

Noncommutative electrodynamics

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In this paper we define a causal Lorentz covariant noncommutative (NC) classical electrodynamics. We obtain an explicit realization of the NC theory by solving perturbatively the Seiberg-Witten map. The action is polynomial in the field strength F , allowing us to preserve both causality and Lorentz covariance. The general structure of the Lagrangian is studied to all orders in the perturbative expansion in the NC parameter θ . We show that monochromatic plane waves are solutions of the equations of motion to all orders. An iterative method has been developed to solve the equations of motion and has been applied to the study of the corrections to the superposition law and to the Coulomb law.

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I. INTRODUCTION

Suggestions on the possibility that nature could allow for noncommuting spatial coordinates came both from the past [1] and more recently in the realm of superstring theory studying low energy excitations of D -branes in a magnetic field [2].

This has stimulated investigations on the noncommutative (NC) versions of gauge field theories and on the behavior of their quantized counterparts. Among these, Maxwell theory is perhaps the easiest example and one where a possible experimental test of this hypothesis could be realizable. Nevertheless, two main problems arise when one tries to implement electromagnetism in a noncommutative geometry: the loss of causality due to the appearance of derivative couplings in the Lagrangian and, more fundamentally, the violation of Lorentz invariance exhibited by plane wave solutions [3]. These problems have been discussed with a different approach in the framework of NC QED [4,5].

In this paper we show that both these problems may be avoided if one allows a nonzero “electrical” component into the tensor θ of the noncommutation relation so including time as a NC coordinate. After application of the Seiberg-Witten (SW) map [2] the theory is perturbative in θ and classical plane waves turn out to be exact solutions. They no longer obey a superposition principle. Finally, a sort of electric-magnetic duality comprehending θ and reminiscent of the known one in commutative Maxwell theory appears between the fields in the equations of motion, reinforcing the interpretation of θ as a sort of background primordial electromagnetic field.

In Sec. II we fix notations and conventions, recall the definition of the SW map [2] and show the explicit solution to second order in θ .

In Sec. III we prove that the Lagrangian of the theory is

polynomial to all orders in the perturbative parameter so that causality is preserved. The equations of motion are derived in Sec. IV where evidence is also given of the mentioned duality.

In Sec. V a general iterative method of solving the equations of motion is outlined.

After proving that plane waves are solutions, the method is applied to the problem of plane wave superposition and to derive corrections to the Coulomb law.

The paper ends with some comments on the results found and on possible experimental settings aimed to directly measure noncommutativity.

II. SW MAP AND SECOND ORDER EXPANTIONS

In the following, a caret over a classical symbol will indicate the same quantity in its NC version. In this fashion, coordinates of flat noncommutative Minkowsky spacetime will be assumed to satisfy

$$[\hat{x}^\mu, \hat{x}^\nu]_* = i \theta^{\mu\nu}, \quad (2.1)$$

where $\theta^{\mu\nu}$ is a real skew tensor whose components are set as follows:

$$\begin{aligned} \theta^{0i} &= \varepsilon^i, \\ \theta^{ij} &= \epsilon^{ijk} \beta_k. \end{aligned} \quad (2.2)$$

Note that we do not impose $\varepsilon^i = 0$. This means that time does not commute with spatial coordinates and θ is a constant tensor field. Besides θ we consider the usual electromagnetic field whose NC action is given by

$$\hat{S} = -\frac{1}{4} \int d^4x \hat{F}^{\mu\nu} *_\hat{F}_{\mu\nu} = -\frac{1}{4} \int d^4x \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}. \quad (2.3)$$

The corresponding Lagrangian and field strength are given by

$$\hat{\mathcal{L}} = -\frac{1}{4} \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}, \quad (2.4)$$

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$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_* . \quad (2.5)$$

Here the star product (*) between NC quantities is defined as usual:

$$(\hat{f} * \hat{g})(x) := e^{i\theta^{\mu\nu} \partial_\mu \partial'_\nu} \hat{f}(x) \hat{g}(x') \Big|_{x=x'} . \quad (2.6)$$

Also, the following conventions will be used for electromagnetic fields:

$$\begin{aligned} E^i &= F^{0i} , \\ B_k &= \frac{1}{2} \epsilon_{ijk} F^{ij} . \end{aligned} \quad (2.7)$$

Now, according to Seiberg and Witten [2], every NC gauge theory \hat{A}_μ has a perturbative description in terms of the non-commuting parameter θ and another commutative theory A_μ possessing the same degrees of freedom as the NC one. The relation between them is established by means of the Seiberg-Witten map:

$$\begin{aligned} \frac{\partial \hat{A}_\mu}{\partial \theta^{\alpha\beta}} &= -\frac{1}{8} \{ \hat{A}_\alpha, \partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu} \}_* - (\alpha \leftrightarrow \beta) , \\ \hat{A}_\mu|_{\theta=0} &= A_\mu . \end{aligned} \quad (2.8)$$

Solving the above equations means determining each piece of the perturbative expansions:

$$\hat{A}_\mu = A_\mu + \hat{A}_\mu^{(1)} + \hat{A}_\mu^{(2)} + \dots , \quad (2.9)$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \hat{F}_{\mu\nu}^{(1)} + \hat{F}_{\mu\nu}^{(2)} + \dots , \quad (2.10)$$

relating at every order in θ the NC quantities with their respective classical counterparts. As is well known [6], one obtains, to first order,

$$\begin{aligned} \hat{A}_\mu^{(1)} &= -\frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) , \\ \hat{F}_{\mu\nu}^{(1)} &= \theta^{\gamma\delta} (F_{\mu\gamma} F_{\nu\delta} - A_\gamma \partial_\delta F_{\mu\nu}) . \end{aligned} \quad (2.11)$$

Considering the second order corrections, we assume $\hat{A}_\mu^{(2)} = \frac{1}{2} \theta^{\alpha\beta} \theta^{\gamma\delta} \mathbf{n}_{\mu\alpha\beta\gamma\delta}$ and substitute the whole expansion of \hat{A}_μ into Eqs. (2.8). We realize that differentiating and then evaluating at $\theta=0$, we end with a recursive relation between second order and first order corrections and their θ derivatives. This leads to computation of the term \mathbf{n} . After careful rearrangements, the expression for the second order correction to A_μ is

$$\begin{aligned} \hat{A}_\mu^{(2)} &= \frac{1}{2} \theta^{\alpha\beta} \theta^{\gamma\delta} \left\{ A_\gamma \partial_\delta A_\alpha \partial_\beta A_\mu + A_\gamma F_{\delta\alpha} F_{\beta\mu} + A_\alpha A_\gamma \partial_\delta F_{\beta\mu} \right. \\ &\quad \left. + \frac{1}{4} \partial_\mu (A_\alpha A_\gamma \partial_\delta A_\beta) \right\} . \end{aligned} \quad (2.12)$$

Similarly, via the relation $\hat{F}_{\mu\nu}^{(2)} = \partial_\mu \hat{A}_\nu^{(2)} + \theta^{\gamma\delta} \partial_\gamma \hat{A}_\mu^{(1)} \partial_\delta A_\nu - (\mu \leftrightarrow \nu)$ one also computes

$$\begin{aligned} \hat{F}_{\mu\nu}^{(2)} &= \theta^{\alpha\beta} \theta^{\gamma\delta} F_{\mu\gamma} F_{\delta\alpha} F_{\beta\nu} - \theta^{\gamma\delta} A_\gamma \partial_\delta \hat{F}_{\mu\nu}^{(1)} \\ &\quad - \frac{1}{2} \theta^{\alpha\beta} \theta^{\gamma\delta} A_\gamma (\partial_\alpha A_\delta + A_\alpha \partial_\delta) \partial_\beta F_{\mu\nu} . \end{aligned} \quad (2.13)$$

III. GENERAL STRUCTURE

We discuss some properties valid to all orders in θ of the perturbative action obtained by means of the SW map.

Proposition 3.1. The Lagrangian $\hat{\mathcal{L}}$ corresponding to the action Eq. (2.3) via the SW map is a polynomial in F only (that is, it does not contain derivatives of F); furthermore, the terms $\hat{\mathcal{L}}^{(n)}$ of order n in θ form a homogeneous polynomial of degree $n+2$ in F .

Proof. From the SW equation (2.8) we have

$$\begin{aligned} \frac{\delta \hat{F}_{\mu\nu}}{\delta \theta^{\alpha\beta}} &= \frac{1}{8} \partial_\nu \{ \hat{A}_\alpha, \partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu} \}_* - (\mu \leftrightarrow \nu) + \frac{i}{8} \{ \{ \hat{A}_\alpha, \partial_\beta \hat{A}_\mu \\ &\quad + \hat{F}_{\beta\mu} \}_* , \hat{A}_\nu \}_* + \frac{i}{8} [\hat{A}_\mu , \{ \hat{A}_\alpha , \partial_\beta \hat{A}_\nu + \hat{F}_{\beta\nu} \}_*]_* \\ &\quad - (\alpha \leftrightarrow \beta) + \delta^* . \end{aligned} \quad (3.1)$$

Here δ^* is supposed to include all the terms arising whenever the derivation acts on the θ s appearing in the * of the * product; they always give rise to total derivatives in the Lagrangian density and so may be neglected. As a consequence, performing an arbitrary number of derivations and then putting $\theta=0$ shows that commutators of the type present in Eq. (3.1) give vanishing contributions.

Then all significant contributions are seen to come from the term $\frac{1}{8} \partial_\nu \{ \hat{A}_\alpha, \partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu} \}_*$, which, evaluated at 0 after k derivations, produces an homogeneous polynomial of order $k+1$ in A with $k+1$ derivatives (with respect to spacetime coordinates) equally distributed on each monomial. Finally, considering $\hat{F}_{\mu\nu}^{(1)}$ at order n in θ , by the same argument, one obtains an homogeneous polynomial of order $n+2$ in A with $n+2$ spacetime derivatives comparing in each monomial.

Now, since the Lagrangian density (obtained from the SW map) is certainly invariant under the usual U(1) gauge transformations, every monomial can be rearranged, modulo integration by parts, so as to depend only on F and possibly its derivatives. But being the number of derivatives exactly equal to the number of A s in every monomial, it follows that derivatives of F cannot appear at all. Q.E.D.

Corollary 3.1. The equations of motion of the U(1) theory take the form

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 , \quad (3.2)$$

where $\tilde{F}^{\mu\nu}$ is the sum of homogeneous polynomials of degree $n+1$ in F and order n in θ (i.e., written symbolically):

$$\tilde{F} = \sum_n \theta^n F^{n+1}. \quad (3.3)$$

As we will see, this property helps to derive a recursive algorithm for their resolution.

The main consequence of the structure (3.3) evidenced above is that the equations of motion for the field strength are of first order. This seems to suggest that the theory is causal even though not requiring time commutativity. In the literature, it is suspected that causality does not survive non-commutativity [7]. In our model though, after undertaking the SW map, the action has been manipulated and integrated by parts to render all terms explicitly gauge invariant. As a by-product, all higher order time derivatives have disappeared. In effect, this task is equivalent to add boundary terms to the Lagrangian: exactly those capable of giving causal consistency to the theory.

Probably this should be the right procedure to follow generally. Furthermore, the fact that preserving causality is no more consequence of imposing zero temporal components in θ allows us to require that it can transform like a tensor in respect to the Lorentz group. It descends that Lorentz covariance is also preserved.

IV. EQUATIONS OF MOTION UP TO SECOND ORDER

Let us expand also the NC Lagrangian density (2.4) into pieces of increasing order in θ :

$$\hat{\mathcal{L}} = \mathcal{L} + \hat{\mathcal{L}}^{(1)} + \hat{\mathcal{L}}^{(2)} + \dots \quad (4.1)$$

The first term here coincides with the classical Maxwell Lagrangian while the other terms are its various corrections. More precisely,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \\ \hat{\mathcal{L}}^{(1)} &= -\frac{1}{2} \hat{F}^{\mu\nu(1)} F_{\mu\nu}, \\ \hat{\mathcal{L}}^{(2)} &= -\frac{1}{4} \{ \hat{F}^{\mu\nu(1)} \hat{F}_{\mu\nu}^{(1)} + 2 F^{\mu\nu} \hat{F}_{\mu\nu}^{(2)} \}. \end{aligned}$$

Recall [8] that up to first order in θ , the NC Lagrangian has the following form:

$$\hat{\mathcal{L}} = -\frac{1}{4} \left[\left(1 - \frac{1}{2} \theta^{\alpha\beta} F_{\alpha\beta} \right) F^2 + 2 \theta^{\alpha\beta} F^{\mu\nu} F_{\mu\alpha} F_{\nu\beta} \right] \quad (4.2)$$

or, upon substitution according to our conventions (2.2), we have

$$\hat{\mathcal{L}} = \frac{1}{2} (1 + \boldsymbol{\beta} \cdot \mathbf{B} - \boldsymbol{\varepsilon} \cdot \mathbf{E}) (\mathbf{E}^2 - \mathbf{B}^2) - (\boldsymbol{\beta} \cdot \mathbf{E} + \boldsymbol{\varepsilon} \cdot \mathbf{B}) (\mathbf{E} \cdot \mathbf{B}). \quad (4.3)$$

Next, looking for the second order term, one finds, with a little effort,

$$\begin{aligned} \hat{\mathcal{L}}^{(2)} &= -\frac{1}{4} \theta^{\alpha\beta} \theta^{\gamma\delta} \left\{ F^{\mu}_{\alpha} F^{\nu}_{\beta} F_{\mu\gamma} F_{\nu\delta} + 2 F^{\mu\nu} F_{\mu\gamma} F_{\beta\nu} F_{\delta\alpha} \right. \\ &\quad \left. + F^{\mu\nu} F_{\mu\alpha} F_{\nu\beta} F_{\delta\gamma} + \frac{1}{8} F_{\beta\alpha} F_{\delta\gamma} F^2 + \frac{1}{4} F_{\beta\gamma} F_{\alpha\delta} F^2 \right\}. \end{aligned} \quad (4.4)$$

Here again, after substitution and accurate computation, you get

$$\begin{aligned} \hat{\mathcal{L}}^{(2)} &= (\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B}) (\boldsymbol{\beta} \cdot \mathbf{E} + \boldsymbol{\varepsilon} \cdot \mathbf{B}) (\mathbf{E} \cdot \mathbf{B}) \\ &\quad + \frac{1}{2} [(\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B})^2 (\mathbf{E}^2 - \mathbf{B}^2) + (\boldsymbol{\varepsilon} \cdot \boldsymbol{\beta}) (\mathbf{E}^2 - \mathbf{B}^2) \\ &\quad \times (\mathbf{E} \cdot \mathbf{B}) - (\mathbf{E} \cdot \mathbf{B})^2 (\boldsymbol{\varepsilon}^2 - \boldsymbol{\beta}^2)]. \end{aligned} \quad (4.5)$$

As already remarked, the second variation of $\hat{\mathcal{L}}$ yields the usual equations of motion

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \quad (4.6)$$

and Eq. (3.3) leads us to write $\tilde{F} = F + \tilde{F}^{(1)} + \tilde{F}^{(2)} + \dots$ where $\tilde{F}^{(n)} \equiv \theta^n F^{n+1}$.

It is now tempting to regard each piece like this as a correction to the classical field strength F due to NC geometry. This is more properly done here than on the expansion (2.10) because we are referring to the equations of motion. Furthermore, the interesting thing [3] is that denoting the content of the NC field \tilde{F} with an electric displacement and magnetic induction (\mathbf{D} , \mathbf{H}) and restating the above expansion as

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + \mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \dots, \\ \mathbf{H} &= \mathbf{B} + \mathbf{H}^{(1)} + \mathbf{H}^{(2)} + \dots, \end{aligned}$$

where the classical fields (\mathbf{E} , \mathbf{B}) in F are recaptured as their zeroth order corrections ($\mathbf{D}^{(0)}$, $\mathbf{H}^{(0)}$), then the equations of motion take the usual Maxwell form

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} &= 0, \\ \nabla \cdot \mathbf{D} &= 0. \end{aligned} \quad (4.8)$$

Note that the first two are simply the Bianchi identities; the other two really describe the behavior of NC electromagnetism in empty space. Working with the first order correction to F , which is

$$\begin{aligned}\tilde{F}^{\mu\nu(1)} = & -\frac{1}{2}(\theta F)F^{\mu\nu} - \frac{1}{4}\theta^{\mu\nu}F^2 + \theta_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} + (\theta^{\mu\beta}F^{\alpha\nu} \\ & - \theta^{\nu\beta}F^{\alpha\mu})F_{\alpha\beta},\end{aligned}\quad (4.9)$$

$$\begin{aligned}\mathbf{H} = & (1 + \boldsymbol{\beta} \cdot \mathbf{B} - \boldsymbol{\varepsilon} \cdot \mathbf{E})\mathbf{B} + (\boldsymbol{\beta} \cdot \mathbf{E} + \boldsymbol{\varepsilon} \cdot \mathbf{B})\mathbf{E} \\ & - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\boldsymbol{\beta} + (\mathbf{E} \cdot \mathbf{B})\boldsymbol{\varepsilon}.\end{aligned}\quad (4.10)$$

we obtain for the NC fields the approximated expressions

$$\begin{aligned}\mathbf{D} = & (1 + \boldsymbol{\beta} \cdot \mathbf{B} - \boldsymbol{\varepsilon} \cdot \mathbf{E})\mathbf{E} - (\boldsymbol{\beta} \cdot \mathbf{E} + \boldsymbol{\varepsilon} \cdot \mathbf{B})\mathbf{B} \\ & - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\boldsymbol{\varepsilon} - (\mathbf{E} \cdot \mathbf{B})\boldsymbol{\beta},\end{aligned}$$

Here the NC tensor θ has been assumed to represent a couple of fields $(\boldsymbol{\varepsilon}, \boldsymbol{\beta})$ in agreement with the conventions (2.2). The second order correction to F reads, explicitly,

$$\begin{aligned}\tilde{F}^{\mu\nu(2)} = & \frac{1}{4}\theta^{\alpha\beta}F_{\delta}^{\gamma}F_{\gamma\beta}(\theta^{\nu\delta}F_{\alpha}^{\mu} - \theta^{\mu\delta}F_{\alpha}^{\nu}) + \frac{1}{4}\{\theta^{\alpha\beta}\theta^{\gamma\delta}F_{\gamma}^{\mu}F_{\beta}^{\nu}F_{\alpha\delta} + \theta^{\alpha\beta}F_{\beta\gamma}F_{\delta\alpha}(\theta^{\nu\delta}F^{\mu\gamma} - \theta^{\mu\delta}F^{\nu\gamma}) + \theta^{\mu\beta}\theta^{\gamma\nu}F^{\delta\alpha}F_{\delta\gamma}F_{\alpha\beta}\} \\ & - \frac{1}{2}\{(\theta F)(\theta^{\gamma\beta}F_{\gamma}^{\mu}F_{\beta}^{\nu} + \theta^{\nu\beta}F^{\mu\gamma}F_{\gamma\beta} - \theta^{\mu\beta}F^{\nu\gamma}F_{\gamma\beta}) + \theta^{\gamma\beta}\theta^{\mu\nu}F^{\alpha\delta}F_{\delta\gamma}F_{\alpha\beta}\} + \frac{1}{8}(\theta F)\{\theta^{\mu\nu}F^2 + (\theta F)F^{\mu\nu}\} \\ & + \frac{1}{4}(\theta^{\alpha\mu}\theta^{\nu\delta}F_{\alpha\delta}F^2 + \theta^{\alpha\beta}\theta^{\phi\delta}F_{\alpha\delta}F_{\beta\gamma}F^{\mu\nu}).\end{aligned}\quad (4.11)$$

This rather involved formula, when reexpressed in terms of the classical fields, gives us the second order terms in θ to be added to the above:

$$\begin{aligned}\mathbf{D}^{(2)} = & [(\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B})^2 - \boldsymbol{\varepsilon}^2\mathbf{B}^2 + (\boldsymbol{\varepsilon} \cdot \mathbf{B})^2 + (\boldsymbol{\varepsilon} \cdot \boldsymbol{\beta})(\mathbf{E} \cdot \mathbf{B})]\mathbf{E} + [(\boldsymbol{\beta} \cdot \mathbf{E})(\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B}) - (\boldsymbol{\varepsilon} \cdot \mathbf{B})(\boldsymbol{\beta} \cdot \mathbf{B}) + \boldsymbol{\beta}^2(\mathbf{E} \cdot \mathbf{B}) \\ & + \frac{1}{2}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\beta})(\mathbf{E}^2 - \mathbf{B}^2)]\mathbf{B} + [(\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B})\mathbf{E}^2 + (\boldsymbol{\beta} \cdot \mathbf{E})(\mathbf{E} \cdot \mathbf{B}) + (\boldsymbol{\beta} \cdot \mathbf{B})\mathbf{B}^2]\boldsymbol{\varepsilon} + (\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B})(\mathbf{E} \cdot \mathbf{B})\boldsymbol{\beta} + [\mathbf{E} \cdot (\boldsymbol{\varepsilon} \times \mathbf{B})]\boldsymbol{\beta} \times \mathbf{E},\end{aligned}\quad (4.12)$$

while for the magnetic induction we get

$$\begin{aligned}\mathbf{H}^{(2)} = & \left[\boldsymbol{\varepsilon}^2(\mathbf{E} \cdot \mathbf{B}) - (\boldsymbol{\varepsilon} \cdot \mathbf{B})(\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B}) - (\boldsymbol{\varepsilon} \cdot \mathbf{E})(\boldsymbol{\beta} \cdot \mathbf{E}) - \frac{1}{2}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\beta})(\mathbf{E}^2 - \mathbf{B}^2) \right]\mathbf{E} + [(\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B})^2 - \boldsymbol{\beta}^2\mathbf{E}^2 + (\boldsymbol{\beta} \cdot \mathbf{E})^2 + (\boldsymbol{\varepsilon} \cdot \boldsymbol{\beta}) \\ & \times (\mathbf{E} \cdot \mathbf{B})]\mathbf{B} - (\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B})(\mathbf{E} \cdot \mathbf{B})(\mathbf{E} \cdot \mathbf{B})\boldsymbol{\varepsilon} + [\mathbf{B} \cdot (\boldsymbol{\beta} \times \mathbf{E})]\boldsymbol{\varepsilon} \times \mathbf{B} + [(\boldsymbol{\varepsilon} \cdot \mathbf{E})\mathbf{E}^2 + (\boldsymbol{\varepsilon} \cdot \mathbf{B})(\mathbf{E} \cdot \mathbf{B}) - (\boldsymbol{\varepsilon} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{B})\mathbf{B}^2]\boldsymbol{\beta}.\end{aligned}\quad (4.13)$$

We end this section observing that applying an electric-magnetic duality directly on the classical fields and reversely on the noncommuting parameter in this way,

$$\begin{aligned}\mathbf{E} & \rightarrow -\mathbf{B}, & \boldsymbol{\varepsilon} & \rightarrow \boldsymbol{\beta} \\ \mathbf{B} & \rightarrow \mathbf{E}, & \boldsymbol{\beta} & \rightarrow -\boldsymbol{\varepsilon},\end{aligned}\quad (4.14)$$

induces, up to second order, an ‘‘electric-magnetic duality’’ on the NC fields (4.10):

$$\begin{aligned}\mathbf{D} & \rightarrow -\mathbf{H}, \\ \mathbf{H} & \rightarrow \mathbf{D}.\end{aligned}\quad (4.15)$$

At present, the meaning of this symmetry is unclear and we suspect it remains true to all orders in the perturbative θ expansion.

V. EXACT SOLUTIONS AND AN ITERATIVE METHOD

We seek solutions to the equations of motion

$$\begin{aligned}\partial_{[\nu}F_{\mu\rho]} & = 0, \\ \partial_{[\nu}\tilde{F}_{\mu\nu]} & = 0,\end{aligned}\quad (5.1)$$

where

$$\tilde{F} = F + \tilde{F}^{(1)} + \tilde{F}^{(2)} + \dots, \quad (5.2)$$

with the structure $\tilde{F}^{(n)} \equiv \theta^n F^{n+2}$ already evident for example in Eqs. (4.9) and (4.11).

The most natural thing to suppose is that also a solution should be written as a sum:

$$F := F^{(0)} + F^{(1)} + F^{(2)} + \dots, \quad (5.3)$$

with pieces $F^{(k)}$ now understood to be corrections to a solution $F^{(0)}$ to the classical Maxwell equations: i.e., $\partial_\nu F^{(0)\mu\nu}=0$ plus the Bianchi identities. We will briefly state this as $\partial F^{(0)}=0$. Furthermore, let $|k$ be the operation of keeping, in a generic expression, all terms up to a given order k in θ , neglecting the others. Then extracting k th order from Eq. (5.2) terms like this,

$$F|_k = \sum_{i=0}^k F^{(i)} \quad (5.4)$$

will be present. Accounting for that, hypothesis (5.3), and the structure (3.3), we get

$$\tilde{F}|_k = F|_k + \theta(FF)|_{k-1} + \cdots + \underbrace{\theta \cdots \theta}_{k} \underbrace{(F \cdots F)}_{k+1} |_0. \quad (5.5)$$

Our purpose is to write down a recursive method of solving the noncommutative Maxwell equation $\partial\tilde{F}=0$ having a classical solution $F^{(0)}$. This is realized order by order noting that $(\partial\tilde{F})|_k = \partial\tilde{F}|_k$. Then, taking first order into the recursive relation (5.5), we have

$$\partial\tilde{F}|_1 = \partial(F^{(0)} + F^{(1)} + \theta(FF)|_0) = 0. \quad (5.6)$$

Now, being $F^{(0)}$ a classical solution, we are led to solve the equation:

$$\partial F^{(1)} = -\partial(\theta F^{(0)} F^{(0)}). \quad (5.7)$$

In exactly the same way, solutions correct up to second order come from

$$\partial F^{(2)} = -\partial[\theta(F^{(0)}F^{(1)} + F^{(1)}F^{(0)}) + \theta\theta F^{(0)}F^{(0)}F^{(0)}]. \quad (5.8)$$

Generally, obtaining the k th term in the expansion (5.3) always reduces to solving an equation of the form

$$\partial_\nu F^{(k)\mu\nu} = J^\mu[F^{(1)}, \dots, F^{(k-1)}], \quad (5.9)$$

where the right member J^μ only involves all the $k-1$ solutions computed in the previous steps. Now, deciding that each two form $F^{(k)}$ comes from a potential $A^{(k)}$ satisfying the Lorentz gauge constraint¹ $\partial_\nu A^{(k)\nu}=0$, then Eq. (5.9) becomes

$$-\square A^\mu = J^\mu. \quad (5.10)$$

This is immediately solved employing the Lienard-Wickert potentials. Then in principle we have got an automatic tool capable of solving the equations of motion in full.

Let us focus, for example, on the single plane wave solution

$$A_\mu = \zeta_\mu e^{ik \cdot x}, \quad (5.11)$$

with $k_\mu k^\mu = \zeta_\mu k^\mu = 0$ in the Lorentz gauge $\partial_\nu A^\nu = 0$. We have

$$F_{\mu\nu}^{(0)} = i(k_\mu \zeta_\nu - k_\nu \zeta_\mu) e^{ik \cdot x}. \quad (5.12)$$

This is a particular case because we will now show that it is an *exact* solution of Eq. (5.1).

Lemma 5.1. Given an antisymmetric matrix $\theta^{\mu\nu}$, a null vector k^α , and a family of vectors $\{\zeta_{(i)}^\beta\}_{i \in I}$ orthogonal to k^α , then any combination of n copies of $\theta^{\mu\nu}$, $(n+1)$ vectors of the given family and $(n+2)$ copies of k^α in which all indices but one are saturated, vanish.

Proof. Try to build a nonvanishing combination. In so doing, you cannot saturate the k vectors with the ζ vectors due to orthogonality. Neither you can saturate two of them with one θ matrix due to its antisymmetry. You are obliged to use one only k vector for each matrix, spending n of them. Of the two remaining, one can be chosen as the free index but the other must necessarily be saturated with one of the ζ vectors or one of the θ matrices giving a vanishing result. Q.E.D.

Proposition 5.1. Monochromatic plane waves solve the field equations (5.1) to every order in θ .

Proof. Let us write the general monochromatic plane wave as

$$A_\mu = \Phi_\mu(K \cdot x), \quad (5.13)$$

with $K^2=0$ and $K^\mu \Phi'_\mu(K \cdot x) = 0$ so that

$$F_{\mu\nu} = K_\mu \Phi'_\nu(K \cdot x) - K_\nu \Phi'_\mu(K \cdot x). \quad (5.14)$$

Let $\tilde{F}^{(n)\mu\nu}$ be the term of order n in θ ; then, $\partial_\mu \tilde{F}^{(n)\mu\nu}$ is the sum of terms obtained by contraction of n copies of $\theta^{\mu\nu}$, n copies of Φ'_α , one copy of Φ'_β , and $n+2$ copies of K_γ . From the Lemma it follows that $\partial_\mu \tilde{F}^{(n)\mu\nu} = 0$. Q.E.D.

The previous property of monochromatic plane waves holds for any Lagrangian having the assumed polynomial structure, independently of the fact that it has been derived from a NC theory using the SW map.

A. Plane wave superposition

While single plane waves turn out to be exact solutions of the field equations, this is no longer valid even for a simple superposition like this,

$$A_\mu := \zeta_\mu e^{ik \cdot x} + \zeta'_\mu e^{ik' \cdot x}, \quad (5.15)$$

corresponding to the classical solution (by linearity):

$$F_{\mu\nu}^{(0)} := i(k_\mu \zeta_\nu - k_\nu \zeta_\mu) e^{ik \cdot x} + i(k'_\mu \zeta'_\nu - k'_\nu \zeta'_\mu) e^{ik' \cdot x}. \quad (5.16)$$

To find out its first order correction in the NC framework we must solve Eq. (5.7), yielding

¹It is easy to show that this can always be done order by order.

$$\begin{aligned}
\partial_\nu F^{(1)\mu\nu} = & i\{[(kk')_\theta(\zeta\zeta') - (\zeta\zeta')_\theta(kk')](k^\mu - k'^\mu) \\
& - [(k\zeta')_\theta(\zeta k') - (\zeta k')_\theta(k\zeta')](k^\mu + k'^\mu) \\
& + 2[(k\zeta')_\theta(kk') - (kk')_\theta(k\zeta')]\zeta^\mu \\
& - 2[(\zeta k')_\theta(kk') - (kk')_\theta(\zeta k')]\zeta'^\mu\}e^{i(k+k')\cdot x},
\end{aligned} \tag{5.17}$$

where the following antisymmetric inner product has been defined: $(vw)_\theta := v^\mu \theta_{\mu\nu} w^\nu$. This equation can be solved assuming

$$F_{\mu\nu}^{(1)} = \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)} \tag{5.18}$$

and $A^{(1)}$ still satisfying an extended Lorentz gauge constraint $\partial_\nu A^{(1)\nu} = 0$.

In fact, rewriting Eq. (5.17) in the form

$$-\square A_\mu^{(1)} = iJ_\mu e^{i(k+k')\cdot x}, \tag{5.19}$$

we realize that J is transverse to $k+k'$ as can be easily proved using the defining relations

$$k^2 = k'^2 = 0, \quad k \cdot \zeta = k' \cdot \zeta' = 0. \tag{5.20}$$

This means that if we put, abruptly,

$$A_\mu^{(1)} = \frac{iJ_\mu}{2k \cdot k'} e^{i(k+k')\cdot x}, \tag{5.21}$$

this solves Eq. (5.17), being also compatible with the extended Lorentz gauge.

Note that this corrected version of the superposition law could be used to reveal a refraction effect suffered by a ray of light in passing from an empty region to one in which a background static and uniform magnetic or electric field is present [3,9]. The incoming and reflected rays propagating in the empty region should be described by a superposition of waves agreeing with the refracted one in the transition region.

B. Coulomb law

All NC theories are characterized by a parameter θ which defines a natural scale of length. From a dimensional analysis, the corrections to the Coulomb law are of order $1/r^4$, but a complete power series expansion in θ/r^2 is expected, so that if L is its convergence ratio (plausibly finite), then non-perturbative contributions should become relevant in the region $r < \sqrt{\theta/L}$ where the perturbative description fails. We can make a sensible study of the NC corrections to the Coulomb law, considering the potential generated by a charged conducting sphere of radius r_0 .

At zeroth order the classical potential is

$$A^{(0)} = \begin{cases} -\frac{e}{r} dt, & r > r_0, \\ -\frac{e}{r_0} dt, & r \leq r_0, \end{cases} \tag{5.22}$$

so that (\hat{x}^i is the radial versor)

$$F_{0i}^{(0)} = -\frac{e}{r^2} \hat{x}^i \theta(r-r_0). \tag{5.23}$$

At first order,

$$\partial_\nu F^{(1)\mu\nu} = -\partial_\nu G^{\mu\nu}, \tag{5.24}$$

with

$$\begin{aligned}
G^{\mu\nu} := & -\frac{1}{2}(\theta F)F^{\mu\nu} - \frac{1}{4}\theta^{\mu\nu}(FF) + \theta_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} + (\theta^{\mu\beta}F^{\alpha\nu} \\
& - \theta^{\nu\beta}F^{\alpha\mu})F_{\alpha\beta}.
\end{aligned} \tag{5.25}$$

A direct computation gives, for the tensor G ,

$$G^{0i} = -\frac{e^2}{r^4} \left[(\epsilon \cdot \hat{x}) \hat{x}^i + \frac{1}{2} \epsilon^i \right] \theta(r-r_0), \tag{5.26}$$

$$G^{ij} = \frac{e^2}{r^4} \left[(\hat{x}^i \theta^{jk} - \hat{x}^j \theta^{ik}) \hat{x}_k + \frac{1}{2} \theta^{ij} \right] \theta(r-r_0), \tag{5.27}$$

so that

$$\partial_\mu F^{(1)\mu\nu} = J^\nu, \tag{5.28}$$

with

$$\begin{aligned}
J = & \frac{e^2}{r^6} (4\vec{\epsilon} \cdot \vec{r}; \vec{r} \wedge \vec{\beta}) \theta(r-r_0) + \frac{e^2}{r_0^5} (3/2\vec{\epsilon} \cdot \vec{r}; -1/2\vec{r} \wedge \vec{\beta}) \\
& \times \delta(r-r_0).
\end{aligned} \tag{5.29}$$

We have solved numerically Eq. (5.28), and we show in Fig. 1 the equipotential level of the zeroth component of $A^{(1)}$, which is symmetrical under rotations about the direction of $\vec{\epsilon}$. The other components have a similar angular behavior, and their precise values depend on the direction and magnitude of $\vec{\beta}$.

The corrections in Fig. 1 give the modification of the Coulomb law in the case of a charged sphere. This interpretation would be well defined with $r_0 \gg \sqrt{\theta}$, being the perturbative solution valid almost everywhere, even inside the conducting sphere. But in the specific example considered in Fig. 1 the sphere coincides with the excluded region, where the perturbative approach fails. This case suggests a different interpretation, as the NC correction to the Coulomb potential of a point charge. In fact, in NC theories it is intuitive to replace pointlike with extended object, whose typical length is $\sqrt{\theta}$.

The corrections to the potential violate the Gauss law and spherical symmetry of the classical solution. As a consequence, we observe in the case of a conducting macroscopic sphere that the potential inside the conductor is not constant. This remark suggests a way to test NC electrodynamics effects. In Fig. 2 we show the relative contribution of the corrections $A^{(1)}$ to the classical Coulomb potential $A^{(0)}$. The size of the corrections is already relevant (e.g., greater than

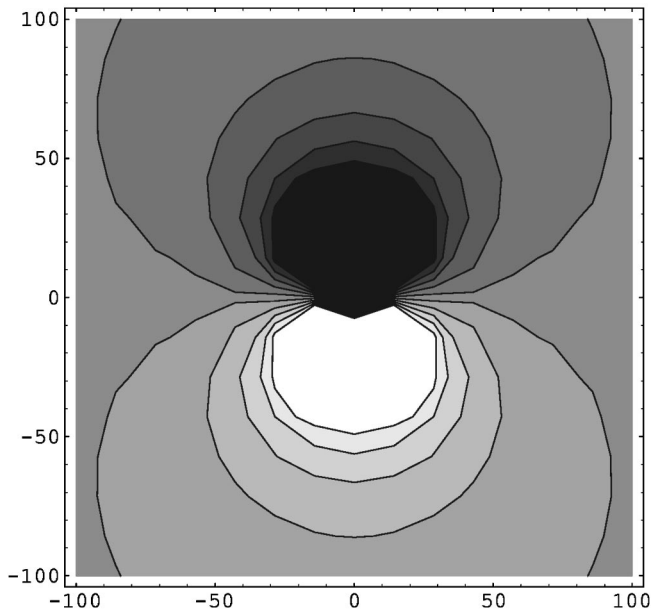


FIG. 1. Correction to the Coulomb law due to $A^{(1)}$. The figure shows the equipotential levels in the (x,z) plane, assuming that $\vec{\epsilon}$ is oriented along the z axis. We have chosen $\epsilon=1$, $r_0=1$. The length unit is r_0 .

10%) at a length scale bigger by more than one order of magnitude with respect to the one determined by the NC parameter.

VI. CONCLUSIONS

In this work we have formulated an explicit perturbative realization of NC electrodynamics, which turns out to be causal and Lorentz invariant. The basic steps to obtain this result have been the use of the SW map and a rearrangement of the action aimed to render every term explicitly gauge invariant by use of careful integration by parts. The resulting expressions do not contain time derivatives of order higher than 2, yielding automatically a causal theory. This latter property is obtained without imposing any constraint on the NC parameter θ , which can be chosen in full generality as a Lorentz tensor, leading to a Lorentz covariant theory.

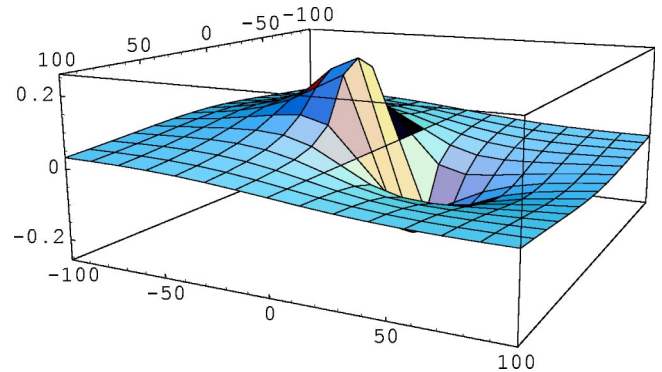


FIG. 2. Ratio $A^{(1)}/A^{(0)}$, which shows the effect of the NC corrections to the Coulomb law, in the (x,z) plane. We have chosen $\vec{\epsilon}$ oriented along the z axis, $\epsilon=1$, $e=1$, and $r_0=1$. The length unit is r_0 .

We have studied the general structure of the Lagrangian to all orders in the perturbative expansion. We have shown that the monochromatic plane wave is solution of the equations of motion to first [9] and even to all orders.

We developed an iterative method to solve the equations of motion. In particular, we applied this method to study the corrections to the superposition law of plane waves and to the electrostatic potential of a spherically symmetric charge distribution. The most relevant qualitative feature of the NC corrections that we calculated is that they have a peculiar signature which makes them, at least in principle, distinguishable from the classical corresponding effects. A possible test of the superposition law could be done by studying the reflection and refraction of light on a magnetic field, using, for instance, the experimental setting described in [10]. Furthermore, the deviations from the Coulomb law could be evidenced by measuring the charge distribution on the surface and the electric field inside an empty conducting sphere.

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