(Super) oscillator on CP^N and a constant magnetic field

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We define the "maximally integrable" isotropic oscillator on CP^N and discuss its various properties, in particular, the behavior of the system with respect to a constant magnetic field. We show that the properties of the oscillator on CP^N qualitatively differ in the $N>1$ and $N=1$ cases. In the former case we construct the "axially symmetric" system which is locally equivalent to the oscillator. We perform the Kustaanheimo-Stiefel transformation of the oscillator on CP^2 and construct some generalized MIC-Kepler problem. We also define a $\mathcal{N}=2$ superextension of the oscillator on CP^N and show that for $N>1$ the inclusion of a constant magnetic field preserves the supersymmetry of the system.

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I. INTRODUCTION

The harmonic oscillator plays a distinguished role in theoretical and mathematical physics, due to its overcomplete symmetry group. The wide number of hidden symmetries provides the oscillator with unique properties, e.g., closed classical trajectories, the degeneracy of the quantum-mechanical energy spectrum, and the separability of variables in a few coordinate systems. The overcomplete symmetry allows one to preserve the exact solvability of the oscillator, even after some deformation of the potential breaking the initial symmetry of the system. Particulary, the oscillator remains exactly solvable after coupling to a constant magnetic field, though the latter removes the hidden symmetries of the system. The reduction of the oscillator to low dimensions allows one to construct new integrable systems with hidden symmetries (in fact, almost all integrable systems of classical and quantum mechanics are related with either the free particle case, or the oscillator) [1]. There is a nontrivial relation between oscillator and Coulomb systems: the $(N+1)$-dimensional Coulomb problem can be obtained from the $2N$-dimensional oscillator by the so-called Levi-Civita (or Bohlin), Kustaanheimo-Stiefel and Hurwitz transformations, when $N=1,2,4$ [2]. The transformations correspond to the reduction of the oscillator by the actions of Z_2, $U(1)$ and $SU(2)$ groups, respectively, and are based on the Hopf maps $S^1/Z_2=S^1$, $S^3/U(1)=\mathrm{CP}^1\cong S^2$, $S^7/SU(2)=\mathbb{H}P^1\cong S^4$ (relating the angular parts of the oscillator and Coulomb problems). Indeed, reducing the oscillators we get some parametric families of Coulomb-like systems, specified by the presence of a magnetic flux for $N=1$; by a Dirac monopole for $N=2$ (the MIC-Kepler system); and by a Yang monopole[1] for $N=4$ (see, respectively, Refs. [4,5,6]). It could be checked easily, that the MIC-Kepler system, initially introduced by Zwanziger for the description of the relative motion of two Dirac dyons, also describes the scattering of two well-separated Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles and dyons. The latter problem was considered in a well-known paper by Gibbons and Manton [7], where the existence of a hidden Coulomb-like symmetry was established (see also [8]). Let us mention also the key role of the Hurwitz transformation (and of the second Hopf map) in the recently proposed higher-dimensional quantum Hall effect [9] (see also [10,11]).

The oscillator is a distinguished system, also with respect to supersymmetrization. A supersymmetric oscillator is specified by the splitting of fermionic and bosonic degrees of freedom. Thus it inherits the hidden symmetries of the initial system. We notice that the construction of integrable supersymmetric mechanics is interesting not only in a field-theoretical context. Being in deep connection with the factorization problem, the supersymmetrization of integrable systems could yield a new set of integrable systems with isospectral potentials. Since the list of references on supersymmetric mechanics is enormous, we refer to the introductory reviews [12] (mostly devoted to the connection of supersymmetric quantum mechanics with the factorization problem) and [13] (containing the most complete list of references on field-theoretical aspects of supersymmetric mechanics).

Recent progress in string theory inspired interest for noncommutative field theories [14] and, in particular, for noncommutative quantum mechanics [15]. The oscillator was found to be a distinguished case in noncommutative quantum mechanics too: at the moment it is the only exactly solved (even in the presence of a constant magnetic field) noncommutative quantum mechanical system with a nonzero potential [16].

There is nontrivial generalization of the oscillator on the sphere and the two-sheet hyperboloid (pseudosphere) [17] given by the potential

$$U_{osc}=\frac{\omega^2 r_0^2}{2}\frac{\mathbf{x}^2}{x_{d+1}^2}. \tag{1.1}$$

Here $\mathbf{x},x_{d+1}$ are the (pseudo)Euclidean coordinates of the

[1]Under "Yang monopole" we mean a five-dimensional $SU(2)$ generalization of a Dirac monopole [3].

ambient space $\mathbb{R}^{d+1}(\mathbb{R}^{d\cdot 1})$: $\epsilon\mathbf{x}^2+x_{d+1}^2=r_0^2$, with $\epsilon=+1$ for the sphere, $\epsilon=-1$ for the pseudosphere.

This system has a nonlinear hidden symmetry algebra, providing it with properties similar to those of a conventional oscillator. Applying to the oscillator on the (pseudo)sphere the standard Levi-Civita, Kustaanheimo-Stiefel and Hurwitz transformations, one can obtain the generalization of flux-Coulomb, MIC-Kepler and Yang-Coulomb systems on the (pseudo)sphere [18].[2] In the present paper we define the oscillator on complex projective spaces CP^N, from the requirement that it possesses hidden symmetries generalizing those of the planar oscillator, and consider its behavior with respect to the coupling to a constant magnetic field.

The oscillator on $CP^1=S^2$ coincides with the Higgs oscillator on the sphere S^2 (note that $CP^1=S^2$). The oscillator on CP^N, $N>1$ is defined by the potential

$$u(z,\bar{z})=\omega^2 r_0^2 z\bar{z}, \tag{1.2}$$

where $z^a,\bar{z}^a$ are inhomogeneous coordinates of CP^N, corresponding to the Fubini-Study metric

$$g_{a\bar{b}}dz^a d\bar{z}^b=r_0^2\frac{dz d\bar{z}}{1+z\bar{z}}-r_0^2\frac{(\bar{z}dz)(zd\bar{z})}{(1+z\bar{z})^2}. \tag{1.3}$$

In contrast to the case of the oscillator on $CP^1=S^2$ which is defined on the disk $|z|<1$, the oscillator on CP^N, $N>1$ is defined on the whole chart. The transition to another chart of CP^N transforms the oscillator into the system with the potential

$$U=\omega^2 r_0^2\left(\frac{1}{z^1\bar{z}^1}+\frac{z^2\bar{z}^2+\cdots z^N\bar{z}^N}{z^1\bar{z}^1}\right),$$

which has the oscillator symmetry algebra.

The Kustaanheimo-Stiefel transformation of the oscillator on CP^2 yields a generalization of the MIC-Kepler system, which can be transformed into the MIC-Kepler system on the three-dimensional hyperboloid.

The oscillator on CP^N admits, because of its Kähler structure, a simple coupling to a constant magnetic field. This can be achieved by carrying out the following replacement of the symplectic structure: $\Omega_0\to\Omega_0+iBg_{a\bar{b}}dz^a\wedge d\bar{z}^b$. The coupling to a constant magnetic field preserves the kinematical $su(N)$ symmetries of the oscillator [for the free particle case, i.e., $\omega=0$, the coupling preserves the whole symmetry algebra $su(N+1)$], although it breaks the hidden symmetries.

Below, we construct the $\mathcal{N}=2$ supersymmetric oscillator on CP^N and study its behavior, with respect to the coupling to a constant magnetic field (the oscillator on CP^N, in contrast with the one on C^N, does not admit the $\mathcal{N}=4$ supersymmetrization). We show that, in contrast with the $\mathcal{N}=2$ superoscillator on $CP^1=S^2$, the $\mathcal{N}=2$ superoscillator on CP^N, $N>1$ allows coupling to a constant magnetic field, without breaking supersymmetry.

II. OSCILLATOR ON CP^N

This section is devoted to the construction of the oscillator system on the complex projective space CP^N. Our consideration essentially exploits the fact that the complex projective space is a constant curvature Kähler manifold. Hence, our model could be easily adopted for the formulation of the oscillator system on the other spaces of that sort.

Let us recall that the Kähler manifold M is equipped with the metric, which could be locally represented in the form

$$g_{a\bar{b}}dz^a d\bar{z}^b=\frac{\partial^2}{\partial z^a\partial\bar{z}^b}dz^a d\bar{z}^b, \tag{2.1}$$

and with the associated Poisson bracket

$$\{f,g\}_0=i\frac{\partial f}{\partial\bar{z}^a}g^{\bar{a}b}\frac{\partial g}{\partial z^b}-i\frac{\partial g}{\partial\bar{z}^b}g^{\bar{a}b}\frac{\partial f}{\partial\bar{z}^a},\quad g^{\bar{a}b}g_{b\bar{c}}=\delta^{\bar{a}}_{\bar{c}}. \tag{2.2}$$

The local real function $K(z,\bar{z})$ is called the Kähler potential.

The complex projective space CP^N could be equipped with the Fubini-Study metric, given by the Kähler potential

$$K=r_0^2\log(1+z\bar{z}). \tag{2.3}$$

The scalar curvature of CP^N is related with the parameter r_0^2 as follows: $R=N(N+1)/r_0^2$. The isometries of the Kähler structure are generated by the *holomorphic Hamiltonian vector fields*

$$\mathbf{V}_\mu=V^a_\mu(z)\frac{\partial}{\partial z^a}+\bar{V}^{\bar{a}}_\mu(\bar{z})\frac{\partial}{\partial\bar{z}^a},\quad[\mathbf{V}_\mu,\mathbf{V}_\nu]=C^\lambda_{\mu\nu}\mathbf{V}_\lambda, \tag{2.4}$$

where

$$\mathbf{V}_\mu=\{h_\mu,\ \}_0,\quad\{h_\mu,h_\nu\}_0=C^\lambda_{\mu\nu}h_\lambda,$$

$$\frac{\partial^2 h_\mu}{\partial z^a\partial z^b}-\Gamma^c_{ab}\frac{\partial h_\mu}{\partial z^c}=0. \tag{2.5}$$

The real functions h_μ are called Killing potentials.

The symmetry algebra of CP^N is $su(N+1)$. This algebra is defined by the Killing potentials

$$h_T=T^{\bar{a}b}h_{\bar{a}b}-\mathrm{tr}\,\hat{T},\quad h^1_a=h^-_a+h^+_a,\quad h^2_a=i(h^-_a-h^+_a), \tag{2.6}$$

where

$$h_{\bar{a}b}=r_0^2\frac{z^a\bar{z}^b}{1+z\bar{z}},\quad h^-_a=r_0^2\frac{z^a}{1+z\bar{z}},\quad h^+_a=r_0^2\frac{\bar{z}^a}{1+z\bar{z}}, \tag{2.7}$$

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[2] Let us remind, that the Coulomb system on the (pseudo)sphere is defined by the potential [19]

$$U_C=-\frac{\gamma}{r_0}\frac{x_{d+1}}{|\mathbf{x}|}.$$

Quantum mechanics of the oscillator and Coulomb system on the D-dimensional sphere and pseudosphere is considered in detail in Ref. [20].

and $\hat{T}$ are $N\times N$ Hermitian matrices: $T^{a\bar{b}}=\overline{T^{b\bar{a}}}$.

The algebra of $h_{a\bar{b}}, h_a^{\pm}$ reads

$$\{h_{a\bar{b}}, h_{c\bar{d}}\}_0 = i\delta_{a\bar{d}}h_{b\bar{c}} - i\delta_{c\bar{b}}h_{a\bar{d}},$$
$$\{h_a^-, h_b^+\}_0 = i\delta_{a\bar{b}}(r_0^2 - \mathrm{tr}\, h_{a\bar{b}}) + ih_{a\bar{b}}, \tag{2.8}$$
$$\{h_a^{\pm}, h_b^{\pm}\}_0 = 0, \quad \{h_a^{\pm}, h_{b\bar{c}}\}_0 = \mp ih_b^{\pm}\delta_{ab}.$$

Let us equip the cotangent bundle $T_*\mathrm{CP}^N$ with the symplectic structure

$$\Omega_B = dz^a\wedge d\pi_a + d\bar{z}^a\wedge d\bar{\pi}_a + iBg_{a\bar{b}}dz^a\wedge d\bar{z}^b, \tag{2.9}$$

which defines, together with the Hamiltonian

$$\mathcal{D} = g^{a\bar{b}}\pi_a\bar{\pi}_b, \tag{2.10}$$

the dynamics of a free particle on CP^N, in the presence of a constant magnetic field B. The isometries of a Kähler structure define the Noether's constants of motion of a free particle

$$\mathcal{J}_\mu = J_\mu + Bh_\mu = V^a_\mu\pi_a + \bar{V}^{\bar{a}}_\mu\bar{\pi}_{\bar{a}} + Bh_\mu : \begin{cases}\{\mathcal{D}, J_\mu\} = 0,\\ \{J_\mu, J_\nu\} = C^\lambda_{\mu\nu}J_\lambda.\end{cases} \tag{2.11}$$

Explicitly, we have

$$J_{a\bar{b}} = -iz^b\pi_a + i\bar{\pi}_b\bar{z}^a, \quad iJ_a^+ = \pi_a + \bar{z}^a(\bar{z}\bar{\pi}),$$
$$-iJ_a^- = \bar{\pi}_a + z^a(z\pi). \tag{2.12}$$

Notice that the vector fields generated by $\mathcal{J}_\mu$ are independent on B

$$\tilde{\mathbf{V}} = V^a(z)\frac{\partial}{\partial z^a} - V^a_{,b}\pi_a\frac{\partial}{\partial \pi_a} + \bar{V}^a(\bar{z})\frac{\partial}{\partial \bar{z}^a} - \bar{V}^a_{,b}\bar{\pi}_a\frac{\partial}{\partial \bar{\pi}_a}. \tag{2.13}$$

Hence, the inclusion of a constant magnetic field preserves the whole symmetry algebra of a free particle moving in a Kähler space.

Now, let us consider the $u(N)$-invariant Hamiltonian

$$\mathcal{H} = g^{a\bar{b}}\pi_a\bar{\pi}_b + U(z\bar{z}), \tag{2.14}$$

and require it to have the hidden symmetry (similar to the one of the oscillator) given by either one of the constants of motion

$$\text{(i)} \quad I^+_{ab} = \mathcal{J}^+_a\mathcal{J}^+_b + f_+(z\bar{z})\bar{z}^a\bar{z}^b,$$
$$\text{(ii)} \quad I_{a\bar{b}} = \mathcal{J}^+_a\mathcal{J}^-_b + f_0(z\bar{z})\bar{z}^a z^b. \tag{2.15}$$

Straightforward calculations immediately yield the following constraints:

$$\text{(i)} \quad B=0,\ N=1,\ U(x) = c_1x/(1-x)^2 + c_0,$$
$$f_+ = c_1/(1-x)^2, \tag{2.16}$$

$$\text{(ii)} \quad B=0,\ N=1,2\ldots,\ U(x) = c_1x + c_0,$$
$$f_0 = c_1.$$

Taking into account that $\mathcal{H} = \mathrm{Tr}\,\hat{I} + \mathrm{Tr}\,\hat{J}^2/2r_0^2$, we get the following generalizations of the oscillator on CP^N.

CP^1. The oscillator is defined by the Hamiltonian system

$$\mathcal{H} = \frac{(1+z\bar{z})^2\pi\bar{\pi}}{r_0^2} + \frac{\omega^2r_0^2z\bar{z}}{(1-z\bar{z})^2}, \quad \Omega_0 = dz\wedge d\pi + d\bar{z}\wedge d\bar{\pi}. \tag{2.17}$$

The symmetry algebra is given by the $U(1)$ generator J and the complex (or vectorial) constant of motion $I^{\pm}$

$$J = i(\pi z - \bar{\pi}\bar{z}), \quad I_+ = \frac{J_+^2}{r_0^2} - \frac{\omega^2r_0^2\bar{z}^2}{(1-z\bar{z})^2}: \ \{J, I_\pm\} = \pm 2iI_\pm,$$
$$\{I_-, I_+\} = 4i\left(\omega^2J + \frac{J\mathcal{H}}{r_0^2} - \frac{J^3}{2r_0^4}\right). \tag{2.18}$$

This is nothing but the well-known Higgs oscillator on the sphere $S^2 = \mathrm{CP}^1$ [17].

CP^N, $N>1$. The oscillator is defined by the Hamiltonian system

$$\mathcal{H} = g^{a\bar{b}}\pi_a\bar{\pi}_b + \omega^2r_0^2z\bar{z}, \quad \Omega_0 = dz^a\wedge d\pi_a + d\bar{z}^a\wedge d\bar{\pi}_a. \tag{2.19}$$

Its symmetries are given by the constants of motion

$$J_{a\bar{b}} = i(z^b\pi_a - \bar{\pi}_b\bar{z}^a), \quad I_{a\bar{b}} = \frac{J_a^+J_b^-}{r_0^2} + \omega^2r_0^2\bar{z}^az^b, \tag{2.20}$$

which define the nonlinear (quadratic) algebra

$$\{J_{a\bar{b}}, J_{c\bar{d}}\} = i\delta_{a\bar{d}}J_{b\bar{c}} - i\delta_{c\bar{b}}J_{a\bar{d}},$$
$$\{I_{a\bar{b}}, J_{c\bar{d}}\} = i\delta_{c\bar{b}}I_{a\bar{d}} - i\delta_{a\bar{d}}I_{c\bar{b}},$$
$$\{I_{a\bar{b}}, I_{c\bar{d}}\} = i\omega^2\delta_{c\bar{b}}J_{a\bar{d}} - i\omega^2\delta_{a\bar{d}}J_{c\bar{b}}$$
$$+ iI_{c\bar{b}}(J_{a\bar{d}} + J_0\delta_{a\bar{d}})/r_0^2 - iI_{a\bar{d}}(J_{c\bar{b}} + J_0\delta_{c\bar{b}})/r_0^2. \tag{2.21}$$

It is convenient to introduce the generators

$$J_i = T_i^{a\bar{b}}J_{a\bar{b}}, \quad J_0 = \mathrm{Tr}\,\hat{J}, \quad I_i = T_i^{a\bar{b}}I_{a\bar{b}}, \quad I_0 = \mathrm{Tr}\,\hat{I}, \tag{2.22}$$

where T_i are traceless $N\times N$ Hermitian matrices [the generators of the $su(N)$ algebra]. The above generators belonging to the center of algebra read

$$J_0 = i(z\pi - \bar{\pi}\bar{z}), \quad \mathcal{H}_{N>1} = I_0 + \frac{\mathrm{Tr}\,\hat{J}^2 + J_0^2}{2r_0^2}. \tag{2.23}$$

Also the following equality holds:

$$\mathrm{Tr}\,\hat{I}^2+\omega^2\mathrm{Tr}\,\hat{J}^2=I_0^2+\omega^2 J_0^2. \tag{2.24}$$

We have got the "maximally integrable" generalization of the oscillator on complex projective spaces, i.e., the system with the highest possible number of functionally independent constants of motion.[3]

We established the following essential properties of the latter system.

The oscillator on CP^N, $N>1$ is well-defined on the whole chart of the complex projective space, $0<|z|<\infty$. The oscillator on $\mathrm{CP}^1\sim S^2$ (as well as on higher-dimensional spheres) is defined on the disc $|z|<1$ only. The constant magnetic field removes the hidden symmetries of the oscillator on CP^N for any N, while it respects them in the case of a free particle, i.e., when $\omega=0$.

The above construction could be easily extended for the noncompact version of CP^N, provided by the Lobachewski space $\mathcal{L}^N=SU(1.N)/U(1)\times SU(N)$. For this purpose, we should replace the Fubini-Study metric with the one generated by the Kähler potential $K=-r_0^2\log(1-z\bar{z})$, and subsequently replace the Killing potentials and Noether constants of CP^N with the ones of $\mathcal{L}^N$. The Killing potentials of $\mathcal{L}^N$ are defined by the functions

$$h_{a\bar{b}}=-r_0^2\frac{z^a\bar{z}^b}{1-z\bar{z}},\quad h_a^-=-r_0^2\frac{z^a}{1-z\bar{z}},\quad h_a^+=-r_0^2\frac{\bar{z}^a}{1-z\bar{z}}. \tag{2.25}$$

Globally, the complex projective space CP^N is covered by $N+1$ charts, marked by the indices $\tilde{a}=0,a$. The transition functions from the $\tilde{b}$th chart to the $\tilde{c}$th one are of the form

$$z^{\tilde{a}}_{(\tilde{c})}=\frac{z^{\tilde{a}}_{(\tilde{b})}}{z^{\tilde{c}}_{(\tilde{b})}},\quad \text{where}\quad z^{\tilde{a}}_{(\tilde{a})}=1. \tag{2.26}$$

On CP^1 the transition functions take the simple form $z\to 1/z$, corresponding to the transition from one hemisphere to the other. The respective transformation of the momenta is $\pi\to -z^2\pi$. The Hamiltonian of the oscillator on CP^1 is obviously invariant under the above transformation. In higher-dimensions we get a rather different picture, since the potential term is not covariant under the transition (2.26). Let us consider this transformation in more details.

The transition functions (2.26) define the following canonical transformation, which is singular on the $z^1=0$ "axes:"

$$z^1\to 1/z^1,\quad \pi_1\to -z^1(z\pi),\quad z^{\hat{a}}\to z^{\hat{a}}/z^1,$$
$$\pi_{\hat{a}}\to z^1\pi_{\hat{a}}\quad \hat{a}=2,\dots,N. \tag{2.27}$$

[3] In the theory of integrable systems such systems are called "maximally superintegrable systems." We prefer to suppress the prefix "super" in this context, in order to avoid any confusion with supersymmetric systems.

The kinetic term is covariant with respect to the above transformation, while the potential term is not. As a result, we get the integrable system on CP^N, $N>1$ defined by the Hamiltonian

$$\mathcal{H}_{\mathrm{Back}}=g^{a\bar{b}}\pi_a\bar{\pi}_b+\omega^2 r_0^2\left(\frac{1}{z^1\bar{z}^1}+\frac{z^2\bar{z}^2+\cdots+z^N\bar{z}^N}{z^1\bar{z}^1}\right). \tag{2.28}$$

This system inherits the whole symmetry algebra of the oscillator, i.e., it is a "maximally integrable" system. Its constants of motion can be obtained by a straightforward transformation of those of the oscillator, given in Eq. (2.20). Note that, in spite of its "maximal integrability," the system is not invariant under "spatial" $u(N)$ rotations.

On the Lobachewski space $\mathcal{L}^N$, $N>1$ there is no analog of this system. The "ambient" space for the Lobachewski plane is $\mathrm{C}^{1\cdot N}$. The transitions (2.26) transform the oscillator on $\mathcal{L}^N$ into a system on the space with the signature $(-,-,+,\dots,+)$.

CP^2: Kustaanheimo-Stiefel transformation

As we mentioned in the Introduction, the oscillator on two-, four-, and eight-dimensional planes and spheres could be reduced to the two-, three- and five-dimensional Coulomb systems, and their generalizations specified by the presence of monopoles. Particularly, the oscillator on $S^2=\mathrm{CP}^1$ and $\mathrm{AdS}_2=\mathcal{L}$ can be reduced, by the so-called Levi-Civita transformation, to the Coulomb systems on two-dimensional hyperboloid (Lobachewski plane) $\mathcal{L}$. Similarly, the Kustaanheimo-Stiefel transformation of the oscillator on a four-dimensional sphere and a four-dimensional two-sheet hyperboloid leads to the generalization of the MIC-Kepler problem on a three-dimensional two-sheet hyperboloid [18].

Let us consider the behavior of the oscillator on CP^2, with respect to the Kustaanheimo-Stiefel transformation. The constants of motion of the oscillator on CP^2 are given by the generators

$$\mathbf{I}=\frac{J_+\boldsymbol{\sigma}J_-}{r_0^2}+\omega^2 r_0^2 z\boldsymbol{\sigma}\bar{z},\quad \mathbf{J}=iz\boldsymbol{\sigma}\pi-i\bar{\pi}\boldsymbol{\sigma}\bar{z},\quad J_0=iz\pi-i\overline{z\pi}, \tag{2.29}$$

where $\boldsymbol{\sigma}$ denotes standard Pauli matrices.

Their algebra reads

$$\{J_0,I_k\}=\{J_0,J_k\}=0,\quad \{J_k,J_l\}=2\epsilon_{klm}J_m,$$
$$\{I_k,J_l\}=2\epsilon_{klm}I_m, \tag{2.30}$$
$$\{I_k,I_l\}=\epsilon_{klm}(2\omega^2 J_m-3I_mJ_0/r_0^2+I_0J_m/r_0^2).$$

In order to reduce this system by the Hamiltonian action of J_0, we have to fix its value

$$J_0=2s, \tag{2.31}$$

and then factorize the level surface by the $U(1)$ group action. The resulting six-dimensional phase space T^*M^{red} can be parametrized by the following $U(1)$-invariant functions:

$$\mathbf{x}=z\boldsymbol{\sigma}\bar{z},\quad \mathbf{p}=\frac{z\boldsymbol{\sigma}\pi+\bar{\pi}\boldsymbol{\sigma}\bar{z}}{2z\bar{z}}:\quad \{\mathbf{x},J_0\}=\{\mathbf{p},J_0\}=0. \tag{2.32}$$

In these coordinates the reduced symplectic structure and the generators of the angular momentum are given by the expressions

$$\Omega_{\rm red}=d\mathbf{p}\wedge d\mathbf{x}+s\,\frac{\mathbf{x}\times d\mathbf{x}\times d\mathbf{x}}{|\mathbf{x}|^3},\quad \mathbf{J}_{\rm red}=\mathbf{J}/2=\mathbf{p}\times\mathbf{x}+s\,\frac{\mathbf{x}}{|\mathbf{x}|}. \tag{2.33}$$

Thus the reduced system is specified by the presence of a Dirac monopole.

The reduced Hamiltonian is given by the expression

$$\mathcal{H}_{\rm red}=\frac{(1+x)}{r_0^2}[x\mathbf{p}^2+(\mathbf{xp})^2]+s^2\frac{(1+x)^2}{r_0^2x}+\omega^2r_0^2x,$$

where

$$x\equiv|\mathbf{x}|. \tag{2.34}$$

Let us fix the constant energy surface

$$\mathcal{H}=E_{\rm osc}. \tag{2.35}$$

Then, dividing by $2r_0^2x$, we can represent it in the form

$$\mathcal{H}_{\rm MIC}=\mathcal{E},\quad \mathcal{H}_{\rm MIC}=\frac{(1+x)}{2r_0^4}\left[\mathbf{p}^2+\frac{(\mathbf{xp})^2}{x}\right]+\frac{s^2}{2r_0^4x^2}-\frac{\gamma}{r_0^2x}, \tag{2.36}$$

where we introduced the notation

$$\gamma=E_{\rm osc}/2-s^2/r_0^2,\quad -2\mathcal{E}=\omega^2+s^2/r_0^4. \tag{2.37}$$

The Hamiltonian $\mathcal{H}_{\rm MIC}$ can be interpreted as the Hamiltonian of some generalized MIC-Kepler problem. Notice that its potential energy term has the same form, as the one of the conventional (flat) MIC-Kepler problem. The hidden symmetries of the system are given by the reduced generators I_i.

Let us perform the canonical transformation $(\mathbf{x},\mathbf{p})\to(\tilde{\mathbf{x}},\tilde{\mathbf{p}})$, going to the coordinates where the metric takes a conformally-flat form:

$$\tilde{\mathbf{x}}=f(x)\mathbf{x},\quad \mathbf{p}=f\tilde{\mathbf{p}}+f'\frac{(\mathbf{x}\tilde{\mathbf{p}})}{x}\mathbf{x}, \tag{2.38}$$

where

$$f(x)=\frac{1}{x}\,\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}. \tag{2.39}$$

In this case, the reduced Hamiltonian reads

$$\mathcal{H}_{\rm red}=\frac{x(1+x)^2\mathbf{p}^2}{4r_0^2}+s^2\frac{(x+1)^4}{4r_0^2x(1-x)^2}+\frac{4\omega^2r_0^2x}{(1-x)^2},\quad x<1, \tag{2.40}$$

while the Hamiltonian of the above obtained generalization of MIC-Kepler problem (2.36) takes the form

$$\mathcal{H}_{MIK}=\frac{(1-x^2)^2}{32r_0^4}\left(\mathbf{p}^2+\frac{s^2}{x^2}\right)-\left(\gamma+\frac{s^2}{2r_0^2}\right)\frac{1+x^2}{4r_0^2x}-\frac{s^2}{4r_0^4}. \tag{2.41}$$

This is nothing but the Hamiltonian of the MIC-Kepler problem on the three-dimensional hyperboloid [18] constructed by the Kustaanheimo-Stiefel transformation of the oscillator on a four-dimensional sphere.

Performing the Kustaanheimo-Stiefel transformation of the system (2.28) on CP^2, we get the following expression for the reduced Hamiltonian:

$$\mathcal{H}_{\rm Back}=\frac{(1+x)}{r_0^2}[x\mathbf{p}^2+(\mathbf{xp})^2]+s^2\frac{1+r}{r_0^2x}+2\omega^2r_0^2\frac{1+x}{x+x_3}-\omega^2r_0^2,\quad x_3\neq x. \tag{2.42}$$

In conformal coordinates (2.38) the latter takes the form

$$\mathcal{H}_{c\rm Back}=\frac{x(1+x)^2\mathbf{p}^2}{4r_0^2}+s^2\frac{(x+1)^4}{4r_0^2x(x-1)^2}+\omega^2r_0^2\frac{(1+x)^2}{2(x+x_3)}-\omega^2r_0^2. \tag{2.43}$$

III. $\mathcal{N}=2$ SUPERSYMMETRIC OSCILLATOR ON CP^N

In this section we construct the $\mathcal{N}=2$ superextension of the oscillator on CP^N coupled to a constant magnetic field. It is well known that any Hamiltonian system of the form

$$\mathcal{H}_0=g^{ij}(p_ip_j+W_{,i}W_{,j}),\quad \Omega^{\rm can}=dp_i\wedge dx^i \tag{3.1}$$

could be easily extended to the system with exact $\mathcal{N}=2$ supersymmetry

$$\{Q^+,Q^-\}=\mathcal{H},\quad \{Q^\pm,Q^\pm\}=0. \tag{3.2}$$

The function $W(x)$ is called superpotential. The oscillator on a sphere S^D belongs to the above class of systems. Its superpotential is given by the expression

$$W=\frac{\omega}{2}\log\frac{2+\mathbf{x}^2}{2-\mathbf{x}^2}, \tag{3.3}$$

where $\mathbf{x}$ denotes the conformal coordinates of the sphere S^D.

For the supersymmerization of the system (3.1), we have to define the supersymplectic structure

$$\Omega=dp_i\wedge dx^i+\frac{1}{2}R_{ijkl}\theta_+^k\theta_-^l dx^i\wedge dx^k+g_{ij}D\theta_+^i\wedge D\theta_-^j,$$

$$D\theta_\pm^i\equiv d\theta_\pm^i+\Gamma_{kl}^i\theta_\pm^k dx^l,\quad \alpha=1,2$$

and the supercharges $Q_\pm=(p_i\pm iW_{,i})\theta_\pm^i$, which obey the condition $\{Q_\pm,Q_\pm\}=0$. Then, we immediately get the $\mathcal{N}=2$ supersymmetric Hamiltonian

$$\mathcal{H}=\{Q_+,Q_-\}=\mathcal{H}_0+W_{i,j}\theta_+^i\theta_-^j+R_{ijkl}\theta_-^i\theta_-^k\theta_+^l.$$

The inclusion of a magnetic field $\Omega\to\Omega+F_{ij}\theta_+^i\theta_-^j$ breaks the $\mathcal{N}=2$ supersymmetry of the system

$$\{Q_\pm, Q_\pm\} = F_{ij}\theta^i_\pm \theta^j_\pm, \quad \{Q_+, Q_-\} = \mathcal{H} + iF_{ij}\theta^i_+\theta^j_-.$$

For the construction of the supersymmetric oscillator on CP^N, let us represent the initial (bosonic) Hamiltonian in the form

$$\mathcal{H} = g^{a\bar{b}}(\pi_a \bar{\pi}_b + \partial_a W \bar{\partial}_b W). \tag{3.4}$$

If the superpotential can be represented in the form $W(z,\bar{z}) = W_+(z) + W_-(\bar{z})$, then one can construct the $\mathcal{N}=4$ supergeneralization of the system on Kähler space [21]. Otherwise, the system can be endowed with $\mathcal{N}=2$ supersymmetry. Hence, we can construct the $\mathcal{N}=4$ supersymmetric oscillator on C^N choosing the superpotential $2W = \omega z^2 + \bar{\omega}\bar{z}^2$. However, we cannot construct the (anti)holomorphic superpotential for the oscillator on CP^N and, consequently, obtain its $\mathcal{N}=4$ superextension. On the other hand, for the oscillators on C^N and CP^N one can find the superpotentials with explicit $su(N)$ symmetry,

$$W = \omega K = \omega z\bar{z} \quad \text{for } \mathrm{C}^N$$

$$2W = \omega r_0 \log(1 - z\bar{z})/(1 + z\bar{z}) \quad \text{for } \mathrm{CP}^1$$

$$W = \omega K = \omega r_0 \log(1 + z\bar{z}) \quad \text{for } \mathrm{CP}^N, N > 1. \tag{3.5}$$

By using such functions, we shall construct the $\mathcal{N}=2$ supersymmetric oscillators on CP^N. We shall see that the linear dependence of the superpotential W on the Kähler potential K leads to an interesting behavior of the supersymmetric system, with respect to a constant magnetic field. Thus, the superoscillator on CP^N, $N>1$ has more similarities with the planar one, than the oscillator on CP^1.

Let us consider a $(2N\cdot 2N)_{\mathrm{C}}$-dimensional phase space equipped with the symplectic structure

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a + i(Bg_{a\bar{b}} + iR_{a\bar{b}c\bar{d}}\eta^c_\alpha \bar{\eta}^d_\alpha) dz^a \wedge d\bar{z}^b + g_{a\bar{b}} D\eta^a_\alpha \wedge D\bar{\eta}^b_\alpha, \tag{3.6}$$

where $D\eta^a_\alpha = d\eta^a_\alpha + \Gamma^a_{bc}\eta^a_\alpha dz^a$, $\alpha = 1,2$, and $\Gamma^a_{bc}, R_{a\bar{b}c\bar{d}}$ are, respectively, the connection and curvature of the Kähler structure. The corresponding Poisson brackets are defined by the following nonzero relations (and their complex conjugates):

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^b_\alpha\} = -\Gamma^b_{ac}\eta^c_\alpha,$$

$$\{\pi_a, \bar{\pi}_b\} = i(Bg_{a\bar{b}} + iR_{a\bar{b}c\bar{d}}\eta^d_\alpha \bar{\eta}^d_\alpha),$$

$$\{\eta^a_\alpha, \bar{\eta}^b_\beta\} = g^{a\bar{b}}\delta_{\alpha\beta}.$$

The symplectic structure (3.6) becomes canonical in the coordinates (p_a, χ^k)

$$p_a = \pi_a - \frac{i}{2}\partial_a \mathbf{g}, \quad \chi^m_i = e^m_b \eta^b_i:$$

$$\Omega_{\mathrm{Scan}} = dp_a \wedge dz^a + d\bar{p}_{\bar{a}} \wedge d\bar{z}^{\bar{a}} + iBg_{a\bar{b}} dz^a \wedge d\bar{z}^b + d\chi^m_\alpha \wedge d\bar{\chi}^{\bar{m}}_\alpha, \tag{3.7}$$

where e^m_a are the einbeins of the Kähler structure: $e^m_a \delta_{m\bar{m}} \bar{e}^{\bar{m}}_{\bar{b}} = g_{a\bar{b}}$.

So, in order to quantize the system, one chooses

$$\hat{p}_a = -i\left(\frac{\partial}{\partial z^a} - iB\frac{\partial K}{\partial z^a}\right), \quad \hat{\bar{p}}_{\bar{a}} = -i\left(\frac{\partial}{\partial \bar{z}^a} + iB\frac{\partial K}{\partial \bar{z}^a}\right),$$

$$[\hat{\chi}^m_\alpha, \hat{\bar{\chi}}^{\bar{n}}_\beta]_+ = \delta^{m\bar{n}}\delta_{\alpha\beta}.$$

In order to construct the system with the exact $\mathcal{N}=2$ supersymmetry (3.2), we have to find the appropriate candidates for $Q^\pm$, which obey the equations $\{Q^\pm, Q^\pm\} = 0$. Let us search the realization of supercharges among the functions

$$Q_\pm = \cos\lambda\, \Theta^\pm_1 + \sin\lambda\, \Theta^\pm_2, \tag{3.8}$$

where

$$\Theta^+_1 = \pi_a \eta^a_1 + i\bar{\partial}_a W \bar{\eta}^a_2, \quad \Theta^+_2 = \bar{\pi}_a \bar{\eta}^a_2 + i\partial_a W \eta^a_1,$$

$$\Theta^-_{1,2} = \bar{\Theta}^+_{1,2}, \tag{3.9}$$

and λ is some parameter. Calculating the Poisson brackets of the functions, we get

$$\{Q^+, Q^+\} = i(\sin 2\lambda B g_{a\bar{b}} + 2\omega \cos 2\lambda W_{a\bar{b}})\eta^a_1 \eta^b_2 \tag{3.10}$$

$$\{Q^+, Q^-\} = \mathcal{H}^0_{SUSY} + \cos 2\lambda B\mathbf{g}/2 - \sin 2\lambda \mathcal{Z}_3. \tag{3.11}$$

Here and in the following, we use the notation

$$\mathcal{H}^0_{SUSY} = \mathcal{H} - R_{a\bar{b}c\bar{d}}\eta^a_1 \bar{\eta}^b_1 \eta^c_2 \bar{\eta}^d_2 - iW_{a;b}\eta^a_1\eta^b_2 + iW_{\bar{a};\bar{b}}\bar{\eta}^a_1\bar{\eta}^b_2 + B\frac{\mathcal{F}_3}{2}, \tag{3.12}$$

where $\mathcal{H}$ denotes the oscillator Hamiltonian on CP^N [see the expressions in Eqs. (2.17),(2.19)], and

$$\mathcal{F}_3 = ig_{a\bar{b}}(\eta^a_1\bar{\eta}^b_1 - \eta^a_2\bar{\eta}^b_2), \quad \mathcal{Z}_3 = iW_{a\bar{b}}(\eta^a_1\bar{\eta}^b_1 - \eta^a_2\bar{\eta}^b_2),$$

$$\mathbf{g} = ig_{a\bar{b}}\eta^a_\alpha \bar{\eta}^b_\alpha. \tag{3.13}$$

In what follows, we will also need the generators

$$\mathcal{F}_+ = ig_{a\bar{b}}\eta^a_1\bar{\eta}^b_2, \quad \mathcal{F}_- = \bar{\mathcal{F}}_+, \tag{3.14}$$

which obey the commutation relations

$$\{\mathcal{F}_\pm, \mathcal{F}_3\} = \mp 2i\mathcal{F}_\pm, \quad \{\mathcal{F}_+, \mathcal{F}_-\} = i\mathcal{F}_3 \tag{3.15}$$

$$\{\Theta^\pm_\alpha, \mathcal{F}_\pm\} = 0, \quad \{\Theta^\pm_\alpha, \mathcal{F}_\mp\} = \pm i\epsilon_{\alpha\beta}\Theta^\mp_\beta,$$

$$\{\Theta^\pm_\alpha, \mathcal{F}_3\} = \pm i\Theta^\pm_\alpha, \tag{3.16}$$

$$\{\mathcal{F}_\pm, \mathbf{g}\} = \{\mathcal{F}_3, \mathbf{g}\} = 0,$$

$$\{\Theta^+_\alpha, \mathbf{g}\} = -i(\pi_a\eta^a_1 - i\bar{\partial}_a W\bar{\eta}^a_2), \quad \text{and so on.} \tag{3.17}$$

Comparing Eqs. (3.10),(3.11) with Eq. (3.2), we can construct the $\mathcal{N}=2$ supersymmetric oscillator on CP^N.

Superoscillator on CP^1. Consider the supersymmetrization of the oscillator on the complex projective plane CP^1. Comparing Eq. (3.10) with Eq. (3.2), we get

$$\{Q^\pm,Q^\pm\}=0\Rightarrow B=0, \quad \cos 2\lambda=0, \quad \sin 2\lambda=\pm 1. \tag{3.18}$$

Hence, we could choose *two* copies of the supercharges and Hamiltonians

$$Q^\pm_\alpha=\frac{\Theta^\pm_1-(-1)^\alpha\Theta^\pm_2}{\sqrt{2}},$$

$$\{Q^+_\alpha,Q^-_\alpha\}\equiv\mathcal{H}_\alpha=\mathcal{H}^0_{SUSY}+(-1)^\alpha\mathcal{Z}_3, \quad \alpha=1,2. \tag{3.19}$$

We constructed two copies of the $\mathcal{N}=2$ supersymmetric oscillator on CP^1. The inclusion of a constant magnetic field B breaks their $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$.

Note that

$$\{Q^\pm_\alpha,Q^\pm_\beta\}=2\epsilon_{\alpha\beta}\mathcal{Z}_\pm \quad Z_\pm=A(z\bar{z})\mathcal{F}_\pm, \quad Z_3=A(z\bar{z})\mathcal{F}_3,$$

where $A(z\bar{z})=\omega([1+(z\bar{z})^2]/(1-z\bar{z})^2)$. Hence, in the planar limit, one has $A\to\omega$, so that the generators $Q^\pm_\alpha, Z_\pm, Z_3, \mathcal{H}$ form a closed Lie superalgebra.

Superoscillator on CP^N, $N>1$. On higher-dimensional complex projective spaces one has

$$W_{a\bar{b}}=\omega g_{a\bar{b}}, \quad \Rightarrow\{Q^\pm,Q^\pm\}=0\Leftrightarrow B\sin 2\lambda+2\omega\cos 2\lambda=0. \tag{3.20}$$

Let us introduce the parameter λ_0

$$\cos 2\lambda_0=\frac{B/2}{\sqrt{\omega^2+(B/2)^2}}, \quad \sin 2\lambda_0=-\frac{\omega}{\sqrt{\omega^2+(B/2)^2}}, \tag{3.21}$$

so that

$$\lambda=\lambda_0+(-1)^\alpha\pi/2, \quad \alpha=1,2. \tag{3.22}$$

Hence, we get the following supercharges:

$$Q^\pm_\alpha=\cos\lambda_0\Theta^\pm_1+(-1)^\alpha\sin\lambda_0\Theta^\pm_2, \tag{3.23}$$

and the pair of corresponding $\mathcal{N}=2$ supersymmetric Hamiltonians

$$\mathcal{H}^\alpha_{SUSY}=\{Q^+_\alpha,Q^-_\alpha\}=\mathcal{H}^0_{SUSY}-(-1)^\alpha \times\left(\cos 2\lambda_0\frac{B}{2}\mathbf{g}-\sin 2\lambda_0\omega\mathcal{F}_3\right). \tag{3.24}$$

We constructed, on higher-dimensional complex projective spaces, two copies of exact $\mathcal{N}=2$ supersymmetric oscillators coupled to a constant magnetic field.

Calculating the commutators of $Q^\pm_1$ and $Q^\pm_2$ we get

$$\{Q^\pm_1,Q^\pm_2\}=2\omega\mathcal{F}_\pm, \quad \{Q^+_1,Q^-_2\}=\cos 2\lambda_0\mathcal{H}^0_{SUSY}+\frac{B}{2}\mathbf{g}, \tag{3.25}$$

where the Poisson brackets between $\mathcal{F}_\pm$, and $Q^\pm_\alpha$ look as follows:

$$\{Q^\pm_\alpha,\mathcal{F}_\pm\}=0, \quad \{Q^\pm_\alpha,\mathcal{F}_\mp\}=\pm\epsilon_{\alpha\beta}Q^\pm_\beta,$$
$$\{Q^\pm_\alpha,\mathcal{F}_3\}=\pm iQ^\pm_\alpha. \tag{3.26}$$

In the absence of a magnetic field, i.e., for $B=0$, $\cos 2\lambda_0=0$, $\sin\lambda_0=-1$, the two systems form the superalgebra

$$\begin{aligned}
&\{Q^\pm_\alpha,Q^\pm_\beta\}=2\omega\epsilon_{\alpha\beta}\mathcal{F}_\pm,\\
&\{Q^\pm_\alpha,Q^\mp_\beta\}=\delta_{\alpha\beta}\mathcal{H}^0_{SUSY}-\sigma^3_{\alpha\beta}\omega^2\mathcal{F}_3,\\
&\{Q^\pm_\alpha,\mathcal{F}_\pm\}=0, \quad \{Q^\pm_\alpha,\mathcal{F}_\mp\}=\pm\epsilon_{\alpha\beta}Q^\pm_\beta,\\
&\{Q^\pm_\alpha,\mathcal{F}_3\}=\pm iQ^\pm_\alpha,\\
&\{\mathcal{F}_\pm,\mathcal{F}_\mp\}=i\mathcal{F}_3, \quad \{\mathcal{F}_\pm,\mathcal{F}_3\}=\pm i\mathcal{F}_\pm.
\end{aligned} \tag{3.27}$$

The symmetry superalgebra of the oscillator on C^N coincides with the above one in any dimension, i.e., once again, we find a quite different behavior for the oscillators on CP^1 and CP^N, $N>1$ spaces, respectively.

Finally, we give the explicit expression of the Noether constants of motion corresponding to the $su(N)$ symmetry

$$\mathcal{J}^{SUSY}_{a\bar{b}}=\mathcal{J}_{a\bar{b}}+\frac{\partial^2 h_{a\bar{b}}}{\partial z^c\partial\bar{z}^d}\eta^c\sigma_3\bar{\eta}^d. \tag{3.28}$$

IV. CONCLUSION

We proposed an integrable system on CP^N, with $4N-1$ functionally independent constants of motion, which could be viewed as the generalization of a $2N$-dimensional oscillator. On the complex projective plane CP$^1=S^2$ this system coincides with the Higgs oscillator; the Kustaanheimo-Stiefel transformation of the system on CP^2 leads to the three-dimensional Coulomb-like system, which is equivalent to the MIC-Kepler problem on the three-dimensional hyperboloid obtained by the Kustaanheimo-Stiefel transformation of the oscillator on S^4. On the other hand, while the spherical oscillator remains unchanged upon transition from one hemisphere to another, the oscillator on CP^N, $N>1$, after transition to another chart, yields a system which, in spite of the absence of a rotational symmetry, remains "maximally integrable."

The oscillators on CP^3 and CP^4, in our opinion, deserve a separate study due to their relevance to the higher-dimensional quantum Hall effect [9]. This theory, based on the quantum mechanics of the particle on S^4 interacting with a $SU(2)$ monopole field, lately has been extended to CP^N spaces in the presence of a constant $U(1)$ (magnetic) field [10]. Since CP^3 can be viewed as a fiber bundle of S^4 with S^2 in the bundle, the four-dimensional quantum Hall system can be formulated as a system on CP^3 [10,11]. Performing the Hurwitz transformation of the oscillator on CP^4, we will get the five-dimensional Coulomb-like system with a $SU(2)$ Yang monopole. This system will have a degenerate ground state, hence it will be suitable for the developing of the five-dimensional quantum Hall effect in the Coulomb field (in the present versions of higher-dimensional quantum Hall theory, the potential field is used for the reduction to lower dimensions).

The Kähler structure makes the study of the coupling of a constant magnetic field to the oscillator on CP^N much simpler than on $2N$-dimensional sphere. In particular, we have shown that the oscillators on C^N and CP^N, $N>1$ coupled with a constant magnetic field behave similarly, with respect to $\mathcal{N}=2$ supersymmetrization. While a constant magnetic field breaks the $\mathcal{N}=2$ supersymmetry of the oscillator on sphere (and on the $\mathrm{CP}^1=S^2$), it preserves the $\mathcal{N}=2$ supersymmetry of the oscillators on CP^N, $N>1$ and C^N. On the other hand, in the absence of a magnetic field, the oscillator on C^N allows us to introduce $\mathcal{N}=4$ supersymmetry, while the oscillators on spheres and CP^N admit only $\mathcal{N}=2$ superextensions. It is easy to see that the similarity of the oscillators on C^N and CP^N, $N>1$, in their behavior with respect to supersymmetrization, is due to the special form of the Hamiltonian

$$\mathcal{H}=g^{a\bar{b}}(\pi_a\bar{\pi}_b+\omega^2\partial_a K\bar{\partial}_b K),$$

where K is a Kähler potential of the metric.

Therefore, from the viewpoint of $\mathcal{N}=2$ supersymmetry, the above Hamiltonian could be viewed as the generalization of the oscillator on an arbitrary Kähler manifold. In that case, the existence of hidden symmetries of the oscillator on CP^N could be viewed as an ‘‘accidental’’ one. Simultaneously, it is clear that the oscillators on other *symmetrical* Kähler spaces, say, on the Lobachewski spaces $\mathcal{L}$, or Grassmanians $\mathrm{Gr}_{N\cdot M}$, will have hidden symmetries, due to the translational invariance of the above spaces.

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