Fermion determinant for general background gauge fields

M. P. Fry
School of Mathematics, University of Dublin, Dublin 2, Ireland
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An exact representation of the Euclidean fermion determinant in two dimensions for centrally symmetric, finite-ranged Abelian background fields is derived. Input data are the wave function inside the field's range and the scattering phase shift with their momenta rotated to the positive imaginary axis and fixed at the fermion mass for each partial wave. The determinant's asymptotic limit for strong coupling and small fermion mass for square-integrable, unidirectional magnetic fields is shown to depend only on the chiral anomaly. The concept of duality is extended from one- to two-variable fields, thereby relating the two-dimensional Euclidean determinant for a class of background magnetic fields to the pair production probability in four dimensions for a related class of electric pulses. Additionally, the "diamagnetic" bound on the two-dimensional Euclidean determinant is related to the negative sign of $\partial\,\mathrm{Im}\,S_{\mathrm{eff}}/\partial m^2$ in four dimensions in the strong coupling, small mass limit, where S_{eff} is the one-loop effective action.

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I. INTRODUCTION

Within the standard model fermion determinants are encountered in the calculation of every physical process. Because of their nonlocal dependence on the gauge fields they are difficult to calculate. Consequently the practice has been to either ignore them—the quenched approximation—or to expand them in power series. Ultimately they will have to be confronted nonperturbatively, as lattice theorists are now doing with faster machines, in order to obtain reliable predictions with known computational error.

The current status of fermion determinants is reviewed in [1]. To indicate just how bad our knowledge of these determinants is, not even the strong coupling limit of the massive Euclidean QED determinant in two dimensions is known except for a constant background magnetic field and for a magnetic field confined to the surface of a cylinder [2]. Therefore, it seems that this is as good a starting point as any to get better insight into the properties of fermion determinants and to understand their physics.

There are other reasons why the two-dimensional QED (QED_2) determinant should be of general interest. Namely, if there were precise nonperturbative information on at least one continuum, infinite-volume determinant then the algorithms of lattice theorists for calculating determinants could be tested by extrapolating their output to zero lattice spacing and infinite volume. Algorithms for determinants can be easily adjusted to any dimensionality, and if some fail to coincide with known results for an Abelian background field in two dimensions then they are certainly useless.

Work in this direction has already begun [3] with the computation of the fermion determinant for massless fermions on a torus using the Neuberger-Dirac operator and the higher-order overlap Dirac operator and the comparison of the results with the exact massless QED_2 determinant on a torus [4]. In massive two-flavor QED_2 the determinant was calculated explicitly to study the masses of the triplet (pion) and singlet (eta) bound states using the overlap and fixed point Dirac operators [5]. Presumably the continuum limit of the determinant itself in the nonperturbative domain discussed below could be used as a sensitive test of the many lattice discretizations of the Dirac operator now in use.

In addition, we develop further the concept of duality and relate the Euclidean QED_2 determinant to the pair production probability in QED_4 for a class of electric pulses. Thus, the Euclidean QED_2 determinant contains nonperturbative physical information in four dimensions.

Fermion determinants are obtained by integrating over the fermion fields to produce the one-loop effective Euclidean action $S_{\mathrm{eff}}=-\ln\det$, where det is formally the ratio $\det(\not{P}-e\not{A}+m)/\det(\not{P}+m)$ of Fredholm determinants of Euclidean Dirac operators. We assume that the continuation to the Euclidean metric has been done. When det is properly defined it is a nonlocal function of the field strength $F_{\mu\nu}$ formed from the potential A_μ, modulo Chern-Simons terms that are absent in two dimensions. Since the determinant is part of the gauge field's action, A_μ and $F_{\mu\nu}$ are random fields. We have discussed elsewhere [2,6,7] how the need to regulate in any dimension above 1 allows one to assume smooth potentials and fields. In order to make further progress we assume in addition that $F_{\mu\nu}$ is centrally symmetric and that it has a finite range a.

This paper is organized as follows. In Sec. II we define the determinant and indicate our strategy for calculating it by first assuming $ma\ll 1$ and then letting $|e\Phi|\gg 1$, where m is the fermion mass and Φ is the flux of the background magnetic field F_{12}. This is the really interesting limit as it takes one deep into the nonperturbative regime. In Sec. III the low-energy scattering phase shifts required to calculate the determinant are obtained. Section IV deals with the small mass, strong coupling expansion of the determinant, while Sec. V presents the explicit form it takes in this limit. Section VI generalizes the concept of duality from one- to two-variable fields, thereby allowing the QED_2 Euclidean determinant to be related to physics in four dimensions. Section VII summarizes our results while the asymptotic form of the determinant given in Sec. V is derived in the Appendix.

II. REPRESENTATION OF THE DETERMINANT

A. Green's functions

The exact calculation of det in QED_2 continued to the Euclidean metric reduces to the scattering problem of a charged particle confined to a plane pierced by a magnetic field, namely [8],

$$\frac{\partial}{\partial e}\ln\det=\frac{e}{\pi}\int d^2r\,\varphi\partial^2\varphi+2m^2\int d^2r\,\varphi(\mathbf{r})\langle\mathbf{r}|(H_+ +m^2)^{-1} -(H_- +m^2)^{-1}|\mathbf{r}\rangle, \tag{1}$$

where the supersymmetric operator pair $H_\pm=(\mathbf{P}-e\mathbf{A})^2\mp eB$ are obtained from the two-dimensional Pauli Hamiltonian $(\mathbf{P}-e\mathbf{A})^2-\sigma_3 eB$. Hence, the subscripts on H in Eq. (1) refer to positive and negative chirality. The auxiliary potential φ is related to the vector potential by $A_\mu=\epsilon_{\mu\nu}\partial_\nu\varphi$ and to the magnetic field by $B=-\partial^2\varphi$ or

$$\varphi(\mathbf{r})=-\frac{1}{2\pi}\int d^2r'\ln|\mathbf{r}-\mathbf{r}'|B(\mathbf{r}'), \tag{2}$$

with $\epsilon_{12}=1$. Expansion of Eq. (1) in powers of e yields the standard one-loop effective action given by the Feynman rules. The first term on the right-hand side of Eq. (1) is $\partial\ln\det/\partial e$ of the massless Schwinger model [9]. Due to the $1/r$ falloff of A_μ when $\Phi\neq 0$ an integration by parts is not justified in this case. As we will see in Sec. V, the presence of the mass dependent term profoundly modifies the determinant, ultimately cancelling the first term when $|e\Phi|\gg 1$. The invariance of Eq. (1) under $\varphi\to\varphi+c$, where c is a constant, gives the index theorem on a two-dimensional Euclidean manifold [8,10].

We now assume that B is centrally symmetric and that $B(r)=0$ for $r>a$. To ensure finite flux we assume B is square integrable in view of the inequality $\Phi^2\leqslant 2\pi^2a^2\int_0^a drr\,B^2(r)$. Referring to Eq. (1), define the Green's function

$$\begin{aligned}\langle r,\theta|(k^2-H_\pm)^{-1}|r',\theta'\rangle &=\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\langle r|(k^2-H_{\pm,l})^{-1}|r'\rangle e^{il(\theta-\theta')}\\ &=\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}G_{\pm,l}(k;r,r')e^{il(\theta-\theta')},\end{aligned} \tag{3}$$

where $A_\theta=\Phi(r)/2\pi r$,

$$H_{\pm,l}=-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{[l-e\Phi(r)/2\pi]^2}{r^2}\mp eB(r), \tag{4}$$

and

$$\Phi(r)=2\pi\int_0^r ds\,sB(s). \tag{5}$$

The calculation is simplified by introducing the Green's function

$$\mathcal{G}_{\pm,l}(k;r,r')=\sqrt{rr'}\,G_{\pm,l}(k;r,r'), \tag{6}$$

where

$$\mathcal{G}(k;r,r')=\langle r|(k^2-\mathcal{H}_{\pm,l})^{-1}|r'\rangle \tag{7}$$

and

$$\mathcal{H}_{\pm,l}=-\frac{d^2}{dr^2}+\frac{[l-e\Phi(r)/2\pi]^2-\frac{1}{4}}{r^2}\mp eB(r). \tag{8}$$

The outgoing-wave Green's functions $\mathcal{G}_{\pm,l}$ are constructed from [11]

$$\mathcal{G}_{\pm,l}(k;r,r')=-\frac{\varphi(k,r_<)f^{(+)}(k,r_>)}{\mathcal{J}(k)}, \tag{9}$$

where φ is a regular solution and $f^{(+)}$ an irregular outgoing-wave solution, of

$$\mathcal{H}_{\pm,l}f=k^2f; \tag{10}$$

$\mathcal{J}$ is the associated Jost function and $r_<,r_>$ denote the lesser and larger values of r,r'. Here and below we will occasionally suppress the subscripts $\pm$ and l to reduce notational clutter.

Regular solutions of Eq. (10) are

$$\varphi_{\pm,l}(k,r)=\frac{e^{i\pi(|l|-W/2)}}{2\sqrt{2}}\begin{cases}\sqrt{a}[H_W^-(ka)+S_\pm H_W^+(ka)]\dfrac{R_\pm(k,r)}{R_\pm(k,a)}, & r<a,\\ \sqrt{r}[H_W^-(kr)+S_\pm H_W^+(kr)], & r>a,\end{cases} \tag{11}$$

where

$$S_{\pm}=e^{i\pi(W-|l|)}e^{2i\delta_l^{\pm}}; \tag{12}$$

H_W^+ and H_W^- denote the Hankel functions $H_W^{(1)}$ and $H_W^{(2)}$, respectively; $\delta_l^{\pm}$ are the scattering phase shifts; $W=|l-e\Phi/2\pi|$; and $\Phi=\Phi(a)$ is the total flux of B. The interior wave functions $R_{\pm,l}$ satisfy the boundary condition $\lim_{r\to 0}r^{-1/2-|l|}R_{\pm,l}=1$. These will be discussed further below. The structure of $\varphi_{\pm,l}$ for $r>a$ ensures that the eigenfunctions of $H_{\pm,l}$ in Eq. (4), $\psi_{\pm,l}=\varphi_{\pm,l}/\sqrt{r}$, correspond to physical wave functions [11]. That is,

$$\psi_{\pm}(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{2}\pi}\sum_{l=-\infty}^{\infty}\psi_{\pm,l}(k,r)e^{il\theta} \tag{13}$$

assumes the asymptotic form for $r\to\infty$,

$$\psi_{\pm}(\mathbf{k},\mathbf{r})\sim\frac{1}{2\pi}e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{1}{2\pi\sqrt{r}}f_{\pm}(k,\theta)e^{ikr}, \tag{14}$$

where

$$f_{\pm}(k,\theta)=\sqrt{\frac{2}{\pi k}}e^{i\pi/4}\sum_{l=-\infty}^{\infty}e^{i\delta_l^{\pm}}\sin\delta_l^{\pm}e^{il\theta}, \tag{15}$$

with the differential scattering cross section $d\sigma/d\Omega=|f_{\pm}(k,\theta)|^2$.

Assuming $R_{\pm,l}$ are known, irregular outgoing-wave solutions of Eq. (10) can be found by standard means, giving

$$f_{\pm,l}^{(+)}(k,r)=\begin{cases}\sqrt{a}\;\;H_W^+(ka)\;\dfrac{R_{\pm}(k,r)}{R_{\pm}(k,a)}+\dfrac{4i}{\pi\sqrt{a}}[H_W^-(ka)+S_{\pm}H_W^+(ka)]^{-1}R_{\pm}(k,a)R_{\pm}(k,r)\displaystyle\int_r^a\frac{ds}{R_{\pm}^2(k,s)}, & r<a,\\ \sqrt{r}\;\;H_W^+(kr), & r>a.\end{cases} \tag{16}$$

Near the regular singular point at $r=0$ of Eq. (10), $f_{\pm,l}^{(+)}\sim\text{const}\times r^{1/2-|l|}$.

Equations (11) and (16) give the Jost function

$$\mathcal{J}=W(f^{(+)},\varphi)=-\frac{i\sqrt{2}}{\pi}e^{i\pi(|l|-W/2)}, \tag{17}$$

which is independent of chirality; W on the left-hand side is the Wronskian. It may be verified that Eqs. (9), (11), (16), and (17) combine to satisfy the basic condition

$$\frac{\partial}{\partial r}\mathcal{G}_{\pm,l}(r,r')\Big|_{r=r'-0}^{r=r'+0}=1. \tag{18}$$

In order to make contact with the determinant in Eq. (1) we now analytically continue k in $\mathcal{G}_{\pm,l}(k,r)$ into the upper half of the complex plane by letting $k=me^{i\pi/2}$. Then Eqs. (1), (3), and (6) give

$$\begin{aligned}\frac{\partial}{\partial e}\text{lndet}=&-2e\int_0^a dr\,r\,\varphi(r)B(r)\\&-2m^2\int_0^{\infty}dr\,\varphi(r)\sum_{l=-\infty}^{\infty}[\mathcal{G}_{+,l}(me^{i\pi/2},r)\\&-\mathcal{G}_{-,l}(me^{i\pi/2},r)],\end{aligned} \tag{19}$$

while Eq. (2) gives

$$\varphi(r)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_r^a ds\,\frac{\Phi(s)}{s}, & r<a,\\ -\dfrac{\Phi}{2\pi}\ln\left(\dfrac{r}{a}\right), & r>a.\end{cases} \tag{20}$$

Because of the invariance of lndet under $\varphi\to\varphi+c$ we have adjusted $\varphi(r)$ so that $\varphi(a)=0$.

For $r>a$, Eqs. (9), (11), (16), and (17) give

$$\begin{aligned}&\mathcal{G}_{+,l}(me^{i\pi/2},r)-\mathcal{G}_{-,l}(me^{i\pi/2},r)\\&\quad=\frac{ir}{\pi}e^{-i\pi|l|}(e^{2i\delta_l^+}-e^{2i\delta_l^-})K_W^2(mr),\end{aligned} \tag{21}$$

where K_W is a modified Bessel function and we used [12]

$$H_W^+(rme^{i\pi/2})=-\frac{2i}{\pi}e^{-i\pi W/2}K_W(mr). \tag{22}$$

The phase shifts in Eq. (21) are understood to be analytically continued as well.

It is convenient to separate the energy-independent Aharonov-Bohm phase shifts [10,13] from $\delta_l^{\pm}$. Without loss of generality we assume $e\Phi>0$. Then, modulo π,

$$\delta_l^+(k)=\begin{cases}\frac{\pi}{2}(|l|-W)+\Delta_l^+(k), & l\neq[e\Phi/2\pi],\\ \frac{\pi}{2}(e\Phi/2\pi)+\Delta_l^+(k), & l=[e\Phi/2\pi],\end{cases} \tag{23}$$

$$\delta_l^-(k)=\frac{\pi}{2}(|l|-W)+\Delta_l^-(k), \quad \text{all } l, \tag{24}$$

where $[x]$ stands for the nearest integer less than x with $[0]=0$. The energy-dependent phase shifts $\Delta_l^{\pm}(k)$ will be calculated in Sec. III.

The Green's function difference on the left-hand side of Eq. (21) for $r<a$ may be dealt with as for $r>a$, this time using [12]

$$H_W^+(ame^{i\pi/2})H_W^-(ame^{i\pi/2})=\frac{4}{\pi^2}e^{-i\pi W}K_W^2(am)-\frac{4i}{\pi}I_W(am)K_W(am), \tag{25}$$

where I_W is a modified Bessel function. For $e\Phi/2\pi=N+\epsilon$, $N=0,1,\ldots,0\leqslant\epsilon<1$, the final result from Eqs. (19), (20), (21), (23), and (24) is

$$\begin{aligned}\frac{\partial}{\partial e}\ln\det={}&-2e\int_0^a dr\,r\varphi(r)B(r)+2am^2\int_0^a dr\,\varphi(r)\sum_l I_W(am)K_W(am)\left[\left(\frac{R_+(r)}{R_+(a)}\right)^2-\left(\frac{R_-(r)}{R_-(a)}\right)^2\right]\\&+\frac{i2am^2}{\pi}\int_0^a dr\,\varphi(r)\sum_{l\neq N}e^{-i\pi W}K_W^2(am)\left[(1-e^{2i\Delta_l^+})\left(\frac{R_+(r)}{R_+(a)}\right)^2-(1-e^{2i\Delta_l^-})\left(\frac{R_-(r)}{R_-(a)}\right)^2\right]\\&+\frac{i2am^2}{\pi}e^{-i\pi\epsilon}K_\epsilon^2(am)\int_0^a dr\,\varphi(r)\left[(1-e^{2i\pi\epsilon}e^{2i\Delta_N^+})\left(\frac{R_+(r)}{R_+(a)}\right)^2-(1-e^{2i\Delta_N^-})\left(\frac{R_-(r)}{R_-(a)}\right)^2\right]\\&+2m^2\int_0^a dr\,\varphi(r)\sum_l\left[R_+^2(r)\int_r^a\frac{ds}{R_+^2(s)}-R_-^2(s)\int_r^a\frac{ds}{R_-^2(s)}\right]+\frac{im^2\Phi}{\pi^2}\int_a^\infty dr\,r\ln(r/a)\sum_{l\neq N}e^{-i\pi W}\\&\times(e^{2i\Delta_l^+}-e^{2i\Delta_l^-})K_W^2(mr)+\frac{im^2\Phi}{\pi^2}(e^{i\pi\epsilon}e^{2i\Delta_N^+}-e^{-i\pi\epsilon}e^{2i\Delta_N^-})\int_a^\infty dr\,r\ln(r/a)K_\epsilon^2(mr).\end{aligned} \tag{26}$$

The interior wave functions $R_\pm$ and the phase shifts $\Delta_l^{\pm}$ are abbreviations for $R_{\pm,l}(me^{i\pi/2},r)$ and $\Delta_l^{\pm}(me^{i\pi/2})$.

The representation (26) is exact. Its advantage over other representations of determinants based on scattering data is that it involves no integration over phase shift energy. It is particularly relevant to a study of the chiral limit $ma\ll1$. Anticipating what follows, the integrals can be interchanged with the sums for the class of fields considered here, allowing the integrals in the exterior region $r>a$ to be done immediately. Only information about the interior wave functions is required to calculate the determinant exactly, and these are known explicitly for $ma\ll1$ as in Eq. (29) below.

The right-hand side of Eq. (26) must be real since it is a Euclidean determinant. This imposes the nontrivial constraints

$$\begin{aligned}&e^{i\pi W}e^{-2i\Delta_l^{\pm}(me^{-i\pi/2})}\\&\quad=-e^{-i\pi W}e^{2i\Delta_l^{\pm}(me^{i\pi/2})}+2\cos\pi W,\\&\qquad\qquad l\neq N \text{ for } + \text{ chirality},\\&e^{-i\pi\epsilon}e^{-2i\Delta_N^+(me^{-i\pi/2})}=-e^{i\pi\epsilon}e^{2i\Delta_N^+(me^{i\pi/2})}+2\cos\pi\epsilon,\\&\Delta_l^{\pm*}(me^{i\pi/2})=\Delta_l^{\pm}(me^{-i\pi/2}).\end{aligned} \tag{27}$$

For the fields considered here and the small mass expansions of $\Delta_l^{\pm}$ made in Sec. III there is complete agreement with Eq. (27).

B. Small mass expansions

We now commence the expansion of lndet when $ma\ll1$. This does not mean an expansion in powers of m^2. Such an expansion does not exist as lndet has a branch beginning at $m=0$ [14]. Rather, we are referring to a collection of leading terms in m such as $m^\nu\ln m$, $\nu>0$, as well as integral powers of m^2.

Since Eq. (10) depends only on k^2 and the boundary condition $\lim_{r\to0}r^{-|l|-1/2}R_{\pm,l}=1$ is independent of $k, R_{\pm,l}(k,r)$ is a regular function of k^2. Therefore we set $R_{\pm,l}(me^{i\pi/2},r)\equiv R_{\pm,l}(m^2,r)$ and begin an expansion in powers of m^2:

$$R_l(m^2,r)=R_l(r)[1+(ma)^2\chi_l(r)+O(ma)^4]. \tag{28}$$

For $m=0$, exact positive chirality solutions of $\mathcal{H}_{+,l}R_{+,l}=0$ are known for $l>0$ [13]; the remaining cases can be dealt with similarly. The results are, up to irrelevant normalization constants that cancel in Eq. (26),

$$R_{+,l}=r^{l+1/2}e^{e\varphi(r)},\quad l\geqslant 0,$$

$$R_{+,-l}=r^{-l+1/2}e^{e\varphi(r)}\int_0^r ds s^{2l-1}e^{-2e\varphi(s)},\quad l>0,$$

$$R_{-,l}=r^{-l+1/2}e^{-e\varphi(r)}\int_0^r ds s^{2l-1}e^{2e\varphi(s)},\quad l>0,$$

$$R_{-,-l}=r^{l+1/2}e^{-e\varphi(r)},\quad l\geqslant 0. \tag{29}$$

Noting Eq. (20), $R_{+,l}$ is square integrable for $l=0,\ldots,N-1$ for $e\Phi/2\pi=N+\epsilon$. This is in accord with the Aharonov-Casher theorem which states that the number of positive (negative) chirality square-integrable zero modes is $[|e\Phi|/2\pi]$, depending on whether $e\Phi>0$ or $(e\Phi<0)$ [15]. These zero modes will be shown to play a dominant role in the strong coupling limit of lndet.

We want to calculate lndet in the limit $ma\ll 1$ followed by $e\Phi\gg 1$. This must be done with care as there may be ratios of terms like $(am)^2e^{e\Phi}/[1+(am)^2e^{e\Phi}]$ which when further expanded in powers of m^2 grows exponentially with $e\Phi$. There is one firm guiding principle here, namely, that the determinant is an entire function of e of order 2 [16,17]. This means that for any complex value of e, $|\det|<A(\epsilon)\exp[K(\epsilon)|e|^{2+\epsilon}]$ for any $\epsilon>0$ and $A(\epsilon),K(\epsilon)$ are constants. Therefore, any growth of lndet faster than quadratic in e means that the expansion one is making is inadmissible. In fact, for real values of e lndet must satisfy the more precise bound

$$-\frac{e^2||B||^2}{4\pi m^2}\leqslant \text{lndet}\leqslant 0, \tag{30}$$

for any B with $||B||^2=\int d^2rB^2(\mathbf{r})<\infty$. There are additional technical assumptions underlying Eq. (30) that the fields considered here satisfy. The right-hand side is the "diamagnetic" bound [17–20] and the left-hand side follows from the general operator structure of det and some standard inequalities [1].

The warning cited above materializes for $0<l<e\Phi/2\pi$ when $B(r)\geqslant 0$. There may be other cases. In the positive chirality sector χ_l^+ in Eq. (28) is

$$\chi_l^+(r)=a^{-2}\int_0^r ds\int_0^s dt(t/s)^{2l+1}e^{2e[\varphi(t)-\varphi(s)]}, \tag{31}$$

for $r<a$. What happens is that the effective potential

$$V(r)=\frac{[l-e\Phi(r)/2\pi]^2-\frac{1}{4}}{r^2}-eB(r) \tag{32}$$

has a high and wide barrier beginning in the range $r<a$ and extending out to $r\sim 2a$ for $e\Phi/2\pi\gg 1$. This gives rise to quasi-stationary states. As a consequence the wave function is enhanced inside $r<a$ and χ_l^+ can become large for strong coupling.

For $l=O(e\Phi/2\pi)$ or larger the barrier in V disappears and the growth of χ_l^+ for $e\Phi/2\pi\gg 1$ slows down. This must happen since $d\chi_l^+/dl<0$ for all e. For $l\gg 1$ the integral in Eq. (31) is dominated in the range $t\lesssim s$, giving $\chi_l^+=O(1/l)$. For the special case of $B(r)=B$, $r<a$ and zero otherwise, we find

$$\chi_l^+(r)\leqslant(4l)^{-1}\ln l+O(1/l), \tag{33}$$

for $l>e\Phi/2\pi-1$, $l>2$, $0\leqslant r\leqslant a$.

To reiterate, care must be taken that every term in the small mass expansion makes sense, either by satisfying the bound (30) or by making sure that the offending term is cancelled by other terms.

III. LOW-ENERGY PHASE SHIFTS

In order to take the small mass limit of det in Eq. (26) we will need the low-energy phase shifts. From here on it is convenient to revert to the solutions of

$$H_{\pm,l}\psi_{\pm,l}=k^2\psi_{\pm,l}, \tag{34}$$

where $H_{\pm,l}$ is defined by Eq. (4) and $\psi_{\pm,l}$ are connected to the regular solutions (11) of Eq. (10) by

$$\psi_{\pm,l}(k,r)=\frac{\varphi_{\pm,l}(k,r)}{\sqrt{r}}. \tag{35}$$

For any chirality the zero-energy solutions (29) of Eq. (10) are related to the zero-energy solutions ψ_l^0 of Eq. (34) by

$$\psi_l^0(r)=\frac{R_l(r)}{\sqrt{r}}. \tag{36}$$

From Eqs. (11), (12), (23), (24), and (35), for $r>a$,

$$\psi_l(k,r)=2^{-1/2}e^{i\delta_l}e^{i\pi|l|/2}[J_W(kr)\cos\Delta_l-Y_W(kr)\sin\Delta_l], \tag{37}$$

where Y_W is a Bessel function of the second kind. This holds for all l and both chiralities except for positive chirality when $l=N$, which has to be dealt with separately. Then

$$\tan\Delta_l=\frac{\gamma_lJ_W(ka)-kaJ'_W(ka)}{\gamma_lY_W(ka)-kaY'_W(ka)}, \tag{38}$$

where

$$\begin{aligned}\gamma_l&=(r\partial_r\psi_l/\psi_l)_a\\&=(r\partial_r\psi_l^0/\psi_l^0)_a-\frac{k^2}{\psi_l^0(a)\psi_l(k,a)}\int_0^a dr r\psi_l^0(r)\psi_l(k,r)\\&\equiv\gamma_l^{(0)}+(ka)^2\gamma_l^{(2)}+(ka)^4\gamma_l^{(4)}+O(ka)^6,\end{aligned} \tag{39}$$

and from Eq. (28)

$$\psi_l(k,r)=\psi_l^0(r)[1-(ka)^2\chi_l(r)+O(ka)^4]. \tag{40}$$

Equations (29) and (36) give

$$\gamma_l^{+(0)}=\begin{cases} l-\dfrac{e\Phi}{2\pi}, & l\geqslant 0\\ l-\dfrac{e\Phi}{2\pi}+\left[\displaystyle\int_0^a\frac{dr}{r}\left(\frac{r}{a}\right)^{-2l}e^{-2e\varphi(r)}\right]^{-1}, & l<0,\end{cases}$$

$$\gamma_l^{-(0)}=\begin{cases} \dfrac{e\Phi}{2\pi}-l+\left[\displaystyle\int_0^a\frac{dr}{r}\left(\frac{r}{a}\right)^{2l}e^{2e\varphi(r)}\right]^{-1}, & l>0,\\ \dfrac{e\Phi}{2\pi}-l, & l\leqslant 0,\end{cases} \tag{41}$$

and Eqs. (39),(40) give, for both chiralities,

$$\gamma_l^{(2)}=-a^{-2}\int_0^a drr\left(\frac{\psi_l^0(r)}{\psi_l^0(a)}\right)^2,$$

$$\gamma_l^{(4)}=a^{-2}\int_0^a drr\left(\frac{\psi_l^0(r)}{\psi_l^0(a)}\right)^2[\chi_l(r)-\chi_l(a)]. \tag{42}$$

The norms of the square-integrable zero modes are, from Eqs. (36), (29) and (20),

$$\begin{aligned}||\psi_l^0||^2&=\int_0^\infty drr|\psi_l^0|^2/a^{2l+2}\\&=\int_0^a\frac{dr}{a}\left(\frac{r}{a}\right)^{2l+1}e^{2e\varphi(r)}\\&\quad+\frac{1}{2(W-1)},\quad l=0,\ldots,N-1.\end{aligned} \tag{43}$$

With Eqs. (38), (39), (41), (42), $e\Phi/2\pi=N+\epsilon\gg 1$, $(ka)^2\ll\epsilon$, $(ka)^2\ll 1-\epsilon$ the following low-energy phase shifts are obtained:

$$\begin{aligned}\Delta_l^+=&-\frac{\pi}{\Gamma^2(W)}\left(\frac{ka}{2}\right)^{2W}\left\{\frac{2}{||\psi_l^0||^2(ka)^2}+\frac{1}{W}\right.\\&+\frac{1}{(1-W)||\psi_l^0||^2}+\frac{[4(W-1)^2(2-W)]^{-1}+2\gamma_l^{(4)}}{||\psi_l^0||^4}\\&\left.+O(ka)^2\right\}+O(ka)^{4W-4},\quad l=0,\ldots,N-2,\end{aligned} \tag{44}$$

$$\begin{aligned}\Delta_{N-1}^+=&-\frac{\pi}{\Gamma^2(1+\epsilon)}\left(\frac{ka}{2}\right)^{2+2\epsilon}\left\{\frac{2}{||\psi_{N-1}^0||^2(ka)^2}+\frac{1}{1+\epsilon}\right.\\&\left.-\frac{1}{\epsilon||\psi_{N-1}^0||^2}+\frac{[4\epsilon^2(1-\epsilon)]^{-1}+2\gamma_{N-1}^{(4)}}{||\psi_{N-1}^0||^4}\right\}\\&-\frac{\pi^2\cot\pi\epsilon}{4\Gamma^4(1+\epsilon)||\psi_{N-1}^0||^4}\left(\frac{ka}{2}\right)^{4\epsilon}+O(ka)^{6\epsilon},\end{aligned} \tag{45}$$

provided $\epsilon>1/|\ln(ka)|$;

$$\begin{aligned}\Delta_N^+=&\frac{\pi}{\Gamma^2(1-\epsilon)}\left[2\int_0^a\frac{dr}{a}\left(\frac{r}{a}\right)^{2N+1}e^{2e\varphi}+\frac{1}{\epsilon-1}\right]\left(\frac{ka}{2}\right)^{2-2\epsilon}\\&+O(ka)^{4-4\epsilon},\end{aligned} \tag{46}$$

provided $1-\epsilon>1/|\ln(ka)|$; and

$$\epsilon>(ka)^2\int_0^a\frac{dr}{a}\left(\frac{r}{a}\right)^{2N+1}e^{2e\varphi}. \tag{47}$$

This may seem impossible to satisfy for large e, but it turns out that the integral in Eq. (47) decreases as a power of N (see the Appendix). Continuing,

$$\begin{aligned}\Delta_l^+=&\frac{2\pi(ka/2)^{2W+2}}{\Gamma^2(1+W)}\int_0^a\frac{dr}{a}\left(\frac{r}{a}\right)^{2l+1}\\&\times e^{2e\varphi}+O(ka)^{2W+4},\quad l=N+1,N+2,\ldots,\end{aligned} \tag{48}$$

and

$$\begin{aligned}\Delta_l^+=&\frac{\pi(ka/2)^{2W}}{\Gamma^2(W)}\left[2\int_0^a\frac{dr}{r}\left(\frac{r}{a}\right)^{2|l|}e^{-2e\varphi}-\frac{1}{W}\right]\\&+O[(ka)^{4W},(ka)^{2W+2}],\quad l=-1,-2,\ldots.\end{aligned} \tag{49}$$

For negative chirality,

$$\begin{aligned}\Delta_l^-=&-\frac{\pi(ka/2)^{2W}}{\Gamma^2(W+1)}\left[2\int_0^a\frac{dr}{r}\left(\frac{r}{a}\right)^{2l}e^{2e\varphi}+\frac{1}{W}\right]^{-1}\\&+O[(ka)^{4W},(ka)^{2W+2}],\quad l=1,\ldots,N,\end{aligned} \tag{50}$$

$$\begin{aligned}\Delta_l^-=&\frac{\pi(ka/2)^{2W}}{\Gamma^2(W)}\left[2\int_0^a\frac{dr}{r}\left(\frac{r}{a}\right)^{2l}e^{2e\varphi}-\frac{1}{W}\right]\\&+O[(ka)^{4W},(ka)^{2W+2}],\quad l=N+1,N+2,\ldots,\end{aligned} \tag{51}$$

$$\begin{aligned}\Delta_l^-=&\frac{2\pi(ka/2)^{2W+2}}{\Gamma^2(W+1)}\int_0^a\frac{dr}{a}\left(\frac{r}{a}\right)^{2|l|+1}e^{-2e\varphi}\\&+O(ka)^{2W+4},\quad l=0,-1,\ldots.\end{aligned} \tag{52}$$

The negative values of $\Delta_l^\pm$ for $l=1,\ldots,N$ can be qualitatively understood as due to the repulsive barrier in V mentioned in Sec. II. The apparent poles in $\Delta_l^\pm$ when W is integral disappear when a careful limit is taken. For example, going back to the basic definition (38),

$$\lim_{\epsilon\to 0}\Delta_{N-1}^{+}=\frac{\pi}{2}\left[\ln(ka/2)+\gamma-\int_0^a\frac{dr}{a}\left(\frac{r}{a}\right)^{2N-1}e^{2e\varphi}\right]^{-1}+O\left(\frac{1}{\ln^3(ka)}\right),$$

$$\lim_{W\to 0}\Delta_l^{-}=\frac{\pi}{2\ln(ka)}+O\left(\frac{1}{\ln^2(ka)}\right), \tag{53}$$

where γ is Euler's constant. The general rule is that simple poles in $\Delta_l^\pm$ when W is integral are replaced with logarithms of the type $\ln(ka)$.

The main observation here is the presence of the $(ka)^{-2}$ factors in Δ_l^+ for $l=0,\ldots,N-1$ which cause each of the corresponding partial-wave Green's functions $\mathcal{G}_{+,l}(me^{i\pi/2},r)$ in Eq. (19) to develop a simple pole in m^2 at the origin. These are of course expected due to the N square-integrable zero modes of $\mathcal{H}_{+,l}$.

We have learned from this calculation that the precise form of these phase shifts is necessary if large cancellations are to go through in the calculation of the determinant. This is further discussed in Sec. IV.

IV. SMALL-MASS, STRONG-COUPLING EXPANSION OF lndet

Because of the rapid falloff of the low-energy phase shifts with l the sums and integrals in Eq. (26) can be interchanged. Using entries 5.54.2 of Ref. [21] and 1.12.3.3 of Ref. [22] one obtains

$$a^{-2}\int_a^\infty dr\,r\ln\left(\frac{r}{a}\right)K_W^2(mr)=\frac{1}{2}K_{W+1}(ma)K_W'(ma)-\frac{1}{2}K_W(ma)K_{W+1}'(ma)+\frac{W}{2ma}\left[K_{W+1}(ma)\frac{\partial}{\partial W}K_W(ma)-K_W(ma)\frac{\partial}{\partial W}K_{W+1}(ma)\right] \tag{54}$$

$$=\frac{\Gamma^2(W-1)}{16}\left(\frac{ma}{2}\right)^{-2W}+\frac{\pi}{16\sin\pi W}\left[2W\psi(W)+\pi W\cot\pi W-2W\ln\left(\frac{ma}{2}\right)-2W+1\right]\times\left(\frac{ma}{2}\right)^{-2}+\frac{\Gamma^2(W)}{8(1-W)(2-W)^2}\left(\frac{ma}{2}\right)^{2-2W}+O[(ma)^{4-2W},(ma)^0],\quad l\neq N \tag{55}$$

$$=\frac{\Gamma^2(\epsilon-1)}{16}\left(\frac{ma}{2}\right)^{-2\epsilon}+\frac{\pi}{16\sin\pi\epsilon}\left[2\epsilon\psi(\epsilon)+\pi\epsilon\cot\pi\epsilon-2\epsilon\ln\left(\frac{ma}{2}\right)-2\epsilon+1\right]\left(\frac{ma}{2}\right)^{-2}-\frac{\pi}{8\epsilon\sin\pi\epsilon}+\frac{\pi^2}{16\Gamma^2(2+\epsilon)(\sin\pi\epsilon)^2}\left(\frac{ma}{2}\right)^{2\epsilon}+\frac{\Gamma^2(\epsilon)}{8(1-\epsilon)(2-\epsilon)^2}\left(\frac{ma}{2}\right)^{2-2\epsilon}+O[(ma)^{2+2\epsilon},(ma)^{4-2\epsilon}],\quad l=N, \tag{56}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ and $0<\epsilon<1$. Apparent singularities in Eqs. (55) and (56) at integral values of W cancel when careful limits are taken. Also required are the following expansions [12]:

$$K_W(z)=\frac{1}{2}\Gamma(W)(z/2)^{-W}\left[1+\frac{(z/2)^2}{1-W}+O(z^4)\right]-\frac{\pi(z/2)^W}{2\Gamma(W+1)\sin\pi W}\left[1+\frac{(z/2)^2}{W+1}+O(z^4)\right] \tag{57}$$

and

$$I_W(z)K_W(z)=\frac{1}{2W}\left[1+\frac{z^2/2}{1-W^2}-\frac{\pi(z/2)^{2W}}{W\Gamma^2(W)\sin\pi W}+O(z^4,z^{2W+2})\right]. \tag{58}$$

The pole at $m^2=0$ in $\mathcal{G}_{+,l}(k=me^{i\pi/2},r)$ from the factors $(ka)^{-2}$ in Δ_l^+ in Eqs. (44) and (45) make the positive chirality terms for $l=0,\ldots,N-1$ in Eq. (26) the dominant ones when $ma\ll 1$. Using Eqs. (44)–(52), (55)–(58) and (28) when it makes sense—as discussed at the end of Sec. II and below—we obtain, from Eq. (26),

$$\frac{\partial}{\partial e}\ln\det = -2e\int_0^a dr\, r\varphi(r)B(r) + \sum_{l=0}^{N-1}\frac{\partial}{\partial e}\ln\|\psi_l^0\|^2 + \frac{\epsilon\Phi}{\pi}\ln(ma) - \frac{\Phi}{2\pi}[2\epsilon\psi(\epsilon)+\pi\epsilon\cot\pi\epsilon+1-2\epsilon + 2\epsilon\ln 2] + O[(ma)^{2\epsilon}\ln(ma),(ma)^{2-2\epsilon} \times\ln(ma),(ma)^2\ln(e\Phi)], \tag{59}$$

provided $|\ln(ma)|^{-1}<\epsilon<1-|\ln(ma)|^{-1}$. Recall that $e\Phi/2\pi = N+\epsilon$.

Regarding the remainder in Eq. (59), there are 12 cases to consider: positive/negative chirality, regions inside/outside the range of B, and the angular momentum ranges $l\leqslant -1$, $0\leqslant l\leqslant N$, $l\geqslant N+1$ for $e\Phi\gg 1$. The terms of order $(ma)^{2\epsilon}\ln(ma)$ and $(ma)^{2-2\epsilon}\ln(ma)$ come from positive chirality, $l=N,N-1$ for $r>a$. The term of order $(ma)^2\ln(e\Phi)$ comes from the $\int_0^a dr\,\varphi(r)R_{\pm,l}^2(r)\int_r^a ds/R_{\pm,l}^2(s)$ terms in Eq. (26) summed over values of l in the neighborhood of $-e\Phi/2\pi$. The presence of the factor $\ln(e\Phi)$ is tentative: there may be subtle cancellations between the positive and negative chirality sectors that will eliminate the logarithm. All of the $O(ma)^2$ remainder estimates are based on what we consider the worst case, namely, $B(r)\geqslant 0$, which causes $\varphi(r)$ to be positive and monotonically decreasing for $0\leqslant r<a$.

Our second comment on Eq. (59) concerns large individual terms in the mass expansion when $e\Phi\gg 1$. Consider the second term in Eq. (26) and the ratio $R_{+,l}(m^2,r)/R_{+,l}(m^2,a)$. As discussed in Sec. II B, $R_{+,l}$ can exponentially increase for $e\Phi\gg 1$ for $0\leqslant l\leqslant e\Phi/2\pi$. However, this ratio at $m^2=0$ $[R_+(0,a)=a^{l+1/2}]$ and its leading correction χ_l^+ in Eq. (31) are cancelled for each l by the third term in Eq. (26). It remains to understand these cancellations and to verify that they continue at order $(ma)^6$ and higher orders.

The terms $R_{+,l}^2(r)\int_r^a ds/R_{+,l}^2(s)$ for $0\leqslant l\leqslant N$ in Eq. (26) have not been expanded since there is no apparent cancellation mechanism. We have found that in one of the worst cases, when $B(r)=B$ for $r<a$ and zero otherwise, these terms when left unexpanded vanish as $e\Phi\to\infty$. For $l>N$ these terms remain bounded when expanded, and for $l\geqslant e\Phi/2\pi$ their leading l behavior is cancelled by the negative chirality sector since the distinction between the two chiralities disappears as $l\to\infty$.

In the exactly solvable case of a magnetic field confined to the surface of a cylinder the mass-dependent terms remain subdominant when $e\Phi\gg 1$ [2]. The study of the cancellation of large terms and the vanishing of ratios of large terms when $e\Phi\to\infty$ is still at a preliminary stage. The control of these terms has much to teach us about the nonperturbative structure of lndet.

Finally, we have previously shown that, for $e\Phi$ fixed and $ma\ll 1$,

$$\ln\det = \frac{|e\Phi|}{2\pi}\ln(ma) + R(m), \tag{60}$$

where $\lim_{m=0}[R/\ln(ma)]=0$ [23]. Now consider the case when $\epsilon=0$ and $e\Phi/2\pi=N$. Then the dominant mass-dependent term in Eq. (26) for $ma\ll 1$ occurs at $l=N-1$, $r>a$:

$$\frac{\partial}{\partial e}\ln\det_{N-1} = \frac{2m^2\Phi}{\pi^2}(\Delta_{N-1}^+ + i\Delta_{N-1}^{+2} - \Delta_{N-1}^- + \cdots)\int_a^\infty dr\, r\ln\left(\frac{r}{a}\right)K_1^2(mr), \tag{61}$$

where $\Delta_{N-1}^{\pm}$ are continued to $k=me^{i\pi/2}$. From Eqs. (50), (53) and

$$a^{-2}\int_a^\infty dr\, r\ln\left(\frac{r}{a}\right)K_1^2(mr) = \frac{[\ln(ma/2)+\gamma]^2+\ln(ma/2)+\gamma+1}{2(ma)^2} + \frac{1}{4}\ln\left(\frac{ma}{2}\right)+\frac{\gamma}{4}-\frac{3}{8}+O[(ma)^2\ln^2(ma)], \tag{62}$$

one gets

$$\frac{\partial}{\partial e}\ln\det_{N-1} = \frac{\Phi}{2\pi}\ln(ma)+O(1), \tag{63}$$

in accord with Eq. (60).

Next, consider the case when $e\Phi/2\pi=N+\epsilon$, $0<\epsilon\leqslant 1$. As $\epsilon\to 1$ a pole at $m^2=0$ begins to develop in $\mathcal{G}_{+,N}$ and $\Delta_N^+(k)\sim\pi/[2\ln(ka)]$. For $e\Phi/2\pi=N+1$ we find

$$\frac{\partial}{\partial e}\ln\det_N = \frac{\Phi}{2\pi}\ln(ma)+O(1), \tag{64}$$

again in accord with Eq. (60). Moreover, the same result is obtained in the limit $\epsilon\to 1$.

In the interval $0<\epsilon<1$ the $(\epsilon\Phi/\pi)\ln(ma)$ term in Eq. (59) comes from the $l=N$, $r>a$ contribution to $\partial\ln\det/\partial e$. This term contradicts Eq. (60) which was derived by holding $e\Phi$ fixed and letting $ma\to 0$. Here we are setting $ma\ll 1$, and then letting $e\Phi$ increase indefinitely. By taking limits in this way the $\ln(ma)$ term becomes an infinitesimal addition to lndet when compared to its growth due to the pileup of normalizable zero modes as $e\Phi$ increases, as we will see in Sec. V. For the present it is assumed that there are other infinitesimal terms not yet found that will result in the shift $(\epsilon\Phi/2\pi)\ln(ma)\to(\Phi/2\pi)\ln(ma)$ in the range of ϵ indicated.

We are confident that Eq. (60) is the leading mass-dependent term in lndet, and it will accordingly be added on to our strong coupling result for lndet in Sec. V.

V. SMALL-MASS, STRONG-COUPLING LIMIT OF lndet

Up to now we have assumed that $B(r)$ is square integrable, centrally symmetric and finite ranged. Further analytic analysis of Eq. (59) requires additional assumptions, namely $B(r)\geqslant 0$ with continuous first and second derivatives. Then we can show that for $e\Phi\to\infty$, the first term in Eq. (59) is cancelled by the zero modes contributing to the second term.

The demonstration is straightforward. Refer to Eqs. (59), (36) and the first lines of Eqs. (43) and (29) and obtain

$$\sum_{l=0}^{N-1}\frac{\partial}{\partial e}\ln\|\psi_l^0\|^2=2\int_0^\infty dr\,r\varphi(r)\sum_{l=0}^{N-1}\frac{r^{2l}e^{2e\varphi(r)}}{\int_0^\infty ds\,s^{2l+1}e^{2e\varphi(s)}}. \tag{65}$$

Now make use of the following theorem of Erdös [24], specialized here to the case of central symmetry: Let $B(r)\geqslant 0$ be a compactly supported magnetic field with a continuous first derivative. Define the ground-state density function

$$P(r)=\sum_{l=0}^{N-1}\frac{r^{2l}e^{2e\varphi(r)}}{\int_0^\infty ds\,s^{2l+1}e^{2e\varphi(s)}}. \tag{66}$$

Then $P(r)/e$ converges to $B(r)$ in L^p for any $1\leqslant p<\infty$ as $e\to\infty$. According to this theorem

$$\sum_{l=0}^{N-1}\frac{\partial}{\partial e}\ln\|\psi_l^0\|^2=2e\int_0^a dr\,r\varphi(r)B(r)+R(e), \tag{67}$$

for $e\Phi\gg 1$ and where $\lim_{e\to\infty}R(e)/e=0$. The r integral in Eq. (65) cuts off due to the finite range of B. Hence, Eq. (67) leads to the promised cancellation in Eq. (59).

The really interesting question now is what is the remainder in Eq. (67)? Erdös' theorem is not yet sharp enough to state what it is. It had better be negative to be in accord with the diamagnetic upper bound in Eq. (30). In the Appendix we investigate this problem by the method of steepest descents assuming $B(r)>0$ with two alternative sets of boundary conditions: $B(a)=0$, $\lim_{r\to a-}B'(r)<0$, and $B(a)>0$. The result in both cases is

$$\lim_{|e\Phi|\gg 1}\ \lim_{ma\ll 1}\ \ln\det=-\frac{|e\Phi|}{4\pi}\ln\left(\frac{|e\Phi|}{(ma)^2}\right)+O(|e\Phi|,(ma)^2|e\Phi|\ln(|e\Phi|)). \tag{68}$$

The case when $eB<0$ is the mirror image of the $eB>0$ case, and so we need only insert absolute value signs to cover both cases. As discussed in Sec. IV, we have inserted the mass-dependent term from Ref. [23]. Comparing Eq. (68) with the constant field result

$$\ln\det=-\frac{eBV}{4\pi}\ln\left(\frac{eB}{m^2}\right)+O(eB), \tag{69}$$

we see that they are formally in accord on setting $V=\pi a^2\to\infty$. Of course we cannot say anything about the remaining mass-dependent terms in Eq. (68) in this limit.

The minus sign in Eq. (68) is a reflection of the paramagnetism of charged fermions in a magnetic field. This is most clearly seen with Schwinger's proper time definition of the determinant [25], namely

$$\ln\det=\frac{1}{2}\int_0^\infty\frac{dt}{t}e^{-tm^2}\mathrm{Tr}[e^{-P^2t}-\exp\{-[(P-eA)^2-\sigma_3B]t\}]. \tag{70}$$

Noting the minus sign in Eq. (68), Eq. (70) indicates that on average the spectrum of the Pauli operator is lowered by B relative to the field-free case. Therefore, the current usage of "diamagnetic" bound to describe the right-hand side of Eq. (30) is a misnomer. The factor $|e\Phi|$ in Eq. (68) multiplying the logarithm is related to the counting of zero modes. More will be said about the physics of Eq. (68) in Sec. VI.

The discussion of the remainder in Eq. (59) in Sec. IV means that we cannot rule out the subdominant term $(ma)^2|e\Phi|\ln(|e\Phi|)$ in Eq. (68); more detailed analysis is required to exclude the $\ln(|e\Phi|)$ factor.

The remarkable thing about Eq. (68) is that the limit is universal for a broad class of fields. Since it only depends on a global property of the background magnetic field—its total flux—we suspect that Eq. (68) is also the limit in the general case of non-central, square-integrable fields.

Finally, the case of zero-flux background fields has not been considered in the literature to the author's knowledge except for the case of massless QED_2 on a torus [4] and a sphere [26]. Our limit seems to indicate that when $\Phi=0$ there are no square-integrable zero modes and hence no mechanism to cancel the first term in Eq. (59). In this case one might suppose that it is this term—the Schwinger term—that is dominant in the small-mass, strong-coupling limit. This is the result in [4].

VI. DUALITY

The purpose of this section is to relate the Euclidean determinant of QED_2 and some of the results of the previous sections to physics in four dimensions. The term duality as used in this section is distinct from Olive-Montonen electric-magnetic duality [27]. It is rather a duality closely related to the analyticity of the one-loop effective action of QED in two and four dimensions.

The Euclidean determinants in QED_4 and QED_2 for the background magnetic field $B=(0,0,B(x_1,x_2))$ are related by

$$-2\pi\frac{\partial}{\partial m^2}\ln\det{}_{\mathrm{QED}_4}=L_3L_4\ln\det{}_{\mathrm{QED}_2}+\frac{L_3L_4\|B\|^2e^2}{12\pi m^2}, \tag{71}$$

where $\|B\|^2=\int dx_1dx_2B^2(x_1,x_2)$, L_3L_4 is the volume of the space-time box for x_3 and x_4, and on-shell charge renormalization is used [6]. Hence B must be at least square integrable in what follows. Assuming one can rotate energy contours in the usual way, continue $\ln\det_{\mathrm{QED}_4}$ to the Lorentz metric by letting $\gamma_4 \to i\gamma_0$, $x_4 \to e^{i(\pi/2-\epsilon)}t$, $\epsilon \to 0+$ and $L_4 \to iT$. On the right $\det_{\mathrm{QED}_2}$ remains a Euclidean determinant and so Eq. (71) now becomes

$$-2\pi\frac{\partial}{\partial m^2}\ln\det{}^L_{\mathrm{QED}_4}(B) = iL_3\ln\det{}^E_{\mathrm{QED}_2}(B) + \frac{iL_3T||B||^2e^2}{12\pi m^2}, \tag{72}$$

with the superscripts E and L denoting Euclidean and Lorentz metrics, respectively. Therefore, given $\det^E_{\mathrm{QED}_2}(B)$ we can calculate $\det^L_{\mathrm{QED}_4}(B)$ for a general unidirectional magnetic field $B(\mathbf{r})$ by integrating Eq. (72) over m^2 as described in Ref. [6].

Now make the duality transformation from the static magnetic field $B(x_1,x_2)$ to the functionally equivalent electric field $E(x_3,t)$ by letting

$$\mathbf{A} = (A_1(x_1,x_2), A_2(x_1,x_2), 0) \to (0,0,A_3(x_3,t)), \tag{73}$$

with $\boldsymbol{\nabla}\times\mathbf{A} = B(x_1,x_2)\hat{\mathbf{k}}$, $\mathbf{E} = -\dot{A}_3\hat{\mathbf{k}} = B(x_3,t)\hat{\mathbf{k}}$ and

$$A_3(x_3,t) = -\int_{t_0}^{t} ds B(x_3,s). \tag{74}$$

A change in t_0 in Eq. (74) results in a gauge transformation and does not affect the determinant. This duality transformation is implemented by the replacement $B(x_1,x_2) \to e^{-i\pi/2}E(x_3,x_4)$ in $\det^E_{\mathrm{QED}_4}$, $\det^E_{\mathrm{QED}_2}$ and $||B||$ in Eq. (71) and the coordinate/momentum relabeling $1\leftrightarrow 3$, $2\leftrightarrow 4$, followed by continuation to the Lorentz metric, including $b \to e^{i\pi/2}\tau$, where b is the range of B in the x_2 direction, and 2τ is the duration of the electric pulse $E(x_3,t)$. An example is given in Eq. (77) below. If B has more than one range parameter in the x_2 direction then all of them must be continued as b. The rule $B\to e^{-i\pi/2}E$ in going from the Euclidean metric back to the Lorentz metric is a consequence of the definition of E above and the rotation $x_4 \to e^{i(\pi/2-\epsilon)}t$. Ultimately it is rooted in the fundamental prescription $m^2 \to m^2 - i\epsilon$. Then Eq. (71) becomes

$$\begin{aligned} -2\pi\frac{\partial}{\partial m^2}&\ln\det{}^{E\to L}_{\mathrm{QED}_4}(B\to e^{-i\pi/2}E) \\ &= L_1L_2\ln\det{}^{E\to L}_{\mathrm{QED}_2}(B\to e^{-i\pi/2}E) \\ &\quad -\frac{iL_1L_2||E||^2e^2}{12\pi m^2}. \end{aligned} \tag{75}$$

As an example consider the last terms in Eqs. (71) and (75) for the case of a magnetic field in a closed region with two range parameters:

$$B(x_1,x_2) = Bf\left(\frac{x_1}{a},\frac{x_2}{b}\right),$$

$$E(x_3,t) = Bf\left(\frac{x_3}{a},\frac{t}{\tau}\right), \tag{76}$$

with $B(x_1=\pm a, x_2) = B(x_1, x_2=\pm b) = 0$. Following the above rules

$$\begin{aligned} ||B||^2 &= B^2\int_{-a}^{a}dx_1\int_{bg_1(x_1/a)}^{bg_2(x_1/a)}dx_2 f^2\left(\frac{x_1}{a},\frac{x_2}{b}\right) \\ &\to -B^2\int_{-a}^{a}dx_3\int_{i\tau g_1(x_3/a)}^{i\tau g_2(x_3/a)}dx_4 f^2\left(\frac{x_3}{a},\frac{x_4}{e^{i\pi/2}\tau}\right) \\ &= -iB^2\int_{-a}^{a}dx_3\int_{\tau g_1(x_3/a)}^{\tau g_2(x_3/a)}dt f^2\left(\frac{x_3}{a},\frac{t}{\tau}\right) \\ &= -i||E||^2, \end{aligned} \tag{77}$$

where $x_2 = bg_i(x_1/a)$, $t = \tau g_i(x_3/a)$, $i=1,2$, define the boundaries of B and E.

Equation (75) may seem to give nothing new, at least when developed in a power-series expansion in E. Its real power enters when $\ln\det^E_{\mathrm{QED}_2}(B)$ is known nonperturbatively as we will now see. Defining the one-loop Lorentz metric effective action by $S_{\mathrm{eff}} = -i\ \ln\det$, Eq. (75) gives

$$\begin{aligned} -2\pi\frac{\partial}{\partial m^2}S^{\mathrm{QED}_4}_{\mathrm{eff}}(E) &= -iL_1L_2\ln\det{}^{E\to L}_{\mathrm{QED}_2}(B\to e^{-i\pi/2}E) \\ &\quad -\frac{L_1L_2||E||^2e^2}{12\pi m^2}. \end{aligned} \tag{78}$$

As an example, consider the finite-range magnetic field

$$B(x_1,x_2) = \left(1-\frac{x_1^2+x_2^2}{a^2}\right)B, \quad x_1^2+x_2^2 \leqslant a^2, \tag{79}$$

and the corresponding electric pulse

$$E(x_3,t) = \left(1-\frac{x_3^2+(ct)^2}{a^2}\right)E, \quad x_3^2+(ct)^2 \leqslant a^2, \tag{80}$$

where B and E are constants, $\Phi = \pi a^2B/2$ and $c\tau = a$. Both fields are directed along the z axis. For $ma\to 0$ and $e\Phi \gg 1$ we found the result (68) for $\ln\det^E_{\mathrm{QED}_2}(B)$. Then following the above rules set

$$\begin{aligned} \ln\det{}^{E\to L}_{\mathrm{QED}_2}&(B\to e^{-i\pi/2}E) \\ &= -\frac{(e\pi a/2)(e^{i\pi/2}\tau)(e^{-i\pi/2}E)}{4\pi}\ln\left(\frac{e\pi e^{-i\pi/2}E}{m^2}\right) \\ &\quad + O(eE) = -\frac{e\pi a\tau E}{8\pi}\ln\left(\frac{eE}{m^2}\right) + O(eE), \end{aligned} \tag{81}$$

where corrections of $O((ma)^2)$ have been ignored. Substituting Eq. (81) in Eq. (78) gives for $eE \gg m^2$

$$2\pi\frac{\partial}{\partial m^2}\operatorname{Im}S_{\text{eff}}^{\text{QED}_4}=-\frac{e\pi a\tau L_1L_2E}{8\pi}\ln\left(\frac{eE}{m^2}\right)+O(eE). \tag{82}$$

As far as we know there is nothing in the literature to directly check Eq. (82) with, or any other class of electric two-variable pulses.

The minus sign in Eq. (82) is universal for the class of fields and their dual pulses considered in this paper. We now see that the physically reasonable result that the pair production probability $1-\exp(-2\operatorname{Im}S_{\text{eff}})$ decreases with increasing fermion mass depends on the paramagnetism of charged fermions in a magnetic field, as indicated by the minus sign in Eq. (68) and discussed afterwards. We take this as direct physical evidence for the validity, at least in the strong-coupling, low-mass domain, of the ‘‘diamagnetic’’ bound on the Euclidean determinant, namely $\det^E_{\text{QED}_2}\leqslant 1$.

The diamagnetic bound also holds in the perturbative domain of large mass and weak coupling since the power series expansion of $\ln\det^E_{\text{QED}_2}$ is asymptotic and the overall sign of the second-order term is negative [14].

A mechanical device that would simulate the pulses implied by the duality transforms on centrally symmetric magnetic fields would be two parallel conducting plates of large extent initially very close together, then pulled apart and then pushed together again. These plates have the unusual property of having opposite surface-charge densities varying with time and their spatial separation.

Duality has been considered recently by Dunne and Hall [28] for nonconstant fields in their study of the exactly solvable single-variable magnetic field $B(x)=B\,\text{sech}^2(x/\lambda)$. Although the asymptotic boundary conditions are different in the magnetic and electric field cases, they allow the analytic continuations required for duality in this example. In a later paper [29] they go beyond exactly solvable background fields by using a WKB approach to approximate the spectrum of the Pauli operator $(\not{P}-e\not{A})^2$. The authors are aware that such an approach cannot prove duality in the single-variable case, but it does give an insight into just how nontrivial duality is. Presumably the final justification of duality in both the one- and two-variable cases is the validity of the Wick rotation in the presence of external fields.

The question arises as to whether there is a duality transformation of the type $B(x_1,x_2)\to e^{-i\pi/2}E(x_1,x_2)$, where E is directed along the third axis. The answer is ‘‘no’’ except for the special case when $E(x_1,x_2)$ is constant within the boundary parallel to the direction of the field. Otherwise, the Bianchi identity excludes such fields. So for B constant over a finite spatial region, duality takes the simple form, from Eq. (72),

$$-2\pi\frac{\partial}{\partial m^2}S_{\text{eff}}^{\text{QED}_4}(B\to e^{-i\pi/2}E)$$
$$=L_3T\ln\det^E_{\text{QED}_2}(B\to e^{-i\pi/2}E)-\frac{L_3T\|E\|^2e^2}{12\pi m^2}. \tag{83}$$

The determinant $\det_{\text{QED}_2}$ retains its Euclidean metric since the background field is static. For the case of a circular boundary of radius a (83) can be checked since there is a reliable semiclassical approximation that is valid for $a^2eE\gg\pi$, namely [30]

$$\operatorname{Im}S_{\text{eff}}^{\text{QED}_4}=\frac{L_3Te^2E^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n\pi m^2/eE}\times\left\{\pi a^2-\pi^2a\left(\frac{n}{eE}\right)^{1/2}\text{erf}\left[a\left(\frac{eE}{n\pi}\right)^{1/2}\right]+\frac{n\pi^2}{eE}(1-e^{-a^2eE/n\pi})\right\}. \tag{84}$$

Then for $a^2eE\gg\pi\gg m^2a^2$,

$$-2\pi\frac{\partial}{\partial m^2}\operatorname{Im}S_{\text{eff}}^{\text{QED}_4}=\frac{L_3T\pi a^2eE}{4\pi}[\ln(eE/m^2)+O(1)], \tag{85}$$

which agrees by inspection with Eq. (83) when combined with Eq. (68), taking $e\Phi>0$ and letting $\Phi=\pi a^2B\to\pi a^2e^{-i\pi/2}E$.

VII. SUMMARY

An exact representation of the Euclidean fermion determinant in two dimensions for centrally symmetric, finite-ranged Abelian background gauge fields has been obtained that depends only on the interior partial-wave functions and scattering phase shifts continued to the upper k plane by setting $k=me^{i\pi/2}$, where m is the fermion mass. In the nonperturbative limit of small fermion mass these are known explicitly, thereby making the determinant amenable to numerical analysis. For the sequence of limits of small fermion mass followed by strong coupling we have been able to obtain the explicit asymptotic limit of the determinant when the background field is unidirectional and nonvanishing except on its boundary. The result is universal, depending only on the two-dimensional chiral anomaly $e\Phi/2\pi$. It should be an easy task to obtain the determinant's asymptotic limit for fluctuating magnetic fields since one only needs to numerically evaluate the second term in Eq. (59). These results should be a useful nonperturbative check on lattice algorithms for fermion determinants when the output is extrapolated to infinite volume and zero lattice spacing.

By extending the concept of duality to two variables we have been able to relate the Euclidean determinant in two dimensions for a wide class of background magnetic fields to the pair production probability in four dimensions for a related class of electric pulses. We have also connected the ‘‘diamagnetic’’ bound on the Euclidean two-dimensional determinant to the negative sign of $\partial\operatorname{Im}S_{\text{eff}}/\partial m^2$ in four dimensions, thereby providing a physical basis for this bound in the strong-coupling, small-mass limit.

Central to this work was the ability to count zero modes in two dimensions. Further analytic progress in three and four dimensions will be hindered, if not blocked, until there are theorems for counting zero modes. In four dimensions more is needed than just the difference of positive and negative chirality zero modes, while in three dimensions there may be some as yet undiscovered topological invariant that will count them.

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APPENDIX

Here we will derive the asymptotic limit (68). Referring to Eq. (43) let

$$I=\int_0^a \frac{dr}{a}\left(\frac{r}{a}\right)^{2l+1} e^{2e\varphi(r)}. \tag{A1}$$

Then,

$$\frac{\partial}{\partial e}\ln\|\psi_l^0\|^2=\frac{\partial}{\partial e}\ln I-\frac{\Phi}{2\pi(W-1)}+\frac{\Phi}{2\pi}\left(W-1+\frac{1}{2I}\right)^{-1} -\frac{1}{2I}\left(W-1+\frac{1}{2I}\right)^{-1}\frac{\partial}{\partial e}\ln I. \tag{A2}$$

Consider the first term in Eq. (A2). Referring to Eq. (59) consider

$$\sum_{l=0}^{N-1}\frac{\partial}{\partial e}\ln I=\sum_{l=0}^{\Lambda}\frac{\partial}{\partial e}\ln I+\sum_{l=\Lambda+1}^{N-1}\frac{\partial}{\partial e}\ln I, \tag{A3}$$

where $\Lambda \gg 1$ and where for $l \leqslant N-1$, $W=e\Phi/(2\pi)-l=N+\epsilon-l$. Refer to the first sum in Eq. (A3). By inspection of Eq. (A1), $I(l=0)=O(e^{2eM})$, where $M=\max\varphi(r)$, $0\leqslant r\leqslant a$, with $\varphi(r)$ given by Eq. (20). Hence, $\partial \ln I/\partial e=O(M)$. For $l=O(N-\gamma N)$, where $\gamma \ll 1$ we find later on in Eq. (A40) with $m=N-l-1=O(\gamma N)$ that $\partial \ln I/\partial e=O(\sqrt{\gamma})$. These two results indicate that I has exponential growth in e for this range of l. Thus, $\partial \ln I/\partial e=O(1)$ or less for $0\leqslant l\leqslant\Lambda$ and

$$\sum_{l=0}^{\Lambda}\frac{\partial}{\partial e}\ln I=O(\Lambda). \tag{A4}$$

Now for the second sum in Eq. (A3). For Λ large enough we can use the method of steepest descents to calculate I except near the point $l=N-1$. Referring to Eq. (A1), let

$$f(r)=(2l+1)\ln\left(\frac{r}{a}\right)+2e\varphi(r). \tag{A5}$$

Assume $B(r)>0$ so that $\Phi(r)$ given by Eq. (5) is monotonically increasing with r. Then $f(r)$ is maximized at point r^* for which

$$l+\frac{1}{2}=e\Phi(r^*)/2\pi, \tag{A6}$$

since $f''(r^*)=-2eB(r^*)<0$. Hence for $l\gg 1$,

$$I=\sqrt{\frac{2\pi}{a^2|f''(r^*)|}}e^{f(r^*)}[1+O(1/N)]. \tag{A7}$$

To calculate the point r^* for each admissible l, note that for $l\to N$, $r^*\to a$. So expand the right-hand side of Eq. (A6) about $r^*=a$ by setting $r^*=a(1-\delta)$. Let $l=N-m-1$, $m\gg 1$, $m\ll N$. Assuming $B(r)$ has continuous first and second derivatives with $B(a)=0$ and $B'(a)<0$ then $\delta=\{2/[a^3|B'(a)|]\}^{1/2}(m/e)^{1/2}+O(m/e)$ and

$$f(r^*)=\frac{4}{3}\left(\frac{\Phi}{\pi a^3|B'(a)|}\right)^{1/2}\frac{m^{3/2}}{\sqrt{N}}+O\left(\frac{m}{N}\right)^{1/2}, \tag{A8}$$

and

$$|f''(r^*)|=4\left(\frac{\pi|B'(a)|}{a\Phi}\right)^{1/2}\sqrt{mN}\left[1+O\left(\frac{m}{N}\right)^{1/2}\right]. \tag{A9}$$

Inserting Eqs. (A8),(A9) in Eq. (A7) gives for $m\ll N$

$$I=\left(\frac{\pi\Phi}{4a^3|B'(a)|}\right)^{1/4}(mN)^{-1/4}\exp\left[\frac{4}{3}\left(\frac{\Phi}{\pi a^3|B'(a)|}\right)^{1/2}\frac{m^{3/2}}{\sqrt{N}} +O\left(\frac{m}{N}\right)^{1/2}\right]\left[1+O\left(\frac{m}{N}\right)^{1/2}\right]. \tag{A10}$$

By definition (A1), $\partial I/\partial m>0$ for $0\leqslant m\leqslant N-1$. This will be true for the estimate (A10) provided

$$m>\left(\frac{\pi a^3|B'(a)|}{64\Phi}\right)^{1/3}N^{1/3}\equiv CN^{1/3}, \tag{A11}$$

in addition to $m\ll N$.

Now return to the second sum in Eq. (A3) and write it as the following sum using Eq. (A7):

$$\sum_{l=\Lambda+1}^{N-1}\frac{\partial}{\partial e}\ln I=\left(\sum_{l=\Lambda+1}^{N-CN^{1/3}}+\sum_{m=0}^{CN^{1/3}}\right)\frac{\partial}{\partial e}\ln I \tag{A12}$$

$$=\sum_{l=\Lambda+1}^{N-CN^{1/3}}\frac{\partial}{\partial e}f(r_l^*)-\frac{1}{2}\sum_{l=\Lambda+1}^{N-CN^{1/3}}\frac{\partial}{\partial e}|f''(r_l^*)| +\sum_{m=0}^{CN^{1/3}}\frac{\partial}{\partial e}\ln I+O\left(\frac{1}{N^2}\right). \tag{A13}$$

Consider the first term in Eq. (A13). We need not rely on Eq. (A8) yet because Eq. (A7) holds irrespective of where the roots r_l^* of Eq. (A6) lie in $(0,a)$. The important point is that they are closely spaced over the entire interval $(0,a)$ for $e\Phi/2\pi\gg 1$ and for $\Phi(r)$ monotonically increasing with r. Hence, the r_l^* can be considered to be nearly continuous across $(0,a)$ for l in the range indicated with

$$dl=\frac{e}{2\pi}\frac{d}{dr_l^*}\Phi(r_l^*)dr_l^*=eB(r_l^*)r_l^*dr_l^*. \tag{A14}$$

Referring to Eqs. (A5), (A6), and (20),

$$\frac{\partial}{\partial e}f(r_l^*)=2\varphi(r_l^*), \tag{A15}$$

and so

$$\sum_{l=\Lambda+1}^{N-CN^{1/3}}\frac{\partial}{\partial e}f(r_l^*)=2\sum_{l=\Lambda+1}^{N-CN^{1/3}}\varphi(r_l^*)$$
$$=2e\int_0^a dr^*\ \ r^*B(r^*)\varphi(r^*)+O(1). \tag{A16}$$

When Eqs. (A16), (A13), (A2) are combined we already see the promised cancellation of the first term in Eq. (59), as guaranteed by Erdös' theorem [24]. We now turn to the calculation of the remainder.

Consider the second sum in Eq. (A13) and break it up into two sums:

$$\sum_{l=\Lambda+1}^{N-CN^{1/3}}\frac{\partial}{\partial e}\ln|f''(r_l^*)|=\left(\sum_{l=\Lambda+1}^{(1-\gamma)N}+\sum_{l=(1-\gamma)N}^{N-CN^{1/3}}\right)\frac{\partial}{\partial e}\ln|f''(r_l^*)|, \tag{A17}$$

where $\gamma\ll 1$. Now deal with the first sum and recall that $f''(r_l^*)=-2eB(r_l^*)$. From Eq. (A6) for $l=\Lambda+1$, $\Phi(r_l^*)/\Phi=(\Lambda+\frac{1}{2})/(N+\epsilon)$ which implies $r_l^*\gtrsim 0$ for $N\gg\Lambda$, and hence $f''(r_l^*)\simeq -2eB(0)$. For the upper limit $l=(1-\gamma)N$, Eq. (A6) gives $\Phi(r_l^*)/\Phi=1-\gamma+O(1/N)$ and hence $r_l^*\lesssim a$. So

$$f''(r_l^*)=-2eB(a)-2eB'(a)(r_l^*-a)+O(r_l^*-a)^2$$
$$=-2e|B'(a)|(a-r_l^*)+O(r_l^*-a)^2 \tag{A18}$$

and

$$\Phi(r_l^*)=\Phi+2\pi aB(a)(r_l^*-a)+\pi[B(a)+aB'(a)]$$
$$\times(r_l^*-a)^2+O(r_l^*-a)^3$$
$$=(1-\gamma)\Phi+O(1/N), \tag{A19}$$

and so $a-r_l^*=\{\gamma\Phi/[\pi a|B'(a)|]\}^{1/2}$. Substituting this result into Eq. (A18) gives

$$f''(r_l^*)=-2e\left(\frac{\gamma\Phi|B'(a)|}{\pi a}\right)^{1/2}+O(e\gamma). \tag{A20}$$

Thus $f''(r_l^*)=O(e)$ for $\Lambda+1<l<(1-\gamma)N$ and so the first sum in Eq. (A17) gives a contribution of $O(1)$.

Next consider the second term in Eq. (A17). With $l=N-m-1$,

$$\sum_{l=(1-\gamma)N}^{N-CN^{1/3}}\frac{\partial}{\partial e}\ln|f''(r_l^*)|=\sum_{m=CN^{1/3}}^{\gamma N}\frac{\partial}{\partial e}\ln|f''(r_l^*)|. \tag{A21}$$

The range of m in (A21) is such that (A9) is valid so that

$$\sum_{l=(1-\gamma)N}^{N-CN^{1/3}}\frac{\partial}{\partial e}\ln|f''(r_l^*)|=\frac{\Phi}{4\pi}\sum_{CN^{1/3}}^{\gamma N}\frac{1}{m}+O(1)$$
$$=\frac{\Phi}{6\pi}\ln N+O(1). \tag{A22}$$

This completes the sum in Eq. (A17) and the second sum in Eq. (A13).

Finally, consider the last sum in Eq. (A13). This requires that I be estimated near the end point $l=N-1$ or $m=0$. For $N\gg 1$ and with $\varphi(r)$ monotonically decreasing to zero $[\varphi'(r)=-\Phi(r)/(2\pi r)]$, the integral in Eq. (A1) is dominated near $r=a$. Since $\varphi'(a)\neq 0$, $\varphi(r)$ has a first-order zero at $r=a$: $\varphi(r)\sim(1-r/a)\Phi/2\pi$, $r\rightarrow a$. Hence, for $N\gg 1$

$$I(m=0)\sim 2^{-2N}(N+\epsilon)^{-2N}e^{2(N+\epsilon)}\int_0^{2(N+\epsilon)}dx x^{2N-1}e^{-x}. \tag{A23}$$

But

$$\int_0^{2(N+\epsilon)}dx x^{2N-1}e^{-x}=(2N-1)!-\Gamma(2N,2(N+\epsilon)), \tag{A24}$$

where $\Gamma(a,x)$ is the incomplete gamma function given by entry 6.5.3 in [12]. Using entries 8.356.2 in [21] and 6.5.35 in [12],

$$\Gamma(2N,2(N+\epsilon))=e^{-2N}(2N)^{2N-1}[\sqrt{\pi N}+O(1)]. \tag{A25}$$

Combining Eqs. (A23)–(A25) with Stirling's formula gives

$$I(m=0)\sim\frac{1}{2}\sqrt{\frac{\pi}{N}}[1+O(1/\sqrt{N})],\quad N\gg 1. \tag{A26}$$

This is an overestimate as we integrated over all of the range $[0,a]$ instead of a patch near $r=a$, and therefore the factor $\sqrt{\pi}/2$ in Eq. (A26) cannot be trusted. However, the result demonstrates that $I(m=0)$ falls off as a power of N and not exponentially. Since $I(m=0)<I(m=CN^{1/3})$ and $I(m=CN^{1/3})=O(N^{-1/3})$ we can state that $\partial\ln I/\partial e=O(1/N)$ for $0\leq m\leq CN^{1/3}$ and so

$$\sum_{m=0}^{CN^{1/3}}\frac{\partial}{\partial e}\ln I=O(N^{-2/3}). \tag{A27}$$

Combining Eqs. (A3), (A4), (A13), (A16), (A17), (A21), (A22), (A27) and intermediate results gives

$$\sum_{l=0}^{N-1}\frac{\partial}{\partial e}\ln I=2e\int_0^a dr\,rB(r)\varphi(r)-\frac{\Phi}{12\pi}\ln\left(\frac{e\Phi}{2\pi}\right)+O(\Lambda),\tag{A28}$$

where $\Lambda\gg 1$ but e independent. This completes the sum of the first term in Eq. (A2).

The sum of the second term in Eq. (A2) is straightforward:

$$\begin{aligned}\sum_{l=0}^{N-1}\frac{1}{W-1}&=\sum_{0}^{N-1}\frac{1}{N+\epsilon-l-1}\\&=\sum_{m=0}^{N-1}\frac{1}{m+\epsilon}\\&=\ln\left(\frac{e\Phi}{2\pi}\right)+O(1).\end{aligned}\tag{A29}$$

Now consider the sum of the third term in Eq. (A2). Letting $m=N-l-1$,

$$\sum_{l=0}^{N-1}\left(\frac{e\Phi}{2\pi}-l-1+\frac{1}{2I}\right)^{-1}=\left(\sum_{m=0}^{CN^{1/3}}+\sum_{CN^{1/3}}^{\gamma N}+\sum_{\gamma N}^{N-1}\right)\times[m+\epsilon+g(m)]^{-1},\tag{A30}$$

where $1/g(m)=2I$ and $g'(m)<0$ for $0\leqslant m\leqslant N-1$, $\gamma\ll 1$, and C is given by Eq. (A11). Consider the first sum:

$$\begin{aligned}\sum_{m=0}^{CN^{1/3}}[m+\epsilon+g(m)]^{-1}&<\sum_{m=0}^{CN^{1/3}}[m+\epsilon+g(CN^{1/3})]^{-1}\\&<\ln\left[\frac{CN^{1/3}+g(CN^{1/3})+\epsilon}{g(CN^{1/3})+\epsilon}\right]\\&\quad+O\left(\frac{1}{g(CN^{1/3})}\right)\\&=O(1),\end{aligned}\tag{A31}$$

since by the definition of g and Eq. (A10), $g(CN^{1/3})=O(N^{1/3})$.

Next consider the second sum in Eq. (A30). Since $g'(m)<0$ we have

$$\begin{aligned}\sum_{CN^{1/3}}^{\gamma N}[m+\epsilon+g(CN^{1/3})]^{-1}&<\sum_{CN^{1/3}}^{\gamma N}[m+\epsilon+g(m)]^{-1}\\&<\sum_{CN^{1/3}}^{\gamma N}[m+\epsilon+g(\gamma N)]^{-1}.\end{aligned}\tag{A32}$$

The last sum is bounded by elementary means by noting that $g(\gamma N)<g(CN^{1/3})=O(N^{1/3})$ and hence $g(\gamma N)/N^{1/3}=O(1)$ or less. Then by inspection the right-hand side is bounded by $\frac{2}{3}\ln N+O(1)$. Likewise so is the first sum, and hence

$$\sum_{CN^{1/3}}^{\gamma N}[m+\epsilon+g(m)]^{-1}=\frac{2}{3}\ln N+O(1).\tag{A33}$$

Finally, consider the last sum in Eq. (A30). Again because $g'(m)<0$,

$$\begin{aligned}\sum_{\gamma N}^{N-1}[m+\epsilon+g(\gamma N)]^{-1}&<\sum_{\gamma N}^{N-1}[m+\epsilon+g(m)]^{-1}\\&<\sum_{\gamma N}^{N-1}[m+\epsilon+g(N-1)]^{-1}.\end{aligned}\tag{A34}$$

As $g(m)=1/(2I)$ and $CN^{1/3}<m=\gamma N\ll N$, we can use Eq. (A10) and conclude

$$g(\gamma N)=\left(\frac{\gamma a^3|B'(a)|}{4\pi\Phi}\right)^{1/4}N^{1/2}\exp\left[-\frac{4}{3}\left(\frac{\Phi\gamma^3}{\pi a^3|B'(a)|}\right)^{1/2}N+O(\gamma^{1/2})\right][1+O(\gamma^{1/2})].\tag{A35}$$

Hence, $g(N-1)<g(\gamma N)=O(N^{1/2}e^{-\lambda N})$ where $\lambda=O(1)$. Simple estimates applied to the first and last sums in Eq. (A34) give

$$\sum_{\gamma N}^{N-1}[m+\epsilon+g(m)]^{-1}=-\ln\gamma+O(1/N).\tag{A36}$$

Combining Eqs. (A30), (A31), (A33) and (A36) gives

$$\sum_{l=0}^{N-1}\left(\frac{e\Phi}{2\pi}-l-1+\frac{1}{2I}\right)^{-1}=\frac{2}{3}\ln N+O(1).\tag{A37}$$

We now turn to the sum of the final term in Eq. (A2). Using previous definitions we can write this as

$$\begin{aligned}&\sum_{l=0}^{N-1}\frac{1}{2I}\left(W-1-\frac{1}{2I}\right)^{-1}\frac{\partial}{\partial e}\ln I\\&=\left(\sum_{m=0}^{CN^{1/3}}+\sum_{CN^{1/3}}^{\gamma N}+\sum_{\gamma N}^{N-1}\right)\frac{g(m)}{m+\epsilon+g(m)}\frac{\partial}{\partial e}\ln I.\end{aligned}\tag{A38}$$

Consider the first sum in Eq. (A38). We have previously noted that $\partial\ln I/\partial e=O(1/N)$ for the range of m indicated. Since $0<g(m)/[m+\epsilon+g(m)]\leqslant 1$, then

$$\sum_{m=0}^{CN^{1/3}}\frac{g(m)}{m+\epsilon+g(m)}\frac{\partial}{\partial e}\ln I=O(N^{-2/3}).\tag{A39}$$

The range of m in the second sum in Eq. (A38) allows the use of Eq. (A10) for I, and hence

$$\frac{\partial}{\partial e}\ln I=-\frac{\Phi}{8\pi N}-\frac{\Phi}{8\pi m}+\frac{\Phi}{24\pi C^{3/2}}\left[3\sqrt{\frac{m}{N}}-\left(\frac{m}{N}\right)^{3/2}\right]+O\left(\frac{1}{\sqrt{mN}},\frac{\sqrt{m}}{N^{3/2}}\right), \tag{A40}$$

where C is defined by Eq. (A11). Then the second sum in Eq. (A38) is

$$\sum_{CN^{1/3}}^{\gamma N}\frac{g(m)}{m+\epsilon+g(m)}\frac{\partial}{\partial e}\ln I=-\frac{\Phi}{8\pi}\sum_{CN^{1/3}}^{\gamma N}\left\{\frac{1}{N}+\frac{1}{m}-C^{-3/2}\left[\sqrt{\frac{m}{N}}-\frac{1}{3}\left(\frac{m}{N}\right)^{3/2}\right]+O\left(\frac{1}{\sqrt{mN}},\frac{\sqrt{m}}{N^{3/2}}\right)\right\}\frac{g(m)}{m+\epsilon+g(m)}. \tag{A41}$$

For the range of m indicated, $g(m)=1/(2I)$ is given by Eq. (A10) and has the functional form $g(m)=\alpha(mN)^{1/4}\exp(-\beta m^{3/2}/\sqrt{N})$, where α, β are constants. Note that $[m+\epsilon+g(m)]^{-1}<m^{-1}$. Then the first sum in Eq. (A41) vanishes as $N\to\infty$ by inspection. Over the range of m indicated, $1/m\leqslant C^{-3/2}\sqrt{m/N}$ and $(m/N)^{3/2}<(m/N)^{1/2}$. Therefore the remaining sums in Eq. (A41) are dominated by $\Sigma^{\gamma N}{}_{CN^{1/3}}g(m)/\sqrt{mN}$ which, when approximated by an integral, is of $O(1)$ and so

$$\sum_{CN^{1/3}}^{\gamma N}\frac{g(m)}{m+\epsilon+g(m)}\frac{\partial}{\partial e}\ln I=O(1). \tag{A42}$$

Finally, we deal with the last sum in Eq. (A38). It is for $0<l<(1-\gamma)N$, and following Eq. (A3) we estimated $\partial\ln I/\partial e=O(1)$ or less for this l range. We have already noted that $g(\gamma N)=O(N^{1/2}e^{-\lambda N})$ and that $g'(m)<0$. Hence we conclude

$$\sum_{\gamma N}^{N-1}\frac{g(m)}{m+\epsilon+g(m)}\frac{\partial}{\partial e}\ln I\leqslant\sum_{\gamma N}^{N-1}\frac{g(m)}{m}\frac{\partial}{\partial e}\ln I=O(N^{1/2}e^{-\lambda N}). \tag{A43}$$

In summary, Eqs. (A38), (A39), (A42), (A43) give

$$\sum_{l=0}^{N-1}\frac{1}{2I}\left(W-1+\frac{1}{2I}\right)^{-1}\frac{\partial}{\partial e}\ln I=O(1). \tag{A44}$$

Combining Eqs. (A2), (A28), (A29), (A37), (A44) gives for $e\Phi/2\pi\gg1$

$$\sum_{l=0}^{N-1}\frac{\partial}{\partial e}\ln||\psi_l^0||^2=2e\int_o^a drrB(r)\varphi(r)-\frac{\Phi}{4\pi}\ln\left(\frac{e\Phi}{2\pi}\right)+O(\Lambda), \tag{A45}$$

where $\Lambda\gg1$ but e independent. Now combine Eq. (A45) with Eq. (59), integrate and combine this with our previous result in Eq. (60) to get for $ma\ll1$ followed by $e\Phi\gg1$

$$\ln\det=-\frac{e\Phi}{4\pi}\ln\left(\frac{e\Phi}{2\pi}\right)+\frac{e\Phi}{4\pi}\ln(ma)^2+O(e\Phi,(ma)^2e\Phi\ln(e\Phi)). \tag{A46}$$

The justification for the inclusion of the $\ln(ma)^2$ term was discussed following Eq. (60). Also, as discussed immediately after Eq. (59), there may be subtle cancellations that will eliminate the $\ln(e\Phi)$ factor in the remainder term $(ma)^2e\Phi\ln(e\Phi)$. The case when $e\Phi<0$ is included by replacing $e\Phi$ in Eq. (A46) everywhere with $|e\Phi|$.

This analysis is for fields $B(r)>0$ for $r<a$ with continuous first and second derivatives and with $B(a)=0$, $B'(a)<0$. For the case $B(a)>0$ the analysis is almost identical to the preceding case and is also a little simpler. The main changes are

$$f(r^*)=\frac{m^2\Phi}{2\pi Na^2B(a)}+O\left(\frac{m^3}{N^2}\right), \tag{A47}$$

$$|f''(r^*)|=\frac{4\pi NB(a)}{\Phi}\left[1+O\left(\frac{m}{N}\right)\right] \tag{A48}$$

and

$$I=\left(\frac{\Phi}{2Na^2B(a)}\right)^{1/2}\exp\left[\frac{m^2\Phi}{2\pi Na^2B(a)}+O\left(\frac{m^3}{N^2}\right)\right]\times\left[1+O\left(\frac{m}{N}\right)\right], \tag{A49}$$

provided $[a^2B(a)/\Phi]^{1/2}N^{1/2}<m\ll N$. The result is the same as Eq. (A45).

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