

## Domain wall lattices

Levon Pogosian

*Theoretical Physics, The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, United Kingdom*

Tanmay Vachaspati

*Department of Physics, Case Western Reserve University, 10900 Euclid Avenue, Cleveland, Ohio 44106-7079*

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We construct lattices with alternating kinks and antikinks. The lattice is shown to be stable in certain models. We consider the forces between kinks and antikinks and find that the lattice dynamics is that of a Toda lattice. Such lattices are exotic metastable states in which the system can get trapped during a phase transition.

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Domain walls are among the simplest topological defects known and have often been used as a test-bed for studying nonperturbative effects. An example of a domain wall is the “ $\phi^4$  kink” trivially extended to three spatial dimensions. This solution is often thought to typify all domain walls. However, it has recently become clear that the properties of  $\phi^4$  kinks do not simply carry over to more complicated systems [1–6], including condensed matter systems such as  $\text{He}^3$  [7].

While kink solutions must exist for topological reasons in models with spontaneously broken discrete symmetries, the precise form of the minimum energy kink solution depends on the details of the model and on any continuous symmetries that may be present. Topology only says that kinks are solutions that connect two points on the disconnected parts of the vacuum manifold. If the disconnected parts of the vacuum manifold contain more than a single point, then topological arguments alone are not sufficient to say which two points should be connected, that is, to specify the boundary conditions that the kink satisfies. More elaborate analysis is needed if one wants to know which of the boundary conditions can lead to the minimal energy solution [3].

As we show in this paper, the enhanced structure of kinks in  $SU(N) \times Z_2$  (for odd  $N$ ) allows for the construction of lattices of kinks and antikinks. In related models the kink lattice is also perturbatively stable. The lattices we are constructing are different from other known lattices such as an Abrikosov lattice [8]. The total topological charge of an Abrikosov lattice is nonvanishing. In contrast, the kink lattice we will construct will have a zero topological charge. This means that we can construct the kink lattice in a box with periodic boundary conditions, and also that the lattice is topologically equivalent to the vacuum. Hence if we start out in an unbroken symmetry phase, with a vanishing net topological charge, there is a chance that, after the symmetry is spontaneously broken, the system will be trapped in the lattice phase instead of the true vacuum. From the lattice phase, the system can then only reach the true vacuum by quantum tunneling.

We start with an  $SU(N) \times Z_2$  field theory whose Lagrangian is

$$L = \text{Tr}(\partial_\mu \Phi)^2 - V(\Phi) \quad (1)$$

where  $\Phi$  is an  $SU(N)$  adjoint and  $V(\Phi)$  is invariant under

$$G \equiv SU(N) \times Z_2. \quad (2)$$

$N \geq 3$  is taken to be odd, and the parameters in  $V$  are such that  $\Phi$  has an expectation value that can be chosen to be

$$\Phi_0 = \eta \sqrt{\frac{2}{N(N^2-1)}} \begin{pmatrix} n\mathbf{1}_{n+1} & \mathbf{0} \\ \mathbf{0} & -(n+1)\mathbf{1}_n \end{pmatrix}, \quad (3)$$

where  $n = (N-1)/2$ ,  $\mathbf{1}_p$  is the  $p \times p$  identity matrix and  $\eta$  is an energy scale determined by the minima of the potential  $V$ . Such an expectation value spontaneously breaks the symmetry down to

$$H = [SU(n+1) \times SU(n) \times U(1)] / Z_{n+1} \times Z_n. \quad (4)$$

We will choose  $V(\Phi)$  to be a quartic polynomial:

$$V(\Phi) = -m^2 \text{Tr}[\Phi^2] + h(\text{Tr}[\Phi^2])^2 + \lambda \text{Tr}[\Phi^4] + V_0 \quad (5)$$

where  $V_0$  is a constant chosen so that the minimum of the potential has  $V=0$ . The Lagrangian is symmetric under  $\Phi \rightarrow -\Phi$  and it is the breaking of this  $Z_2$  symmetry that gives rise to topological domain wall solutions. We could also extend the model by making it locally gauge invariant. The solutions described below will still be valid with the gauge fields set to zero; the stability analysis will change.

While our analysis can easily be carried out for general  $N$ , the physics is more transparent for a specific choice of  $N$ . Hence we will choose  $N=5$  and, where relevant, remark on the case of general  $N$ .<sup>1</sup> Then the desired symmetry breaking to

$$H = [SU(3) \times SU(2) \times U(1)] / [Z_3 \times Z_2] \quad (6)$$

is achieved in the parameter range

<sup>1</sup>The corresponding quartic model with  $N=3$  has an accidental  $SO(8)$  symmetry. We could consider  $N=3$  if  $(\text{Tr}(\Phi^3))^2$  and  $\text{Tr}(\Phi^6)$  terms were added to the potential. We have chosen to work with quartic potentials and with the larger value of  $N$ .

$$\frac{h}{\lambda} > -\frac{N^2+3}{N(N^2-1)} \Big|_{N=5} = -\frac{7}{30}. \quad (7)$$

The vacuum expectation value (VEV)  $\Phi_0$  is (up to any gauge rotation)

$$\Phi_0 = \frac{\eta}{\sqrt{60}} \text{diag}(2,2,2,-3,-3) \quad (8)$$

with  $\eta \equiv m/\sqrt{\lambda'}$  and

$$\lambda' \equiv h + \frac{N^2+3}{N(N^2-1)} \Big|_{N=5} \lambda = h + \frac{7}{30} \lambda. \quad (9)$$

In Refs. [1–3] it was found that there are several domain wall solutions in this model but a solution with least energy is achieved if  $\Phi(-\infty) \equiv \Phi_- = \Phi_0$  and

$$\Phi(+\infty) \equiv \Phi_+ = -\frac{\eta}{\sqrt{60}} \text{diag}(2,-3,-3,2,2). \quad (10)$$

Two features of  $\Phi_+$  are worthy of note. First, there is a minus sign in front. This puts  $\Phi_+$  and  $\Phi_-$  in disconnected parts of the vacuum manifold. The second feature is that two blocks of entries of  $\Phi_+$  are permuted with respect to those of  $\Phi_-$ . In other words,  $\Phi_-$  and  $-\Phi_+$  are related by a non-trivial gauge rotation. Furthermore, the kink solution (or, domain wall solution, in more than one dimension) can be written down explicitly in the case when  $h/\lambda = -3/20$  [1,2]:

$$\Phi_k = \frac{1 - \tanh(\sigma x)}{2} \Phi_- + \frac{1 + \tanh(\sigma x)}{2} \Phi_+ \quad (11)$$

where  $\sigma = m/\sqrt{2}$ . For other values of the coupling constants, the solution has been found numerically [1].

The topological charge of a kink can be defined as

$$Q = \sqrt{\frac{60}{\eta}} (\Phi_R - \Phi_L), \quad (12)$$

where  $\Phi_R$  and  $\Phi_L$  are the asymptotic values of the Higgs field to the right ( $R$ ) and left ( $L$ ) of the kink. (The rescaling has been done for convenience.) Then the charge of the kink in Eq. (11) is

$$Q^{(1)} = \text{diag}(-4,1,1,1,1). \quad (13)$$

Similarly, one can construct kinks with charge matrices  $Q^{(i)}$  ( $i=1, \dots, 5$ ) which have  $-4$  as the  $ii$  entry and  $+1$  in the remaining diagonal entries. Hence there are kink solutions with 5 different topological charge matrices. Individually, the kinks can be gauge rotated into one another. But when two kinks are present, the different charges are physically relevant. This is most easily seen by noting that the interaction between a kink with charge  $Q^{(i)}$  and an antikink with charge  $\bar{Q}^{(j)} = -Q^{(j)}$  is proportional to  $\text{Tr}(Q^{(i)}\bar{Q}^{(j)})$  [4]. Then we have

$$\text{Tr}(Q^{(i)}\bar{Q}^{(j)}) = -20 \quad \text{if } i=j$$

$$= +5 \quad \text{if } i \neq j. \quad (14)$$

The sign of the trace tells us if the force between the kink and antikink is attractive (minus) or repulsive (plus). Hence the force between a kink and an antikink with different orientations ( $i \neq j$ ) is repulsive. This observation is key to the construction of kink lattices.

In Ref. [4], the repulsive potential between a kink and an antikink at rest was derived. When the kink and antikink separation,  $R$ , is large, the result reduces to

$$U(R) = 4 \sqrt{\frac{2m^3}{\lambda}} e^{-2\sqrt{2}mR}. \quad (15)$$

To construct a kink lattice, we now need to arrange a periodic sequence of kink charges such that the nearest neighbor interactions are repulsive. Kinks that are not nearest neighbors but are further apart will also interact, and perhaps even attract each other. However the forces between kinks and antikinks fall off exponentially fast and just taking nearest-neighbor interactions into account should be sufficient, at least for lattice spacing larger than the kink width. So now we can write down a sequence of charges that can form a kink lattice. This is

$$\dots Q^{(1)} \bar{Q}^{(5)} Q^{(3)} \bar{Q}^{(1)} Q^{(5)} \bar{Q}^{(3)} \dots \quad (16)$$

and the sequence just repeats itself. Alternately, we could have a finite lattice if the kinks were in a compact space, such as a compact higher dimension, or the  $S^1$  that arises in evaluating the partition function in statistical mechanics.

The sequence listed above is the minimum sequence for which the nearest neighbor interactions are repulsive. The repeating length of 6 kinks is independent of  $N$  in  $SU(N)$  since it is clear that we need at least, and no more than, 3 different kinds of kink charges.

Another way to write the kink sequence is to write it as a sequence of Higgs field expectation values. We write this sequence for the above minimal lattice:

$$\begin{aligned} \dots \rightarrow & +(2,2,2,-3,-3) \rightarrow -(2,-3,-3,2,2) \\ & \rightarrow +(-3,2,2,-3,2) \rightarrow -(2,-3,2,2,-3) \\ & \rightarrow +(2,2,-3,-3,2) \rightarrow -(-3,-3,2,2,2) \\ & \rightarrow +(2,2,2,-3,-3) \rightarrow \dots \end{aligned} \quad (17)$$

We have constructed the solution for the minimal kink lattice numerically in one spatial dimension with periodic boundary conditions. In Fig. 1 we show the total energy of the minimal lattice as a function of lattice spacing.

The minimal lattice of 6 kinks is easily generalized to longer sequences. A sequence of 10 kinks in the  $N=5$  case is aesthetic in the sense that it uses all the 5 different charge matrices democratically:

$$\dots Q^{(1)} \bar{Q}^{(5)} Q^{(3)} \bar{Q}^{(4)} Q^{(2)} \bar{Q}^{(1)} Q^{(5)} \bar{Q}^{(3)} Q^{(4)} \bar{Q}^{(2)} \dots \quad (18)$$

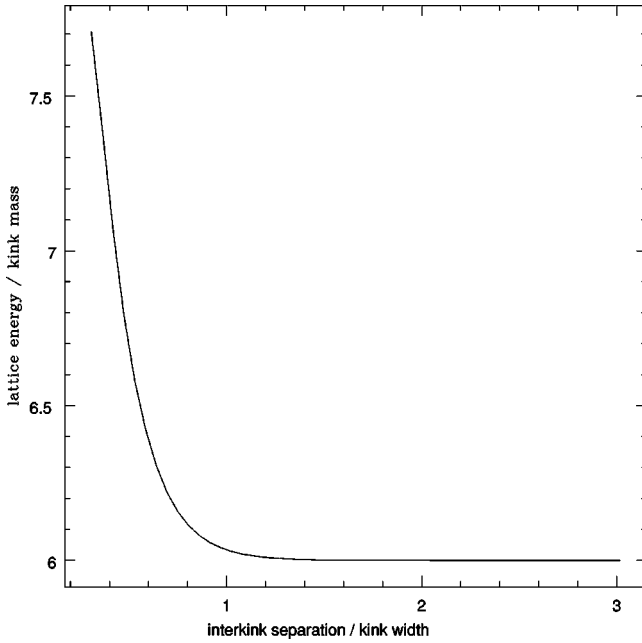


FIG. 1. The energy of the minimal lattice versus lattice spacing for  $h/\lambda = -3/20$ ,  $\lambda = 0.5$  and  $\eta = 1$ .

There can be longer sequences as well. Similarly one can construct sequences in the general  $N$  case.

We have also numerically studied the dynamics of the lattice by giving one of the kinks an initial velocity. We find that the kink scatters elastically on the neighboring antikink, and the motion propagates down the lattice. Indeed, lattices of masses interacting via exponentially decaying repulsive forces [see Eq. (15)] have been studied in the literature and are known as Toda lattices [9]. Hence the kink lattice is a Toda lattice.

We now discuss the stability of the lattice. A detailed stability analysis shows that the lattice in Eq. (16) has three unstable modes, corresponding to rotations in the 1-3, 1-5, 3-5 blocks. To clarify the instability, we draw an analogy between the kink lattice and a lattice of bar magnets (Fig. 2). Each bar magnet on its own has a zero mode that corresponds to rotations in three dimensional space. When placed in the lattice shown in Fig. 2, rotational zero modes turn into unstable modes. Similarly, an isolated kink has zero modes corresponding to rotations in gauge space—for example, a kink with charge  $Q_1$  can be rotated into the kink with charge  $Q_3$  without any cost in energy. When a kink of charge  $Q_1$  is placed near an antikink of charge  $\bar{Q}_3$ , the zero mode becomes an unstable mode, making it favorable for  $Q_1$  to rotate

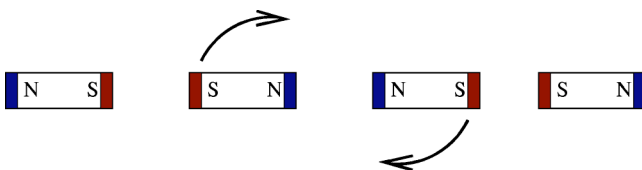


FIG. 2. A linear row of bar magnets placed North to North and South to South has an instability towards rotations in the transverse directions as shown.

into  $Q_3$  after which the kink and antikink can annihilate.<sup>2</sup>

This understanding of the instability also suggests a resolution: if the rotational zero modes are sufficiently suppressed or absent, the lattice will become stable. To suppress the rotational zero modes, we could break the symmetry further so that the kink is no longer invariant under rotations. We can also consider a case where the zero modes are completely absent right from the start. We will discuss this latter case as it is simpler to deal with and provides an explicit example of a field theory with stable kink lattices.

Consider the model with four real scalar fields,

$$L = \frac{1}{2} \sum_{i=1}^4 (\partial_\mu f_i)^2 + V(f_1, f_2, f_3, f_4) \quad (19)$$

and

$$\begin{aligned} V = & -\frac{m^2}{2} \sum_{i=1}^4 f_i^2 + \frac{h}{4} \left( \sum_{i=1}^4 f_i^2 \right)^2 + \frac{\lambda}{8} \sum_{a=1}^3 f_a^4 \\ & + \frac{\lambda}{4} \left[ \frac{7}{30} f_4^4 + f_1^2 f_2^2 \right] + \frac{\lambda}{20} [4(f_1^2 + f_2^2) + 9f_3^2] f_4^2 \\ & + \frac{\lambda}{\sqrt{5}} f_2 f_4 \left( f_1^2 - \frac{f_2^2}{3} \right) + \frac{m^2}{4} \eta^2. \end{aligned} \quad (20)$$

This model has been obtained by truncating the field  $\Phi$  occurring in Eq. (1) to its diagonal elements. The fields  $f_1$  and  $f_2$  correspond to the diagonal generators  $\lambda_3$  and  $\lambda_8$  of  $SU(3)$  [see Eq. (6)] in the Gell-Mann basis,  $f_3$  corresponds to the diagonal generator  $\tau_3$  of  $SU(2)$ , and  $f_4$  corresponds to the generator of  $U(1)$ . Now our four field model does not have the continuous  $SU(5)$  symmetry of the model in Eq. (1). The only remnant of the  $SU(5)$  symmetry corresponds to the permutation of the five diagonal entries of  $\Phi$ . In addition, the model also has the  $Z_2$  symmetry under which  $f_i \rightarrow -f_i$ . Hence the model has an  $S_5 \times Z_2$  symmetry.

A vacuum of the model is given by  $f_1 = 0 = f_2 = f_3$  and  $f_4 \neq 0$ . This breaks the symmetry to  $S_3 \times S_2$ , corresponding to permutations of  $\Phi$  in the  $SU(3)$  and  $SU(2)$  blocks. The vacuum manifold consists of  $5! \times 2/3! \times 2! = 20$  discrete points. If we fix the vacua at  $x = -\infty$ , this implies that there are 20 kink solutions in the model. All these 20 kink solutions have been described in Ref. [3].

The construction of kink lattices proceeds exactly as in the  $SU(5)$  case above because the off-diagonal components of  $\Phi$  vanish there. Hence the  $S_5 \times Z_2$  model contains kink lattice solutions. Furthermore, these lattices are stable because the dangerous rotational perturbations are absent by the very construction of the model.

The occurrence of stable kink lattices with net vanishing topological charge implies that there are metastable states in the field theory. Generally metastable states are present in

<sup>2</sup>If the  $SU(5)$  was a local gauge symmetry, these unstable modes would still be present. In addition, unstable modes due to the gauge fields could arise.

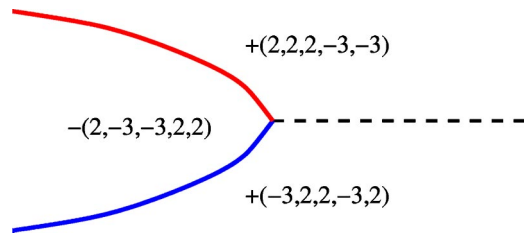


FIG. 3. The distribution of Higgs expectation values in three domains can lead to a wall junction [5]. In the  $SU(5)$  model, the dashed line is a nontopological wall [3]. In the  $S_5$  model the dashed line denotes a topological wall but without  $Z_2$  charge.

field theories due to features in the potential. Here, however, the metastable states are non-perturbative features of the model.

The existence of domain wall lattices is of interest in the context of phase transitions. What is the probability that a domain wall lattice will form during a phase transition? The answer depends on the complicated dynamics of a domain wall network in three spatial dimensions. For example, the model admits wall junctions of the kind shown in Fig. 3 [5] and different walls can have different tensions. We first note that the kinks with charge given in Eq. (13) (and permutations thereof) are the lightest kinks in the system having  $Z_2$  topology. Hence, it is likely that only they will survive at the end of the phase transition. Furthermore, since there are more repelling kink-antikink pairs than attracting ones, all attracting walls will eventually annihilate and remaining walls would be repelling. In future work we plan to study lattice formation in the  $S_5$  model in three spatial dimension by computer simulations.

A simpler situation to consider is the formation in one spatial dimension for the  $S_5$  model. We only need to consider the lightest kinks in the model—other kinks will decay into these kinks upon evolution. So we can restrict our attention to a sequence of kinks with charges given in Eq. (13) and permutations. Now let us assume that we have a kink with charge  $Q_1$ . A neighboring antikink can have charge  $\bar{Q}_1$ ,  $\bar{Q}_4$  or  $\bar{Q}_5$  [see Eq. (17)]. Of these only the first is unsuitable for a lattice and has a probability  $1/3$ . Therefore if the phase transition produces  $2n$  kinks, then the probability of having exactly  $2j$  kinks that annihilate and  $2n - 2j$  survive to form a lattice is derived by finding the number of ways of choosing

the  $j$  annihilating pairs and  $n - j$  surviving pairs and multiplying by the probability of annihilation ( $1/3$ ) and survival ( $2/3$ ). The result is that the probability of exactly  $j$  pairs annihilating is:  ${}^n C_j (1/3)^j (2/3)^{n-j}$ . Summing this expression from  $j=0$  to  $n-3$  gives the total probability for obtaining a lattice provided we have  $2n$  kinks. The sum can easily be evaluated. The interesting limit is when  $2n$  is large. In that case, the probability tends to unity. Hence a kink lattice is certain to form if there are a large number of kinks. Further, the number of kinks is large if a large number of correlation domains are produced during the phase transition.

It would be interesting to test these ideas in a laboratory systems in which a kink lattice can exist. Periodic boundary conditions could be achieved if a toroidal sample were to undergo a phase transition.

Finally we mention the implications of a domain wall lattice produced during a cosmological phase transition. If spacetime is  $R^4 \times S^1$  and the wall lattice resides in the (small) compact dimension, there will be an effective cosmological constant in the  $R^4$  due to invariance under Lorentz boosts of the wall Refs. [10,11]. The effective cosmological constant may be time dependent if the coupling constant  $\lambda$  were to run with energy scale, or to depend on the dynamics of the spacetime, or on another field. Yet another source of time dependence can come via the number of walls in the lattice since the wall lattice is not protected by topology or any conserved number. So the number of walls in the lattice can cascade down and eventually become zero. The difficulty with this cosmological scenario is that the extra compact dimension will not be static and will lead to an effective Newton's gravitational constant that is time dependent. Since the metric of the system is not yet known, it is not possible to say if the time variation can be slow enough for the scenario to be viable.

In conclusion, we have shown that stable lattices of domain walls can exist in a wide class of field theories. These are exotic metastable states in which the system can get trapped with high probability during a phase transition.

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