer $\mathcal{F}(x,t)$ reduces to the usual valence parton density $f(x)=\mathcal{F}(x;t=0)$. Furthermore, there is the usual form factor normalization condition $F(t=0)=1$. Finally, for the valence quark distributions, the integral of $f(x)$ over x is 1. These conditions are satisfied in the simplest way by the factorized *Ansatz* $\mathcal{F}(x,t)=f(x)F(t)$, in which there is no interplay between x and t dependence of $\mathcal{F}(x,t)$. One may expect that in reality the situation is more complicated. Consider a wave function with a Gaussian dependence on the transverse momentum $k_\perp$ (cf. [10])

$$\psi(x,k_\perp)=\varphi(x)e^{-k_\perp^2/2x(1-x)\Lambda^2} \tag{3.3}$$

[note that $k_\perp^2/4x(1-x)$ is essentially $\vec{k}^2$ written in the light-cone variables $x,k_\perp$]. Taking the Gaussian integral over $x,k_\perp$ we get

$$\mathcal{F}^{(2b)}(x,t)=f^{(2b)}(x)e^{(1-x)t/4x\Lambda^2}, \tag{3.4}$$

where

$$f^{(2b)}(x)=\pi x(1-x)\Lambda^2\varphi^2(x)=\mathcal{F}^{(tb)}(x,t=0) \tag{3.5}$$

is the two-body part of the relevant parton density $f(x)$. To get the total result for either the usual $f(x)$ or nonforward parton densities $\mathcal{F}(x,t)$, one should add the contributions due to higher Fock components. These contributions are not small, e.g., with the Gaussian *Ansatz* the valence $\overline{du}$ contribution to the normalization of the π^+ form factor for $t=0$ is about 25% [10]. The problem is that we do not have a formalism providing explicit expressions for an infinite tower of light-cone wave functions. However, the parton densities $f(x)$ are known from experiment. In this situation, one can treat Eq. (3.4) as a guide for fixing the interplay between the x and t dependence of NDs and model them by

$$\mathcal{F}^a(x,t)=f_a(x)e^{\bar{x}t/4x\Lambda^2}. \tag{3.6}$$

The functions $f_a(x)$ here are the usual valence a-quark parton densities. One can take them from existing parametrizations of parton densities such as Glück-Reya-Vogt (GRV), Martin-Roberts-Stirling (MRS), CTEQ densities, etc. This model (originally proposed in Ref. [18]) was successfully applied in Ref. [7] to describe the proton form factor $F_1(t)$ in a wide region $1<-t<10$ GeV^2 of momentum transfer by fitting the only parameter Λ^2 characterizing the effective proton size.

IV. POWER-LAW WAVE FUNCTIONS

GPDs give the most general parametrization of nonforward matrix elements. Furthermore, both of them, the DDs $F(x,y;t)$ and SPDs $\mathcal{F}_\zeta(X;t)$ are functions of three variables: in addition to the invariant momentum transfer t they depend on two "longitudinal" variables x,y or X,ζ. However, the Gaussian model of the previous section gives a representation for the form factor in terms of a one-dimensional x integral of the function $\mathcal{F}(x,t)$ depending on only two variables, x and t. One may suspect that the Gaussian *Ansatz* is a degenerate case failing to reveal the richer structure present in more general situations. In what follows, our goal is to study a model based on power-law wave functions. As we will see, although this model is more complicated, we are still able to get most of the results in analytic form, which allows us to use it for building nontrivial *Ansätze* for generalized parton distributions.

The $q\bar{q}$ wave function of the pion found numerically in [11] was parametrized analytically by a power-law fit

$$\varphi(\vec{k})\sim\left(\frac{1}{1+k^2/\Lambda^2}\right)^{\kappa}, \tag{4.1}$$

with $\kappa\sim2$ rather than by a Gaussian $\varphi(\vec{k})\sim\exp(-k^2/\Lambda_G^2)$. Here $k^2=k_z^2+k_\perp^2$ is the square of the relative three-momentum and Λ is the parameter characterizing the width of the k^2 distribution, i.e., the size of the system. In the case of equal quark masses, there is a simple relation [19] between the usual variables $(k_z,k_\perp)$ and the infinite momentum frame (IMF) variables $(x,k_\perp)$

$$x=\frac{1}{2}\left[1+\frac{k_z}{\sqrt{m^2+k_z^2+k_\perp^2}}\right], \tag{4.2}$$

where m is the effective quark mass. The relation between $\varphi(\vec{k})$ and the IMF wave function $\psi(x,k_\perp)$ is given by

$$\psi(x,k_\perp)=\varphi(\vec{k})\frac{(1+k^2/m^2)^{1/4}}{\sqrt{x(1-x)}}. \tag{4.3}$$

For light quarks, one may expect that the size parameter Λ is close to the effective quark mass m. Then the factor $(1+k^2/m^2)^{1/4}$ can be essentially absorbed into a redefinition of the power κ, whose precise value, in fact, is not critical for our purposes. Thus, in what follows, we will consider a simplified power-law IMF wave function

$$\psi(x,k_\perp)=\frac{N}{\sqrt{x(1-x)}[a+bk_\perp^2]^2}, \tag{4.4}$$

where $a+bk_\perp^2$ is the IMF version of $(1+k^2/\Lambda^2)$ with

$$a=1+\frac{s^2\left(x-\frac{1}{2}\right)^2}{x(1-x)},\quad b=\frac{s^2}{4m^2x(1-x)},\quad s=\frac{m}{\Lambda}, \tag{4.5}$$

and N is the normalization constant. For the two-body Fock component, it can be fixed from the requirement that the integral of $\psi(x,k_\perp)$ over x and $k_\perp$ should give the pion decay constant

$$\sqrt{\frac{3}{2\pi^3}}\int \psi^{(2b)}(x,k_\perp)dxd^2k_\perp=f_\pi. \tag{4.6}$$

As noted in the previous section, the knowledge of the two-body wave function is not sufficient to calculate the pion form factor. To get it, we should add the contribution from all higher Fock components. Just as in the case of the Gaussian wave function, the two-body component is responsible only for some portion of "1" in the normalization condition $F(0)=1$ [10,12]. Again, the structure of higher Fock components can only be guessed. To avoid making too many guesses, we will analyze the simplest "one-guess" model in which a single two-body-like function $\psi(x,k_\perp)$ (4.4) imitates the contribution of all Fock components into the pion form factor. Thus, we take

$$F(Q^2)=\int \psi(x,k_\perp+(1-x)q_\perp)\psi(x,k_\perp)dxd^2k_\perp, \tag{4.7}$$

and normalize the effective wave function $\psi(x,k_\perp)$ by

$$\int |\psi(x,k_\perp)|^2\,dxd^2k_\perp=1. \tag{4.8}$$

This condition (4.8) gives an explicit expression for the normalization constant N of the effective wave function:

$$N^2=\frac{3}{4\pi}\left(\frac{s}{m}\right)^2\frac{1}{A(s)}, \tag{4.9}$$

where

$$A(s)=\int_0^1\frac{dx}{a^3}=\int_0^1\frac{(1-z^2)^3}{[1-(1-s^2)z^2]^3}\,dz. \tag{4.10}$$

Before proceeding further, we would like to make it clear that substituting the total contribution of higher Fock components by a two-body type term is just a toy model, and we do not expect it to adequately describe all the aspects of the pion structure. In particular, the total parton density in the toy model has the same ($x\to1-x$ symmetric) shape as its two-body part, and it vanishes at $x=0$. One would expect, however, that the contributions of higher Fock components are shifted to smaller and smaller x values, producing eventually the experimentally observed $\sim1/\sqrt{x}$ behavior. We do not know how much each term of the infinite tower of Fock components contributes to the parton density, but we know (from experiment) what is the total result. Thus, our ultimate strategy, just as in the case of the Gaussian wave function, is to calculate GPDs in the toy model, identify the factor corresponding to the usual parton density, and substitute it by the experimental one. On the other hand, one may expect that the form factor, being an integral of the relevant GPD, should not be too sensitive to the details of its x dependence, at least in some range of momentum transfer t. Thus, we study first the form factor in our toy model. We show that, despite its crudeness, the toy model can easily fit the form factor data by adjusting the two parameters of the model. Then we incorporate the main advantage of the toy model, the possibility to do calculations analytically, and obtain the representation for the form factor in terms of DDs. Finally, we "correct" the latter in such a way that, after integration, they produce experimental parton densities. We also show that this model gives DDs with a nontrivial "profile" dependence on the y variable.

V. FORM FACTOR IN TOY MODEL

The $k_\perp$ integral in the expression for the form factor

$$F(Q^2)=N^2\int\frac{dx\,d^2k_\perp}{x(1-x)[a+b(k_\perp+(1-x)q_\perp)^2]^2[a+bk_\perp^2]^2} \tag{5.1}$$

can be done using either the Feynman parameters or the Schwinger α-representation method briefly described below. To this end, we use

$$\frac{1}{A^\kappa}=\frac{1}{\Gamma(\kappa)}\int_0^\infty \alpha^{\kappa-1}e^{-\alpha A}d\alpha, \tag{5.2}$$

where $\kappa=2$ in our case. After calculating the Gaussian integral over $k_\perp$, we arrive at the representation for the form factor in terms of two parameters α_1 and α_2:

$$F(Q^2)=\pi N^2\int_0^\infty\int_0^\infty\frac{\alpha_1 d\alpha_1\alpha_2 d\alpha_2}{(\alpha_1+\alpha_2)}\times\int_0^1 e^{-a(\alpha_1+\alpha_2)}e^{-b(1-x)^2Q^2\alpha_1\alpha_2/(\alpha_1+\alpha_2)}\times\frac{dx}{bx(1-x)}, \tag{5.3}$$

where $Q^2=q_\perp^2$. Changing the variables

$$\alpha_1+\alpha_2=\lambda,\quad \alpha_1=\gamma\lambda,\quad \alpha_2=(1-\gamma)\lambda,\quad d\alpha_1 d\alpha_2=\lambda d\lambda d\gamma, \tag{5.4}$$

we obtain the parametric representation

$$F(Q^2)=\pi N^2\int_0^1\frac{dx}{bx(1-x)}\int_0^1 d\gamma\,\gamma(1-\gamma)\times\int_0^\infty\lambda^2 d\lambda\, e^{-[a\lambda+b\lambda(1-x)^2\gamma(1-\gamma)Q^2]}. \tag{5.5}$$

Integration over λ is easily performed to give

$$F(Q^2)=2\pi N^2\int_0^1\frac{dx}{bx(1-x)}\times\int_0^1 d\gamma\frac{\gamma(1-\gamma)}{[a+b(1-x)^2\gamma(1-\gamma)Q^2]^3}. \tag{5.6}$$

Incorporating the normalization condition, we get the final result

$$F_\pi(Q^2)=\frac{1}{A(s)}\int_0^1 dx\int_0^1 d\gamma\frac{6\gamma(1-\gamma)}{[a+b(1-x)^2\gamma(1-\gamma)Q^2]^3}. \tag{5.7}$$

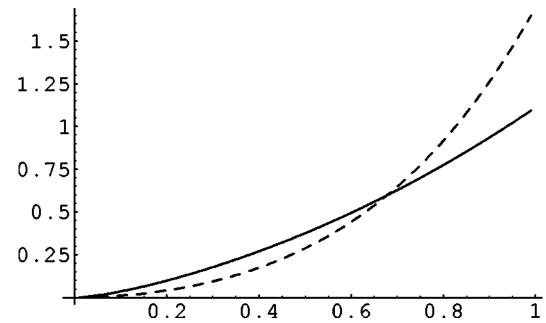

FIG. 1. Combinations $s^2B(s)/A(s)$ (solid) and $s^4C(s)/A(s)$ (dashed) as functions of the parameter s.

By construction, the form factor has the correct value at $Q^2=0$. However, its slope at this point depends on the values of the model parameters m and s. To obtain the analytic expression for the slope we note that, in the small Q^2 limit, one can expand the denominator of the γ integration

$$F_\pi(Q^2)|_{Q^2\to 0}=\frac{6}{A(s)}\int_0^1\frac{dx}{a^3}\int_0^1 d\gamma\,\gamma(1-\gamma)\times\left[1-\frac{3b}{a}\gamma(1-\gamma)Q^2(1-x)^2+\frac{6b^2}{a^2}[\gamma(1-\gamma)]^2Q^4(1-x)^4+\cdots\right]. \tag{5.8}$$

Using the normalization condition (4.8) for the wave function, taking the integrals over γ, and introducing the variable $z=2x-1$ [such as in Eq. (4.10)] we obtain

$$F_\pi(Q^2)|_{Q^2\to 0}=1-\frac{3}{20}\frac{Q^2}{m^2}s^2\frac{B(s)}{A(s)}+\frac{9}{280}\frac{Q^4}{m^4}s^4\frac{C(s)}{A(s)}+\cdots, \tag{5.9}$$

where

$$B(s)=\int_0^1\frac{(1+z^2)(1-z^2)^3}{(1-(1-s^2)z^2)^4}\,dz \tag{5.10}$$

and

$$C(s)=\int_0^1\frac{(1+z^4+6z^2)(1-z^2)^3}{(1-(1-s^2)z^2)^5}\,dz. \tag{5.11}$$

The integrals $A(s)$, $B(s)$, and $C(s)$ can be calculated in elementary functions, although the results are rather lengthy. Figure 1 shows the plot of the combinations $s^2B(s)/A(s)$ and $s^4C(s)/A(s)$ as functions of the parameter s. The combination $s^2B(s)/A(s)$ is monotonically increasing from zero to infinity. Hence, after choosing the effective mass m we can always find such a parameter s that the slope $dF_\pi(Q^2)/dQ^2$ of the pion form factor at $Q^2=0$ has the experimental value $dF_\pi^{expt}(Q^2)/dQ^2\approx 1/m_\rho^2$ [20]. For masses $m=0.2$, 0.3, and

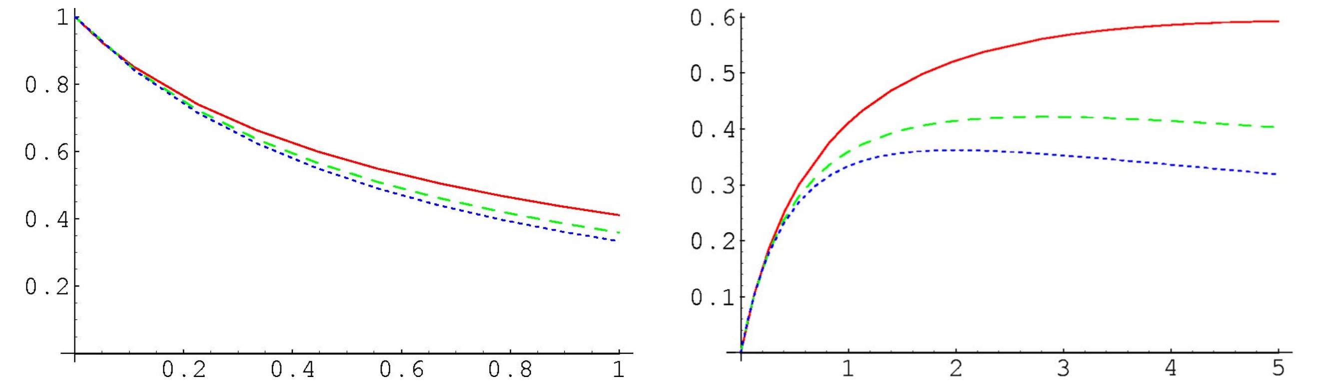

FIG. 2. (Color online) Left: Form factor $F_\pi(Q^2)$ for three different parametrizations of the wave function: with $m=0.2$ GeV (solid), 0.3 GeV (dashed), and 0.4 GeV (dotted). Right: $Q^2F_\pi(Q^2)$, with the same wave functions as in the left panel.

0.4 GeV, the parameters s fitting the pion charge radius are 0.56, 0.95, and 1.33, respectively.

In Fig. 2 (left) we have plotted the form factor as a function of Q^2 in the low Q^2 region $Q^2<1$ GeV^2 for these three different parametrizations of the wave function. Since they have the same slope at $Q^2=0$, the curves are rather close to each other. However, the difference between the curves becomes more pronounced as Q^2 increases. In Fig. 2 (right) the form factor calculated with these three effective wave functions is shown in the Q^2 region up to 5 GeV^2 relevant to future experiments at Jefferson Lab. Among these three choices, the closest to existing experimental data [21] is the curve corresponding to $m=0.3$ GeV and $s=0.95$.

VI. ASYMPTOTIC BEHAVIOR OF THE PION FORM FACTOR

According to Fig. 2 (right) in the accessible energy range $Q^2\lesssim 5$ GeV^2, the model curves show the behavior close to the $1/Q^2$ scaling. Since the mass scales involved are rather small, $m^2,\Lambda^2\sim 0.1$ GeV^2, one may think that the form factor is already in the asymptotic region.

To analyze the asymptotic behavior of the form factor one can follow the approach described in Ref. [22]. The basic idea is that in the Drell-Yan formula (4.7) we deal with an overlap of two functions $\psi(x,k_\perp)$ and $\psi(x,k_\perp-(1-x)q_\perp)$ whose $k_\perp$ arguments are separated by a gap $(1-x)Q$ in magnitude. Furthermore, $\psi(x,k_\perp)$ rapidly decreases with increasing $k_\perp$. Hence, when Q^2 is large, the integral over $k_\perp$ in the form factor expression is dominated by two regions of phase space [22]:

(1) $|k_\perp|\ll(1-x)Q$, where $\psi(x,k_\perp)$ is large;

(2) $|k_\perp+(1-x)q_\perp|\ll(1-x)Q$, where $\psi(x,k_\perp+(1-x)q_\perp)$ is large.

In the first case, $k_\perp$ can be neglected compared to $(1-x)q_\perp$ in the wave function. The contribution from this region is then approximated by

$$F_\pi^{(1)}(Q^2)\sim\int_0^1\psi(x,(1-x)Q)dx \times\int\theta(|k_\perp|<(1-x)Q)\psi(x,k_\perp)d^2k_\perp. \quad (6.1)$$

Since the wave function falls off rapidly at large transverse momenta, the major contribution to the integral comes from the region where $|k_\perp|$ is much smaller than $(1-x)Q$, and one may hope that the $k_\perp$ integral of $\psi(x,k_\perp)$ can be approximated by the pion distribution amplitude $\varphi(x)$. The next statement usually made is that the large-Q behavior of the function $\psi(x,(1-x)Q)$ is determined by the large-$k_\perp$ behavior of $\psi(x,k_\perp)$ and, hence, the large-Q behavior of the form factor just repeats the large-$k_\perp$ behavior of the wave function; i.e., if $\psi(x,k_\perp)\sim 1/k_\perp^n$, then $F\sim 1/Q^n$. Note that the last statement is true only if, after these substitutions, the integral over x converges. However, it is easy to derive that, after the $k_\perp$ integration in Eq. (6.1), the remaining integrand for the x integration is proportional to

$$\frac{x^3(1-x)^5Q^2}{[x(1-x)+s^2(x-1/2)^2+(1-x)^2Q^2/4\Lambda^2]^3}. \quad (6.2)$$

Neglecting the $x(1-x)$ and $s^2(x-1/2)^2$ terms compared to $(1-x)^2Q^2/4\Lambda^2$, one would get the integral $dx/(1-x)$ logarithmically diverging in the $x\to 1$ region. Of course, this approximation is only true when $(1-x)^2Q^2/4\Lambda^2\gg 1$ or $x\ll 1-2\Lambda/Q$. This cut-off converts the logarithmic divergence into $\ln(Q^2/\Lambda^2)$. Hence, the asymptotic behavior is $\sim\ln(Q^2/\Lambda^2)/Q^4$. This result can also be obtained from our representation (5.7) for the form factor, which we write now as

$$F_\pi(Q^2)=\frac{1}{A(s)}\int_0^1 dx\int_0^1 d\gamma \times\frac{6\ \gamma(1-\gamma)x^3(1-x)^3}{[x(1-x)+s^2(x-1/2)^2+\gamma(1-\gamma)(1-x)^2Q^2/4\Lambda^2]^3}. \quad (6.3)$$

Again, there are two regions, $\gamma\ll 4\Lambda^2/[(1-x)^2Q^2]$ and $1-\gamma\ll 4\Lambda^2/[(1-x)^2Q^2]$ producing the leading large-Q^2

contribution $\sim\Lambda^4/[(1-x)^2Q^2]^2$. Combined with other x-dependent factors, this gives the $dx/(1-x)$ divergence or, after a more accurate calculation, the $\ln(Q^2/\Lambda^2)/Q^4$ asymptotic behavior. This behavior is not yet visible in the curves shown in Fig. 2 (right). The curves suggest, in fact, the $1/Q^2$ behavior. The slow approach to asymptopia can be traced to the rather small numerical factor $\gamma(1-\gamma)/4 \sim 1/16$ accompanying the Q^2 term. As a result, the effective scale governing the Q^2 behavior is something like $16\Lambda^2 \sim 1.5\ \mathrm{GeV}^2$ rather than simply Λ^2. Thus, the quark mass squared $m^2\sim 0.1\ \mathrm{GeV}^2$ is small compared to the effective scale, and it is worth investigating what happens when quarks are massless, i.e., when $s=0$. Then

$$F_\pi(Q^2)|_{s=0}=\int_0^1 dx\int_0^1 d\gamma\,\frac{6\gamma(1-\gamma)x^3}{[x+\gamma(1-\gamma)(1-x)Q^2/4\Lambda^2]^3}, \tag{6.4}$$

and the situation changes drastically: the large-Q^2 behavior is dominated by integration over the $1-x\ll 4\Lambda^2/[\gamma(1-\gamma)Q^2]$ region. The remaining integral over γ has no singularities, and we get $F_\pi(Q^2)\sim\Lambda^2/Q^2$ for the asymptotic behavior. Clearly, the asymptotic behavior for massless quarks is governed by the soft (or Feynman) mechanism. One can easily check that the same asymptotic power law $F_\pi(Q^2)\sim 1/Q^2$ holds for the $\psi(x,k_\perp)\sim 1/[\sqrt{x(1-x)}(1+bk_\perp^2)^\kappa]$ wave functions with any power κ, and also for the exponential wave function $\psi(x,k_\perp)\sim e^{-bk_\perp^2}/\sqrt{x(1-x)}$. This puzzling observation has a rather simple explanation: the valence parton densities in these models with massless quarks are constant, $f(x)=1$, and it is this singular $f(x)|_{x\to 1}\to$const behavior which is responsible for the $1/Q^2$ contribution to the form factor. If we "corrected" the model so that $f(x)$ has a more realistic $\sim(1-x)$ behavior for x close to 1, the Feynman mechanism contribution would have a $1/Q^4$ asymptotic behavior.

An efficient way to obtain the asymptotic expansion in powers and logarithms of Λ^2/Q^2 is based on the Mellin representation for the denominator factor:

$$\left[x+\gamma(1-\gamma)(1-x)\frac{Q^2}{4\Lambda^2}\right]^{-3} =\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\Gamma(-J)\Gamma(J+3) \times\gamma^J(1-\gamma)^J(1-x)^J x^{-J-3}\left(\frac{Q^2}{4\Lambda^2}\right)^J dJ. \tag{6.5}$$

Now, the γ and x integrals can be calculated in Γ functions to give

$$F_\pi(Q^2)=\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}6\,\frac{\Gamma(-J)\Gamma(1-J)}{(J+1)\Gamma(2J+5)} \times\Gamma^2(J+2)\Gamma^2(J+3)\left(\frac{Q^2}{4\Lambda^2}\right)^J dJ. \tag{6.6}$$

The integrand has poles at integer J in the left half plane. We explicitly displayed the rightmost of these poles $1/(J+1)$. It corresponds to $x\sim 1$ integration and gives the leading asymptotic contribution equal to $12\Lambda^2/Q^2$. Expanding the integrand in the vicinity of $J=-2,-3$, etc., we can get subleading contributions. Note that the singularity at $J=-2$ is a double pole $1/(J+2)^2$. Hence, this contribution will have the $(\Lambda^2/Q^2)^2\ln(Q^2/\Lambda^2)$ term. One can also close the integration contour in the right half plane. This procedure gives the small-Q^2 expansion of $F_\pi(Q^2)$, the first terms of which are explicitly written in Eq. (5.8).

VII. DOUBLE DISTRIBUTIONS IN THE TOY MODEL

Let us now analyze the connection of our expression for the pion form factor

$$F_\pi(Q^2)=\frac{1}{A(s)}\int_0^1 dx\int_0^1 d\gamma\,\frac{6\gamma(1-\gamma)}{[1+s^2(x-1/2)^2/x(1-x)+\gamma(1-\gamma)s^2(1-x)Q^2/4m^2x]^3} \tag{7.1}$$

with generalized parton distributions. Introducing the variable $y=(1-x)\gamma$, we can rewrite this formula as

$$F_\pi(Q^2)=\frac{1}{A(s)}\int_0^1 dx\int_0^1 dy\,\theta(x+y\leqslant 1)\,\frac{6y(1-x-y)/(1-x)^3}{[1+s^2(x-1/2)^2/x(1-x)+y(1-x-y)s^2Q^2/4m^2x(1-x)]^3}. \tag{7.2}$$

It may be treated as the standard representation [3]

$$F_\pi(Q^2)=\int_0^1 dx\int_0^{1-x} F(x,y;-Q^2)\,dy \tag{7.3}$$

of the pion form factor in terms of the double distribution[1]

$$F(x,y;t)=\theta(x+y\leqslant 1)\ \frac{6y(1-x-y)/(1-x)^3}{A(s)[1+s^2(x-1/2)^2/x(1-x)-ty(1-x-y)s^2/4m^2x(1-x)]^3} \tag{7.4}$$

(we switched to $t\equiv -Q^2$ to conform with the standard notation for generalized parton distributions). This double distribution has correct spectral properties [3,5]: it vanishes outside the triangle $0\leqslant x,y,x+y\leqslant 1$. It also satisfies the Munich symmetry condition [16]

$$F(x,y;t)=F(x,1-x-y;t). \tag{7.5}$$

Furthermore, for $t=0$, it has the factorized form suggested in Ref. [9]:

$$F(x,y;t=0)=\theta(x+y\leqslant 1)h(x,y)f(x), \tag{7.6}$$

in which the y dependence appears only in the normalized profile function

$$h(x,y)=6y(1-x-y)/(1-x)^3 \tag{7.7}$$

satisfying

$$\int_0^{1-x} h(x,y)dy=1. \tag{7.8}$$

The remaining factor

$$f(x)=\frac{1}{A(s)[1+s^2(x-1/2)^2/x(1-x)]^3} \tag{7.9}$$

depends on x only, and may be interpreted as the parton distribution for the valence quarks inside the pion. For nonzero t, the profile function $h(x,y)$ also factorizes out in the expression for the double distribution, Eq. (7.4). However, it is multiplied by a function that has a nontrivial dependence on all three variables x, y, and t. In Fig. 3, we plot $F(x,y;t)$ as a function of x and y for a few values of t.

The form of the double distribution presented above corresponds to the parametrization $k=xp_1+yr$ of the active quark momentum k in terms of the initial pion momentum p_1 and the momentum transfer $r=p_1-p_2$. The final state pion has then the momentum $p_2=p_1-r$: the initial and final pions are not treated symmetrically in this formalism. To reinforce the symmetry, one should introduce the average momentum $P=(p_1+p_2)/2$ (the initial and final pion momenta are then $P\pm r/2$; see Sec. II) and write the active quark momentum as $k=\beta P+(1+\alpha)r/2$. To get the relevant double distribution $f(\beta,\alpha;t)$, we write the γ variable as $\gamma=(1+\eta)/2$ and then introduce α by $\alpha=(1-\beta)\eta$. This gives

$$f(\beta,\alpha;t)=\theta(|\alpha|\leqslant 1-\beta)\ \frac{\frac{3}{4}[(1-\beta)^2-\alpha^2]/(1-\beta)^3}{A(s)[1+s^2(\beta-1/2)^2/\beta(1-\beta)-[(1-\beta)^2-\alpha^2]s^2t/16m^2\beta(1-\beta)]^3}. \tag{7.10}$$

The normalized profile function in this case is $\frac{3}{4}[(1-\beta)^2-\alpha^2]/(1-\beta)^3$.

Integration over $k_\perp$ can be performed in a similar way for a more general power-law LC wave function $\psi(x,k_\perp)\sim(a+bk_\perp^2)^{-\kappa}/\sqrt{x(1-x)}$. In this case, one obtains DDs with κ-dependent profiles $\sim[(1-\beta)^2-\alpha^2]^{\kappa-1}/(1-\beta)^{2\kappa-1}$. The power of the denominator factor in Eq. (7.10) also changes from 3 to $2\kappa-1$. Note that the faster the decrease of $\psi(x,k_\perp)$ with $k_\perp$, the narrower is the α profile of the resulting DD and the faster its decrease with $-t$. The purely exponential wave function ($\kappa\to\infty$) gives an infinitely narrow profile function $\delta(\alpha)$ (or $\delta[y-(x-1)/2]$ in the case of $F(x,y;t)$). The integral over α (or y) is trivial, and this is the formal reason why the Gaussian model gives a one-dimensional integral representation for the form factor. As suspected, the Gaussian model is "too narrow": it cannot reveal the $y(\alpha)$ profile feature inherent to DDs in the general case.

VIII. SKEWED DISTRIBUTIONS IN TOY MODEL

Having the expression for the double distribution, we can construct the nonforward distributions [5] in the standard way from

$$\mathcal{F}_\zeta(X,t)=\int_0^1 dx\int_0^{1-x} F(x,y;t)\,\delta(X-x-\zeta y)dy. \tag{8.1}$$

Note that for $F(x,y;t)$ given by Eq. (7.4) the integrations again can be performed in elementary functions. Hence, in this particular model, one can analytically study the interplay

[1]In Appendix A, it is demonstrated that the variables x,y in Eq. (7.4) have the same meaning as in the DD definition.

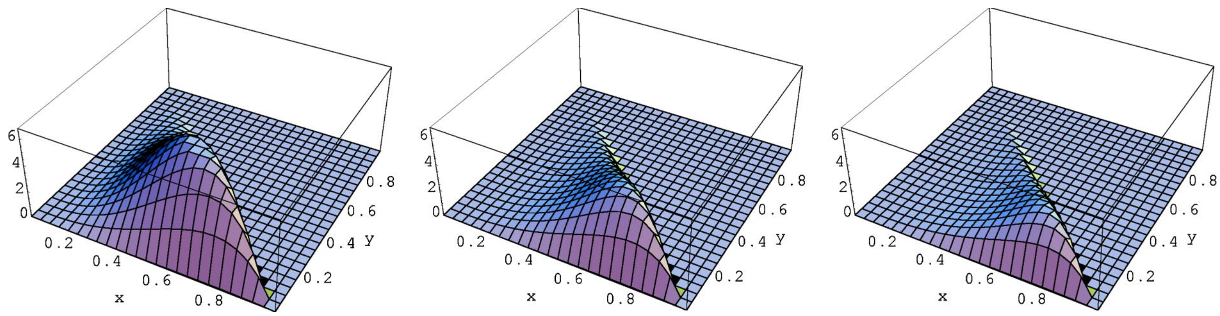

FIG. 3. (Color online) $F(x,y;t)$ as a function of x and y for three values of $t=0$, -0.5, and -1 GeV^2.

between X, ζ, and t dependence of the nonforward parton distributions (although the expressions are now really lengthy).

In Fig. 4, we plot $\mathcal{F}_\zeta(X,t)$ as a function of X and t for some values of ζ. Note that for each value of ζ the nonforward distributions satisfy the reduction formula

$$\int_0^1 \mathcal{F}_\zeta(X,t)\,dX = F_\pi(-t). \tag{8.2}$$

An important point is that the right hand side here has no dependence on the skewedness parameter ζ.

One can also use the symmetric double distribution $f(\beta,\alpha;t)$ and the relation

$$H^{\mathrm{val}}(\tilde{x},\xi;t) = \int_0^1 d\beta \int_{-1+\beta}^{1-\beta} f^{\mathrm{val}}(\beta,\alpha;t)\,\delta(\tilde{x}-\beta-\xi\alpha)\,d\alpha \tag{8.3}$$

to obtain Ji's off-forward parton distributions $H^{\mathrm{val}}(\tilde{x},\xi;t)$. Note that for the infinitely narrow profile function $f^{\{\infty\}}(\beta,\alpha;t) = F(\beta,t)\,\delta(\alpha)$corresponding to the purely Gaussian wave function, the OFPD $H(\tilde{x},\xi;t)$ is given by the ξ-independent function $H^{\{\infty\}}(\tilde{x},\xi;t) = F(\tilde{x},t)$: there are no skewedness effects. Thus, the SPDs obtained from the DDs based on a power-law *Ansatz* have a richer structure.

IX. "REALISTIC" MODEL

The function $f(x)$, Eq. (7.9), was interpreted above as the toy model version of the valence quark distribution in the pion. However, its form strongly differs from the usual phenomenological parametrizations. At a normalization point $\mu\sim1$ GeV, the latter have a form close to

$$f^R(x) = \frac{3}{4}(1-x)/\sqrt{x},$$

with the $1/\sqrt{x}$ reflecting the Regge behavior due to exchanges that are not taken into account in the toy model. In the latter, we assumed that the contribution from the higher Fock components to $f(x)$ has the same shape as the two-body one. Also, the expression that we use for the two-body wave function is again just a model guess. In particular, as shown in Appendix A, the double distribution of our toy model can be obtained from the scalar triangle diagram taken at spacelike virtualities of the external currents imitating the pions. In the spin-1/2 case, there are extra x-dependent factors originating from the numerators of propagators. Thus, one should not take the x dependence of the toy model DD too seriously. On the other hand, we observed that the y dependence of the model DD has a rather universal structure: for $t=0$, it is given by the profile function $\sim[y(1-x-y)]^{\kappa-1}$ only. For $t\neq0$, the y dependence appears also in the denominator factor, but it has a simple structure basically resulting from kinematics. These observations suggest "minimally correcting" the model DD: to change its x shape at $t=0$ without changing the pattern of its y and t dependence. To preserve the analytic structure of the interplay between the t vs x and y dependence of DDs dictated by the simplest *Ansatz* (7.4) we take the model

$$F^R(x,y;t) = F(x,y;t)\,\frac{f^R(x)}{f(x)}. \tag{9.1}$$

In terms of the effective two-body-like LC wave function, this corresponds to the change

$$\psi(x,k_\perp) \to \psi^R(x,k_\perp) = \psi(x,k_\perp)\sqrt{\frac{f^R(x)}{f(x}}. \tag{9.2}$$

The parameters m and s of the new model should again be fixed by fitting the slope of the pion form factor at $t=0$ and its behavior in the $-t\sim1$ GeV^2 region. In Fig. 5, we show the curve for the pion form factor obtained with the *Ansatz* (9.1) and the values $m=0.46$ and $s=0.81$. It practically coincides with the curve obtained within the original model for $m=0.3$ GeV and $s=0.95$ in the region of "not-so-high" transfer.

With the new DDs, one can obtain a realistic model for the SPDs $\mathcal{F}_\zeta^R(X,t)$ via Eq. (8.1). The SPD is presented in Fig. 6 as a function of X and t for some values of ζ.

As a more explicit illustration of the t dependence of SPDs, in Fig. 7 we show SPDs $F_\zeta(X;t)$ with different ζ's for two different values of t, for both the original and the "realistic" model.

When ζ increases, the maxima of SPDs shift to higher values of X. The rate of change is more drastic in the case of the realistic model, where the SPD changes from a monotonically decreasing curve for $\zeta=0$ (corresponding to the

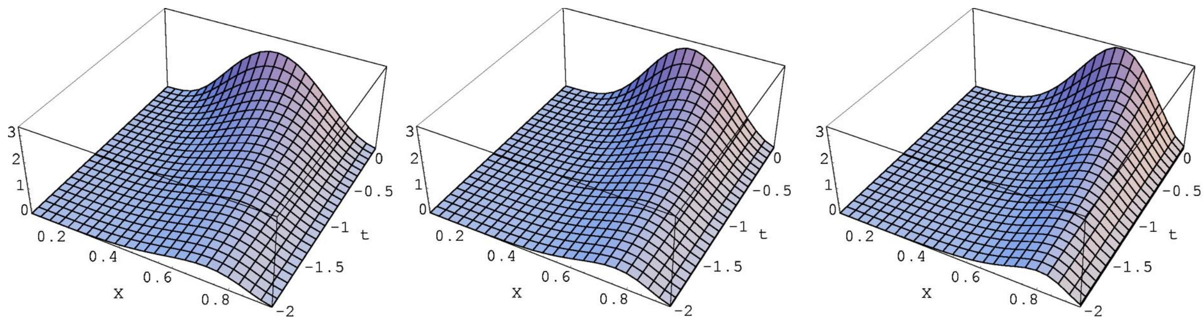

FIG. 4. (Color online) $\mathcal{F}_\zeta(X,t)$ as a function of X and t for three values of $\zeta=0.2$, 0.4, and 0.6.

usual parton density) to a shape resembling that of distribution amplitudes, as ζ tends to 1. It is interesting to analyze the limiting case $\zeta\to 1$. As demonstrated by Polyakov [23], in the soft pion limit, $m_\pi^2\to 0,\zeta\to 1,t=0$, the isovector part of the pion SPD should coincide with the pion distribution amplitude. To check if this constraint is satisfied by our models, we take the function $F_{\zeta=1}(X;t=0)$ and symmetrize it with respect to $X\leftrightarrow 1-X$ to project onto the isovector component. In Fig. 8, we show the results both for the toy model and the realistic model. For the toy model, we obtain a double humped curve, resembling the Chernyak-Zhitnitsky (CZ) model $\varphi^{CZ}(X)=30(1-2X)^2X(1-X)$. More precisely, this curve can be fit, with good accuracy, by the sum $0.43\ \varphi^{as}(X)+0.57\ \varphi^{CZ}(X)$ of the CZ and the asymptotic distribution amplitude $\varphi^{as}(X)=6X(1-X)$. The pion distribution amplitude (DA) in the toy model can be obtained directly by integrating the two-body wave function $\psi(x,k_\perp)$ over the transverse momentum. The result is close to the asymptotic amplitude, so one can say that the toy model does not satisfy the constraint imposed by the Polyakov soft pion theorem. On the other hand, in the realistic model, the function $[F_{\zeta=1}(X;t=0)+F_{\zeta=1}(1-X;t=0)]/2$ is very close to the asymptotic form and to the distribution amplitude obtained from the two-body wave function. Experimentally, the pion DA is known to be rather close to the asymptotic form. Thus, the realistic model *de facto* satisfies the constraint imposed by the soft pion theorem.

X. CONCLUSIONS

In this paper, we demonstrated how to obtain a model for the valence (or C-odd) pion double distribution $F(x,y;t)$ and the skewed parton distribution $\mathcal{F}_\zeta(X;t)$ satisfying, by construction, such important constraints as reduction relations to usual parton densities and form factors and spectral and polynomiality conditions. The SPDs derived in our model have a nontrivial interplay between X, ζ, and t dependence. Furthermore, they were adjusted to describe pion form factors for all available t, so we expect that our model describes the t dependence of the pion GPDs of the valence quarks for both small and large t. The ability to have a unified model for GPDs from $t=0$ to $|t|\sim 10\ \text{GeV}^2$ is especially important in (future) applications to nucleons, for which GPDs are already being studied experimentally for both small (DVCS) and large t (wide-angle Compton scattering).

ACKNOWLEDGMENTS

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APPENDIX A: DD IN A SCALAR MODEL

Our conversion of the integral representation for the form factor $F(t)$ into a function $F(x,y;t)$ of three variables may

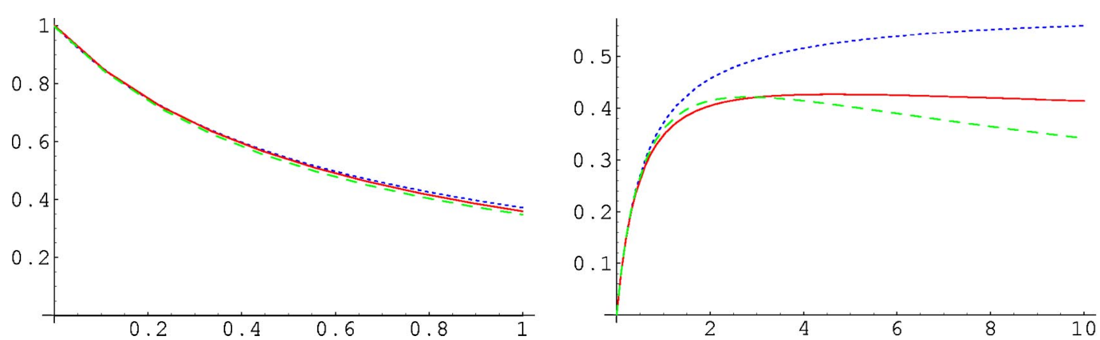

FIG. 5. (Color online) Form factor $F_\pi(Q^2)$ (left) and $Q^2F_\pi(Q^2)$ GeV^2 (right) obtained with the "realistic" model for double distributions (solid line). For comparison, we present the results for the original model (dashed) and "ρ-meson fit" $Q^2/[1+Q^2/(0.77\ \text{GeV})^2]$ (dotted).

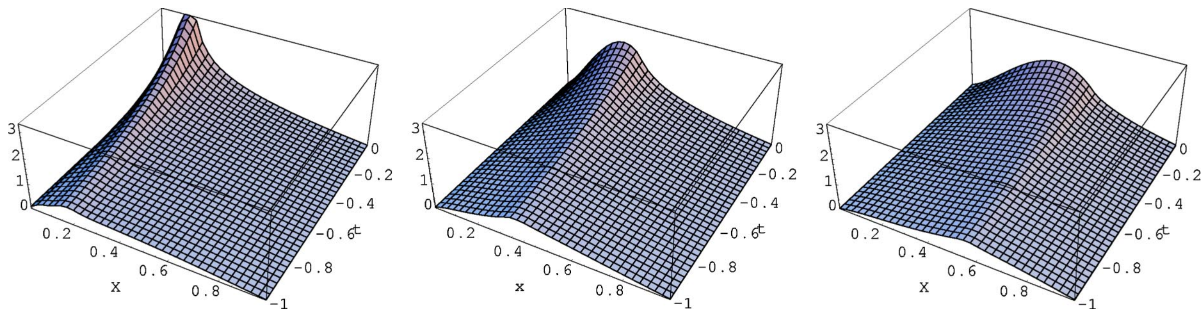

FIG. 6. (Color online) $\mathcal{F}^R_\zeta(X,t)$ obtained with the "realistic" DD as a function of X and t for three values of $\zeta=0.2$, 0.4, and 0.6.

look like a rather ambiguous exercize. Below, by a covariant field-theoretic calculation, we demonstrate that x and y really have the meaning of the variables of a double distribution.

First, consider a one-loop box diagram for the scalar analogue of deeply virtual Compton scattering amplitude (see Fig. 9). The initial and final "pions" are imitated by scalar currents Π corresponding to spacelike momenta p_1 and p_2, the initial "photon" momentum is q_1, and that of the final one is q_2. The momentum invariants describing this four-point function are

$$p_1^2, p_2^2, Q^2=-q_1^2, \quad q_2^2=0, \quad s=(p_1+q_1)^2,$$

$$t=(p_1-p_2)^2. \tag{A1}$$

Using the α representation

$$\frac{1}{m^2-k^2-i\epsilon}=i\int_0^\infty e^{i\alpha(k^2-m^2+i\epsilon)}d\alpha \tag{A2}$$

for each of four scalar propagators and calculating the resulting Gaussian integral over the loop momentum k we obtain

$$T(p_1,p_2,q_1)=-\int_0^\infty \exp\left\{ i\left[\frac{\alpha_1(\alpha_3 s-\alpha_4 Q^2)+\alpha_2\alpha_4 t+\alpha_3(\alpha_4 p_1^2+\alpha_2 p_2^2)}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}-\rho(m^2-i\epsilon)\right]\right\}\frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\rho^2}, \tag{A3}$$

where $\rho\equiv\alpha_1+\alpha_2+\alpha_3+\alpha_4$. We are interested in the Bjorken kinematics when there are two large variables s and Q^2 which have the same order of magnitude $s\sim(1/x_{Bj}-1)Q^2$, while other invariants are small. The large-Q^2 asymptotics in this situation is determined by integration over the region where the coefficients accompanying s and Q^2 vanish simultaneously. Otherwise, the integrand rapidly oscillates and the result of integration is exponentially suppressed. Integration over $\alpha_1\sim 0$ region is the simplest (and, as inspection shows, the leading) possibility. It corresponds to hard momentum flow through the propagator connecting the photon vertices. Performing the $\alpha_1\sim 0$ integration, we obtain

$$T(p_1,p_2,q_1)=-i\int_0^\infty \frac{d\alpha_2 d\alpha_3 d\alpha_4/\lambda^2}{s\alpha_3/\lambda-Q^2\alpha_4/\lambda+i\epsilon} \times\exp\left\{\frac{i}{\lambda}[\alpha_2\alpha_4 t+\alpha_3(\alpha_4 p_1^2+\alpha_2 p_2^2)] -i\lambda(m^2-i\epsilon)\right\}+O(1/Q^4), \tag{A4}$$

where $\lambda\equiv\alpha_2+\alpha_3+\alpha_4$. Denoting $\nu=2(p_1q_1)$ and writing $s=\nu-Q^2$ (p_1^2 is neglected compared to Q^2), we represent the Q^2-dependent term in the denominator as $\nu\alpha_3/\lambda-Q^2(\alpha_3+\alpha_4)/\lambda$, or finally as $\nu[\alpha_3/\lambda-\zeta(1-\alpha_2/\lambda)]$, where $\zeta=Q^2/\nu$. Introducing the double distribution

$$F(x,y;t,p_1^2,p_2^2)=i\int_0^\infty \delta\left(x-\frac{\alpha_3}{\lambda}\right)\delta\left(y-\frac{\alpha_2}{\lambda}\right) \times\exp\left\{\frac{i}{\lambda}[\alpha_2\alpha_4 t+\alpha_3(\alpha_4 p_1^2+\alpha_2 p_2^2)] -i\lambda(m^2-i\epsilon)\right\}\frac{d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2}, \tag{A5}$$

we can write the scalar DVCS amplitude in the partonic form

$$T(p_1,p_2,q_1)=-\frac{1}{\nu}\int_0^1\int_0^1 \frac{F(x,y;t,p_1^2,p_2^2)}{x+y\zeta-\zeta+i\epsilon}dx\,dy. \tag{A6}$$

A few comments are in order. First, we reemphasize that there is no ζ dependence in the definition of DDs. The second comment concerns the spectral properties of DDs. It is

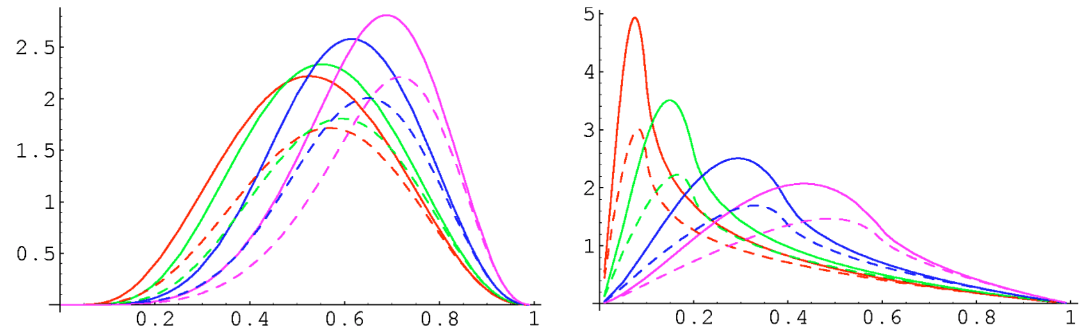

FIG. 7. (Color online) SPDs $F_{\zeta}(X;t)$ with $\zeta=0.1,0.2,0.4,0.6$ are shown for two different values $t=0$ (solid lines) and $t=-0.2$ GeV^2, for both the original (left) and "realistic" (right) model. The curves corresponding to larger ζ have maxima at higher X.

easy to see that both variables x,y vary between 0 and 1. Furthermore, their sum is also confined within these limits: $0\leqslant x+y\leqslant 1$. Finally, the hard amplitude depends on the DD variables x,y through the combination $x+y\zeta$ only, so one can treat it as a new variable $X=x+y\zeta$ and use nonforward parton distributions $\mathcal{F}_{\zeta}(X;t)$ instead of the DDs $F(x,y;t)$.

With the same technique, one can calculate the nonforward matrix element $\langle p_2|\mathcal{O}_n|p_1\rangle$ of the composite operator $\mathcal{O}_n=\varphi(iz\vec{\partial})^n\varphi$ and obtain its α representation

$$i\int_0^{\infty}\left(\frac{\alpha_3(p_1z)+\alpha_2(rz)}{\lambda}\right)^n \times\exp\left\{\frac{i}{\lambda}[\alpha_2\alpha_4 t+\alpha_3(\alpha_4p_1^2+\alpha_2p_2^2)] -i\lambda(m^2-i\epsilon)\right\}\frac{d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2}. \tag{A7}$$

Note that the derivative $(iz\vec{\partial})$ acting on the field φ is expected to give the factor (kz), where k is the momentum of φ. Equation (A7) shows that $(kz)=\alpha_3(p_1z)/\lambda+\alpha_2(rz)/\lambda$, i.e., α_3/λ and α_2/λ should be interpreted as the variables x and y of the double distribution $F(x,y;t)$. Alternatively, using the binomial expansion for the $(\cdots)^n$ factor, one can see that the coefficients in front of $(p_1z)^{n-k}(rz)^k$ in this expansion are given by the $x^{n-k}y^k$ moments of the DD (A5). Integrating over λ, we get the relevant DD explicitly:

$$F(x,y;t,p_1^2,p_2^2)=-\{y(1-x-y)t +x[(1-x-y)p_1^2+yp_2^2]-m^2\}^{-1}. \tag{A8}$$

Putting the "pions" on equal footing by setting $p_1^2=p_2^2=-M^2$, we get a DD

$$F^{(1)}(x,y;t,M^2)=\{-y(1-x-y)t+x(1-x)M^2+m^2\}^{-1} \tag{A9}$$

satisfying the $y\leftrightarrow 1-x-y$ *Munich symmetry* condition. This expression can be rewritten in the form

$$F^{(1)}(x,y;t,M^2)=\frac{1}{x(1-x)M^2}\left\{1+\frac{m^2}{x(1-x)M^2} -\frac{y(1-x-y)t}{x(1-x)M^2}\right\}^{-1} \tag{A10}$$

resembling the DDs obtained from the power-law wave function. Introducing $\Lambda^2=M^2/4+m^2$, we get the expression

$$F^{(1)}(x,y;t,M^2)|_{M^2=4(\Lambda^2-m^2)} =\frac{1}{4x(1-x)\Lambda^2}\left\{1+\frac{m^2(x-1/2)^2}{4x(1-x)\Lambda^2}-\frac{y(1-x-y)t}{4x(1-x)\Lambda^2}\right\}^{-1} \tag{A11}$$

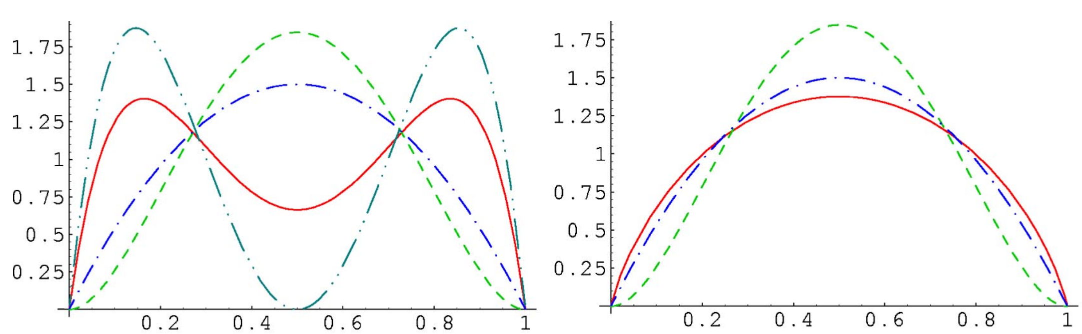

FIG. 8. (Color online) Left: isovector part of $F_{\zeta=1}(X;t=0)$ for the toy model (solid) compared to the asymptotic pion DA (dash-dotted), CZ distribution amplitude (dash-double-dotted), and DA obtained from the two-body wave function (dashed). Right: isovector part of $F_{\zeta=1}(X;t=0)$ for the realistic model (solid) compared to the asymptotic pion DA (dash-dotted) and DA obtained from two-body wave function (dashed).

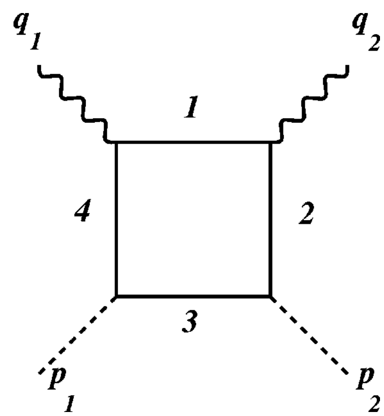

FIG. 9. Box diagram for DVCS in a scalar model.

whose denominator factor has the structure close to that of the DDs obtained in the model with the power-law wave functions. However, the denominator power is (-1) instead of (-3). Applying $(p_1^2\partial/\partial p_1^2)(p_2^2\partial/\partial p_2^2)$ to $F(x,y;t,p_1^2,p_2^2)$, Eq. (A8),[2] and setting $p_1^2=p_2^2=-M^2$, we obtain

$$F^{(2)}(x,y;t,M^2)=\frac{2y(1-x-y)}{x(1-x)^3M^2}\left\{1+\frac{m^2}{x(1-x)M^2}-\frac{y(1-x-y)t}{x(1-x)M^2}\right\}^{-3}. \tag{A12}$$

Now, using $\Lambda^2=M^2/4+m^2$, we end up with a DD differing from the toy model DD, Eq. (7.4), just by the x-dependent factor $1/x$ and an overall normalization. Note that our toy DD was based on the formula for the vector form factor; hence, for full correspondence, we should consider a DD related to operators containing the extra $i\overset{\leftrightarrow}{\partial}{}^\mu$ derivative. This results in the extra factor of

$$\frac{\alpha_3}{\lambda}(p_1^\mu+p_2^\mu)+\frac{\alpha_2-\alpha_4}{\lambda}(p_1^\mu-p_2^\mu)=2xP^\mu+(2y-x-1)r^\mu.$$

As expected, the P^μ part contains the missing factor of x. Since the r^μ part is Munich antisymmetric, it does not contribute to form factor and forward densities (see also [24]). In general, such terms are not restricted by the reduction relations (2.13),(2.14). However, they contribute to SPDs for $\zeta\neq 0$, and their modeling deserves separate consideration.

APPENDIX B: SPDs IN THE IMPACT PARAMETER REPRESENTATION

In this appendix, we investigate the properties of our model SPDs in the impact parameter representation. For the scalar triangle diagram, such an analysis was recently performed by Pobylitsa [25]. He also uses the α representation and double distributions for the triangle diagram, but takes the version of double distributions in which the plus component of the momentum of the spectator system is written as $up_1^+ + vp_2^+$. Instead of Eq. (A5), we have then

$$P(u,v;t,p_1^2,p_2^2)=i\int_0^\infty \delta\left(u-\frac{\alpha_4}{\lambda}\right)\delta\left(v-\frac{\alpha_2}{\lambda}\right)\times\exp\left\{\frac{i}{\lambda}[\alpha_2\alpha_4 t+\alpha_3(\alpha_4p_1^2+\alpha_2p_2^2)]-i\lambda(m^2-i\epsilon)\right\}\frac{d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2}. \tag{B1}$$

This DD is related to the SPD $H(\tilde{x},\xi;t)$ by

$$H(\tilde{x},\xi;t,p_1^2,p_2^2)=\int_0^1\int_0^1 \delta(1-\tilde{x}-u(1+\xi)-v(1-\xi))\times P(u,v;t,p_1^2,p_2^2)\,\theta(0\leqslant u+v\leqslant 1)\,du\,dv. \tag{B2}$$

Now, one should take $p_1^2=p_2^2=m_\pi^2$, $t=-(|\Delta_\perp|^2+4\xi^2m_\pi^2)/(1-\xi^2)$ and calculate the double Fourier transform

$$B(\tilde{x},\xi;b_\perp)=\int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{i(\Delta_\perp b_\perp)}H\left(\tilde{x},\xi;-\frac{|\Delta_\perp|^2+4\xi^2m_\pi^2}{1-\xi^2}\right). \tag{B3}$$

The δ function in Eq. (B2) can be rewritten as $\delta(1-u/r_1-v/r_2)/(1-\tilde{x})$, where the parameters r_1,r_2 given by $r_1=(1-\tilde{x})/(1+\xi)$, $r_2=(1-\tilde{x})/(1-\xi)$ have the meaning of the spectator's plus momentum measured in units of the initial or final pion plus momenta. Due to this δ function, we can write $u=zr_1, v=(1-z)r_2$, with $0\leqslant z\leqslant 1$ in the $\tilde{x}>\xi$ region. Finally, the integral over $d\lambda dz$ can be transformed into integration over the variables $\sigma_1=z\lambda$ and $\sigma_2=(1-z)\lambda$. A remarkable fact is that the resulting integrand $I(\sigma_1,\sigma_2)$ factorizes $I(\sigma_1,\sigma_2)=J_1(\sigma_1)J_2(\sigma_2)$. As a consequence, the expression for $B(\tilde{x},\xi;b_\perp)$ also has a factorized form [25]

$$B(\tilde{x},\xi;b_\perp)=\frac{1-x}{4\pi}V_0(r_1,(1-\xi)b_\perp)V_0(r_2,(1+\xi)b_\perp), \tag{B4}$$

where the generalized impact-parameter LC wave functions

$$V_0(r,c_\perp)=\frac{1}{4\pi r}\int_0^\infty\frac{d\sigma}{\sigma}\exp\left[-\frac{c_\perp^2}{4\sigma r^2}-\sigma(m^2-r(1-r)m_\pi^2)\right]=\frac{1}{2\pi r}K_0\left(\frac{|c_\perp|}{r}\sqrt{m^2-r(1-r)m_\pi^2}\right) \tag{B5}$$

can be expressed through the modified Bessel function K_0. As demonstrated in Refs. [25], the factorized representation

[2]It is easy to see that the $\Pi\varphi\varphi$ scalar vertex differentiated with respect to the virtuality of the scalar Π current corresponds to the $\kappa=2$ power-law wave function (4.4).

(B4) guarantees that the positivity bounds [26,9,27,14,28–31] for this SPD are satisfied in the model with scalar quarks. The same SPD multiplied by $(1-\tilde{x})$ satisfies the positivity bounds for spin-1/2 quarks.

If we proceed as in Appendix A, i.e., first differentiate Eq. (B1) with respect to p_1^2 and p_2^2 and then take $p_1^2=p_2^2=m_\pi^2$, we get an extra factor $\alpha_2\alpha_4\alpha_3^2/\lambda^2$. After the transformations described above, this results in the factor $r_1r_2\sigma_1\sigma_2[1-(r_1\sigma_1+r_2\sigma_2)/(\sigma_1+\sigma_2)]^2$, and the integral over σ_1,σ_2 cannot be factorized into a product of two separate integrals over σ_1,σ_2. The unfactorizable piece comes from the α_3^2 factor resulting from differentiation with respect to external virtualities. To avoid this factor, but still preserve the $\sim 1/k_\perp^4$ behavior of the effective IMF wave function, one can perform differentiation with respect to the squares of the quark masses (i.e., take all the masses different, differentiate with respect to m_i^2's corresponding to lines 2 and 4, and then take all the masses equal). This produces the factor $\alpha_2\alpha_4$, or eventually $r_1r_2\sigma_1\sigma_2$, which does not violate the factorized structure of the integrand. In the impact parameter representation, the result has the structure of Eq. (B4), but with $V_0(r,c_\perp)$ substituted by the expression

$$V_1(r,c_\perp)=\frac{|c_\perp|m}{4\pi\sqrt{1-r(1-r)m_\pi^2/m^2}}\times K_1\left(\frac{|c_\perp|m}{r}\sqrt{1-r(1-r)m_\pi^2/m^2}\right) \quad \text{(B6)}$$

involving the modified Bessel function K_1. For the original IMF wave function, differentiation with respect to the active quark mass is equivalent to choosing the *Ansatz*

$$\psi(x,k_\perp)=\frac{\mathcal{N}}{x\sqrt{x(1-x)}[a+bk_\perp^2]^2} \quad \text{(B7)}$$

instead of Eq. (4.4). It has the extra $1/x$ factor enhancing the wave function at small x. In the spirit of our discussion of the x dependence in the main text of the paper, we may say that such a function more adequately models the contribution of higher Fock components.

The positivity bounds are satisfied also in a more general case when $B(x,\xi;b_\perp)$ is given by the sum [32,25]

$$B(\tilde{x},\xi;b_\perp)=(1-\tilde{x})^{N+1}\sum_n Q_n(r_1,(1+\xi)b_\perp)\times Q_n(r_2,(1-\xi)b_\perp), \quad \text{(B8)}$$

where $N=0$ for "scalar quarks" and $N=1$ for the spin-1/2 case. This opens the possibility of building models for GPDs consistent with both the polynomiality and positivity constraints. The simplest idea is to start with the α representation (A5) for the DD corresponding to the scalar triangle diagram, and modify it by multiplying the integrand by a function $R(m^2\sqrt{\alpha_2\alpha_4})$ depending only on the product $\alpha_2\alpha_4$ [choosing the argument as $\sqrt{\alpha_2\alpha_4}$ we get eventually a function of $(1-\tilde{x})$; the parameter m^2 was included to make the argument of $R(a)$ dimensionless]. If this function can be represented as

$$R(a)=\sum_n R_n a^n \quad \text{(B9)}$$

with all R_n positive (the sum should be understood in a wide sense, it can involve integrations), the function $B(\tilde{x},\xi;b_\perp)$ in such a model has the structure of Eq. (B8), and positivity constraints are satisfied. The model double distribution based on $R(a)$ gives the following expression for $H(\tilde{x},\xi;t)$ in the $\tilde{x}>\xi$ region:

$$H(\tilde{x},\xi;t)|_{\tilde{x}>\xi}=(1-\tilde{x})^{N-1}r_1r_2\int_0^\infty d\rho\int_0^1 dz R(\rho\sqrt{z\bar{z}\bar{r}_1\bar{r}_2})\times\exp\{-\rho[1-(1-zr_1-\bar{z}r_2)(zr_1+\bar{z}r_2)\times m_\pi^2/m^2-z\bar{z}r_1r_2t/m^2]\} \quad \text{(B10)}$$

(we use the notation $\bar{z}\equiv 1-z$ here, and later we also use $\bar{x}\equiv 1-x$, etc.). Note that we performed Wick rotation $\alpha_j\to -i\alpha_j$ in the original α representation (B1), which is justified if the pion stability condition $m_\pi^2<4m^2$ is satisfied. In the forward limit ($\xi=0,t=0$) this gives

$$f(x)=(1-x)^{N+1}\int_0^1 dz\int_0^\infty d\rho R(\rho\sqrt{z\bar{z}\,x})e^{-\rho(1-x\bar{x}m_\pi^2/m^2)}. \quad \text{(B11)}$$

The function $R(a)$ should be adjusted to fit experimental forward distribution $f^{\text{expt}}(x)$, and then it can be used for calculation of $H(\tilde{x},\xi;t)$. In particular, for massless pions, the coefficients R_n can be expressed directly as

$$R_n=\frac{n+1}{[\Gamma(n/2+1)]^2}A_{n+1+N} \quad \text{(B12)}$$

in terms of the coefficients of the $\bar{x}^k$ expansion of the forward distribution

$$f^{\text{expt}}(x)=\sum_k A_k(1-x)^k. \quad \text{(B13)}$$

Note that for the simple model $f^R(x)=\frac{3}{4}(1-x)/\sqrt{x}$ of Sec. IX, all the coefficients $A_k=\Gamma(k-1/2)/\Gamma(1/2)(k-1)!$ are positive. Alternatively, since both Eqs. (B10) and (B11) in the $t=0,m_\pi^2=0$ limit involve the same integral of the R function, the only change being $\bar{x}\to\sqrt{r_1r_2}$, in this case we can directly write H through $f(x)$

$$H(\tilde{x},\xi;t=0)|_{\tilde{x}>\xi;m_\pi^2=0}=\left(\frac{1-\tilde{x}}{\sqrt{r_1r_2}}\right)^{N-1}f(1-\sqrt{r_1r_2})$$
$$=(1-\xi^2)^{(N-1)/2}f\left(1-\frac{1-\tilde{x}}{\sqrt{1-\xi^2}}\right). \tag{B14}$$

In the case of nonforward distributions, we have

$$\mathcal{F}(X,\zeta;t=0)|_{X>\zeta;m_\pi^2=0}=(1-\zeta)^{(N-1)/2}f\left(1-\frac{1-X}{\sqrt{1-\zeta}}\right). \tag{B15}$$

Taking $f(x)=f^R(x)$ and $N=1$ (spinor quarks), we get curves that are very close to the $X>\zeta$ parts of the realistic model curves shown in Fig. 7. In the region $X\leqslant\zeta$, the functions can be obtained only from formulas explicitly involving double distributions. In particular, the scalar $R(a)$ model gives

$$F(x,y;t)=\int_0^\infty e^{-\rho p(x,y,t)}R[\rho\sqrt{y(1-x-y)}]d\rho, \tag{B16}$$

where $p(x,y,t)=1-y(1-x-y)t/m^2-x(1-x)m_\pi^2/m^2$. Fixing $R(a)$, e.g., by the requirement that $f(x)=f^R(x)$ in the $m_\pi\to 0$ limit (which allows one to get the result in analytic form) and using Eq. (B12) we get

$$F(x,y;t)=\sum_n A_{n+1}\frac{(n+1)![y(1-x-y)]^{n/2}}{\Gamma^2(n/2+1)[p(x,y,t)]^{n+1}}. \tag{B17}$$

This expression can also be written as

$$F(x,y;t)=\sum_n A_{n+1}h^{(n/2)}(x,y)\left[\frac{(1-x)}{p(x,y,t)}\right]^{n+1} \tag{B18}$$

where $h^{(n/2)}(x,y)$ is the normalized profile function of order $n/2$. Thus, the DD $F(x,y;t)$ in this model is given by a sum of powerlike terms similar to those discussed in Sec. VII. The main difference is that the y profile now is not universal: one should expand the forward distribution $f(x)$ into a power series over $(1-x)$ and supplement the $(1-x)^{n+1}$ term by the n-dependent profile function $h^{(n/2)}(x,y)$. Because of the correlation between the power of $(1-x)$ and the order of the profile function, the shape of the y profile of the double distribution changes with x. Since all parton distributions $f(x)$ tend to infinity as x goes to 0, the small-x region is dominated by terms with large n, and the profile is more narrow when $x\to 0$, becoming infinitely narrow as $x\to 0$. Note that in the case when the profile is infinitely narrow for all x, i.e., when $F(x,y;t=0)=\delta[y-(1-x)/2]f(x)$, the $t=0$ skewed distribution is given by $\mathcal{F}_\zeta(X)=f[(X-\zeta/2)/(1-\zeta/2)]/\sqrt{1-\zeta}$: it repeats the form of the forward distribution $f(x)$ and is infinite for $X=\zeta/2$. To check if this also happens for the model (B18) with the changing profile and realistic $\sim x^{-0.5}$ behavior for small x, we took $f^{\rm expt}(x)=f^R(x)$ [or, which is the same, $A_{n+1}=\Gamma(n+1/2)/\Gamma(1/2)n!$ in Eq. (B17)] and constructed model skewed distributions $\mathcal{F}_\zeta(X)$ in both the $X>\zeta$ and $X<\zeta$ regions (see Fig. 10). The resulting functions are indeed singular for $X=\zeta/2$, but finite otherwise.

We plan to perform a more detailed study of GPDs within the $R(a)$ model in a separate paper.

APPENDIX C: TWO-PION DISTRIBUTION AMPLITUDE IN SCALAR MODEL

Let us now apply our approach to processes of two-pion production in $\gamma^*\gamma\to\pi\pi$ or $\gamma\gamma\to\pi\pi$ collisions. The $\gamma^*\gamma\to\pi\pi$ process in the kinematics when γ^* is highly virtual, while $s\equiv m_{\pi\pi}^2$ is small [33] is the crossed-channel reaction to DVCS while $\gamma\gamma\to\pi\pi$ process in the kinematics when $s,|t|,|u|$ are large [34] is the crossed-channel reaction to wide-angle Compton scattering. Here we are going to consider only the simpler case of the kinematics of the first process.

Originally [33], it was proposed to describe the nonperturbative stage of this process by the two-pion distribution amplitude $(2\pi{\rm DA})\Phi(z,\zeta;s)$ which describes the conversion of two quarks with plus momenta zP^+ and $\bar{z}P^+$ into two pions with momenta $p_1^+=\zeta P^+$ and $p_2^+=\bar{\zeta}P^+$ (see Fig. 11), where P is the total momentum of the pion pair (recall that the invariant mass of the pair is small: $P^2\equiv s\ll 1$ GeV^2). Later, Teryaev [35] proposed to use the double distribution description (see also Ref. [36] for further developments) corresponding to parametrization of the plus component of the spectator momentum as $up_1^+-vp_2^+$ (note that both momenta are now outgoing, and this is reflected in the change of the relative sign of the p_1 and p_2 parts of the spectator momentum compared to the DVCS case). In P^+ units, the spectator plus momentum can be written either as $(\zeta-z)P^+$ (using the 2πDA variable z) or as $u\zeta P^+-v\bar{\zeta}P^+$ (using the DD variables u,v). Thus, the connection between the two sets of variables is given by

$$z=(1-u-v)\zeta+v.$$

So we can express the 2πDA $\Phi(z,\zeta;s)$ in terms of the DD $M(u,v;s)$:

$$\Phi(z,\zeta;s)=\int_0^1 du\int_0^1 dv\,\theta(u+v\leqslant 1)$$
$$\times\delta(z-\zeta(1-u-v)-v)M(u,v;s). \tag{C1}$$

The DD representation for $\Phi(z,\zeta;s)$ can be used to derive some general properties of 2πDAs, such as the polynomiality condition

$$\int_0^1 z^n\Phi(z,\zeta;s)dz=\sum_{l=0}^n K_l\zeta^l, \tag{C2}$$

which states that the z^n moment of $\Phi(z,\zeta;s)$ is the nth order polynomial of ζ.

To model 2πDAs by superposition of perturbative contributions, we start with the α representation for the relevant scalar diagram. The double distribution can be written similarly to Eq. (B1):

$$M(u,v;s)=im^2\int_0^\infty \delta\left(u-\frac{\alpha_4}{\lambda}\right)\delta\left(v-\frac{\alpha_2}{\lambda}\right)\times\exp\left\{\frac{i}{\lambda}[\alpha_2\alpha_4 s+\alpha_3(\alpha_4+\alpha_2)m_\pi^2]-i\lambda(m^2-i\epsilon)\right\}\frac{d\alpha_2 d\alpha_3 d\alpha_4}{\lambda^2}. \quad \text{(C3)}$$

Here from the beginning we take $p_1^2=p_2^2=m_\pi^2$, and add the overall factor m^2 to make the function dimensionless. Note that for the triangle diagram we have $(1-u-v)=\alpha_3/\lambda \equiv\beta_3$. So we write the scalar triangle version of the 2πDA as

$$\Phi(z,\zeta;s)=im^2\int_0^\infty d\lambda\int_0^1 d\beta_2\int_0^1 d\beta_3\,\theta(\beta_2+\beta_3\leqslant 1)\times\delta(z-\zeta\beta_3-\beta_2)\exp\{i\lambda[(1-\beta_2-\beta_3)\beta_2 s+\beta_3(1-\beta_3)m_\pi^2-m^2+i\epsilon]\}. \quad \text{(C4)}$$

Integrating over λ and incorporating the δ function to calculate the β_2 integral, we get

$$\Phi(z,\zeta;s)=\int_0^{\min\{z/\zeta,\overline{z}/\overline{\zeta}\}} d\beta_3[1-(z-\beta_3\zeta)(\overline{z}-\beta_3\overline{\zeta})s/m^2-\beta_3(1-\beta_3)m_\pi^2/m^2-i\epsilon]^{-1}. \quad \text{(C5)}$$

This representation explicitly demonstrates the well-known fact (see, e.g., [37,36]) that $\Phi(z,\zeta;s)$ is nonanalytic at the point $z=\zeta$. The integral can be taken in the general case, but it is instructive to analyze the simplest limit $s=0$, $m_\pi^2=0$. In this case, the result is the function

$$\Phi(z,\zeta;s=0)_{m_\pi^2=0}=\frac{z}{\zeta}\,\theta(z<\zeta)+\frac{1-z}{1-\zeta}\,\theta(z>\zeta) \quad \text{(C6)}$$

which coincides with a part of the pion DA evolution kernel. Its eigenfunctions are the Gegenbauer polynomials $C_n^{3/2}(2\zeta-1)$ and the eigenvalues are $1/(n+1)(n+2)$ [38,39]. Hence, we can write

$$\Phi(z,\zeta;s=0)_{m_\pi^2=0}=4z(1-z)\sum_{n=0}^\infty\frac{2n+3}{(n+1)^2(n+2)^2}\times C_n^{3/2}(2z-1)C_n^{3/2}(2\zeta-1). \quad \text{(C7)}$$

It is convenient to write 2πDA as a sum over $z(1-z)C_n^{3/2}(2z-1)$, since these are the eigenfunctions of the evolution kernel. On the other hand, the combination $(2\zeta-1)$ is related to the cosine of the angle between the pions' momenta, so it is natural to expand the ζ dependence of $\Phi(z,\zeta;s)$ in the Legendre polynomials $P_l(2\zeta-1)$. Using the formula

$$C_l^{3/2}(x)-C_{l-2}^{3/2}(x)=(2l+1)P_l(x),$$

we can write $C_n^{3/2}(2\zeta-1)$ as a sum of $P_l(2\zeta-1)$ and obtain

$$\Phi(z,\zeta;s=0)_{m_\pi^2=0}=4z(1-z)\sum_{n=0}^\infty\frac{2n+3}{(n+1)^2(n+2)^2}\times C_n^{3/2}(2z-1)\sum_{l=0}^n(2l+1)\times P_l(2\zeta-1)\frac{1+(-1)^{n-l}}{2}. \quad \text{(C8)}$$

This expansion has the structure of the general representation for 2πDAs proposed by Polyakov [23]. In specific models, only the first terms of the expansion are included. It is easy to check that, in our case, the exact result (C6) is well approximated by the first few terms of the Gegenbauer expansion (C7), even in the vicinity of the nonanalyticity point $z=\zeta$. Polyakov [23] considers also a more complicated $s\neq 0$ case, using the $\pi\pi$ scattering information to model the s dependence. Recently, Kivel and Polyakov [36] used chiral perturbation theory to include $O(m_\pi^2)$ corrections to the chiral limit.

Now we want to show how one can incorporate information about the usual (forward) parton densities to build models for 2πDAs. To this end, it is convenient to write the pion momenta as $p_1=P/2+r$ and $p_2=P/2-r$ (the plus components are implied, but we omit the + superscript here and below). The quark momenta can be written then as

$$k_1=\frac{1+\alpha}{2}P+xr,\quad k_2=\frac{1-\alpha}{2}P-xr, \quad \text{(C9)}$$

where the variables α and x are related to u,v by $x=1-u-v$ and $\alpha=v-u$. The support region is $|\alpha|\leqslant 1-x$. The 2πDA $\Phi(z,\zeta;s)$ is related to the double distribution $F(x,\alpha;s)$ by

$$\Phi(z,\zeta;s)=\int_0^1 dx\int_{-1+x}^{1-x}d\alpha\,\delta(z-1/2-x(\zeta-1/2)-\alpha/2)\times F(x,\alpha;s). \quad \text{(C10)}$$

In this description, the total pair momentum P is shared by the quarks in the fractions $(1+\alpha)/2$ and $(1-\alpha)/2$, while the relative momentum r is carried by active quarks in the fractions x and $-x$. Hence, the relevant double distribution $F(x,\alpha;s)$ is the timelike analogue of the function $f(x,\alpha;t)$ considered in Sec. II. In the forward limit, it reduces to the usual parton densities

$$\int_{-1+x}^{1-x}F(x,\alpha;s=0)d\alpha=f(x). \quad \text{(C11)}$$

In the $m_\pi^2=0$ case, we have $F(x,\alpha;s=0)=\frac{1}{2}\theta(|\alpha|\leqslant 1-x)$; hence the integral in Eq. (C11) gives $(1-x)$, which is exactly the forward distribution for the scalar massless tri-

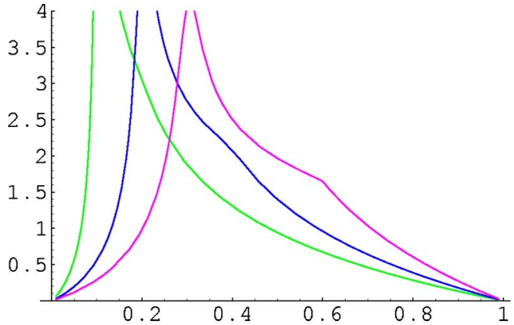

FIG. 10. (Color online) SPDs $F_\zeta(X;t=0)$ with $\zeta=0.2,0.4,0.6$ obtained from Eqs. (B18) and (B13). The forward distribution was modeled by $f^R(x)=(3/4)(1-x)/\sqrt{x}$. The curves tend to ∞ for $X=\zeta/2$.

angle. To get a more realistic $f(x)$, we can use the $R(a)$ model described in Appendix B. It gives

$$\Phi^R(z,\zeta;s)=\int_0^{\min\{z/\zeta,\overline{z}/\overline{\zeta}\}}dx \times\int_0^\infty \exp\left\{-\rho\left[1-(z-x\zeta)(\overline{z}-x\overline{\zeta})\times\frac{s}{m^2}-x\overline{x}\frac{m_\pi^2}{m^2}\right]\right\}R(\rho\sqrt{(z-x\zeta)(\overline{z}-x\overline{\zeta})})d\rho. \tag{C12}$$

According to Appendix B, the $R(a)$ model is equivalent to the sum of "wave function overlap" contributions of Eq. (B8) type, similar to those obtained within the light-cone approaches (see, e.g., [40,31,41]). However, the $R(a)$ construction has the advantage that it also provides a model for 2πDA. In the standard light-cone formalisms, the 2πDA would involve the $q\to\pi q$ vertices which cannot be interpreted as light-cone wave functions.

In the scalar triangle model, the 2πDA $\Phi(z,\zeta;s=0)$ is obtained from the same DD $F(x,\alpha;s=0)=f(x,\alpha;t=0)$ which produces the $t=0$ OFPD $H(\tilde{x},\xi;t=0)$. Comparing Eqs. (C10) and (8.3), we can formally write

$$\Phi(z,\zeta;s=0)=H\left(\frac{2z-1}{2\zeta-1},\frac{1}{2\zeta-1};\ t=0\right). \tag{C13}$$

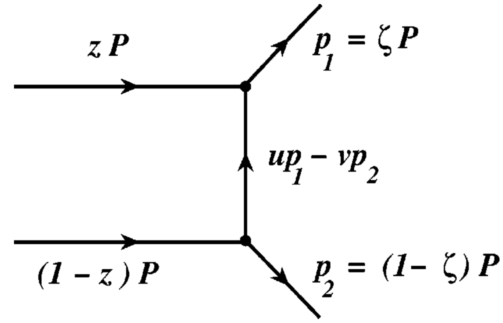

FIG. 11. Plus-momentum flux structure of the two-pion distribution amplitude.

The relation is even simpler,

$$\phi(\tilde{x},\tilde{\xi};s=0)=H(\tilde{x}/\tilde{\xi},1/\tilde{\xi};t=0) \tag{C14}$$

for the 2πDA $\phi(\tilde{x},\tilde{\xi};s=0)$ written in the symmetric variables $\tilde{x}=2z-1$ and $\tilde{\xi}=2\zeta-1$. Since $|\tilde{\xi}|\leqslant 1$, the OFPD $H(\tilde{x}/\tilde{\xi},1/\tilde{\xi};t=0)$ is taken at skewedness values with absolute magnitude larger than 1. Hence, as suggested by Teryaev [35], the 2πDA may be treated as a continuation of OFPD into the $|\xi|>1$ region. More precisely, $H|_{t=0}$ and $\Phi|_{s=0}$ may be treated as $|\xi|<1$ and $|\xi|>1$ components of the same function.

To make parallel with $I=0$ and $I=1$ components of 2πDAs in QCD, one should take the combinations

$$\Phi^{\pm}(z,\zeta;s)=\frac{1}{2}[\Phi(z,\zeta;s)\pm\Phi(1-z,\zeta;s)], \tag{C15}$$

which are symmetric or antisymmetric with respect to the middle point $z=1/2$. They are given by summation over even or odd n in Eq. (C8). Taking $s=0,m_\pi^2=0$ and fixing $R(a)$ in the same way as in Appendix B, we obtained the curves shown in Fig. 12.

At $z=1/2$, the symmetric function in this model is infinite. This result is similar to the singularity of SPDs $\mathcal{F}_\zeta(X)$ for $X=\zeta/2$ observed in Appendix B. It reflects the fact that the profile of $F(x,\alpha,s=0)$ in the $R(a)$ model becomes infinitely narrow as $x\to 0$. Indeed, for DDs with infinitely narrow profile for all x, i.e., for $F(x,\alpha,s=0)=f(x)\delta(\alpha)$, we would have $\Phi(z,\zeta;s=0)=f[(z-1/2)/(\zeta-1/2)]/(1-\zeta/2)$,

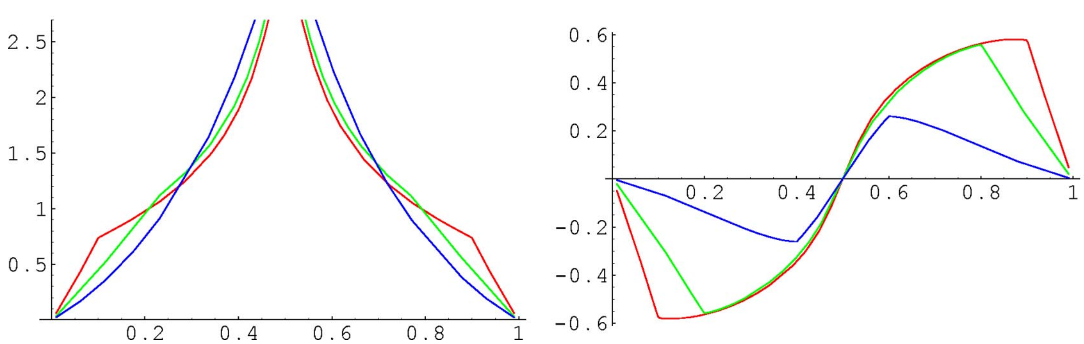

FIG. 12. (Color online) Two-pion distribution amplitudes $\Phi^+(z,\zeta;s=0)$ and $\Phi^-(z,\zeta;s=0)$ with $\zeta=0.1$, 0.2, 0.4 obtained in the scalar $R(a)$ model. The forward distribution was modeled by $f^R(x)=(3/4)(1-x)/\sqrt{x}$.

which gives an infinite result for $z=1/2$ if $f(0)\to\infty$. The curves also have cusps for $z=\zeta$ and $z=1-\zeta$. They appear because the DD $F(x,\alpha,s=0)$ of the $R(a)$ model does not vanish at the upper corner $x=0,\alpha=1$ of the support region. This is because the profile function for the lowest term of the $R(a)$ expansion is $h^{(0)}(x,y)=1/(1-x)$: unlike the profile functions $h^{(n)}(x,y)$ with $n>0$, it does not vanish at the borderlines $x+|\alpha|=1$.

Note that 2πDAs of the purely scalar model are symmetric with respect to the change $\{z\to 1-z,\zeta\to 1-\zeta\}$ while in QCD the 2πDAs describing the transition of spin-1/2 quarks into pions changes sign after this transformation. The triangle perturbative contributions for this case were considered a few years ago by Polyakov and Weiss [37]. We plan to extend their calculation by combining it with the ideas of the present paper.

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