Neutrino oscillations from relativistic flavor currents

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By resorting to recent results on the relativistic currents for mixed (flavor) fields, we calculate a space-time dependent neutrino oscillation formula in quantum field theory. Our formulation provides an alternative to existing approaches for the derivation of space dependent oscillation formulas and it also accounts for the corrections due to the nontrivial nature of the flavor vacuum. By exploring different limits of our formula, we recover already known results. We study in detail the case of one-dimensional propagation with Gaussian wave packets both in the relativistic and in the nonrelativistic regions: in the last case, numerical evaluations of our result show significant deviations from the standard formula.

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I. INTRODUCTION

The recent experimental evidence [1–5] of neutrino mixing and oscillations [6] is the first clear sign of physics beyond the standard model [7]. Thus much effort is currently devoted to the full understanding of such phenomenon, from the issue of its origin to more phenomenological ones.

In this framework, there has been recently remarkable progress in the study of the problem of field mixing in quantum field theory (QFT) [8–20]. It is important to remark that, despite the fact that it is usually treated in quantum mechanics (QM), the mixing between states of different masses is not even allowed in a nonrelativistic theory due to the Bargmann superselection rules [21]. The problem of constructing a Hilbert space for flavor states is indeed a long standing one [22] which has even been claimed to be impossible to achieve [23] (see however Ref. [18] for a criticism of that argument). As first shown in Ref. [8], the difficulty lies in the fact that the Hilbert spaces for particles with definite flavor and those with definite mass are actually orthogonal in the infinite volume limit. This result has been subsequently generalized for any number of generations and for different types (fermionic or bosonic) of fields [16,17,10]. It has also emerged that the use of the correct Hilbert space (i.e., the one for the flavor fields) leads to corrections to the usual Pontecorvo oscillation formula.

The necessity for a full QFT treatment of mixing also stems from a more phenomenological perspective, namely from the calculation of (flavor) oscillation formulas. The usual treatment gives, indeed, in a very simple way an oscillation formula in time, which is actually not of great use when discussing current experiments where the distance source detector is measured rather than time (data being collected over large time intervals) [24]. In order to derive an oscillation formula in space, one usually converts the time oscillation formula by means of some assumptions (equal time assumption, classical propagation, etc.) which are however questionable in many respects (see Ref. [25] for a review). Thus several QFT approaches have been developed [26–30], such as the *external wave packet models* or the *stationary boundary conditions models*, which attempt a more realistic description of neutrino oscillations. It seems indeed that the main features of such formula are now quite well understood [25], including the concepts of coherence length, localization terms, dispersion times, etc.

On the other hand, all these calculation schemes have been developed for highly relativistic neutrinos and/or for nearly degenerate masses and in most cases the spin structure of the wave functions was neglected (i.e., neutrinos were treated as scalars). Also, the identification of the Hilbert spaces for mass and flavor neutrinos was implicitly assumed in the choice of the propagators (see Ref. [9] for a discussion of flavor propagators).

A completely different approach to the problem is indeed possible and was advocated in Refs. [31,32]: an oscillation formula in space could be derived in a straightforward way, by considering the flux of neutrinos through the detector as the integral over the measurement time and detector surface of the relativistic flavor current for the oscillating neutrinos. An attempt in this direction is contained in Ref. [31], which however deals with QM and therefore uses a probability current derived from Schrödinger equation. As pointed out in Ref. [32], the obstacle to the extension of this result to QFT lies in the difficulty of constructing a relativistic current for mixed particles, which boils down again to the problem of the definition of the Hilbert space for such (not on-shell) particles.

The solution to this problem was recently given in Ref. [13]: it turned out that it is indeed possible to define these (nonconserved) flavor currents in a consistent way by use of previous results on the flavor Hilbert space. An analysis of the currents associated to mixing for the case of three flavor neutrino mixing with CP violation and for boson mixing can be found in Refs. [10,11].

In this paper, we calculate the oscillation formula by using the relativistic flavor currents above mentioned. For simplicity, we only treat the case of mixing among two generations of Dirac neutrinos: the extension to Majorana fields and to three flavors will be given elsewhere. The formula we obtain in a very direct way has the full space-time dependence and contains previous results obtained by use of the flavor charges. We then obtain a general expression for the electron neutrino flux in three dimensions with spherical symmetry. Although a fully three dimensional analysis is possible within our formulation, we study in detail the one-dimensional case with Gaussian wave packets, which is indeed the one most frequently considered in literature. We show how in the highly relativistic limit and by integrating over an infinite time, the standard space dependent oscillation formula [25,26] is obtained, which includes the coherence length and the localization term. However our formula is also valid for nonrelativistic neutrinos and it accounts for flavor vacuum effects. Numerical evaluations of our result show deviations from the usual formula in the nonrelativistic regime.

The paper is organized as follows: in Sec. II we present a derivation of the flavor currents in the line of Ref. [13] and give some details on the construction of the flavor Hilbert space. In Sec. III we first obtain the expectation value of the flavor current in the most general case and then a more explicit expression in the case of a spherically symmetric emission. Then, in Sec. IV, we specialize to the one-dimensional case and obtain the space dependent expression for the electron neutrino flux. Further analysis is done by choosing Gaussian wave packets. Sec. V is devoted to conclusions.

II. RELATIVISTIC CURRENTS FOR MIXED FIELDS

Following Ref. [13], let us consider the following Lagrangian density describing two Dirac fields with a mixed mass term:

$$\mathcal{L}(x)=\bar{\Psi}_f(x)(i\not\partial-M)\Psi_f(x), \tag{1}$$

where $\Psi_f^T=(\nu_e,\nu_\mu)$ and

$$M=\begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}.$$

The mixing transformations

$$\Psi_f(x)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\Psi_m(x), \tag{2}$$

with θ being the mixing angle and $\Psi_m^T=(\nu_1,\nu_2)$, diagonalize the quadratic form of Eq. (1) to the Lagrangian for two free Dirac fields, with masses m_1 and m_2:

$$\mathcal{L}(x)=\bar{\Psi}_m(x)(i\not\partial-M_d)\Psi_m(x), \tag{3}$$

where $M_d=\mathrm{diag}(m_1,m_2)$. One also has $m_e=m_1\cos^2\theta+m_2\sin^2\theta$, $m_\mu=m_1\sin^2\theta+m_2\cos^2\theta$, and $m_{e\mu}=(m_2-m_1)\sin\theta\cos\theta$. We assume $m_2\geqslant m_1$ and θ in $[0,\pi/4]$.

The Lagrangian Eq. (1) is invariant under global $U(1)$ phase transformations of the type $\Psi_m'=e^{i\alpha}\Psi_m$: as a result, we have the conservation of the Noether charge $Q=\int d^3\mathbf{x}\, I^0(x)$ [with $I^\mu(x)=\bar{\Psi}_m(x)\gamma^\mu\Psi_m(x)$] which is indeed the total charge of the system (i.e., the total lepton number).

Consider now the $SU(2)$ transformations acting on Ψ_m:

$$\Psi_m'(x)=e^{i\alpha_j\tau_j}\Psi_m(x), \quad j=1,2,3, \tag{4}$$

with α_j real constants, $\tau_j=\sigma_j/2$ and σ_j being the Pauli matrices.

For $m_1\neq m_2$, the Lagrangian is not generally invariant under Eq. (4) and we obtain, by use of the equations of motion,

$$\delta\mathcal{L}(x)=i\alpha_j\bar{\Psi}_m(x)[\tau_j,M_d]\Psi_m(x)=-\alpha_j\partial_\mu J^\mu_{m,j}(x),$$

$$J^\mu_{m,j}(x)=\bar{\Psi}_m(x)\gamma^\mu\tau_j\Psi_m(x), \quad j=1,2,3. \tag{5}$$

The above analysis is valid at classical level. We now quantize the free fields ν_1 and ν_2 in the usual way (see Appendix A for conventions).

Then the charge operators $Q_{m,j}(t)\equiv\int d^3\mathbf{x}\, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra $[Q_{m,j}(t),Q_{m,k}(t)]=i\epsilon_{jkl}Q_{m,l}(t)$. The Casimir operator is proportional to the total charge. From Eq. (5) we also see that $Q_{m,3}$ is conserved as M_d is diagonal. Let us define the combinations:

$$Q_1\equiv\frac{1}{2}Q+Q_{m,3}, \quad Q_2\equiv\frac{1}{2}Q-Q_{m,3} \tag{6}$$

$$Q_i=\sum_r\int d^3\mathbf{k}(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}-\beta^{r\dagger}_{-\mathbf{k},i}\beta^r_{-\mathbf{k},i}), \quad i=1,2, \tag{7}$$

where the last expression has been normal ordered, as usual. These are nothing but the Noether charges associated with the noninteracting fields ν_1 and ν_2: in the absence of mixing, they are the flavor charges, separately conserved for each generation. Observe now that the transformation

$$\Psi_f(x)=e^{-2i\theta Q_{m,2}(t)}\Psi_m(x)e^{2i\theta Q_{m,2}(t)} \tag{8}$$

is just the mixing Eq. (2). Thus $G_\theta(t)\equiv e^{2i\theta Q_{m,2}(t)}$ is the generator of the mixing transformations (see Appendix A). In Ref. [8], it has been shown that the action of $G_\theta(t)$ on the vacuum state $|0\rangle_{1,2}$ results in a new vector (flavor vacuum) $|0(t)\rangle_{e,\mu}\equiv G_\theta^{-1}(t)|0\rangle_{1,2}$, orthogonal to $|0\rangle_{1,2}$ in the infinite volume limit.

Following Ref. [13], in accordance with Eq. (6), we define the *flavor charges* for mixed fields as

$$Q_\sigma(t)\equiv G_\theta^{-1}(t)Q_iG_\theta(t) \tag{9}$$

with $(\sigma,i)=(e,1),(\mu,2)$ and $Q_e(t)+Q_\mu(t)=Q$. They have a simple expression in terms of the flavor ladder operators:

$$Q_\sigma(t)=\int d^3\mathbf{x}\,\nu_\sigma^\dagger(x)\nu_\sigma(x)$$

$$=\sum_r \int d^3\mathbf{k}(\alpha_{\mathbf{k},\sigma}^{r\dagger}(t)\alpha_{\mathbf{k},\sigma}^{r}(t)-\beta_{-\mathbf{k},\sigma}^{r\dagger}(t)\beta_{-\mathbf{k},\sigma}^{r}(t)),$$

$$\sigma=e,\mu. \tag{10}$$

These charge operators then act on the flavor Hilbert space $\mathcal{H}_{e,\mu}$ (see Appendix A). For the single neutrino and antineutrino states of definite momentum and helicity:[1]

$$|\nu_\sigma(t)\rangle\equiv\alpha_{\mathbf{k},\sigma}^{r\dagger}(t)|0(t)\rangle_{e,\mu}, \quad |\overline{\nu}_\sigma(t)\rangle\equiv\beta_{\mathbf{k},\sigma}^{r\dagger}(t)|0(t)\rangle_{e,\mu}, \tag{11}$$

one obtains

$$Q_\sigma(t)|\nu_\sigma(t)\rangle=|\nu_\sigma(t)\rangle, \quad Q_\sigma(t)|\overline{\nu}_\sigma(t)\rangle=-|\overline{\nu}_\sigma(t)\rangle. \tag{12}$$

In the following, we will use the flavor currents:

$$J_\sigma^\mu(x)\equiv\overline{\nu}_\sigma(x)\gamma^\mu\nu_\sigma(x)=G_\theta^{-1}(t)J_i^\mu(x)G_\theta(t), \tag{13}$$

with $(\sigma,i)=(e,1),(\mu,2)$. From the global $U(1)$ invariance follows the continuity equation:

$$\partial_\mu[J_e^\mu(x)+J_\mu^\mu(x)]=0. \tag{14}$$

III. SPACE-TIME DEPENDENT NEUTRINO OSCILLATION FORMULA

In this section, we consider the calculation of the general (i.e. space-time dependent) neutrino oscillation formula, by use of the flavor currents introduced in Sec. II. In Sec. IV, we then extract from this the space dependent formula by integrating over time.

As already discussed in Refs. [31,32], although in the context of QM, what one is ultimately interested in current experiments is actually the flux of neutrinos of a given flavor through the surface of the detector Ω in a (large) measurement time T. In the case of electron neutrinos, this quantity is given by

$$\Phi_{\nu_e\to\nu_e}(L)=\int_{t_0}^T dt\int_\Omega\langle\nu_e|J_e^i(\mathbf{x},t)|\nu_e\rangle d\mathbf{S}^i, \tag{15}$$

where L is the distance source-detector. Our aim is now to calculate the expectation value of the flavor four-current density on a localized neutrino state, defined as a wave packet in flavor Hilbert space.

A. Expectation value of the flavor current density

We define an initial,[2] localized state $|\nu_e(\mathbf{x}_0,t_0)\rangle$ with definite flavor e:

$$|\nu_e(\mathbf{x}_0,t_0)\rangle=A\int d^3\mathbf{k}e^{-i(\omega_{k,1}t_0-\mathbf{k}\cdot\mathbf{x}_0)}f(\mathbf{k})\alpha_{\mathbf{k},e}^{r\dagger}(t_0)|0\rangle_{e,\mu}, \tag{16}$$

where A is a normalization constant, $f(\mathbf{k})$ is the form of the wave packet and $|0\rangle_{e,\mu}$ the flavor vacuum at time $t=t_0$. This corresponds to an electron neutrino being emitted at $(t_0,\mathbf{x}_0)$. For convenience we set $(t_0,\mathbf{x}_0)=(0,0,0,0)$ in the following. Note that the wave packet is defined in the Hilbert space of flavor fields.

The normalization of this state is given by

$$1=\langle\nu_e|\nu_e\rangle=|A|^2\int d^3\mathbf{k}\int d^3\mathbf{p}f^*(\mathbf{k})f(\mathbf{p})\{\alpha_{\mathbf{k},e}^r(0),\alpha_{\mathbf{p},e}^{r\dagger}(0)\}$$

$$=|A|^2\int d^3\mathbf{k}|f(\mathbf{k})|^2.$$

From Sec. II, the electron neutrino four-current is given as

$$J_e^\mu(\mathbf{x},t)=\overline{\nu}_e(\mathbf{x},t)\gamma^\mu\nu_e(\mathbf{x},t)=\nu_e^\dagger(\mathbf{x},t)\Gamma^\mu\nu_e(\mathbf{x},t), \tag{17}$$

where $\Gamma^\mu=\gamma^0\gamma^\mu$. In the following we will use the chiral representation of the Dirac matrices (see Appendix B). The expansion for the flavor fields given in Eq. (A9) leads to the following explicit form of the current operator:

$$\begin{aligned}J_e^\mu(\mathbf{x},t)=\int\int &\frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{x}}\\ &\times\sum_{r,s}[u_{\mathbf{p},1}^{s\dagger}(t)\Gamma^\mu u_{\mathbf{k},1}^r(t)\alpha_{\mathbf{p},e}^{s\dagger}(t)\alpha_{\mathbf{k},e}^r(t)\\ &-v_{-\mathbf{p},1}^{s\dagger}(t)\Gamma^\mu v_{-\mathbf{k},1}^r(t)\beta_{-\mathbf{k},e}^{r\dagger}(t)\beta_{-\mathbf{p},e}^s(t)\\ &+u_{\mathbf{p},1}^{s\dagger}(t)\Gamma^\mu v_{-\mathbf{k},1}^r(t)\alpha_{\mathbf{p},e}^{s\dagger}(t)\beta_{-\mathbf{k},e}^{r\dagger}(t)\\ &+v_{-\mathbf{p},1}^{s\dagger}(t)\Gamma^\mu u_{\mathbf{k},1}^r(t)\beta_{-\mathbf{p},e}^s(t)\alpha_{\mathbf{k},e}^r(t)].\end{aligned} \tag{18}$$

In Appendix C, we prove that ${}_{e,\mu}\langle 0|J^\mu(\mathbf{x},t)|0\rangle_{e,\mu}=0$. This result leads to (see Appendix A)

$$\langle\nu_e|J_e^\mu(\mathbf{x},t)|\nu_e\rangle=\Psi^\dagger(\mathbf{x},t)\Gamma^\mu\begin{pmatrix}1&1\\1&1\end{pmatrix}\Psi(\mathbf{x},t), \tag{19}$$

with

$$\Psi(\mathbf{x},t)\equiv A\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{k})\times\begin{pmatrix}u_{\mathbf{k},1}^r X_{\mathbf{k},e}(t)\\ \sum_s v_{-\mathbf{k},1}^s(\vec{\sigma}\cdot\mathbf{k})^{sr}Y_{\mathbf{k},e}(t)\end{pmatrix}, \tag{20}$$

$$X_{\mathbf{k},e}(t)=\cos^2\theta e^{-i\omega_{k,1}t}+\sin^2\theta[e^{-i\omega_{k,2}t}|U_\mathbf{k}|^2+e^{i\omega_{k,2}t}|V_\mathbf{k}|^2], \tag{21}$$

[1] Note that there is no ambiguity here in defining the flavor neutrino states with definite momentum.

[2] We work in the Heisenberg picture; the time evolution is thus borne by the operators.

$$Y_{\mathbf{k},e}(t)=\sin^2\theta|U_{\mathbf{k}}|\chi_1\chi_2\left[\frac{1}{\omega_{k,2}+m_2}-\frac{1}{\omega_{k,1}+m_1}\right]\times[e^{-i\omega_{k,2}t}-e^{i\omega_{k,2}t}], \tag{22}$$

where

$$\vec{\sigma}\cdot\mathbf{k}=\begin{pmatrix} k_3 & k_- \\ k_+ & -k_3 \end{pmatrix}$$

and $\chi_i\equiv((\omega_{k,i}+m_i)/4\omega_{k,i})^{1/2}$. We made use of the following relations:

$$\{\alpha^r_{\mathbf{k},e}(t),\alpha^{s\dagger}_{\mathbf{p},e}(0)\}=\delta^{rs}X_{\mathbf{k},e}(t)e^{i\omega_{k,1}t}\delta^3(\mathbf{k}-\mathbf{p}),$$

$$\{\beta^{r\dagger}_{-\mathbf{k},e}(t),\alpha^{s\dagger}_{\mathbf{p},e}(0)\}=(\vec{\sigma}\cdot\mathbf{k})^{rs}Y_{\mathbf{k},e}(t)e^{-i\omega_{k,1}t}\delta^3(\mathbf{k}-\mathbf{p}). \tag{23}$$

The expression in Eq. (19) contains the most general information about neutrino oscillations and can explicitly evaluated once the form of the wave packet is specified. A similar expression can be easily obtained for the other quantity of interest, namely $\langle\nu_e|J^\mu_\mu(\mathbf{x},t)|\nu_e\rangle$.

For comparison with previous results and for better understanding the expression (19), let us consider the limit situation in which the wave packet becomes a plane wave, with definite momentum: $f(\mathbf{k})=(2\pi)^{3/2}\delta^3(\mathbf{k}-\mathbf{q})$. This obviously means that we lose information about localization and we need to integrate over the entire volume to get the flavor oscillations (in time). We indeed find ($\rho_e\equiv J^0_e$)

$$\begin{aligned}\langle\nu_e|Q_e(t)|\nu_e\rangle&=\int d^3\mathbf{x}\langle\nu_e|\rho_e(x)|\nu_e\rangle\\&=1-\sin^2(2\theta)\left[|U_{\mathbf{q}}|^2\sin^2\left(\frac{\omega_{q,2}-\omega_{q,1}}{2}t\right)\right.\\&\quad\left.+|V_{\mathbf{q}}|^2\sin^2\left(\frac{\omega_{q,2}+\omega_{q,1}}{2}t\right)\right],\end{aligned} \tag{24}$$

which agrees with the result obtained in Refs. [9]. Notice the presence of the nonstandard oscillation term and of the momentum dependent amplitudes (the Bogoliubov coefficients satisfying $|U_{\mathbf{q}}|^2+|V_{\mathbf{q}}|^2=1$).

We now consider the situation in which we have a spherically symmetric emission of the neutrinos from the source at (x_0,t_0). This allows us to limit our investigation to the radial flux, which can be identified without loss of generality with the z component of the current. In this case, Eq. (19) takes it simplest form, since Γ^3 is diagonal in the chosen (chiral) representation.

The matrix Γ^3 can be expanded as a linear combination of spinor outer products:

$$\Gamma^3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}=\gamma^0\gamma^3=\sum_{j=1}^{4}(-1)^{j+1}\eta_j\eta_j^\dagger, \tag{25}$$

where

$$\eta_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \eta_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \eta_3=\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\quad \eta_4=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{26}$$

This decomposition permits a remarkable rearrangement, since Eq. (19) then becomes

$$\begin{aligned}\langle\nu_e|J^3_e(\mathbf{x},t)|\nu_e\rangle&=\sum_{j=1}^{4}(-1)^{j+1}\Psi_j^\dagger(\mathbf{x},t)\begin{pmatrix}1&1\\1&1\end{pmatrix}\Psi_j(\mathbf{x},t)\\&=\sum_{j=1}^{4}(-1)^{j+1}|\mathcal{A}^r_j+\mathcal{B}^r_j|^2,\end{aligned} \tag{27}$$

where (see also Appendix D)

$$\Psi_j(\mathbf{x},t)=\eta_j^\dagger\Psi(\mathbf{x},t)\equiv\begin{pmatrix}\mathcal{A}^r_j(\mathbf{x},t)\\\mathcal{B}^r_j(\mathbf{x},t)\end{pmatrix}. \tag{28}$$

Equation (27) gives a compact expression for the flavor current expectation value in the case of a spherically symmetric emission. It can indeed be integrated to give a useful expression for the electron neutrino flux at a given distance from the source. Although this calculation is of interest, because it contains information on the three-dimensional nature of the propagation, it will be not developed further here, and in the following section we will consider the case of one-dimensional propagation. Note that in the present case (spherical symmetry), the expectation values of J^1_e and J^2_e at any point can be obtained from the above expectation value of J^3_e. In the general case Eq. (19), however, the expectation values of J^1_e and J^2_e have to be calculated independently and this can be done by use of the decomposition of Γ^1 and Γ^2 given in Appendix B.

It remains to consider ρ_e, which is the relevant measurable quantity for experiments without angular resolution. It is indeed easy to get its expectation value:

$$\langle\nu_e|\rho_e(\mathbf{x},t)|\nu_e\rangle=\sum_{j=1}^{4}|\mathcal{A}^r_j+\mathcal{B}^r_j|^2, \tag{29}$$

which is related to the expectation value of J^3_e as follows:

$$\begin{aligned}\langle\nu_e|J^3_e(\mathbf{x},t)|\nu_e\rangle&=\langle\nu_e|\rho(\mathbf{x},t)|\nu_e\rangle-2|\mathcal{A}^r_2+\mathcal{B}^r_2|^2\\&\quad-2|\mathcal{A}^r_4+\mathcal{B}^r_4|^2.\end{aligned} \tag{30}$$

This is a significant result in the sense that it directly contradicts the usual $x=t$ approach: if the neutrino really travelled on a straight line then, for points for which the Cartesian z direction coincides with the polar radial direction (as defined with respect to the source), one should have[3] $\langle\nu_e|J^3_e(x)|\nu_e\rangle=\langle\nu_e|\rho(x)|\nu_e\rangle$.

[3]The result would be the same up to a constant if one replaced the velocity of light by the (constant) average velocity of the neutrinos.

B. One-dimensional case

Given the complexity of the three-dimensional analysis, we now limit the problem to one spatial dimension (as do most of other treatments in the literature).

The wave packet now takes the form $f(\mathbf{k}) = (2\pi)^{3/2}\delta(k_1)\delta(k_2)f(k_3)$, which corresponds to a packet with definite momentum in the x and y direction and a momentum distribution $f(k_3)$ along the z direction. The uncertainty principle then dictates that the corresponding spatial distribution is that of a beam of infinite radius and uniform surface density moving in the z direction (a neutrino sheet).

In the one-dimensional case, we have (see Appendix D) $\mathcal{A}_2^r = \mathcal{A}_3^r = 0$ and $\mathcal{B}_2^r = \mathcal{B}_3^r = 0$ for all r. In order to perform explicit evaluations, we choose in the following $r=1$: this does not imply any loss of generality since the final result is independent of r. We also put $\mathcal{A}_1^1 + \mathcal{B}_1^1 = \psi_+ + \psi'_+$ and $\mathcal{A}_4^1 + \mathcal{B}_4^1 = \psi_- + \psi'_-$.

Equation (27) then becomes

$$\langle \nu_e | J_e^3(z,t) | \nu_e \rangle = |\psi_+(z,t) + \psi'_+(z,t)|^2 - |\psi_-(z,t) + \psi'_-(z,t)|^2, \tag{31}$$

with

$$\psi_+(z,t) = \frac{A}{2}\int_0^\infty dk [\cos^2\theta e^{-i\omega_{k,1}t} + \sin^2\theta e^{-i\omega_{k,2}t}] \mathrm{T}_k(z), \tag{32}$$

$$\psi'_+(z,t) = iA\int_0^\infty dk \sin^2\theta \sin(\omega_{k,2}t)|V_k| \times [\mathrm{T}_k(z)|V_k| - \mathrm{T}_{-k}(z)|U_k|], \tag{33}$$

$$\psi_-(z,t) = \frac{A}{2}\int_0^\infty dk [\cos^2\theta e^{-i\omega_{k,1}t} + \sin^2\theta e^{-i\omega_{k,2}t}] \mathrm{T}_{-k}(z), \tag{34}$$

$$\psi'_-(z,t) = iA\int_0^\infty dk \sin^2\theta \sin(\omega_{k,2}t)|V_k| \times [\mathrm{T}_{-k}(z)|V_k| + \mathrm{T}_k(z)|U_k|] \tag{35}$$

where

$$\mathrm{T}_k(z) = e^{ikz} f(k)\left[\left(1+\frac{m_1}{\omega_{k,1}}\right)^{1/2} + \left(1-\frac{m_1}{\omega_{k,1}}\right)^{1/2}\right] + e^{-ikz} f(-k)\left[\left(1+\frac{m_1}{\omega_{k,1}}\right)^{1/2} - \left(1-\frac{m_1}{\omega_{k,1}}\right)^{1/2}\right]. \tag{36}$$

The above relations are the main result of this work: starting with an arbitrary wave packet $f(k)$ we can now determine the expected one-dimensional flux of electron neutrinos at a given distance from the source as

$$\Phi_{\nu_e\to\nu_e}(z) = \int_0^\infty dt \langle \nu_e | J_e^3(z,t) | \nu_e \rangle. \tag{37}$$

Note that the $\psi'_\pm$ terms are the corrections due to the use of the flavor Hilbert space in the definition of neutrino state. Equation (37) is an exact expression for the space oscillation formula in 1D and will be in the following evaluated numerically with the choice of a Gaussian wave packet. In the following section, however, we digress on the relativistic limit, which allows us to extract an (approximated) analytical form of Eq. (37) and to compare it with the existing results in the literature.

IV. APPROXIMATIONS AND NUMERICAL ANALYSIS

In this section, we explicitly evaluate Eq. (37) by choosing a Gaussian wave packet. We first consider the relativistic limit and show how the full result Eq. (37) reduces to the standard formula given in [26,25]. Then we proceed with numerical evaluations of the exact formula Eq. (37), which allow us to test the nonrelativistic regime.

We choose $f(k)$ to be a Gaussian wave packet with momentum spread σ_k:

$$f(k) = H \exp\left[-\frac{(k-q)^2}{4\sigma_k^2}\right], \tag{38}$$

with $H = (2\pi\sigma_k^2)^{-1/4}$.

A. Recovering the standard oscillation formula: sharp Gaussian wave packets in relativistic limit

To recover previous results in the literature [26,25], we consider the case of sharp Gaussian wave packet in relativistic limit, satisfying the following conditions:

(1) The mean momentum q of both mass species lies in the relativistic regime such that $\omega_{q,i} \gg m_i$ and $|V_q| \to 0$. Hence ψ'_+ and ψ'_- can be ignored.

(2) The Gaussian is sharply peaked, $q \gg \sigma_k$: the largest contributions to the integrals come from around the mean momentum q, i.e., Laplace's method [33] can be used.

Using the above conditions, one has $(1 \pm m_i/\omega_{k,i})^{1/2} \approx 1 \pm m_i/2\omega_{q,i}$. Thus, to first order in $m_1/\omega_{q,1}$,

$$\psi_+(z,t) \approx AH\int_0^\infty dk [\cos^2\theta e^{-i(\omega_{k,1}t - kz)} + \sin^2\theta e^{-i(\omega_{k,2}t-kz)}] \exp\left[-\frac{(k-q)^2}{4\sigma_k^2}\right] + AH\frac{m_1}{2\omega_{q,1}}\int_0^\infty dk [\cos^2\theta e^{-i(\omega_{k,1}t+kz)} + \sin^2\theta e^{-i(\omega_{k,2}t+kz)}] \exp\left[-\frac{(k+q)^2}{4\sigma_k^2}\right]. \tag{39}$$

For q positive, the integral on the second line can be neglected. On the other hand, we obtain for ψ_- the following expression:

$$\psi_-(z,t)\approx AH\frac{m_1}{2\omega_{q,1}}\int_0^\infty dk[\cos^2\theta e^{-i(\omega_{k,1}t-kz)} + \sin^2\theta e^{-i(\omega_{k,2}t-kz)}]\exp\left[-\frac{(k-q)^2}{4\sigma_k^2}\right] + AH\int_0^\infty dk[\cos^2\theta e^{-i(\omega_{k,1}t+kz)} + \sin^2\theta e^{-i(\omega_{k,2}t+kz)}]\exp\left[-\frac{(k+q)^2}{4\sigma_k^2}\right]. \quad (40)$$

Again, the second term is exponentially suppressed and can be neglected. The first term of ψ_- is also small for relativistic neutrinos and will be neglected in the following. We thus have

$$\langle \nu_e|J_e^3(z,t)|\nu_e\rangle \simeq |\psi_+(z,t)|^2. \quad (41)$$

1. Momentum integration

We now turn towards the evaluation of ψ_+ using Laplace's method. Let us first define the *dispersion* times $T_i^{disp}=\omega_{q,i}/2\sigma_k^2$, which set the time scale for the spatial dispersion of the wave packets.

Expanding the integrand about the mean momentum to second order in $\delta k=k-q$, we obtain[4]

$$\psi_+(z,t)\approx AH\cos^2\theta e^{-i(\omega_{q,1}t-qz)}\exp\left[-\frac{(z-v_{q_1}t)^2}{4S_1(t)}\right]\times\int_{-q}^\infty d(\delta k)\exp\left[-S_1(t)\left(\delta k-i\frac{(z-v_{q,1}t)}{2S_1(t)}\right)^2\right] + AH\sin^2\theta e^{-i(\omega_{q,2}t-qz)}\exp\left[-\frac{(z-v_{q_2}t)^2}{4S_2(t)}\right]\times\int_{-q}^\infty d(\delta k)\exp\left[-S_2(t)\left(\delta k-i\frac{(z-v_{q_2}t)}{2S_2(t)}\right)^2\right], \quad (42)$$

with $S_i(t)\equiv\sigma_x^2(1+it/T_i^{disp})$. Also, σ_x is a spread in the configuration space, defined by $\sigma_x\sigma_k=\frac{1}{2}$ and $v_{q,i}\equiv q/\omega_{q,i}$. To a good approximation we can set $\int_{-q}^\infty\approx\int_{-\infty}^\infty$ and obtain

$$\int_{-\infty}^\infty d(\delta k)e^{-S_i(t)(\delta k-i[(z-v_{q,1}t)/2S_i(t)])^2} = \sqrt{\frac{4\pi\sigma_k^2}{1+it/T_i^{disp}}}\equiv I_i(t).$$

Following the usual treatment [25], we now consider propagation for $t\ll T_i^{disp}$. Then $I_i(t)$ becomes a constant: $I=2\sqrt{\pi}\sigma_k$. For propagation times beyond T_i^{disp}, Laplace's method cannot be applied and one has to resort to the method of stationary phase [33,25].

We are now in a position to calculate the interference term. We get, after some rearrangements,

$$\langle \nu_e|J_e^3(z,t)|\nu_e\rangle \simeq |A|^2H^2|I|^2\left(\cos^4\theta\exp\left[-\frac{(z-v_{q,1}t)^2}{2\sigma_x^2}\right] + \sin^4\theta\exp\left[-\frac{(z-v_{q,2}t)^2}{2\sigma_x^2}\right]\right) + 2|A|^2H^2|I|^2\mathrm{Re}[\cos^2\theta\sin^2\theta e^{-i\phi(t)-f(t)}], \quad (43)$$

where the phase $\phi(t)$ and $f(t)$ are given by

$$\phi(t)=(\omega_{q,1}-\omega_{q,2})t,$$

$$f(t)=\frac{1}{4\sigma_x^2}[(z-v_{q,1}t)^2+(z-v_{q,2}t)^2]. \quad (44)$$

We note that $\phi(t)$ and $f(t)$ are the usual terms obtained in the external wave packet model in QFT [25,26]. We now proceed with time integration in order to finally get an expression involving distance only.

2. Time average

When calculating the interference term, spatial oscillations results from the cross term. We show this explicitly by considering the first term in Eq. (43): in the Laplace regime where $(t/T_i^{disp})\approx 0$, if we assume that the experiment runs for a very long time,

$$\int_{-\infty}^\infty \exp\left[-\frac{(z-v_{q,i}t)^2}{2\sigma_x^2}\right]dt=\frac{\sqrt{2\pi}}{v_{q,i}}\sigma_x.$$

The first and second terms of Eq. (43) are thus merely a collection of constants.

---

[4]Using the expansion $\omega_{k,i}\approx\omega_{q,i}+\delta k^2/2\omega_{q,i}+v_{q,i}\delta k$ and making a change of variable $k-q=\delta k$, we get

$$\psi_+\approx AH\cos^2\theta e^{i(qz-\omega_{q,1}t)}\int_{-q}^\infty d(\delta k)e^{i\delta k(z-v_{q,1}t)}\times\exp\left[-\delta k^2\left(\frac{1}{4\sigma_k^2}+i\frac{t}{2\omega_{q,1}}\right)\right]+AH\sin^2\theta e^{i(qz-\omega_{q,2}t)}\times\int_{-q}^\infty d(\delta k)e^{i\delta k(z-v_{q,2}t)}\exp\left[-\delta k^2\left(\frac{1}{4\sigma_k^2}+i\frac{t}{2\omega_{q,2}}\right)\right],$$

which is brought in the given form by completing the square for the argument of the Gaussian.

Let us now turn to the oscillation term. The dominant contributions in the time integral come from $\exp[-f(t)]$, more precisely in the neighborhood of the maximum value of $f(t)$, which occurs at

$$t_{max}=\left(\frac{v_{q,1}+v_{q,2}}{v_{q,1}^2+v_{q,2}^2}\right)z. \tag{45}$$

Thus the integrand can be approximated as

$$e^{-i\phi(t)-f(t)}\simeq\exp\left[-i\phi_{max}-f_{max}-it\frac{d\phi}{dt}\bigg|_{t_{max}}-\frac{t^2}{2}\frac{d^2f}{dt^2}\bigg|_{t_{max}}\right], \tag{46}$$

$$f_{max}=f(t_{max})=\frac{z^2\sigma_k^2(v_{q,1}-v_{q,2})^2}{v_{q,1}^2+v_{q,2}^2}\approx z^2\left(\frac{\sqrt{2}\pi\sigma_k}{L^{osc}q}\right)^2=\left(\frac{z}{L^{coh}}\right)^2, \tag{47}$$

$$\phi_{max}=\phi(t_{max})=\frac{zq(v_{q,2}^2-v_{q,1}^2)}{v_{q,1}v_{q,2}(v_{q,1}^2+v_{q,2}^2)}\approx-2\pi\frac{z}{L^{osc}} \tag{48}$$

to first order.[5] A coherence length $L^{coh}=(L^{osc}q)/(\sqrt{2}\pi\sigma_k)$ and an oscillation length $L^{osc}=4\pi q/\triangle m^2$ with $\Delta m^2=m_2^2-m_1^2$, having the same form as in the standard approach, are thus recovered. The other terms can be factorized as follows:

$$\frac{v_{q,1}^2+v_{q,2}^2}{4\sigma_x^2}\left(t-i\frac{(\omega_{q,1}-\omega_{q,2})}{v_{q,1}^2+v_{q,2}^2}2\sigma_x^2\right)^2+\sigma_x^2\frac{(\omega_{q,1}-\omega_{q,2})^2}{v_{q,1}^2+v_{q,2}^2}. \tag{49}$$

The first term integrates to a constant: $\sqrt{2\pi}\sigma_x$. As for the second term, it becomes $(\sigma_x^2/2)(\omega_{q,1}-\omega_{q,2})^2\simeq2\pi^2(\sigma_x/L^{osc})^2$ which gives the localization term [25]. Hence we can finally write down the (normalized) highly-relativistic space oscillation formula (compare to Ref. [26]):

$$\Phi_{\nu_e\to\nu_e}(z)\simeq1-\frac{1}{2}\sin^2(2\theta)\left\{1-\cos\left(2\pi\frac{z}{L^{osc}}\right)\times\exp\left[-\left(\frac{z}{L^{coh}}\right)^2-2\pi^2\left(\frac{\sigma_x}{L^{osc}}\right)^2\right]\right\}. \tag{50}$$

Thus we have shown that the exact formula Eq. (37) reproduces the usual oscillation formula in the relativistic limit.

B. Numerical calculations

In this section, we perform numerical evaluations of the exact oscillation formula Eq. (37) in order to compare it with the approximate result Eq. (50) for sample values of the parameters. We plot the two expressions in the maximal mixing case for a given σ_k and for different values of q and of the masses: in the relativistic case (Fig. 1) we observe a perfect agreement of the two formulas, as expected.

We then explore the nonrelativistic region and observe that relevant deviations from the standard formula do indeed appear, both in the oscillation amplitude and in the phase. This is already evident in Figs. 2,3 where we use $q=100$ with $m_1=1$, $m_2=3$ in the first case and $m_1=1$, $m_2=10$ in

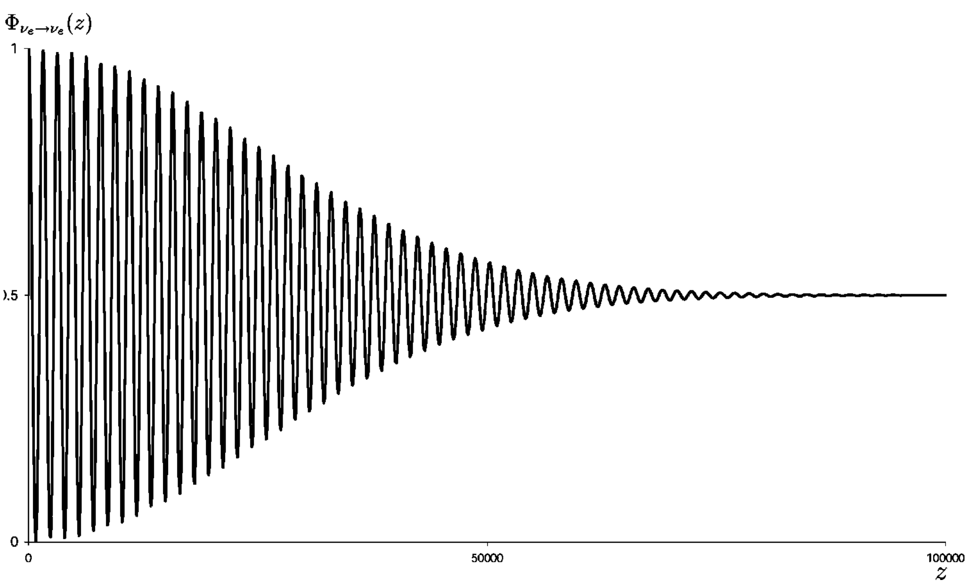

FIG. 1. Plot of the QFT flavor flux (solid line) against the standard oscillation formula (dashed line) for $\theta=\pi/4$, $\sigma_k=10$, $m_1=1$, $m_2=3$, and $q=1000$.

[5] We assume that $v_{q,i}\approx1-m_i^2/2q^2$. Thus $v_{q,1}^2+v_{q,2}^2\approx2$, $v_{q,2}^2-v_{q,1}^2\approx-\triangle m^2/q^2$ and $v_{q,1}v_{q,2}\approx1$.

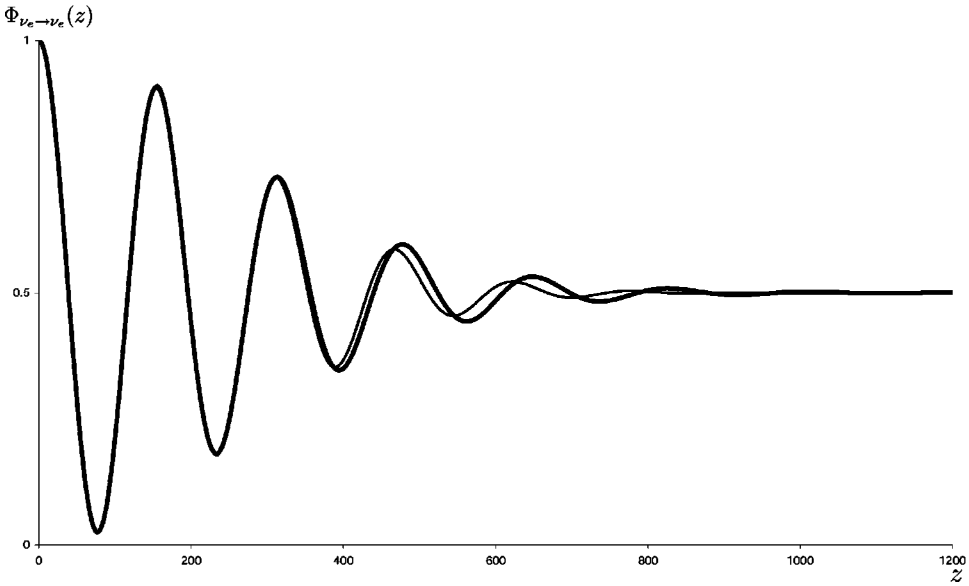

FIG. 2. Plot of the QFT flavor flux (thick line) against the standard oscillation formula (light line) for $\theta=\pi/4$, $\sigma_k=10$, $m_1=1$, $m_2=3$, and $q=100$.

the second case. The fact that the two plots are simply scaled with respect to each other, indicates that for these values of parameters the usual relation coherence length vs oscillation length: $L^{coh}=(L^{osc}q)/(\sqrt{2}\pi\sigma_k)$ is still valid with$L^{osc}=4\pi q/\triangle m^2$. In Fig. 4 we use $q=50$ with $m_1=1$, $m_2=3$ and as expected we observe a larger deviation from the usual formula.

Note that the value of the Bogoliubov coefficient $|V_k|$ is very small in all the considered cases,[6] so the observed corrections originate from the "standard" terms $\psi_\pm$ Eqs. (32),(34) in the oscillation formula Eq. (37) rather than from the flavor vacuum contribution, which thus turns out to be very difficult to detect when considering (time-)averaged oscillation formulas like in the present case.

A more complete discussion of these corrections and of their possible phenomenological relevance will be given elsewhere [34].

V. CONCLUSIONS

In this paper, we have for the first time derived a space-time dependent oscillation formula directly from the relativistic currents for flavor fields. These currents have been recently studied in Ref. [13], a result which served as basis for the present analysis, together with previous results on the Hilbert space for mixed fields [8,9].

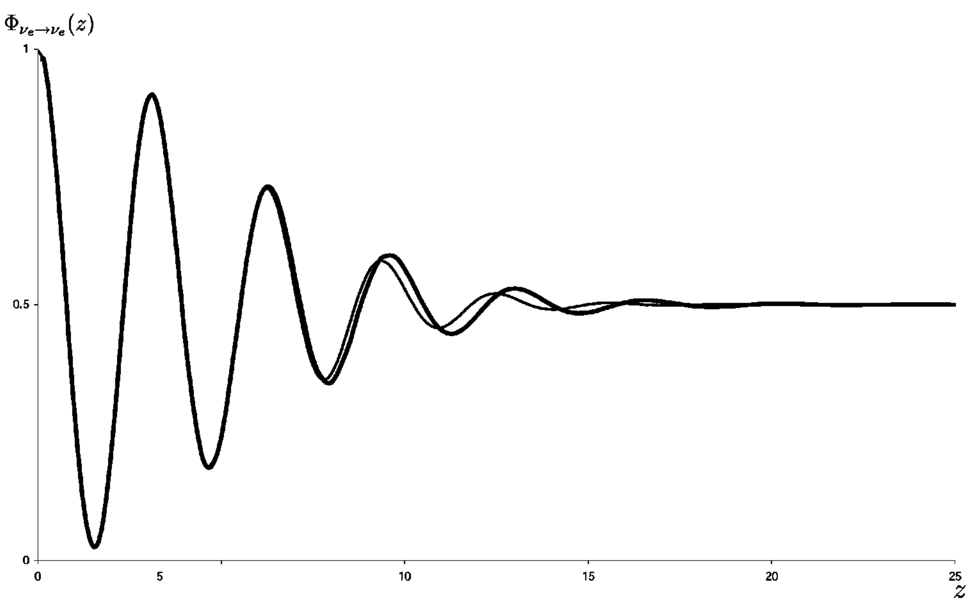

FIG. 3. Plot of the QFT flavor flux (thick line) against the standard oscillation formula (light line) for $\theta=\pi/4$, $\sigma_k=10$, $m_1=1$, $m_2=10$, and $q=100$.

[6]The values of $|V_q|$ are 1×10^{-3} (Fig. 1), 1×10^{-2} (Fig. 2), 4×10^{-2} (Fig. 3), and 2×10^{-2} (Fig. 4).

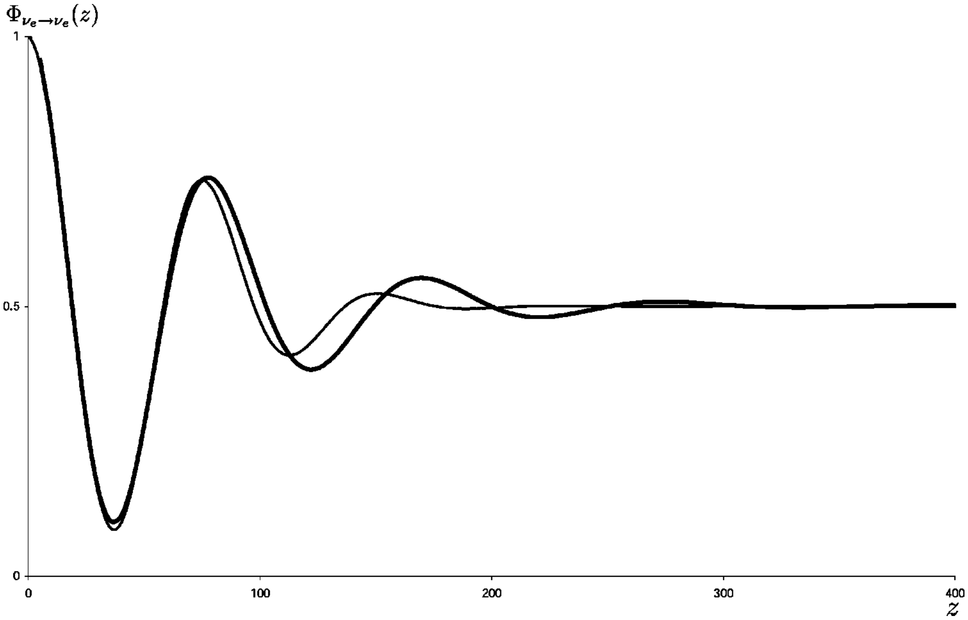

FIG. 4. Plot of the QFT flavor flux (thick line) against the standard oscillation formula (light line) for $\theta = \pi/4$, $\sigma_k = 10$, $m_1 = 1$, $m_2 = 3$, and $q = 50$.

We first presented a general expression for the electron neutrino flux in three dimensions and then specialized to the case with spherical symmetry, for which we were able to find a more explicit expression. In order to perform further analysis and numerical evaluations, we then considered the one-dimensional case with Gaussian wave packets, which is also the one most frequently treated in literature.

Our formulation presents several advantages with respect to existing treatments of neutrino oscillations in quantum field theory: it is a very straightforward approach which is easy to relate to practical experimental situations; it takes into account the nontrivial nature of the flavor vacuum and flavor states are thus consistently defined. It takes explicitly into account the full spin structure of neutrino states and does not resort to relativistic limit and/or assumption of nearly degenerate masses for the energy eigenstates.

We have shown how, in different limits, our formula reproduce existing results. Thus, in the case of relativistic neutrinos with nearly degenerate masses, we recover analytically the standard space dependent expression for neutrino oscillations [26,25], which is thus once again confirmed from an independent approach. Also, previous results on the flavor charges [9] have been recovered, exhibiting the non-standard oscillation terms.

The numerical analysis shows that in the nonrelativistic regime, our formula predicts significant deviations from the standard oscillation formula [26,25], which on the other hand cannot be expected to be valid in that region. An analysis of the phenomenological implications of the results obtained in this paper will be presented elsewhere [34].

ACKNOWLEDGMENTS

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APPENDIX A: FLAVOR HILBERT SPACE

The free fields $\nu_1(x)$ and $\nu_2(x)$ are written as $(t \equiv x_0)$

$$\nu_i(x) = \sum_{r=1,2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} [u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i} + v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i}], \quad i=1,2, \tag{A1}$$

where $u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}u^r_{\mathbf{k},i}$ and $v^r_{\mathbf{k},i}(t) = e^{i\omega_{k,i}t}v^r_{\mathbf{k},i}$, with $\omega_{k,i} = \sqrt{|\mathbf{k}|^2 + m_i^2}$. The $\alpha^r_{\mathbf{k},i}$ and the $\beta^r_{\mathbf{k},i}$, $i,r=1,2$ are the annihilation operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{\mathbf{k},i}|0\rangle_{12} = \beta^r_{\mathbf{k},i}|0\rangle_{12} = 0$. The anticommutation relations are

$$\{\nu_i^\alpha(\mathbf{x}), \nu_j^{\beta\dagger}(\mathbf{y})\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\,\delta_{\alpha\beta}\delta_{ij}, \quad \alpha,\beta = 1,\ldots,4, \tag{A2}$$

$$\{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{p},j}\} = \delta^3(\mathbf{k}-\mathbf{p})\,\delta_{rs}\delta_{ij},$$

$$\{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{p},j}\} = \delta^3(\mathbf{k}-\mathbf{p})\,\delta_{rs}\delta_{ij}, \quad i,j=1,2. \tag{A3}$$

All other anticommutators are zero. The orthonormality and completeness relations are

$$u^{r\dagger}_{\mathbf{k},i}u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i}v^s_{\mathbf{k},i} = \delta_{rs}, \quad u^{r\dagger}_{\mathbf{k},i}v^s_{-\mathbf{k},i} = v^{r\dagger}_{-\mathbf{k},i}u^s_{\mathbf{k},i} = 0,$$

$$\sum_r (u^r_{\mathbf{k},i}u^{r\dagger}_{\mathbf{k},i} + v^r_{-\mathbf{k},i}v^{r\dagger}_{-\mathbf{k},i}) = \mathbb{1}_2. \tag{A4}$$

where $\mathbb{1}_n$ is the $n\times n$ unit matrix. Equation (2) can be recast in the form

$$\nu_\sigma(x) \equiv G_\theta^{-1}(t)\,\nu_i(x)\,G_\theta(t), \tag{A5}$$

$$G_\theta(t)=\exp\left[\theta\int d^3\mathbf{x}(\nu_1^\dagger(x)\nu_2(x)-\nu_2^\dagger(x)\nu_1(x))\right], \quad \text{(A6)}$$

with $(\sigma,i)=(e,1),(\mu,2)$. The generator $G_\theta(t)$ does not leave invariant the vacuum $|0\rangle_{1,2}$:

$$|0(t)\rangle_{e,\mu}=G_\theta^{-1}(t)|0\rangle_{1,2}. \quad \text{(A7)}$$

We will refer to $|0(t)\rangle_{e,\mu}$ as the flavor vacuum: it is orthogonal to $|0\rangle_{1,2}$ in the infinite volume limit [8]. We define the flavor annihilators, relative to the fields $\nu_e(x)$ and $\nu_\mu(x)$ as[7]

$$\alpha^r_{\mathbf{k},\sigma}(t)\equiv G_\theta^{-1}(t)\alpha^r_{\mathbf{k},i}(t)G_\theta(t),$$
$$\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\equiv G_\theta^{-1}(t)\beta^{r\dagger}_{-\mathbf{k},i}(t)G_\theta(t) \quad \text{(A8)}$$

with $(\sigma,i)=(e,1),(\mu,2)$. The flavor fields can be expanded as

$$\nu_\sigma(x)=\sum_{r=1,2}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},\sigma}(t)+v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)]e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \text{(A9)}$$

with $(\sigma,i)=(e,1),(\mu,2)$. The flavor annihilation operators are defined as operators

$$\alpha^r_{\mathbf{k},e}(t)=\cos\theta\,\alpha^r_{\mathbf{k},1}+\sin\theta\sum_s[u^{r\dagger}_{\mathbf{k},1}(t)u^s_{\mathbf{k},2}(t)\alpha^s_{\mathbf{k},2}+u^{r\dagger}_{\mathbf{k},1}(t)v^s_{-\mathbf{k},2}(t)\beta^{s\dagger}_{-\mathbf{k},2}], \quad \text{(A10)}$$

$$\alpha^r_{\mathbf{k},\mu}(t)=\cos\theta\,\alpha^r_{\mathbf{k},2}-\sin\theta\sum_s[u^{r\dagger}_{\mathbf{k},2}(t)u^s_{\mathbf{k},1}(t)\alpha^s_{\mathbf{k},1}+u^{r\dagger}_{\mathbf{k},2}(t)v^s_{-\mathbf{k},1}(t)\beta^{s\dagger}_{-\mathbf{k},1}], \quad \text{(A11)}$$

$$\beta^r_{-\mathbf{k},e}(t)=\cos\theta\,\beta^r_{-\mathbf{k},1}+\sin\theta\sum_s[v^{s\dagger}_{-\mathbf{k},2}(t)v^r_{-\mathbf{k},1}(t)\beta^s_{-\mathbf{k},2}+u^{s\dagger}_{\mathbf{k},2}(t)v^r_{-\mathbf{k},1}(t)\alpha^{s\dagger}_{\mathbf{k},2}], \quad \text{(A12)}$$

$$\beta^r_{-\mathbf{k},\mu}(t)=\cos\theta\,\beta^r_{-\mathbf{k},2}-\sin\theta\sum_s[v^{s\dagger}_{-\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t)\beta^s_{-\mathbf{k},1}+u^{s\dagger}_{\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t)\alpha^{s\dagger}_{\mathbf{k},1}]. \quad \text{(A13)}$$

In the reference frame where $\mathbf{k}$ is collinear with $\hat{\mathbf{k}}\equiv(0,0,1)$, the spins decouple and we have

$$\alpha^r_{\mathbf{k},e}(t)=\cos\theta\,\alpha^r_{\mathbf{k},1}+\sin\theta(U^*_\mathbf{k}(t)\alpha^r_{\mathbf{k},2}+\epsilon^r_\mathbf{k}V_\mathbf{k}(t)\beta^{r\dagger}_{-\mathbf{k},2}), \quad \text{(A14)}$$

$$\alpha^r_{\mathbf{k},\mu}(t)=\cos\theta\,\alpha^r_{\mathbf{k},2}-\sin\theta(U_\mathbf{k}(t)\alpha^r_{\mathbf{k},1}-\epsilon^r_\mathbf{k}V_\mathbf{k}(t)\beta^{r\dagger}_{-\mathbf{k},1}), \quad \text{(A15)}$$

$$\beta^r_{-\mathbf{k},e}(t)=\cos\theta\,\beta^r_{-\mathbf{k},1}+\sin\theta(U^*_\mathbf{k}(t)\beta^r_{-\mathbf{k},2}-\epsilon^r_\mathbf{k}V_\mathbf{k}(t)\alpha^{r\dagger}_{\mathbf{k},2}), \quad \text{(A16)}$$

$$\beta^r_{-\mathbf{k},\mu}(t)=\cos\theta\,\beta^r_{-\mathbf{k},2}-\sin\theta(U_\mathbf{k}(t)\beta^r_{-\mathbf{k},1}+\epsilon^r_\mathbf{k}V_\mathbf{k}(t)\alpha^{r\dagger}_{\mathbf{k},1}), \quad \text{(A17)}$$

where $\epsilon^r_\mathbf{k}\equiv(-1)^{r+\mathbf{k}\cdot\hat{\mathbf{k}}/|\mathbf{k}|+1}$ and $U_\mathbf{k}(t)$, $V_\mathbf{k}(t)$ are Bogoliubov coefficients given by

$$U_\mathbf{k}(t)\equiv u^{r\dagger}_{\mathbf{k},2}(t)u^r_{\mathbf{k},1}(t)=v^{r\dagger}_{-\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t)=|U_\mathbf{k}|e^{i(\omega_{k,2}-\omega_{k,1})t}, \quad \text{(A18)}$$

$$V_\mathbf{k}(t)\equiv\epsilon^r_\mathbf{k}u^{r\dagger}_{\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t)=-\epsilon^r_\mathbf{k}u^{r\dagger}_{\mathbf{k},2}(t)v^r_{-\mathbf{k},1}(t)=|V_\mathbf{k}|e^{i(\omega_{k,2}+\omega_{k,1})t}, \quad \text{(A19)}$$

$$|U_\mathbf{k}|=\left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{1/2}\left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{1/2}\times\left(1+\frac{|\mathbf{k}|^2}{(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}\right), \quad \text{(A20)}$$

$$|V_\mathbf{k}|=\left(\frac{\omega_{k,1}+m_1}{2\omega_{k,1}}\right)^{1/2}\left(\frac{\omega_{k,2}+m_2}{2\omega_{k,2}}\right)^{1/2}\times\left(\frac{|\mathbf{k}|}{(\omega_{k,2}+m_2)}-\frac{|\mathbf{k}|}{(\omega_{k,1}+m_1)}\right), \quad \text{(A21)}$$

satisfying $|U_\mathbf{k}|^2+|V_\mathbf{k}|^2=1$.

APPENDIX B: CHIRAL REPRESENTATION AND USEFUL RELATIONS

$$u^r_{k,i}=\chi_i\begin{pmatrix}\left(1+\dfrac{\vec{\sigma}\cdot\mathbf{k}}{\omega_{k,i}+m_i}\right)\xi^r\\ \left(1-\dfrac{\vec{\sigma}\cdot\mathbf{k}}{\omega_{k,i}+m_i}\right)\xi^r\end{pmatrix},$$

$$v^r_{k,i}=\chi_i\begin{pmatrix}\left(1+\dfrac{\vec{\sigma}\cdot\mathbf{k}}{\omega_{k,i}+m_i}\right)\xi^r\\ \left(-1+\dfrac{\vec{\sigma}\cdot\mathbf{k}}{\omega_{k,i}+m_i}\right)\xi^r\end{pmatrix},\quad r=1,2, \quad \text{(B1)}$$

$$\xi^1=\begin{pmatrix}1\\0\end{pmatrix},\quad \xi^2=\begin{pmatrix}0\\1\end{pmatrix},\quad \chi_i\equiv\left(\frac{\omega_{k,i}+m_i}{4\omega_{k,i}}\right)^{1/2},\quad i=1,2, \quad \text{(B2)}$$

$$v^{1\dagger}_{-k,1}u^1_{k,2}=-v^{2\dagger}_{-k,1}u^2_{k,2}=2\chi_1\chi_2\left(\frac{-k_3}{\omega_{k,1}+m_1}+\frac{k_3}{\omega_{k,2}+m_2}\right), \quad \text{(B3)}$$

$$v^{1\dagger}_{-k,1}u^2_{k,2}=(v^{2\dagger}_{-k,1}u^1_{k,2})^*=2\chi_1\chi_2\left(\frac{-k_-}{\omega_{k,1}+m_1}+\frac{k_-}{\omega_{k,2}+m_2}\right), \quad \text{(B4)}$$

and $u^{r\dagger}_{k,i}u^s_{k,j}=0$ for $r\neq s$. We define

[7]The annihilation of the flavor vacuum at each time is expressed as $\alpha^r_{\mathbf{k},e}(t)|0(t)\rangle_{e,\mu}=G_\theta^{-1}(t)\alpha^r_{\mathbf{k},1}|0\rangle_{1,2}=0$.

$$\Gamma^0 = \mathbb{1}_4, \quad \Gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad i=1,2,3. \tag{B5}$$

Notice that Γ^1 and Γ^2 can be decomposed as follows:

$$\Gamma^i = \sum_{j=1}^{4} \eta_j \eta_j^\dagger - \lambda_i \lambda_i^\dagger - \vartheta_i \vartheta_i^\dagger, \tag{B6}$$

with

$$\lambda_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \vartheta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix}, \quad \vartheta_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}. \tag{B7}$$

APPENDIX C: VACUUM EXPECTATION VALUE OF THE CURRENT

We show here that ${}_{e,\mu}\langle 0|J_e^\mu(\mathbf{x},t)|0\rangle_{e,\mu}=0$. Let us consider for example the following quantity:

$$\begin{aligned} &{}_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{p},e}(t)\alpha^s_{\mathbf{k},e}(t)|0\rangle_{e,\mu} \\ &\quad = {}_{1,2}\langle 0|\,G_\theta(0)\alpha^{r\dagger}_{\mathbf{p},e}(t)\alpha^s_{\mathbf{k},e}(t)G_\theta^{-1}(0)|0\rangle_{1,2}. \end{aligned} \tag{C1}$$

Now let us define $\tilde{\alpha}^r_{\mathbf{k},e}(t) \equiv G_\theta(0)\alpha^r_{\mathbf{k},e}(t)G_\theta^{-1}(0)$ and H.c. Of course, $\tilde{\alpha}^r_{\mathbf{k},e}(0)=\alpha^r_{\mathbf{k},1}$. It is easy to realize that [see Eqs. (A11)]

$$\{\tilde{\alpha}^r_{\mathbf{k},e}(t), \tilde{\alpha}^{s\dagger}_{\mathbf{p},e}(t')\}=0 \quad \text{for } \mathbf{k}\neq\mathbf{p}, \quad \forall\ t,t',r,s. \tag{C2}$$

This in turn implies that the quantity (C1) vanishes for $\mathbf{k} \neq \mathbf{p}$. Since this is valid for any two flavor operators, we only need to calculate the various terms for the (flavor) VEV of the current for equal momenta. We thus write

$${}_{e,\mu}\langle 0|J_e^\mu(\mathbf{x},t)|0\rangle_{e,\mu} = \int d^3\mathbf{k}\, {}_{e,\mu}\langle 0|J_e^\mu(\mathbf{k},t)|0\rangle_{e,\mu}=0. \tag{C3}$$

The flavor vacuum is invariant under rotations, so we can choose without loss of generality the most suitable spatial reference frame for each of the terms in the momentum integration. In our case, this is the one for which $\mathbf{k}$ is collinear to $(0,0,1)$. In this frame, we obtain the relations

$${}_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},e}(t)\alpha^s_{\mathbf{k},e}(t)|0\rangle_{e,\mu} = {}_{e,\mu}\langle 0|\beta^{r\dagger}_{-\mathbf{k},e}(t)\beta^s_{-\mathbf{k},e}(t)|0\rangle_{e,\mu} = \delta^{rs}4|V_\mathbf{k}|^2\sin^2\theta\left[\cos^2\theta\sin^2\left(\frac{\omega_1+\omega_2}{2}t\right)+\sin^2\theta|U_\mathbf{k}|^2\sin^2(\omega_2 t)\right], \tag{C4}$$

$$\begin{aligned} {}_{e,\mu}\langle 0|\beta^r_{-\mathbf{k},e}(t)\alpha^s_{\mathbf{k},e}(t)|0\rangle_{e,\mu} &= ({}_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},e}(t)\beta^{r\dagger}_{-\mathbf{s},e}(t)|0\rangle_{e,\mu})^* \\ &= \delta^{rs}\epsilon^r_\mathbf{k}\sin^2\theta e^{2i\omega_1 t}|U_\mathbf{k}||V_\mathbf{k}|[(1+e^{-2i\omega_1 t}-2e^{-2i(\omega_1+\omega_2)t})\cos^2\theta+2i\sin(\omega_2 t)(e^{-i\omega_2 t}|U_\mathbf{k}|^2 \\ &\quad + e^{i\omega_2 t}|V_\mathbf{k}|^2)\sin^2\theta]. \end{aligned} \tag{C5}$$

Because of relation (C4) and the orthonormality relations for the spinors, we easily realize that ${}_{e,\mu}\langle 0|J_e^0(\mathbf{k},t)|0\rangle_{e,\mu}=0$.

As for ${}_{e,\mu}\langle 0|J_e^3(\mathbf{k},t)|0\rangle_{e,\mu}$, we need to consider

$$u^{r\dagger}_{\mathbf{k},i}(t)\Gamma^3 u^s_{\mathbf{k},i}(t) = -v^{r\dagger}_{-\mathbf{k},i}(t)\Gamma^3 v^s_{-\mathbf{k},i}(t) = \delta^{rs}\frac{k_3}{\omega_{k,i}}, \tag{C6}$$

$$\begin{aligned} u^{r\dagger}_{\mathbf{k},i}(t)\Gamma^3 v^r_{-\mathbf{k},i}(t) &= (v^{r\dagger}_{-\mathbf{k},i}(t)\Gamma^3 u^r_{\mathbf{k},i}(t))^* \\ &= -(-1)^r\delta^{rs}\frac{m_i}{\omega_{k,i}}e^{2i\omega_{k,i}t}. \end{aligned} \tag{C7}$$

As a consequence of relations (C4), (C5) and (C6), (C7), the expectation value ${}_{e,\mu}\langle 0|J_e^3(\mathbf{k},t)|0\rangle_{e,\mu}$ is an odd function of $\mathbf{k}$ and therefore its integral vanishes.

Note that ${}_{e,\mu}\langle 0|J_e^{1,2}(\mathbf{k},t)|0\rangle_{e,\mu}$ vanish identically in the chosen reference frame, because of

$$\begin{aligned} u^{r\dagger}_{\mathbf{k},i}(t)\Gamma^j u^s_{\mathbf{k},i}(t) &= -v^{r\dagger}_{-\mathbf{k},i}(t)\Gamma^j v^s_{-\mathbf{k},i}(t) \\ &= u^{r\dagger}_{\mathbf{k},i}(t)\Gamma^j v^s_{-\mathbf{k},i}(t)=0, \quad j=1,2. \end{aligned} \tag{C8}$$

APPENDIX D: EXPLICIT FORMS

$$\mathcal{A}^r_j(\mathbf{x},t) = A\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{k})\sum_s [\eta_j^\dagger u^s_{\mathbf{k},1}(t)]\delta^{sr}X_{\mathbf{k},e}(t), \tag{D1}$$

$$\begin{aligned} \mathcal{B}^r_j(\mathbf{x},t) = A\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{k})\sum_s [\eta_j^\dagger v^s_{-\mathbf{k},1}(t)] \\ \times(\sigma\cdot\mathbf{k})^{sr}Y_{\mathbf{k},e}(t), \end{aligned} \tag{D2}$$

$$\mathcal{A}_1^1 = A\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})\chi_1(1+k_3/\Omega_1)X_{\mathbf{k},e}(t),$$

$$\mathcal{B}_1^1 = A\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})\chi_1(k_3-|\mathbf{k}|^2/\Omega_1)Y_{\mathbf{k},e}(t),$$

$$\mathcal{A}_2^1 = -\mathcal{A}_3^1 = A\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})\chi_1(k_+/\Omega_1)X_{\mathbf{k},e}(t),$$

$$\mathcal{B}_2^1 = -\mathcal{B}_3^1 = A\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})\chi_1 k_+ Y_{\mathbf{k},e}(t),$$

$$\mathcal{A}_4^1 = A\int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})\chi_1(1-k_3/\Omega_1)X_{\mathbf{k},e}(t),$$

$$\mathcal{B}_4^1 = -A\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})\chi_1(|\mathbf{k}|^2/\Omega_1+k_3)Y_{\mathbf{k},e}(t), \tag{D3}$$

where $\Omega_1 \equiv \omega_{k,1}+m_1$ and $k_+ = k_1 + ik_2$, $k_- = k_1 - ik_2$.

For $\mathbf{k}$ collinear with $\hat{\mathbf{k}} \equiv (0,0,1)$ or in the one-dimensional case, we have

$$\{\beta_{-\mathbf{k},e}^{s\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0)\} = \delta^{sr}\epsilon_{\mathbf{k}}^r |U_{\mathbf{k}}||V_{\mathbf{k}}|\sin^2\theta [e^{i\omega_{k,2}t} - e^{-i\omega_{k,2}t}]e^{-i\omega_{k,1}t}. \tag{D4}$$

[1] J. Davis, D. S. Harmer, and K. C. Hoffmann, Phys. Rev. Lett. **20**, 1205 (1968).
[2] M. Koshiba, in ''Erice 1998, From the Planck length to the Hubble radius,'' p. 170; Super-Kamiokande Collaboration, S. Fukuda *et al.*, Phys. Rev. Lett. **86**, 5656 (2001).
[3] SNO Collaboration, Q. R. Ahmad *et al.*, Phys. Rev. Lett. **87**, 071301 (2001); **89**, 011301 (2002).
[4] KamLAND Collaboration, K. Eguchi *et al.*, Phys. Rev. Lett. **90**, 021802 (2003).
[5] K2K Collaboration, M. H. Ahn *et al.*, Phys. Rev. Lett. **90**, 041801 (2003).
[6] B. Pontecorvo, Zh. Éksp. Teor. Fiz. **33** (1958) [Sov. Phys. JETP **6**, 429 (1958)]; Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962); V. Gribov and B. Pontecorvo, Phys. Lett. **28B**, 493 (1969); S. M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978).
[7] T. Cheng and L. Li, *Gauge Theory of Elementary Particle Physics* (Clarendon, Oxford, 1989).
[8] M. Blasone and G. Vitiello, Ann. Phys. (N.Y.) **244**, 283 (1995); **249**, 363(E) (1995).
[9] M. Blasone, P. A. Henning, and G. Vitiello, Phys. Lett. B **451**, 140 (1999); M. Blasone, in ''Erice 1998, From the Planck length to the Hubble radius,'' pp. 584–593, hep-ph/9810329.
[10] M. Blasone, A. Capolupo, O. Romei, and G. Vitiello, Phys. Rev. D **63**, 125015 (2001); M. Blasone, P. A. Henning, and G. Vitiello, in ''La Thuile 1996, Results and perspectives in particle physics,'' pp. 139–152, hep-ph/9605335.
[11] M. Blasone, A. Capolupo, and G. Vitiello, Phys. Rev. D **66**, 025033 (2002).
[12] M. Blasone and G. Vitiello, Phys. Rev. D **60**, 111302 (1999).
[13] M. Blasone, P. Jizba, and G. Vitiello, Phys. Lett. B **517**, 471 (2001).
[14] M. Blasone, A. Capolupo, and G. Vitiello, in ''Zhang-Jia-Jie 2001, Flavor physics,'' pp. 425–433, hep-th/0107125; M. Blasone, A. Capolupo, and G. Vitiello, hep-ph/0107183.
[15] M. Blasone, P. A. Henning, and G. Vitiello, Phys. Lett. B **466**, 262 (1999); X. B. Wang, L. C. Kwek, Y. Liu, and C. H. Oh, Phys. Rev. D **63**, 053003 (2001).
[16] K. C. Hannabuss and D. C. Latimer, ''Fermion mixing in quasifree states,'' hep-th/0207268; J. Phys. A **33**, 1369 (2000).
[17] M. Binger and C. R. Ji, Phys. Rev. D **60**, 056005 (1999); C. R. Ji and Y. Mishchenko, *ibid.* **64**, 076004 (2001); **65**, 096015 (2002).
[18] K. Fujii, C. Habe, and T. Yabuki, Phys. Rev. D **59**, 113003 (1999); **60**, 099903(E) (1999).
[19] K. Fujii, C. Habe, and T. Yabuki, Phys. Rev. D **64**, 013011 (2001).
[20] K. Fujii, C. Habe, and M. Blasone, ''Operator relation among neutrino fields and oscillation formulas in matter,'' hep-ph/0212076.
[21] V. Bargmann, Ann. Math. **59**, 1 (1954); A. Galindo and P. Pascual, *Quantum Mechanics* (Springer-Verlag, Berlin, 1990).
[22] T. Kaneko, Y. Ohnuki, and K. Watanabe, Prog. Theor. Phys. **30**, 521 (1963); K. Fujii, Nuovo Cimento **34**, 722 (1964).
[23] C. Giunti, C. W. Kim, and U. W. Lee, Phys. Rev. D **45**, 2414 (1992); C. W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics* (Harwood Academic, Chur, Switzerland, 1993).
[24] B. Kayser, Phys. Rev. D **24**, 110 (1981); B. Kayser, F. Gibrat-Debut, and D. Perrier, *The Physics of Massive Neutrinos* (World Scientific, Singapore, 1989).
[25] M. Beuthe, Phys. Rev. D **66**, 013003 (2002); Phys. Rep. **375**, 105 (2003).
[26] C. Giunti, J. High Energy Phys. **11**, 017 (2002); C. Giunti and C. W. Kim, Phys. Rev. D **58**, 017301 (1998).
[27] W. Grimus and P. Stockinger, Phys. Rev. D **54**, 3414 (1996); W. Grimus, P. Stockinger, and S. Mohanty, *ibid.* **59**, 013011 (1999).
[28] C. Y. Cardall, Phys. Rev. D **61**, 073006 (2000); C. Y. Cardall and D. J. H. Chung, *ibid.* **60**, 073012 (1999).
[29] K. Kiers and N. Weiss, Phys. Rev. D **57**, 3091 (1998).
[30] T. Yabuki and K. Ishikawa, Prog. Theor. Phys. **108**, 347 (2002).
[31] B. Ancochea, A. Bramon, R. Munoz-Tapia, and M. Nowakowski, Phys. Lett. B **389**, 149 (1996).
[32] M. Zralek, Acta Phys. Pol. B **29**, 3925 (1998).
[33] A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956).
[34] M. Blasone, P. Pires Pachêco, and H. W. C. Tseung (in preparation).