Two-loop renormalization group equations in general gauge field theories

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The complete set of two-loop renormalization group equations in general gauge field theories is presented. This includes the β functions of parameters with and without a mass dimension.

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I. INTRODUCTION

Renormalization group equations (RGEs) provide a unique method in the analysis of particle physics. A comprehensive analysis of RGEs confirmed the behavior of asymptotic freedom in QCD, which played a pivotal role in establishing a non-Abelian gauge theory for the strong interaction [1]. The running of the coupling constants and mass parameters is crucial in the global analysis of high precision electroweak experiments [2]. On the other hand, a RGE analysis extrapolated to extremely high energy provides a possible, in some cases the only feasible, test for physics beyond the standard model (SM). RGEs are a natural ingredient in the analysis of grand unification theories and string theories. For more than ten years, it has been known that gauge couplings do not unify within the SM. This gives extra evidence against simple grand unification theories such as SU(5) without supersymmetry, in addition to the nonobservation of proton decay. On the other hand, gauge couplings seem to unify at a scale $\sim 2\times 10^{16}$ GeV in the minimal supersymmetric standard model, which can be interpreted as indirect evidence for supersymmetry as well as unification theories [3–5].

Computations of RGEs in gauge theories have been performed for various models to different orders of perturbation. Persistent efforts yielded recently a four-loop result for the β function of the strong coupling constant [6]. Two-loop RGEs of dimensionless couplings in general gauge theory as well as in the specific case of the SM were calculated long ago in a series of classic papers by Machacek and Vaughn [7–9]. By introducing a nonpropagating gauge-singlet "dummy" scalar field, two-loop RGEs of dimensional couplings can be readily inferred from dimensionless results [10,11]. These were used to derive the RGEs of supersymmetric theories a decade later [10].

Recently [11], we recalculated the two-loop RGEs in the SM, using a combination of the general results of [7–9] and direct calculations from Feynman diagrams. A new coefficient was found in the β function of the quartic coupling and a class of gauge invariants were found to be absent in the β functions of hadronic Yukawa couplings. We also presented the two-loop β function of the Higgs boson mass parameter in complete form, which provided a partial but useful check on the calculation of the quartic coupling. Whenever discrepancy with results in the literature appeared, we carefully inspected the relevant Feynman diagrams to ensure consistency. In this paper, we present the complete set of two-loop renormalization group equations in general gauge field theories. This includes the β functions of parameters with and without a mass dimension. The results in [11] can then be readily reproduced.

In Sec. II, we present essential notations and definitions, along with a discussion of the differences between [7–9] and our analysis. In Sec. III, we present the γ functions of the scalar and fermion fields. In Sec. IV, we present the β functions of dimensionless parameters and in Sec. V those of dimensional parameters. In Sec. VI the results are extended to semisimple groups. We conclude in Sec. VII.

II. NOTATION AND DEFINITIONS

We start with a general renormalizable field theory with gauge fields V_μ^A of a compact simple group G, scalar fields, ϕ_a, and two-component fermion fields ψ_j. In Sec. VI, these results will be extended to semisimple groups. The Lagrangian of the theory can be conveniently divided into three parts,

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1+(\text{gauge fixing + ghost terms}), \tag{1}$$

where $\mathcal{L}_0$ contains no dimensional parameters and $\mathcal{L}_1$ includes all terms with dimensional parameters. Explicitly,

$$\mathcal{L}_0=-\frac{1}{4}F_A^{\mu\nu}F_{\mu\nu}^A+\frac{1}{2}D^\mu\phi_a D_\mu\phi_a+i\psi_j^+\sigma^\mu D_\mu\psi_j -\frac{1}{2}(Y_{jk}^a\psi_j\zeta\psi_k\phi_a+\text{H.c.})-\frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d, \tag{2}$$

where $\zeta=\pm i\sigma_2$. Unlike [7–9], we have included an overall $\frac{1}{2}$ factor in the Yukawa coupling terms. The gauge field strengths are defined to be

$$F_{\mu\nu}^A=\partial_\mu V_\nu^A-\partial_\nu V_\mu^A+gf^{ABC}V_\mu^B V_\nu^C, \tag{3}$$

where f^{ABC} are the structure constants of the gauge group and g is the gauge coupling constant. Choosing the standard R_ξ gauge, the gauge field propagator is

$$D_{\mu\nu}^{AB}(k)=\delta^{AB}\left(-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\frac{i}{k^2}, \tag{4}$$

where ξ is the gauge parameter. The covariant derivatives of the matter fields are

$$D_\mu \phi_a = \partial_\mu \phi_a - ig\,\theta^A_{ab} V^A_\mu \phi_b, \tag{5}$$

$$D_\mu \psi_j = \partial_\mu \psi_j - ig t^A_{jk} V^A_\mu \psi_k, \tag{6}$$

where both θ^A_{ab} and t^A_{jk} are Hermitian matrices, which form representations of the gauge group on the scalar and fermion fields, respectively. Since any complex scalar fields can always be decomposed in terms of real ones, scalars in this paper are assumed to be real. θ^A are thus purely imaginary and antisymmetric. For later convenience, we define the following gauge invariants:

$$C_2^{ab}(S) = \theta^A_{ac}\theta^A_{cb}, \quad S_2(S)\delta^{AB} = \mathrm{Tr}[\theta^A \theta^B], \tag{7}$$

$$C_2^{ab}(F) = t^A_{ac} t^A_{cb}, \quad S_2(F)\delta^{AB} = \mathrm{Tr}[t^A t^B], \tag{8}$$

$$C_2(G)\delta^{AB} = f^{ACD} f^{BCD}. \tag{9}$$

$C_2^{ab}(R)$ is block diagonal for each irreducible representation $R(=S,F)$ of eigenvalues of $C_2(R)$.

In this paper, we use dimensional regularization and the modified minimal subtraction algorithm. The renormalized coupling constants x_k in $d=4-2\epsilon$ are related to the corresponding bare coupling constants x_k^0 by

$$x_k^0 \mu^{-\rho_k \epsilon} = x_k + \sum_{n=1}^{\infty} a_k^{(n)} \frac{1}{\epsilon^n}, \tag{10}$$

where μ is an arbitrary mass scale parameter, $\rho_k = 1(2)$ for gauge and Yukawa (scalar quartic) coupling constants, and $a_k^{(n)}$ are to be calculated perturbatively. The β functions of x_k are defined to be

$$\beta_{x_k} = \mu \frac{dx_k}{d\mu}\bigg|_{\epsilon=0}. \tag{11}$$

It is easy to see that

$$\beta_{x_k} = \sum_l \rho_l x_l \frac{\partial a_k^{(1)}}{\partial x_l} - \rho_k a_k^{(1)}. \tag{12}$$

Perturbatively, one has

$$\beta_{x_k} = \frac{1}{(4\pi)^2}\beta^I_{x_k} + \frac{1}{(4\pi)^4}\beta^{II}_{x_k} + \cdots, \tag{13}$$

where $\beta^I_{x_k}$ and $\beta^{II}_{x_k}$ are one- and two-loop contributions, respectively. The wave function renormalization constant Z_i of the ith field can be expressed as

$$Z_i = 1 + \sum_{n=1}^{\infty} C_i^{(n)} \frac{1}{\epsilon^n}. \tag{14}$$

The corresponding anomalous dimension is

$$\gamma_i = \frac{1}{2}\mu \frac{d}{d\mu}\log Z_i = -\frac{1}{2}\sum_l \rho_l x_l \frac{\partial C_i^{(1)}}{\partial x_l}. \tag{15}$$

Also perturbatively, one has

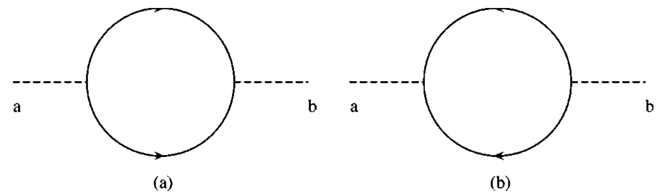

FIG. 1. Fermion radiative corrections to the scalar propagator

$$\gamma_i = \frac{1}{(4\pi)^2}\gamma_i^I + \frac{1}{(4\pi)^4}\gamma_i^{II} + \cdots, \tag{16}$$

where γ_i^I and γ_i^{II} are one- and two-loop contributions, respectively.

The constraint on the Yukawa coupling matrices imposed by gauge invariance is given by

$$Y^b_{jk}\theta^A_{ba} + Y^a_{jl} t^A_{lk} + t^{A*}_{jl} Y^a_{lk} = 0. \tag{17}$$

When a fermion is involved simultaneously with a Yukawa coupling and a gauge coupling, one usually has the combination $Y^a_{jl}t^A_{lk} + t^{A*}_{jl}Y^a_{lk}$. That is to say, in these cases, Y^a is always preceded by a t^{A*} or followed by a t^A. Equation (17) can thus be used to simplify the gauge structure of Feynman diagrams. In [7–9], fermion fields are implicitly assumed to be real so t^A are pure imaginary and antisymmetric. $t^{A\star}_{jl}$ was replaced by $-t^A_{jl}$ and Eq. (17) is reduced to

$$Y^b_{jk}\theta^A_{ba} + Y^a_{jl}t^A_{lk} - t^A_{jl}Y^a_{lk} = 0. \tag{18}$$

However, in most physical theories, fermions are in general Weyl and complex, with only the possible exceptions of neutrinos and gauginos as real Majorona fermions. Fortunately, close inspections of Feynman diagrams show that both approaches yield the same final results in most cases. In cases where they differ, we will retain $t^{A\star}_{jl}$.

By assuming real fermion fields, the direction of a fermion propagator need not be discriminated either. Shown in Fig. 1 are two fermion loop diagrams that contribute to the propagators of scalar fields. The two diagrams are different for complex fermions and the total result is proportional to

$$Y_2^{ab}(S) = \frac{1}{2}\mathrm{Tr}(Y^{+a}Y^b + Y^{+b}Y^a). \tag{19}$$

$Y_2^{ab}(S)$ forms a Hermitian matrix and is block diagonal for each irreducible representation of the eigenvalue $Y_2(S)$. For real fermions, the two terms in $Y_2^{ab}(S)$ become equal and they reduce to a common factor $\mathrm{Tr}(Y^{+a}Y^b)$. Again, we will retain Eq. (19) and other similar combinations.

The second part of the Lagrangian is

$$\mathcal{L}_1 = -\frac{1}{2}[(m_f)_{jk}\psi_j \zeta \psi_k + \mathrm{H.c.}] - \frac{m^2_{ab}}{2!}\phi_a\phi_b - \frac{h_{abc}}{3!}\phi_a\phi_b\phi_c. \tag{20}$$

In principle, the β functions of $(m_f)_{jk}$, m^2_{ab}, and h_{abc} can be calculated directly. But this would be tedious and can be avoided. To do so, we introduce a nonpropagating dummy real scalar field $\phi_{\hat{d}}$ with no gauge interactions and rewrite the Lagrangian as

$$\mathcal{L}_1 = -\frac{1}{2}(Y^{\hat{d}}_{jk}\psi_j\zeta\psi_k\phi_{\hat{d}} + \text{H.c.}) - \frac{\lambda_{ab\hat{d}\hat{d}}}{4!}\phi_a\phi_b\phi_{\hat{d}}\phi_{\hat{d}} - \frac{\lambda_{abc\hat{d}}}{4!}\phi_a\phi_b\phi_c\phi_{\hat{d}} \tag{21}$$

with the substitutions of $Y^{\hat{d}}_{ij} = (m_f)_{ij}$, $\lambda_{ab\hat{d}\hat{d}} = 2m^2_{ab}$, and $\lambda_{abc\hat{d}} = h_{abc}$. The β functions of m_f, m^2_{ab}, and h_{abc} are equal to those of the new Yukawa coupling $Y^{\hat{d}}$, the quartic scalar coupling $\lambda_{ab\hat{d}\hat{d}}$, and $\lambda_{abc\hat{d}}$. The latter can be readily obtained from the β functions of Y^a and λ_{abcd} by suppressing both the summations of the $\hat{d}$-type indices and related gauge couplings.

III. WAVE FUNCTION RENORMALIZATION

A. Scalar wave function renormalization

To one loop, the anomalous dimensions of scalars are

$$\gamma^{sI}_{ab} = 2\kappa Y^{ab}_2(S) - g^2(3-\xi)C^{ab}_2(S). \tag{22}$$

To two loops,

$$\gamma^{sII}_{ab} = -g^4 C^{ab}_2(S)\left[\left(\frac{35}{3} - 2\xi - \frac{1}{4}\xi^2\right)C_2(G) - \frac{10}{3}\kappa S_2(F) - \frac{11}{12}S_2(S)\right] + \frac{1}{2}\Lambda^2_{ab}(S) + \frac{3}{2}g^4 C^{ac}_2(S)C^{cb}_2(S) - 3\kappa H^2_{ab}(S) - 2\kappa\overline{H}^2_{ab}(S) + 10\kappa g^2 Y^{2F}_{ab}(S), \tag{23}$$

where $\kappa = 1/2$ (1) for two- (four-)component fermions, and

$$\Lambda^2_{ab}(S) = \frac{1}{6}\lambda_{acde}\lambda_{bcde}, \tag{24}$$

$$H^2_{ab}(S) = \frac{1}{2}\text{Tr}(Y^a Y^{+b} Y^c Y^{+c} + Y^{+a} Y^b Y^{+c} Y^c), \tag{25}$$

$$\overline{H}^2_{ab}(S) = \frac{1}{2}\text{Tr}(Y^a Y^{+c} Y^b Y^{+c} + Y^{+a} Y^c Y^{+b} Y^c), \tag{26}$$

$$Y^{2F}_{ab}(S) = \frac{1}{2}\text{Tr}[C_2(F)(Y^a Y^{+b} + Y^b Y^{+a})]. \tag{27}$$

In contrast with [7], $Y^{ab}_2(S)$, $H^2_{ab}(S)$, $\overline{H}^2_{ab}(S)$, and $Y^{2F}_{ab}(S)$ are modified so that they are Hermitian for complex fermions. As mentioned above, this is because complex fermion lines have two distinct directions in Feynman diagrams. The contributions of the two parts are Hermitian conjugate to each other. For real fermions, both contributions are real. So they are equal and can be combined. A similar structure appears in later results. Here and hereafter, we express the Casimir factors as $C^{ab}_2(S)$ instead of $C_2(S)\delta_{ab}$ and $C^{ac}_2(S)C^{cb}_2(S)$ instead of $C^2_2(S)\delta_{ab}$, to accommodate reducible representations.

B. Fermion wave function renormalization

To one loop, the anomalous dimensions of fermions are

$$\gamma^F_I = \frac{1}{2}Y^a Y^{+a} + g^2 C_2(F)\xi. \tag{28}$$

To two loops,

$$\gamma^F_{II} = -\frac{1}{8}Y^a Y^{+b} Y^b Y^{+a} - \frac{3}{2}\kappa Y^a Y^{+b} Y^{ab}_2(S) + g^2\left[\frac{9}{2}C^{ab}_2(S)Y^a Y^{+b} - \frac{7}{4}C_2(F)Y^a Y^{+a} - \frac{1}{4}Y^a C_2(F) Y^{+a}\right] + g^4 C_2(F)\left[\left(\frac{25}{4} + 2\xi + \frac{1}{4}\xi^2\right)C_2(G) - 2\kappa S_2(F) - \frac{1}{4}S_2(S)\right] - \frac{3}{2}g^4[C_2(F)]^2. \tag{29}$$

Here the only modification is in the $Y^{ab}_2(S)$, which appears in the first line of Eq. (29). This is due to one-loop fermion subdiagrams in scalar field propagators.

IV. DIMENSIONLESS PARAMETERS

A. The gauge coupling constant

The β function for the gauge coupling constants has been extensively studied. Recent results are up to the fourth order. Here we include the two-loop results for completeness:

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left\{\frac{11}{3}C_2(G) - \frac{4}{3}\kappa S_2(F) - \frac{1}{6}S_2(S) + \frac{2\kappa}{(4\pi)^2}Y_4(F)\right\} - \frac{g^5}{(4\pi)^4}\left\{\frac{34}{3}[C_2(G)]^2 - \kappa\left[4C_2(F) + \frac{20}{3}C_2(G)\right]S_2(F) - \left[2C_2(S) + \frac{1}{3}C_2(G)\right]S_2(S)\right\}, \tag{30}$$

where $Y_4(F)$ is defined through

$$Y_4(F) = \frac{1}{d(G)}\text{Tr}[C_2(F)Y^a Y^{+a}]. \tag{31}$$

$d(G)$ is the dimension of the group. Note that, although the Yukawa couplings are normalized differently, Eq. (30) assumes the same form as Eq. (6.1) of [7].

B. The Yukawa couplings

The β functions of the Yukawa couplings can be expressed as

$$\beta^a = \gamma^a + \gamma^{+F} Y^a + Y^a \gamma^F + \gamma^S_{ab} Y^b, \tag{32}$$

where γ^a are the anomalous dimensions of the operators $\phi_a \psi_j \zeta \psi_k$, and γ^F and γ^S_{ab} are the anomalous dimensions of the corresponding fermions and bosons, respectively. To one loop,

$$\beta^a_I = \frac{1}{2}[Y^+_2(F) Y^a + Y^a Y_2(F)] + 2 Y^b Y^{+a} Y^b + 2\kappa Y^b Y^{ab}_2(S) - 3g^2\{C_2(F), Y^a\}, \tag{33}$$

where

$$Y_2(F) = Y^{+a} Y^a. \tag{34}$$

To two loops,

$$\begin{aligned}\beta^a_{II} ={}& 2Y^c Y^{+b} Y^a (Y^{+c} Y^b - Y^{+b} Y^c) - Y^b [Y_2(F) Y^{+a} + Y^{+a} Y^+_2(F)] Y^b - \frac{1}{8}[Y^b Y_2(F) Y^{+b} Y^a + Y^a Y^{+b} Y^+_2(F) Y^b] \\ &- 4\kappa Y^{ac}_2(S) Y^b Y^{+c} Y^b - 2\kappa Y^b \overline{H}^2_{ab}(S) - \frac{3}{2}\kappa Y^{bc}_2(S)(Y^b Y^{+c} Y^a + Y^a Y^{+c} Y^b) - 3\kappa Y^b H^2_{ab}(S) - 2\lambda_{abcd} Y^b Y^{+c} Y^d \\ &+ \frac{1}{2}\Lambda^2_{ab}(S) Y^b + 3g^2\{C_2(F), Y^b Y^{+a} Y^b\} + 5g^2 Y^b \{C_2(F), Y^{+a}\} Y^b - \frac{7}{4}g^2[C_2(F) Y^+_2(F) Y^a + Y^a Y_2(F) C_2(F)] \\ &- \frac{1}{4}g^2[Y^b C_2(F) Y^{+b} Y^a + Y^a Y^{+b} C_2(F) Y^b] + 6g^2 H^a_{2t} + 10\kappa g^2 Y^b Y^{2F}_{ab}(S) + 6g^2[C^{bc}_2(S) Y^b Y^{+a} Y^c - 2C^{ac}_2(S) Y^b Y^{+c} Y^b] \\ &+ \frac{9}{2}g^2 C^{bc}_2(S)(Y^b Y^{+c} Y^a + Y^a Y^{+c} Y^b) - \frac{3}{2}g^4\{[C_2(F)]^2, Y^a\} + 6g^4 C^{ab}_2(S)\{C_2(F), Y^b\} \\ &+ g^4\left[-\frac{97}{6}C_2(G) + \frac{10}{3}\kappa S_2(F) + \frac{11}{12}S_2(S)\right]\{C_2(F), Y^a\} - \frac{21}{2}g^4 C^{ab}_2(S) C^{bc}_2(S) Y^c \\ &+ g^4 C^{ab}_2(S)\left[\frac{49}{4}C_2(G) - 2\kappa S_2(F) - \frac{1}{4}S_2(S)\right] Y^b,\end{aligned} \tag{35}$$

where

$$H^a_{2t} = t^{A*} Y^a Y^{+b} t^{A*} Y^b + Y^b t^A Y^{+b} Y^a t^A. \tag{36}$$

The second term in the fourth line, the first term in the sixth line, and the second term in the tenth line, are expressed in terms of the Hermitian gauge invariants $\overline{H}^2_{ab}(S)$, $H^2_{ab}(S)$, and $Y^{2F}_{ab}(S)$, respectively. They are all from the γ functions of the scalar fields. They differ from the fourth line and the ninth line of (3.3) in [8], which included only parts of the expressions. In H^a_{2t}, in front of Y^a and Y^b one has t^{A*} instead of t^A, again due to the fact that fermions are complex.

C. The scalar quartic couplings

The β functions of the scalar quartic couplings can be expressed as

$$\beta_{abcd} = \gamma_{abcd} + \sum_i \gamma^S(i) \lambda_{abcd}, \tag{37}$$

where γ_{abcd} are the anomalous dimensions of the operators $\phi_a \phi_b \phi_c \phi_d$, and $\gamma^S(i)$ is the anomalous dimension of the scalar field i. To one loop,

$$\beta^I_{abcd} = \Lambda^2_{abcd} - 8\kappa H_{abcd} + 2\kappa \Lambda^Y_{abcd} - 3g^2 \Lambda^S_{abcd} + 3g^4 A_{abcd}, \tag{38}$$

where

$$\Lambda^2_{abcd} = \frac{1}{8}\sum_{perms} \lambda_{abef}\lambda_{efcd}, \tag{39}$$

$$H_{abcd} = \frac{1}{4}\sum_{perms} \mathrm{Tr}(Y^a Y^{+b} Y^c Y^{+d}), \tag{40}$$

$$\Lambda^Y_{abcd} = \sum_i Y_2(i) \lambda_{abcd}, \tag{41}$$

$$\Lambda^S_{abcd} = \sum_i C_2(i) \lambda_{abcd}, \tag{42}$$

$$A_{abcd} = \frac{1}{8}\sum_{perms} \{\theta^A, \theta^B\}_{ab} \{\theta^A, \theta^B\}_{cd}. \tag{43}$$

To two loops,

$$\beta^{II}_{abcd}=\frac{1}{2}\sum_i \Lambda^2(i)\lambda_{abcd}-\overline{\Lambda}^3_{abcd}-4\kappa\overline{\Lambda}^{2Y}_{abcd}+\kappa\left\{8\overline{H}^{\lambda}_{abcd}-\sum_i [3H^2(i)+2\overline{H}^2(i)]\lambda_{abcd}\right\}+4\kappa(H^Y_{abcd}+2\overline{H}^Y_{abcd}+2H^3_{abcd})$$
$$+g^2\left\{2\overline{\Lambda}^{2S}_{abcd}-6\Lambda^{2g}_{abcd}+4\kappa(H^S_{abcd}-H^F_{abcd})+10\kappa\sum_i Y^{2F}(i)\lambda_{abcd}\right\}-g^4\left\{\left[\frac{35}{3}C_2(G)-\frac{10}{3}\kappa S_2(F)\right.\right.$$
$$\left.-\frac{11}{12}S_2(S)\right]\Lambda^S_{abcd}-\frac{3}{2}\Lambda^{SS}_{abcd}-\frac{5}{2}A^{\lambda}_{abcd}-\frac{1}{2}\overline{A}^{\lambda}_{abcd}+4\kappa(B^Y_{abcd}-10\overline{B}^Y_{abcd})\Bigg\}$$
$$+g^6\left\{\left[\frac{161}{6}C_2(G)-\frac{32}{3}\kappa S_2(F)-\frac{7}{3}S_2(S)\right]A_{abcd}-\frac{15}{2}A^S_{abcd}+27A^g_{abcd}\right\}, \tag{44}$$

where $\Lambda^2(i)$, $H^2(i)$, $\overline{H}^2(i)$, and $Y^{2F}(i)$ are the eigenvalues of the invariants of $\Lambda^2_{ab}(S)$, $H^2_{ab}(S)$, $\overline{H}^2_{ab}(S)$, and $Y^{2F}_{ab}(S)$, respectively. The other invariants are defined by

$$\overline{\Lambda}^3_{abcd}=\frac{1}{4}\sum_{perms}\lambda_{abef}\lambda_{cegh}\lambda_{dfgh}, \tag{45}$$

$$\overline{\Lambda}^{2Y}_{abcd}=\frac{1}{8}\sum_{perms}Y^{fg}_2(S)\lambda_{abef}\lambda_{cdeg}, \tag{46}$$

$$\overline{H}^{\lambda}_{abcd}=\frac{1}{8}\sum_{perms}\lambda_{abef}\,\mathrm{Tr}(Y^cY^{+e}Y^dY^{+f}+Y^{+c}Y^eY^{+d}Y^f), \tag{47}$$

$$H^Y_{abcd}=\sum_{perms}\mathrm{Tr}[Y_2(F)Y^{+a}Y^bY^{+c}Y^d], \tag{48}$$

$$\overline{H}^Y_{abcd}=\frac{1}{2}\sum_{perms}\mathrm{Tr}(Y^eY^{+a}Y^eY^{+b}Y^cY^{+d}+Y^{+e}Y^aY^{+e}Y^bY^{+c}Y^d), \tag{49}$$

$$H^3_{abcd}=\frac{1}{2}\sum_{perms}\mathrm{Tr}(Y^aY^{+b}Y^eY^{+c}Y^dY^{+e}), \tag{50}$$

$$\overline{\Lambda}^{2S}_{abcd}=\frac{1}{8}\sum_{perms}C^{fg}_2(S)\lambda_{abef}\lambda_{cdeg}, \tag{51}$$

$$\Lambda^{2g}_{abcd}=\frac{1}{8}\sum_{perms}\lambda_{abef}\lambda_{cdgh}\theta^A_{eg}\theta^A_{fh}, \tag{52}$$

$$H^S_{abcd}=\sum_i C_2(i)H_{abcd}, \tag{53}$$

$$H^F_{abcd}=\sum_{perms}\mathrm{Tr}[\{C_2(F),Y^a\}Y^{+b}Y^cY^{+d}], \tag{54}$$

$$\Lambda^{SS}_{abcd}=\sum_i [C_2(i)]^2\lambda_{abcd}, \tag{55}$$

$$A^{\lambda}_{abcd}=\frac{1}{4}\sum_{perms}\lambda_{abef}\{\theta^A,\theta^B\}_{ef}\{\theta^A,\theta^B\}_{cd}, \tag{56}$$

$$\overline{A}^{\lambda}_{abcd}=\frac{1}{4}\sum_{perms}\lambda_{abef}\{\theta^A,\theta^B\}_{ce}\{\theta^A,\theta^B\}_{df}, \tag{57}$$

$$B^Y_{abcd}=\frac{1}{4}\sum_{perms}\{\theta^A,\theta^B\}_{ab}\,\mathrm{Tr}[t^{A*}t^{B*}Y^cY^{+d}+Y^ct^At^BY^{+d}], \tag{58}$$

$$\overline{B}^Y_{abcd}=\frac{1}{4}\sum_{perms}\{\theta^A,\theta^B\}_{ab}\,\mathrm{Tr}(t^{A*}Y^ct^BY^{+d}), \tag{59}$$

$$A^S_{abcd}=\sum_i C_2(i)A_{abcd}, \tag{60}$$

$$A^g_{abcd}=\frac{1}{8}f^{ACE}f^{BDE}\sum_{perms}\{\theta^A,\theta^B\}_{ab}\{\theta^C,\theta^D\}_{cd}. \tag{61}$$

In the first term of B^Y_{abcd}, Y^c is preceded by $t^{A*}t^{B*}$ instead of t^At^B, since the fermions are complex. Similarly, in $\overline{B}^Y_{abcd}$, Y^c is preceded by t^{A*} instead of t^A. Since there is only one t factor in this case, this introduces one extra minus sign. Therefore, the relative sign between B^Y_{abcd} and $\overline{B}^Y_{abcd}$ in Eq. (44) is minus while it was plus in [9]. In addition to $H^2_{ab}(S)$, $\overline{H}^2_{ab}(S)$, and $Y^{2F}_{ab}(S)$, $\overline{H}^Y_{abcd}$ is also reexpressed to be Hermitian.

V. DIMENSIONAL PARAMETERS

A. Fermion mass

The β functions of the fermion mass can be inferred from those of the Yukawa couplings by taking the a indices to be dummies. The trilinear scalar terms start to contribute from two loops. The one-loop result is

$$\beta^I_{m_f}=\frac{1}{2}[Y^+_2(F)m_f+m_fY_2(F)]+2Y^bm^+_fY^b+\kappa Y^b\,\mathrm{Tr}(m^+_fY^b+m_fY^{+b})-3g^2\{C_2(F),m_f\}. \tag{62}$$

The two-loop result is

$$\beta^{II}_{m_f}=2Y^cY^{+b}m_f(Y^{+c}Y^b-Y^{+b}Y^c)-Y^b[Y_2(F)m_f^++m_f^+Y_2^+(F)]Y^b-\frac{1}{8}[Y^bY_2(F)Y^{+b}m_f+m_fY^{+b}Y_2^+(F)Y^b]$$
$$-2\kappa Y^bY^{+c}Y^b\,\mathrm{Tr}(m_f^+Y^c+m_fY^{+c})-\frac{3}{2}\kappa Y_2^{bc}(S)(Y^bY^{+c}m_f+m_fY^{+c}Y^b)-\frac{3}{2}\kappa Y^b\,\mathrm{Tr}[Y_2(F)Y^{+b}m_f+m_f^+Y_2^+(F)Y^b]$$
$$-\kappa Y^b\,\mathrm{Tr}(Y^cm_f^+Y^cY^{+b}+Y^{+c}m_fY^{+c}Y^b)-2h_{bcd}Y^bY^{+c}Y^d+\frac{1}{12}h_{cde}\lambda_{bcde}Y^b+3g^2\{C_2(F),Y^bm_f^+Y^b\}$$
$$+5g^2Y^b\{C_2(F),m_f^+\}Y^b-\frac{7}{4}g^2[C_2(F)Y_2^+(F)m_f+m_fY_2(F)C_2(F)]-\frac{1}{4}g^2[Y^bC_2(F)Y^{+b}m_f$$
$$+m_fY^{+b}C_2(F)Y^b]+6g^2[t^{A*}m_fY^{+b}t^{A*}Y^b+Y^bt^AY^{+b}m_ft^A]+5\kappa g^2Y^b\,\mathrm{Tr}[C_2(F)(m_fY^{+b}+Y^bm_f^+)]$$
$$+6g^2C_2^{bc}(S)Y^bm_f^+Y^c-\frac{3}{2}g^4\{[C_2(F)]^2,m_f\}+\frac{9}{2}g^2C_2^{bc}(S)(Y^bY^{+c}m_f+m_fY^{+c}Y^b)$$
$$+g^4\left[-\frac{97}{6}C_2(G)+\frac{10}{3}\kappa S_2(F)+\frac{11}{12}S_2(S)\right]\{C_2(F),m_f\}. \qquad (63)$$

B. Trilinear scalar couplings

The β functions of trilinear scalar couplings can be inferred from those of the quartic couplings by taking one of the four indices to be a dummy. The fermion masses contribute from one loop. The one-loop result is

$$\beta^I_{h_{abc}}=\Lambda^2_{abc}-8\kappa H_{abc}+2\kappa\Lambda^Y_{abc}-3g^2\Lambda^S_{abc}, \qquad (64)$$

where the invariants are defined as

$$\Lambda^2_{abc}=\frac{1}{2}\sum_{perms}\lambda_{abef}h_{efc}, \qquad (65)$$

$$H_{abc}=\frac{1}{2}\sum_{perms}\mathrm{Tr}(m_fY^{+a}Y^bY^{+c}+Y^am_f^+Y^bY^{+c}), \qquad (66)$$

$$\Lambda^Y_{abc}=\sum_i Y_2(i)h_{abc}, \qquad (67)$$

$$\Lambda^S_{abc}=\sum_i C_2(i)h_{abc}. \qquad (68)$$

The two-loop result is

$$\beta^{II}_{h_{abc}}=\frac{1}{2}\sum_i\Lambda^2(i)h_{abc}-\overline{\Lambda}^3_{abc}-4\kappa\overline{\Lambda}^{2Y}_{abc}$$
$$+\kappa\left\{8\overline{H}^{\lambda m}_{abc}+8\overline{H}^h_{abc}-\sum_i[3H^2(i)+2\overline{H}^2(i)]h_{abc}\right\}$$
$$+4\kappa(H^Y_{abc}+2\overline{H}^Y_{abc}+2H^3_{abc})$$
$$+g^2\left\{2\overline{\Lambda}^{2S}_{abc}-6\Lambda^{2g}_{abc}+4\kappa(H^S_{abc}-H^F_{abc})+10\kappa\sum_i Y^{2F}(i)h_{abc}\right\}$$
$$-g^4\left\{\left[\frac{35}{3}C_2(G)-\frac{10}{3}\kappa S_2(F)-\frac{11}{12}S_2(S)\right]\Lambda^S_{abc}-\frac{3}{2}\Lambda^{SS}_{abc}-\frac{5}{2}A^\lambda_{abc}-\frac{1}{2}\overline{A}^\lambda_{abc}+4\kappa(B^Y_{abc}-10\overline{B}^Y_{abc})\right\}, \qquad (69)$$

where the invariants are defined as

$$\overline{\Lambda}^3_{abc}=\frac{1}{2}\sum_{perms}(\lambda_{abef}\lambda_{cegl}h_{fgl}+\lambda_{aegl}\lambda_{bfgl}h_{cef}), \qquad (70)$$

$$\overline{\Lambda}^{2Y}_{abc}=\frac{1}{2}\sum_{perms}Y_2^{fg}(S)\lambda_{abef}h_{ceg}, \qquad (71)$$

$$\overline{H}^{\lambda m}_{abc}=\frac{1}{8}\sum_{perms}\lambda_{abef}\,\mathrm{Tr}(Y^cY^{+e}m_fY^{+f}+m_fY^{+e}Y^cY^{+f}+Y^{+c}Y^em_f^+Y^f+m_f^+Y^eY^{+c}Y^f), \qquad (72)$$

$$\overline{H}^h_{abc}=\frac{1}{4}\sum_{perms}h_{aef}\,\mathrm{Tr}(Y^bY^{+e}Y^cY^{+f}+Y^{+b}Y^eY^{+c}Y^f), \qquad (73)$$

$$H^Y_{abc}=\sum_{perms}\mathrm{Tr}[Y_2(F)(m_f^+Y^aY^{+b}Y^c+Y^{+a}m_fY^{+b}Y^c+Y^{+a}Y^bm_f^+Y^c+Y^{+a}Y^bY^{+c}m_f)], \qquad (74)$$

$$\overline{H}^{Y}_{abc}=\frac{1}{2}\sum_{perms}\mathrm{Tr}(Y^e m_f^+ Y^e Y^{+a} Y^b Y^{+c} + Y^e Y^{+a} Y^e m_f^+ Y^b Y^{+c} + Y^e Y^{+a} Y^e Y^{+b} m_f Y^{+c} + Y^e Y^{+a} Y^e Y^{+b} Y^c m_f^+ + \mathrm{H.c.}), \tag{75}$$

$$H^{3}_{abc}=\frac{1}{2}\sum_{perms}\mathrm{Tr}(m_f Y^{+a} Y^e Y^{+b} Y^c Y^{+e} + Y^a m_f^+ Y^e Y^{+b} Y^c Y^{+e} + Y^a Y^{+b} Y^e m_f^+ Y^c Y^{+e} + Y^a Y^{+b} Y^e Y^{+c} m_f Y^{+e}), \tag{76}$$

$$\overline{\Lambda}^{2S}_{abc}=\frac{1}{2}\sum_{perms} C_2^{fg}(S) h_{aef}\lambda_{bceg}, \tag{77}$$

$$\Lambda^{2g}_{abc}=\frac{1}{2}\sum_{perms} h_{aef}\lambda_{bcgl}\theta^A_{eg}\theta^A_{fl}, \tag{78}$$

$$H^{S}_{abc}=\sum_i C_2(i) H_{abc}, \tag{79}$$

$$H^{F}_{abc}=\sum_{perms}\mathrm{Tr}[\{C_2(F),m_f\}Y^{+a}Y^bY^{+c} + \{C_2(F),Y^a\}m_f^+ Y^b Y^{+c} + \{C_2(F),Y^a\}Y^{+b} m_f Y^{+c} + \{C_2(F),Y^a\}Y^{+b}Y^c m_f], \tag{80}$$

$$\Lambda^{SS}_{abc}=\sum_i [C_2(i)]^2 h_{abc}, \tag{81}$$

$$A^{\lambda}_{abc}=\frac{1}{2}\sum_{perms} h_{aef}\{\theta^A,\theta^B\}_{ef}\{\theta^A,\theta^B\}_{bc}, \tag{82}$$

$$\overline{A}^{\lambda}_{abc}=\frac{1}{2}\sum_{perms} h_{aef}\{\theta^A,\theta^B\}_{be}\{\theta^A,\theta^B\}_{cf}, \tag{83}$$

$$B^{Y}_{abc}=\frac{1}{4}\sum_{perms}\{\theta^A,\theta^B\}_{ab}\,\mathrm{Tr}(t^{A*}t^{B*}m_f Y^{+c} + m_f t^A t^B Y^{+c} + t^{A*}t^{B*}Y^c m_f^+ + Y^c t^A t^B m_f^+), \tag{84}$$

$$\overline{B}^{Y}_{abc}=\frac{1}{4}\sum_{perms}\{\theta^A,\theta^B\}_{ab}\,\mathrm{Tr}(t^{A*}m_f t^B Y^{+c} + t^{A*}Y^c t^B m_f^+). \tag{85}$$

C. Scalar mass

The β functions of scalar masses can also be inferred from those of the quartic couplings by taking two of the four indices to be dummies. From one loop, both fermion masses and trilinear terms contribute. The one-loop result is

$$\beta^{I}_{m^2_{ab}}=m^2_{ef}\lambda_{abef}+h_{aef}h_{bef}-4\kappa H_{ab}-3g^2\Lambda^S_{ab}+2\kappa\Lambda^Y_{ab}, \tag{86}$$

where the invariants are defined as

$$H_{ab}=\mathrm{Tr}[(Y^aY^{+b}+Y^bY^{+a})m_f m_f^+ + (Y^{+a}Y^b + Y^{+b}Y^a)m_f^+ m_f + Y^a m_f^+ Y^b m_f^+ + m_f Y^{+a} m_f Y^{+b}], \tag{87}$$

$$\Lambda^S_{ab}=\sum_i C_2(i) m^2_{ab}, \tag{88}$$

$$\Lambda^Y_{ab}=\sum_i Y_2(i) m^2_{ab}. \tag{89}$$

The two-loop contribution is

$$\begin{aligned}\beta^{II}_{m^2_{ab}}=&\frac{1}{2}\sum_i \Lambda^2(i)m^2_{ab}-\frac{1}{2}\overline{\Lambda}^3_{ab}-4\kappa\overline{\Lambda}^{2Y}_{ab}\\&+\kappa\left\{4\overline{H}^{\lambda}_{ab}-\sum_i[3H^2(i)+2\overline{H}^2(i)]m^2_{ab}\right\}\\&+2\kappa(H^Y_{ab}+2\overline{H}^Y_{ab}+2H^3_{ab})\\&+g^2\left\{2\overline{\Lambda}^{2S}_{ab}-6\Lambda^{2g}_{ab}+2\kappa(H^S_{ab}-H^F_{ab})\right.\\&\left.+10\kappa\sum_i Y^{2F}(i)m^2_{ab}\right\}\\&-g^4\left\{\left[\frac{35}{3}C_2(G)-\frac{10}{3}\kappa S_2(F)-\frac{11}{12}S_2(S)\right]\Lambda^S_{ab}\right.\\&\left.-\frac{3}{2}\Lambda^{SS}_{ab}-\frac{5}{2}A^{\lambda}_{ab}-\frac{1}{2}\overline{A}^{\lambda}_{ab}+2\kappa(B^Y_{ab}-10\overline{B}^Y_{ab})\right\},\end{aligned} \tag{90}$$

where the invariants are defined as

$$\overline{\Lambda}^3_{ab}=\lambda_{abef}h_{egl}h_{fgl}+2m^2_{ef}\lambda_{aegl}\lambda_{bfgl}+2h_{aef}h_{fgl}\lambda_{begl}+2h_{bef}h_{fgl}\lambda_{aegl}, \tag{91}$$

$$\overline{\Lambda}^{2Y}_{ab}=Y_2^{fg}(S)(m^2_{eg}\lambda_{abef}+h_{aef}h_{beg}), \tag{92}$$

$$\begin{aligned}\overline{H}^{\lambda}_{ab}=&\frac{1}{2}\lambda_{abef}\,\mathrm{Tr}(m_fY^{+e}m_fY^{+f}+\mathrm{H.c.})\\&+m^2_{ef}\,\mathrm{Tr}(Y^aY^{+e}Y^bY^{+f}+\mathrm{H.c.})\\&+h_{aef}\,\mathrm{Tr}(Y^bY^{+e}m_fY^{+f}+\mathrm{H.c.})\\&+h_{bef}\,\mathrm{Tr}(Y^aY^{+e}m_fY^{+f}+\mathrm{H.c.}),\end{aligned} \tag{93}$$

$$H_{ab}^{Y}=2\,\mathrm{Tr}[\{Y_2(F),m_f^+m_f\}(Y^{+a}Y^b+Y^{+b}Y^a)] +2\,\mathrm{Tr}[Y_2(F)Y^{+a}m_f(Y^{+b}m_f+m_f^+Y^b) +Y_2(F)m_f^+Y^a(Y^{+b}m_f+m_f^+Y^b) +Y_2(F)Y^{+b}m_f(Y^{+a}m_f+m_f^+Y^a) +Y_2(F)m_f^+Y^b(Y^{+a}m_f+m_f^+Y^a)], \tag{94}$$

$$\overline{H}_{ab}^{Y}=\mathrm{Tr}[(Y^eY^{+a}Y^eY^{+b}+Y^eY^{+b}Y^eY^{+a})m_fm_f^+ +Y^em_f^+Y^em_f^+(Y^aY^{+b}+Y^bY^{+a}) +Y^eY^{+a}Y^em_f^+(Y^bm_f^++m_fY^{+b}) +Y^em_f^+Y^eY^{+a}(Y^bm_f^++m_fY^{+b}) +Y^eY^{+b}Y^em_f^+(Y^am_f^++m_fY^{+a}) +Y^em_f^+Y^eY^{+b}(Y^am_f^++m_fY^{+a})+\mathrm{H.c.}], \tag{95}$$

$$H_{ab}^{3}=\mathrm{Tr}[(Y^aY^{+b}+Y^bY^{+a})Y^em_f^+m_fY^{+e} +m_fm_f^+Y^e(Y^{+a}Y^b+Y^{+b}Y^a)Y^{+e} +Y^am_f^+Y^e(Y^{+b}m_f+m_f^+Y^b)Y^{+e} +m_fY^{+a}Y^e(Y^{+b}m_f+m_f^+Y^b)Y^{+e} +Y^bm_f^+Y^e(Y^{+a}m_f+m_f^+Y^a)Y^{+e} +m_fY^{+b}Y^e(Y^{+a}m_f+m_f^+Y^a)Y^{+e}], \tag{96}$$

$$\overline{\Lambda}_{ab}^{2S}=C_2^{fg}(S)\lambda_{abef}m_{eg}^2+C_2^{fg}(S)h_{aef}h_{beg}, \tag{97}$$

$$\Lambda_{ab}^{2g}=\lambda_{abef}m_{gl}^2+h_{aef}h_{bgl}\theta_{eg}^A\theta_{fl}^A, \tag{98}$$

$$H_{ab}^{S}=\sum_i C_2(i)H_{ab}, \tag{99}$$

$$H_{ab}^{F}=2\,\mathrm{Tr}[\{C_2(F),Y^a\}Y^{+b}m_fm_f^+ +\{C_2(F),Y^b\}Y^{+a}m_fm_f^+ +\{C_2(F),m_f\}m_f^+(Y^aY^{+b}+Y^bY^{+a}) +\{C_2(F),Y^a\}m_f^+(Y^bm_f^++m_fY^{+b}) +\{C_2(F),m_f\}Y^{+a}(Y^bm_f^++m_fY^{+b}) +\{C_2(F),Y^b\}m_f^+(Y^am_f^++m_fY^{+a}) +\{C_2(F),m_f\}Y^{+b}(Y^am_f^++m_fY^{+a})], \tag{100}$$

$$\Lambda_{ab}^{SS}=\sum_i [C_2(i)]^2m_{ab}^2, \tag{101}$$

$$A_{ab}^{\lambda}=m_{ef}^2\{\theta^A,\theta^B\}_{ef}\{\theta^A,\theta^B\}_{ab}, \tag{102}$$

$$\overline{A}_{ab}^{\lambda}=m_{ef}^2\{\theta^A,\theta^B\}_{ae}\{\theta^A,\theta^B\}_{bf}, \tag{103}$$

$$B_{ab}^{Y}=\{\theta^A,\theta^B\}_{ab}\,\mathrm{Tr}(t^{A*}t^{B*}m_fm_f^++m_ft^At^Bm_f^+), \tag{104}$$

$$\overline{B}_{ab}^{Y}=\{\theta^A,\theta^B\}_{ab}\,\mathrm{Tr}(t^{A*}m_ft^Bm_f^+). \tag{105}$$

VI. EXTENSION TO NONSIMPLE GROUPS

So far, the gauge group was assumed to be simple. These results can be extended to semisimple groups by assigning appropriate substitution rules, based upon close inspection of the relevant Feynman diagrams [7–9].

Assume the gauge group is a direct product of simple groups, $G_1\times\cdots\times G_n$, with corresponding gauge coupling constants $g_1,\ldots,g_n$. The substitution rules for the gauge coupling constants are

$$g^3C_2(G)\to g_k^3C_2(G_k), \tag{106}$$

$$g^3S_2(R)\to g_k^3S_2^k(R), \tag{107}$$

$$g^5[C_2(G)]^2\to g_k^5[C_2(G_k)]^2, \tag{108}$$

$$g^5C_2(G)S_2(R)\to g_k^5C_2(G_k)S_2^k(R), \tag{109}$$

$$g^5C_2(R)S_2(R)\to\sum_l g_k^3g_l^2C_2^l(R)S_2^k(R). \tag{110}$$

Here and hereafter, k and l are subgroup indices; R can be either S or F.

For other β and γ functions, we first have the following general substitution rules:

$$g^2C_2(R)\to\sum_k g_k^2C_2^k(R), \tag{111}$$

$$g^4C_2(G)C_2(R)\to\sum_k g_k^4C_2(G_k)C_2^k(R), \tag{112}$$

$$g^4S_2(R)C_2(R')\to\sum_k g_k^4S_2^k(R)C_2^k(R'), \tag{113}$$

$$g^4C_2(R)C_2(R')\to\sum_{k,l} g_k^2g_l^2C_2^k(R)C_2^l(R'). \tag{114}$$

In H_{2t}^a, B_{abcd}^Y, B_{abc}^Y, B_{ab}^Y, $\overline{B}_{abcd}^Y$, $\overline{B}_{abc}^Y$, and $\overline{B}_{ab}^Y$, the substitution rules are

$$\theta^A\to\theta_k^A,\quad \theta^B\to\theta_l^B,$$

$$t^A\to t_k^A,\quad t^B\to t_l^B,$$

$$(t^{A*}\to t_k^{A*},\quad t^{B*}\to t_l^{B*}),$$

$$g^4\to g_k^2g_l^2. \tag{115}$$

For example, $g^4 B^Y_{abcd}$ is substituted by

$$\frac{1}{4}\sum_{k,l} g_k^2 g_l^2 \sum_{perms} \{\theta_k^A,\theta_l^B\}_{ab}\,\mathrm{Tr}(t_k^{A*} t_l^{B*} Y^c Y^{+d} + Y^c t_k^A t_l^B Y^{+d}). \tag{116}$$

In quartic coupling, trilinear coupling and scalar mass-squared terms, further substitution rules are needed. We introduce a new tensor

$$\Lambda_{ab,cd} = (\theta^A)_{ac}(\theta^A)_{bd} \tag{117}$$

so the gauge invariants A_{abcd} can be rewritten as

$$A_{abcd} = \frac{1}{4}\sum_{perms}(\Lambda_{ac,ef}\Lambda_{ef,bd} + \Lambda_{ae,fd}\Lambda_{eb,cf}). \tag{118}$$

The substitution rule for $\Lambda_{ab,cd}$ is

$$g^2\Lambda_{ab,cd} \to \sum_k g_k^2 \Lambda^k_{ab,cd}. \tag{119}$$

Thus, $g^4 A_{abcd}$ can be substituted by

$$\frac{1}{4}\sum_{k,l} g_k^2 g_l^2 \sum_{perms}(\Lambda^k_{ac,ef}\Lambda^l_{ef,bd} + \Lambda^k_{ae,fd}\Lambda^l_{eb,cf}), \tag{120}$$

$g^6 S_2(R) A_{abcd}$ by

$$\frac{1}{4}\sum_{k,l} g_k^4 g_l^2 S_2^k(R) \sum_{perms}(\Lambda^k_{ac,ef}\Lambda^l_{ef,bd} + \Lambda^k_{ae,fd}\Lambda^l_{eb,cf}), \tag{121}$$

$g^6 C_2(G) A_{abcd}$ by

$$\frac{1}{4}\sum_{k,l} g_k^4 g_l^2 C_2(G_k) \sum_{perms}(\Lambda^k_{ac,ef}\Lambda^l_{ef,bd} + \Lambda^k_{ae,fd}\Lambda^l_{eb,cf}), \tag{122}$$

and $g^6 A^S_{abcd}$ by

$$\sum_k \sum_i g_k^2 C_2^k(i) \frac{1}{4}\sum_{l,m} g_l^2 g_m^2 \sum_{perms}(\Lambda^l_{ac,ef}\Lambda^m_{ef,bd} + \Lambda^l_{ae,fd}\Lambda^m_{eb,cf}). \tag{123}$$

Finally, one has

$$g^6 A^g_{abcd} \to \sum_k g_k^6 A^g_{abcd}(k). \tag{124}$$

VII. CONCLUSIONS

We have now presented the complete set of two-loop renormalization group equations in general gauge theories. This includes the β functions of parameters with and without a mass dimension. We have so far restricted the gauge groups to be semisimple. If the gauge group contains more than one U(1) group, the situation is subtle. If there are no kinetic mixings between the U(1) groups after renormalization, these results can be applied by straightforward extension. If there are kinetic mixings, these results have to be applied with caution. In general, modifications will be warranted [12]. In the case of the SM where one has only one U(1) group, one readily reproduces the results in [11].

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