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Bootstrapping perturbative perfect actions

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We study the exact renormalization group of the four dimensional ϕ^4 theory perturbatively. We reformulate the differential renormalization group equations as integral equations that define the continuum limit of the theory directly with no need for a bare theory. We show how the self-consistency of the integral equations leads to the determination of the interaction vertices in the continuum limit. The inductive proof of the existence of a solution to the integral equations amounts to a proof of perturbative renormalizability, and it consists of nothing more than counting the scale dimensions of the interaction vertices. Universality is discussed within a context of the exact renormalization group.

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I. INTRODUCTION

Renormalization theory has more than fifty years of history starting from the studies of ultraviolet divergences in QED. Originally thought of as a cookbook recipe for obtaining finite results free of ultraviolet divergences, the idea of renormalization took a long time for its full acceptance until its physical meaning was clarified and its relation to universality in critical phenomena was understood. In his series of lectures [1] Wilson explains how to construct the continuum limit of a quantum field theory by taking a classical statistical model to the limit of criticality. The so-called scaling functions are what particle physicists call renormalized Green functions from which scattering cross sections of elementary particles can be computed.

One thread of development, lattice simulation of field theory, was started immediately after Wilson's work on renormalization, and the field thrives to this day. Another development started somewhat later when Polchinski applied Wilson's exact renormalization group (ERG) equation to the study of perturbative renormalization [2].

The central idea in Wilson's renormalization theory is the theory space that consists of all possible theories with a fixed cutoff scheme. Flows of the renormalization group are generated in the theory space under the rescaling of distance. To keep track of the renormalization group flows exactly, the theory space must contain an infinite number of dimensions allowing for all possible interaction vertices. But only a finite dimensional subspace, denoted as $S(\infty)$ in Sec. 12 of Ref. [1], is of fundamental importance. This is the space of flows originating from an ultraviolet fixed point. It is parametrized by a finite number of parameters, called relevant parameters. Any theory in this subspace can be traced backward along a renormalization group flow to the fixed point, and the theory gives a continuum limit. In more recent literature, the theories in $S(\infty)$ are called (quantum) perfect actions, implying that they contain the physics of continuous space despite the use of a finite momentum cutoff. For a review on perfect actions, see Ref. [3] and references therein.

In Ref. [2] Polchinski rendered Wilson's exact differential renormalization group equations to a form more manageable for perturbative studies. Polchinski used his form of equations to obtain a quantitative estimate for how the flows from bare theories approach $S(\infty)$. Strictly speaking the theory considered, ϕ^4, has no ultraviolet fixed point, but $S(\infty)$ exists perturbatively, and the distance between the flows and $S(\infty)$ has been shown to behave as e^{-2t}, where t is the logarithmic momentum scale so that the physical cutoff momentum is e^t times the physical renormalization scale [10]. Polchinski's work brought Wilson's physical insight into renormalization to the perturbative renormalization theory which had been mostly diagrammatic and calculational.

The purpose of the present work is to simplify the perturbative study of the exact renormalization group (ERG) even further by reformulating the ERG differential equations as integral equations that define the continuum limit $S(\infty)$ directly. As is well known, an integral equation is nothing more than a differential equation together with an initial (or asymptotic) condition, but the rewriting brings a great advantage in this case. The advantage is that the integral equations incorporate "renormalizability" of the theory manifestly. If the equations have a solution, the theory is renormalizable automatically. The issue is not the ultraviolet finiteness of the theory, but it is the existence of a solution.

The existence of a solution to the integral equations is proved using perturbation theory. A recursive solution of the integral equations is what we call perturbation theory. The integral equations are self-contained, and can determine themselves. Using the word "bootstrap" as a mnemonic for the self-determining nature, we can say that the integral equations bootstrap themselves.

A careful examination of the original work of Polchinski has been made in Ref. [4] where the issues of perturbative analyticity, unitarity, and causality have also been studied. The emphasis of the present paper is on the new formulation of the ERG in terms of integral equations and on the new insights given by the formulation, and we do not aim at the rigor exemplified in Ref. [4]. Our "proof" in Sec. IV is a "physicist's proof" which should be acceptable to almost any physicist.

The present paper is organized as follows. In Sec. II we review the perturbative treatment of Wilson's ERG by Polchinski. In Sec. III we introduce the reformulation of the

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ERG as integral equations. In Sec. IV we prove inductively that the integral equations have a solution. In Sec. V we discuss universality in the context of the integral ERG equations. Finally in Sec. VI we conclude the paper with comments for further developments of the integral equation approach.

II. EXACT RENORMALIZATION GROUP EQUATIONS

A large amount of literature is available on the exact renormalization group (see Ref. [5] and references therein), and the main purpose of this section is to set the notation for the rest of the paper. Please note that unlike what is common in the field theory literature, a rescaling is done after each step of renormalization to keep the momentum cutoff constant.

A. Brief review of Polchinski's rendition of Wilson's differential ERG equation

We consider a $\mathbf{Z}_2$ invariant scalar field theory in four dimensional Euclidean space. The propagator of the scalar field ϕ is given by

$$\frac{K(p)}{p^2+m^2} \tag{1}$$

where the momentum cutoff function $K(p)$ is a smooth scalar function that is monotonically decreasing in p^2 and has the property

$$K(p)=\begin{cases} 1 & \text{for} \quad p^2<1, \\ 0 & \text{for} \quad p^2>2^2. \end{cases} \tag{2}$$

The cutoff function $K(p)$ is fixed once and for all for all the theories in the theory space.

Each point in the theory space corresponds to an interaction action given in the following form:

$$-S_{\rm int}[\phi] = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \times \int_{p_1,\ldots,p_{2n-1}} \phi(p_1)\ldots\phi(p_{2n})\mathcal{V}_{2n}(p_1,\ldots,p_{2n}) \tag{3}$$

where $p_{2n} \equiv -(p_1+\cdots+p_{2n-1})$, and the momentum integrals are taken only over the $2n-1$ independent momenta. The notation

$$\int_p f(p) \equiv \int \frac{d^4p}{(2\pi)^4} f(p) \tag{4}$$

is used for the momentum integral. We will call $\mathcal{V}_{2n}$ an interaction vertex from now on. $\phi(p)$ is the Fourier transform of the scalar field in the momentum space. Once the interaction action $S_{\rm int}$ is given, the generating functional of the Green functions is obtained as

$$Z[J] = \exp\left(\frac{1}{2}\int_p \frac{K(p)}{p^2+m^2}\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}\right) \times e^{-S_{\rm int}[\phi]+\int_p J(-p)\phi(p)}\Big|_{\phi=0} \tag{5}$$

and the $2n$-point Green function is given by

$$\langle\phi(p_1)\ldots\phi(p_{2n-1})\phi\rangle_{m^2,\mathcal{V}} = \exp\left(\frac{1}{2}\int_p \frac{K(p)}{p^2+m^2}\frac{\delta}{\delta\phi(p)}\frac{\delta}{\delta\phi(-p)}\right) \times \phi(p_1)\ldots\phi(p_{2n-1})\phi e^{-S_{\rm int}[\phi]}\Big|_{\phi=0}. \tag{6}$$

Since the cutoff function K is fixed, a theory is specified by the choice of the squared mass m^2 and the interaction action $S_{\rm int}$. The latter is characterized by an infinite number of interaction vertices $\{\mathcal{V}_{2n}\}$, and the theory space is infinite dimensional.

The ERG transformation by scale $e^{\Delta t}$, where Δt is an infinitesimal positive constant, is defined so that the momentum p corresponds to the higher momentum $pe^{\Delta t}$ under the transformation. In the case of free theory, all the vertices $\{\mathcal{V}_{2n}\}$ vanish, and only the squared mass scales as

$$m^2 \to m^2 e^{2\Delta t} \tag{7}$$

under the renormalization. In the presence of interactions, we must transform the vertices $\{\mathcal{V}_{2n}\}$ to $\{\mathcal{V}_{2n}+\Delta\mathcal{V}_{2n}\}$ so that the Green functions are related by

$$\langle\phi(p_1)\ldots\phi(p_{2n-1})\phi\rangle_{m^2,\mathcal{V}} = e^{(4n-y_{2n})\Delta t}\langle\phi(p_1e^{\Delta t})\ldots\phi(p_{2n-1}e^{\Delta t})\phi\rangle_{m^2e^{2\Delta t},\mathcal{V}+\Delta\mathcal{V}} \tag{8}$$

where we define

$$y_{2n} \equiv 4-2n. \tag{9}$$

The infinitesimal change of the interaction action $\Delta S_{\rm int}[\phi]$ corresponding to the infinitesimal change of the interaction vertices $\{\Delta\mathcal{V}_{2n}\}$ was given by Wilson in Ref. [1]. In this paper we will consider the particular form given by Polchinski in Ref. [2]:

$$-\Delta S_{\rm int} = \Delta t \cdot \frac{1}{2}\int_p \frac{-2p^2\dfrac{dK(p)}{dp^2}}{p^2+m^2} \times \left\{\frac{\delta S_{\rm int}}{\delta\phi(-p)}\frac{\delta S_{\rm int}}{\delta\phi(p)} - \frac{\delta^2 S_{\rm int}}{\delta\phi(p)\delta\phi(-p)}\right\}. \tag{10}$$

[This is exactly the same as Eq. (18) of Ref. [2] rewritten in our notation.] We can obtain the ERG equations for the individual vertices $\{\mathcal{V}_{2n}\}$ by substituting the perturbative expansion (3) into the above.

Let us introduce a logarithmic scale parameter t to define a one-parameter family of vertices $\{\mathcal{V}_{2n}(t)\}$. At t, the squared mass is given by $m^2 e^{2t}$. Polchinski's equation (10) implies the following differential equations for the vertices:

$$\begin{aligned}\frac{d}{dt}&(e^{-y_{2n}t}\mathcal{V}_{2n}(t;p_1e^t,\ldots,p_{2n}e^t))\\ &=\sum_{k=0}^{[(n-1)/2]}\sum_{\substack{\text{partitions:}\\ I+J=\{2n\}}} e^{-y_{2(k+1)}t}\mathcal{V}_{2(k+1)}(t;p_{I_1}e^t,\ldots,p_{I_{2k+1}}\\ &\quad\times e^t,-(p_{I_1}+\cdots+p_{I_{2k+1}})e^t)\\ &\quad\times\frac{\Delta((p_{I_1}+\cdots+p_{I_{2k+1}})^2e^{2t})}{(p_{I_1}+\cdots+p_{I_{2k+1}})^2+m^2}e^{-y_{2(n-k)}t}\mathcal{V}_{2(n-k)}\\ &\quad\times(t;p_{J_1}e^t,\ldots,p_{J_{2(n-k)-1}}e^t,(p_{I_1}+\cdots+p_{I_{2k+1}})e^t)\\ &\quad+\frac{1}{2}\int_q\frac{\Delta(qe^t)}{q^2+m^2}e^{-y_{2(n+1)}t}\mathcal{V}_{2(n+1)}\\ &\quad\times(t;qe^t,-qe^t,p_1e^t,\ldots,p_{2n}e^t)\end{aligned}\tag{11}$$

where the sum over partitions is the sum over all possible ways of splitting $p_1,\ldots,p_{2n}$ into two groups, and I and J stand for the groups of $2k+1$, $2(n-k)-1$ elements, respectively. Later we will introduce a shorthand notation p_I to mean either the list of $p_{I_1},\ldots,p_{I_{2k+1}}$ or the sum $p_{I_1}+\cdots+p_{I_{2k+1}}$. The same goes for p_J. The function $\Delta(p)$ is defined by

$$\Delta(p)\equiv-2p^2\frac{d}{dp^2}K(p).\tag{12}$$

The Gauss symbol $[(n-1)/2]$ denotes the largest integer less than or equal to $(n-1)/2$.

Two comments are in order:

(1) Given the requirement (8), the ERG transformation is not uniquely determined due to a potential change of fields, and it depends on a choice of convention. Here we have adopted a particular convention so that all the renormalization effects, including the renormalization of the squared mass and wave function, are included in the renormalization of the interaction vertices $\{\mathcal{V}_{2n}\}$.

(2) The constant y_{2n} defined by Eq. (9) is the scale dimension of the vertex $\mathcal{V}_{2n}$. For example, we find

$$y_2=2,\quad y_4=0,\quad y_6=-2,\quad y_8=-4,\ldots.\tag{13}$$

The left-hand side of the ERG equation (11) implies that the effect of $\mathcal{V}_{2n}(0)$ on the vertices $\{\mathcal{V}_{2n}(t)\}$ at the logarithmic scale t is of order $e^{y_{2n}t}$. Hence, $\mathcal{V}_2$ is relevant, $\mathcal{V}_4$ is marginal, and $\mathcal{V}_{2n\geqslant 6}$ are irrelevant. This point will be explained again at the end of Sec. III.

B. Conventional use of the ERG equations

In Ref. [2] the ERG equation (10) was used to prove the perturbative renormalizability of the ϕ^4 theory. As the starting point of a renormalization group flow we choose a bare theory defined by the squared mass $m^2e^{2t_0}$ and the vertices

$$\mathcal{V}_2(t_0;p)=a_2(t_0;\lambda)+m^2\mathrm{e}^{2t_0}z_m(t_0;\lambda)+p^2z_\phi(t_0;\lambda)\tag{14}$$

$$\mathcal{V}_4(t_0;p_1,\ldots,p_4)=(-\lambda)[1+z_\lambda(t_0;\lambda)]\tag{15}$$

$$\mathcal{V}_{2n\geqslant 6}(t_0;p_1,\ldots,p_{2n})=0\tag{16}$$

where t_0 is a large *negative* constant, and a_2, z_m, z_ϕ, and z_λ are perturbative series in the coupling constant λ. We run the ERG to obtain $\{\mathcal{V}_{2n}(t=0)\}$. If $\{\mathcal{V}_{2n}(0)\}$ exist in the limit $t_0\to-\infty$, we call the theory renormalizable. Polchinski showed that the limit exists if we choose a_2, z_m, z_ϕ, and z_λ as appropriate power series in λ and t_0, and that the approach to the limit behaves as e^{2t_0} with power corrections in t_0 at each order of perturbation theory.

In the next section we will introduce a more direct way of obtaining the continuum limit $\{\mathcal{V}_{2n}(0)\}$. We will not define the continuum limit by taking the infrared limit of a bare theory as above, but we will define it in terms of integral equations that the continuum limit must obey. The integral equations construct $S(\infty)$ without the help of any bare theory.

III. CONSTRUCTION OF INTEGRAL EQUATIONS

We construct integral equations from the differential equation (10) [or equivalently Eqs. (11)] by following a standard procedure. An integral equation is a combination of a differential equation with an initial condition, and in our case it is the ultraviolet asymptotic behavior of the vertex functions that plays the role of the initial condition.

If the theory had a good honest ultraviolet fixed point, the asymptotic behavior of the vertex functions would be simply

$$\mathcal{V}_{2n}(-t;p_1,\ldots,p_{2n})\to\mathcal{V}^*_{2n}(p_1,\ldots,p_{2n})\quad\text{as}\quad t\to+\infty\tag{17}$$

where $\{\mathcal{V}^*_{2n}\}$ are the fixed-point vertices. The perturbative ϕ^4 theory does not have an ultraviolet fixed point, and we must replace the above asymptotic conditions by alternative conditions. We impose that the vertices be given by

$$\begin{aligned}\mathcal{V}_{2n}(-t;p_1,\ldots,p_{2n})\to{}&A_{2n}(-t;p_1,\ldots,p_{2n})\\ &+m^2e^{-2t}B_{2n}(-t;p_1,\ldots,p_{2n})\\ &+\cdots\end{aligned}\tag{18}$$

as $t\to+\infty$ where

(1) The right-hand side is an expansion in powers of m^2e^{-2t}.

(2) A_{2n} and B_{2n}, independent of m^2, are finite order polynomials of t at each order in perturbation theory.

(3) A_{2n} and B_{2n} are local, i.e., they can be expanded in powers of momenta if the momenta are small compared to 1.

We will construct integral equations using the above assumptions, and in Sec. IV we will justify the assumptions using the integral equations themselves.

Here is a comment on the sign convention for the parameter t. The parameter t was introduced in the previous section to denote the logarithmic renormalization scale. It grows as we go downstream toward infrared on the RG flow. Since we will need to go upstream to write down integral equations, we will mainly deal with negative t in the rest of the paper. Since we easily forget that t is negative, we denote it explicitly as $-t$ so that $t>0$ when we go upstream on the RG flow.

The asymptotic behavior (18) implies in particular

$$e^{2t}\mathcal{V}_2(-t;pe^{-t})\to e^{2t}A_2(-t;0)+p^2\frac{d}{dp^2}A_2(-t;p)\bigg|_{p^2=0}+m^2B_2(-t;0) \tag{19}$$

$$\mathcal{V}_4(-t;p_1e^{-t},\dots,p_4e^{-t})\to A_4(-t;0,0,0,0) \tag{20}$$

$$e^{y_{2n}t}\mathcal{V}_{2n}(-t;p_1e^{-t},\dots,p_{2n}e^{-t})\to 0 \quad \text{for} \quad 2n\geqslant 6 \tag{21}$$

as $t\to+\infty$ where the corrections are suppressed by e^{-2t} (with powers of t). Here we recall that the squared mass that goes with the vertices $\{\mathcal{V}_{2n}(-t)\}$ is m^2e^{-2t}, and the above expansions are taken in powers of m^2e^{-2t} and momenta pe^{-t}. The last equation is valid because of $y_{2n}<0$ and the assumed polynomial behavior of $\mathcal{V}_{2n}(-t)$.

The above asymptotic behavior makes the following equations trivially valid:

$$\begin{aligned} e^{2t}\mathcal{V}_2(-t;pe^{-t}) &= \lim_{T\to\infty}\bigg[(e^{2t}\mathcal{V}_2(-t;pe^{-t})-e^{2(t+T)}\mathcal{V}_2(-t-T;pe^{-t-T})) \\ &\quad +e^{2(t+T)}A_2(-t-T;0)+p^2\frac{d}{dp^2}A_2(-t-T;p)\bigg|_{p^2=0} \\ &\quad +m^2B_2(-t-T;0)\bigg] \end{aligned} \tag{22}$$

$$\begin{aligned} \mathcal{V}_4(-t;p_1e^{-t},\dots,p_4e^{-t}) &= \lim_{T\to\infty}[(\mathcal{V}_4(-t;p_1e^{-t},\dots,p_4e^{-t}) \\ &\quad -\mathcal{V}_4(-t-T;p_1e^{-t-T},\dots,p_4e^{-t-T})) \\ &\quad +A_4(-t-T;0,0,0,0)] \end{aligned} \tag{23}$$

$$\begin{aligned} e^{y_{2n}t}\mathcal{V}_{2n}(-t;p_1e^{-t},\dots,p_{2n}e^{-t}) &= \lim_{T\to\infty}[e^{y_{2n}t}\mathcal{V}_{2n}(-t;p_1e^{-t},\dots,p_{2n}e^{-t}) \\ &\quad -e^{y_{2n}(t+T)}\mathcal{V}_{2n}(-t-T;p_1e^{-t-T},\dots,p_{2n}e^{-t-T})]. \end{aligned} \tag{24}$$

The differences of the vertices on the right-hand sides are obtained by integrating the differential ERG equations (11) of the previous section. We obtain

$$\begin{aligned} e^{2t}\mathcal{V}_2(-t;pe^{-t}) &= \lim_{T\to\infty}\bigg[\int_0^T dt'\bigg\{e^{2(t+t')}\mathcal{V}_2(-t-t';pe^{-t-t'})\frac{\Delta(pe^{-t-t'})}{p^2+m^2}e^{2(t+t')}\mathcal{V}_2(-t-t';pe^{-t-t'}) \\ &\quad +\frac{1}{2}\int_q\frac{\Delta(qe^{-t-t'})}{q^2+m^2}\mathcal{V}_4(-t-t';qe^{-t-t'},-qe^{-t-t'},pe^{-t-t'},-pe^{-t-t'})\bigg\} \\ &\quad +e^{2(t+T)}A_2(-t-T)+p^2C_2(-t-T)+m^2B_2(-t-T)\bigg] \end{aligned} \tag{25}$$

$$\begin{aligned} \mathcal{V}_4(-t;p_1e^{-t},\dots,p_4e^{-t}) &= \lim_{T\to\infty}\bigg[\int_0^T dt'\bigg\{\sum_{i=1}^4 e^{2(t+t')}\mathcal{V}_2(-t-t';p_ie^{-t-t'})\frac{\Delta(p_ie^{-t-t'})}{p_i^2+m^2}\mathcal{V}_4(-t-t';p_1e^{-t-t'},\dots,p_4e^{-t-t'}) \\ &\quad +\frac{1}{2}\int_q\frac{\Delta(qe^{-t-t'})}{q^2+m^2}e^{-2(t+t')}\mathcal{V}_6(-t-t';qe^{-t-t'},-qe^{-t-t'},p_1e^{-t-t'},\cdots,p_4e^{-t-t'})\bigg\} \\ &\quad +A_4(-t-T)\bigg] \end{aligned} \tag{26}$$

$$e^{y_{2n}t}\mathcal{V}_{2n}(-t;p_1\mathrm{e}^{-t},\ldots,p_{2n}e^{-t})=\lim_{T\to\infty}\int_0^T dt'\Bigg\{\sum_{k=0}^{[(n-1)/2]}\sum_{\substack{\text{partitions:}\\ I+J=\{2n\}}} e^{y_{2(k+1)}(t+t')}\mathcal{V}_{2(k+1)}(-t-t';p_I e^{-t-t'})\times\frac{\Delta(p_I e^{-t-t'})}{p_I^2+m^2}e^{y_{2(n-k)}(t+t')}\mathcal{V}_{2(n-k)}(-t-t';p_J e^{-t-t'})+\frac{1}{2}\int_q\frac{\Delta(qe^{-t-t'})}{q^2+m^2}e^{y_{2(n+1)}(t+t')}\times\mathcal{V}_{2(n+1)}(-t-t';qe^{-t-t'},-qe^{-t-t'},p_1e^{-t-t'},\ldots)\Bigg\} \tag{27}$$

where we have introduced the notation

$$A_2(-t)\equiv A_2(-t;0),\quad B_2(-t)\equiv B_2(-t;0),$$

$$C_2(-t)\equiv\frac{\partial}{\partial p^2}A_2(-t;p)\bigg|_{p^2=0} \tag{28}$$

$$A_4(-t)\equiv A_4(-t;0,0,0,0) \tag{29}$$

and the short-hand notations p_I,p_J have been used.

The above integral equations are not self-contained yet, since the right-hand sides depend on the asymptotic forms A_2, B_2, C_2, and A_4 which are known only after the left-hand sides, i.e., $\mathcal{V}_2$ and $\mathcal{V}_4$, are known.

A crucial observation is to be made now: the requirement that the limits $T\to+\infty$ to exist for the above equations determines the asymptotic forms. We will explain this in the remainder of this section.

For large T, we find the following asymptotic behavior using Eq. (18):

$$\frac{1}{2}\int_q\frac{\Delta(qe^{-t-T})}{q^2+m^2}\mathcal{V}_4(-t-T;qe^{-t-T},-qe^{-t-T},pe^{-t-T},-pe^{-t-T})$$
$$\to\frac{1}{2}e^{2(t+T)}\int_q\frac{\Delta(q)}{q^2}A_4(-t-T;q,-q,0,0)+p^2\frac{1}{2}\left(\frac{d}{dp^2}\int_q\frac{\Delta(q)}{q^2}A_4(-t-T;q,-q,p,-p)\right)_{p^2=0}$$
$$+m^2\frac{1}{2}\int_q\Delta(q)\left(\frac{1}{q^2}B_4(-t-T;q,-q,0,0)-\frac{1}{q^4}A_4(-t-T;q,-q,0,0)\right), \tag{30}$$

$$\frac{1}{2}\int_q\frac{\Delta(qe^{-t-T})}{q^2+m^2}e^{-2(t+T)}\mathcal{V}_6(-t-T;qe^{-t-T},-qe^{-t-T},p_1e^{-t-T},\ldots,p_4e^{-t-T})$$
$$\to\frac{1}{2}\int_q\frac{\Delta(q)}{q^2}A_6(-t-T;q,-q,0,0,0,0). \tag{31}$$

In deriving this it is important to note that $\Delta(q)$ is nonvanishing only for $1<|q|<2$.

The above asymptotic behavior determines the t dependence of the asymptotic forms A_2, B_2, C_2, and A_4 so that the limit $T\to+\infty$ of the integral equations exist. We must therefore obtain

$$-\frac{d}{dt}(e^{2t}A_2(-t))=e^{2t}\frac{1}{2}\int_q\frac{\Delta(q)}{q^2}A_4(-t;q,-q,0,0) \tag{32}$$

$$-\frac{d}{dt}B_2(-t)=\frac{1}{2}\int_q \Delta(q)\left(\frac{1}{q^2}B_4(-t;q,-q,0,0)-\frac{1}{q^4}A_4(-t;q,-q,0,0)\right) \tag{33}$$

$$-\frac{d}{dt}C_2(-t)=\frac{1}{2}\frac{\partial}{\partial p^2}\int_q \frac{\Delta(q)}{q^2}A_4(-t;q,-q,p,-p)\bigg|_{p^2=0} \tag{34}$$

$$-\frac{d}{dt}A_4(-t)=\frac{1}{2}\int_q \frac{\Delta(q)}{q^2}A_6(-t;q,-q,0,0,0,0). \tag{35}$$

Thus, the asymptotic forms are determined by the asymptotic forms of the higher point vertices up to t-independent constants. Hence, we obtain

$$e^{2t}A_2(-t)=-\int^t dt' e^{2t'}\frac{1}{2}\int_q \frac{\Delta(q)}{q^2}A_4(-t';q,-q,0,0) \tag{36}$$

$$B_2(-t)=\int_0^t dt'\frac{1}{2}\int_q \Delta(q)\left(-\frac{1}{q^2}B_4(-t';q,-q,0,0)+\frac{1}{q^4}A_4(-t';q,-q,0,0)\right)+B_2(0) \tag{37}$$

$$C_2(-t)=-\int_0^t dt'\frac{1}{2}\frac{d}{dp^2}\int_q \frac{\Delta(q)}{q^2}\times A_4(-t';q,-q,p,-p)\bigg|_{p^2=0}+C_2(0) \tag{38}$$

$$A_4(-t)=-\int_0^t dt'\frac{1}{2}\int_q \frac{\Delta(q)}{q^2}A_6(-t';q,-q,0,0,0,0)+A_4(0). \tag{39}$$

Here, $B_2(0)$, $C_2(0)$, and $A_4(0)$ are t-independent constants which cannot be determined by the differential Eqs. (33),(34),(35). The constants $B_2(0),C_2(0)$ have to do with finite renormalization of the squared mass and wave function, respectively. The constant $A_4(0)$ is the self-coupling constant, and together with m^2 it parametrizes the space of continuum limit $S(\infty)$. We will discuss more about these finite constants in Sec. V.

The determination of $A_2(-t)$ by Eq. (36) needs an explanation. As it is, the integral over t' is ambiguous by a constant, which implies that $A_2(-t)$ is ambiguous by a constant multiple of e^{-2t}. From Eq. (19) the large t behavior of the two-point vertex at zero momentum is given by

$$\mathcal{V}_2(-t;0)\to A_2(-t)+m^2e^{-2t}B_2(-t). \tag{40}$$

Hence, the ambiguity of order e^{-2t} in $A_2(-t)$ has the same order of magnitude as the $B_2(-t)$ term. We wish to remove the ambiguity in such a way that only the term proportional to m^2 gives the order e^{-2t} contribution to the asymptotic form of $\mathcal{V}_2(-t;0)$ above. This choice is equivalent to the mass independent scheme, and it turns out that with this choice the massless theory is given by $m^2=0$ [11]. To complete the definition of $A_2(-t)$, we must first define a kth order polynomial $T_k(t)$ by the condition

$$\frac{d}{dt}(e^{2t}T_k(t))=e^{2t}t^k. \tag{41}$$

Imposing that $T_k(t)$ be a polynomial, we have removed the potential ambiguity of order e^{-2t}. Now, given a power series expansion in t

$$\frac{1}{2}\int_q \frac{\Delta(q)}{q^2}A_4(-t;q,-q,0,0)=\sum_{k=0}^{\infty} t^kP_k \tag{42}$$

we define $A_2(-t)$ unambiguously by

$$A_2(-t)\equiv-\sum_{k=0}^{\infty} T_k(t)P_k. \tag{43}$$

This is the precise meaning of Eq. (36). For a concrete expression of the polynomial $T_k(t)$, please refer to Appendix B.

We have thus obtained the following integral equations:

$$e^{2t}\mathcal{V}_2(-t;pe^{-t})=\int_0^\infty dt'\Bigg[e^{2(t+t')}\mathcal{V}_2(-t-t';pe^{-t-t'})\frac{\Delta(pe^{-t-t'})}{p^2+m^2}e^{2(t+t')}\mathcal{V}_2(-t-t';pe^{-t-t'})+\frac{1}{2}\int_q\Delta(qe^{-t-t'})$$
$$\times\Bigg\{\frac{1}{q^2+m^2}\mathcal{V}_4(-t-t';qe^{-t-t'},-qe^{-t-t'},pe^{-t-t'},-pe^{-t-t'})-\frac{1}{q^2}$$
$$\times A_4(-t-t';qe^{-t-t'},-qe^{-t-t'},0,0)-p^2e^{-2(t+t')}\frac{1}{q^2}\frac{\partial}{\partial p^2}A_4(-t-t';qe^{-t-t'},-qe^{-t-t'},p,-p)\Big|_{p^2=0}$$
$$-m^2e^{-2(t+t')}\left(\frac{1}{q^2}B_4(-t-t';qe^{-t-t'},-qe^{-t-t'},0,0)-\frac{1}{q^4}A_4(-t-t';qe^{-t-t'},-qe^{-t-t'},0,0)\right)\Bigg\}\Bigg]$$
$$+e^{2t}A_2(-t)+p^2C_2(-t)+m^2B_2(-t), \tag{44}$$

$$\mathcal{V}_4(-t;p_1e^{-t},\ldots,p_4e^{-t})=\int_0^\infty dt'\Bigg[\sum_{i=1}^{4}e^{2(t+t')}\mathcal{V}_2(-t-t';p_ie^{-t-t'})\frac{\Delta(p_ie^{-t-t'})}{p_i^2+m^2}\mathcal{V}_4(-t-t';p_1e^{-t-t'},\ldots,p_4e^{-t-t'})$$
$$+\frac{1}{2}\int_q\Delta(qe^{-t-t'})\Bigg\{\frac{1}{q^2+m^2}e^{-2(t+t')}\mathcal{V}_6(-t-t';qe^{-t-t'},-qe^{-t-t'},p_1e^{-t-t'},\ldots,p_4e^{-t-t'})$$
$$-\frac{1}{q^2}e^{-2(t+t')}A_6(-t-t';qe^{-t-t'},-qe^{-t-t'},0,\ldots,0)\Bigg\}\Bigg]+A_4(-t), \tag{45}$$

$$e^{y_{2n}t}\mathcal{V}_{2n}(-t;p_1e^{-t},\ldots,p_{2n}e^{-t})=\int_0^\infty dt'\Bigg\{\sum_{k=0}^{[(n-1)/2]}\sum_{\substack{\text{partitions:}\\ I+J=\{2n\}}}e^{y_{2(k+1)}(t+t')}\mathcal{V}_{2(k+1)}(-t-t';p_Ie^{-t-t'})\frac{\Delta(p_Ie^{-t-t'})}{p_I^2+m^2}$$
$$\times e^{y_{2(n-k)}(t+t')}\mathcal{V}_{2(n-k)}(-t-t';p_Je^{-t-t'})+\frac{1}{2}\int_q\frac{\Delta(qe^{-t-t'})}{q^2+m^2}$$
$$\times e^{y_{2(n+1)}(t+t')}\mathcal{V}_{2(n+1)}(-t-t';qe^{-t-t'},-qe^{-t-t'},p_1e^{-t-t'},\ldots)\Bigg\} \tag{46}$$

where $A_2(-t)$ is given by Eqs. (36),(43), $B_2(-t)$ by Eq. (37), $C_2(-t)$ by Eq. (38), and $A_4(-t)$ by Eq. (39). The constants $B_2(0)$, $C_2(0)$, and $A_4(0)$ are input parameters to be discussed further in Sec. V. These integral equations are self-contained in the sense that they admit a perturbative solution as explained in the next section.

Before we end this section, let us make two observations. We first observe how the relevance, marginality, and irrelevance of vertices manifest themselves in the above integral equations. We notice that the $2n$-point vertex $\mathcal{V}_{2n}(-t-t')$ always appears multiplied by the exponential factor $e^{y_{2n}(t+t')}$. Thus, at scale $-t$ the effect of the two-point vertex $\mathcal{V}_2(-t-t')$ at scale $-t-t'$ is of order $e^{2t'}\gg 1$ if $t'\gg 1$, and it is relevant. The effect of the four-point vertex $\mathcal{V}_4(-t-t')$ at scale $-t$ is unsuppressed or marginal. But the effect of $\mathcal{V}_{2n\geqslant 6}(-t-t')$ is only of order $e^{y_{2n}t'}\ll 1$, and it is irrelevant.

We also observe the mechanism behind the finiteness of the integrals over t' in the integral equations. The right-hand sides of the integral equations have two parts. The first part consists of products of two vertices. For any external momentum p, $\Delta(pe^{-t'})$ vanishes for large t' since $\Delta(p)=0$ for $|p|<1$. Hence, the first part is finite upon integration over t'. The second part consists of a loop integral over the momentum q. For large t' the integrand of the t' integral behaves as $e^{-2t'}$ either by the exponential factor $e^{y_{2(n+1)}t'}$ or by the subtractions of asymptotic forms. Thus, the second part is also finite upon integration over t'.

IV. SOLUTION OF INTEGRAL EQUATIONS

In this section we wish to show that the integral equations (44)–(46) derived in the previous section determine the vertex functions order by order in perturbation theory.

A. Flow of perturbative solutions

To solve the integral equations (44)–(46) for the vertex functions $\{\mathcal{V}_{2n}(t=0)\}$ at scale $t=0$, we must determine the vertex functions along the entire renormalization group flow from $t=+\infty$ leading up to the end point $t=0$. Each renormalization flow is parametrized by the squared mass m^2, and the constants $B_2(0)$, $C_2(0)$, and $A_4(0)$. In order to solve the integral equations perturbatively in powers of the coupling constant λ, we must assume that these constants can be expanded in powers of λ as

$$B_2(0)=\sum_{k=1}^{\infty} (-\lambda)^k z_m^{(k)} \tag{47}$$

$$C_2(0)=\sum_{k=1}^{\infty} (-\lambda)^k z_\phi^{(k)} \tag{48}$$

$$A_4(0)=-\lambda+\sum_{k=1}^{\infty} (-\lambda)^{1+k} z_\lambda^{(k)} \tag{49}$$

where $z_m^{(k)}$, $z_\phi^{(k)}$, and $z_\lambda^{(k)}$ are arbitrary constants. The choice of these constants correspond to a convention or a renormalization scheme as will be discussed more fully in Sec. V. One choice convenient for explicit calculations is our version of the "minimal subtraction" scheme defined by

$$B_2(0)=C_2(0)=0, \quad A_4(0)=-\lambda. \tag{50}$$

In the following discussion, however, we will not choose the minimal subtraction scheme, and we will keep our choice of $B_2(0),C_2(0),A_4(0)$ arbitrary [12].

Let us recall the recursive solution of an integral equation

$$f(x)=\lambda+\int dy G(x,y) f(y)^2 \tag{51}$$

where G is a known integration kernel. The recursive solution gives $f(x)$ as a power series in λ:

$$f(x)=\lambda+\sum_{k=1}^{\infty} \lambda^{1+k} f_k(x). \tag{52}$$

If $f(x)$ is determined to order λ^{n-1}, we can use it to compute the right-hand side up to order λ^n. Thus, $f(x)$ is obtained to order λ^n. The recursive method works because the integral is quadratic in f.

The structure of our integral equations (44)–(46) is similar to the simple integral equation above. The starting point of the perturbative calculations of the vertices is the four-point vertex $\mathcal{V}_4$ at order λ:

$$\mathcal{V}_4(-t;p_1,\ldots,p_4)=A_4(-t)=-\lambda. \tag{53}$$

Everything bootstraps from this.

Let us briefly sketch the perturbative procedure leaving details to the next subsection. (Lowest order calculations are given in Appendix A.) Suppose we have computed all the vertices up to order λ^{n-1} at which only $\mathcal{V}_2,\ldots,\mathcal{V}_{2n}$ are nonvanishing. At order $\lambda^n (n\geqslant 1)$, we must start from $\mathcal{V}_{2(n+1)}$ which is given explicitly by the tree-level Feynman diagrams with n vertices $(-\lambda)$ and $n-1$ internal propagators

$$\frac{1-K(p)}{p^2+m^2 e^{-2t}}.$$

For example, we find

$$e^{-2t}\mathcal{V}_6(-t;p_1e^{-t},\ldots,p_6e^{-t}) =(-\lambda)^2\left(\frac{1-K((p_1+p_2+p_3)e^{-t})}{(p_1+p_2+p_3)^2+m^2}+9\text{ permutations}\right) \tag{54}$$

$$e^{-4t}\mathcal{V}_8(-t;p_1e^{-t},\ldots,p_8e^{-t}) =(-\lambda)^3\left(\frac{1-K((p_1+p_2+p_3)e^{-t})}{(p_1+p_2+p_3)^2+m^2} \times\frac{1-K((p_4+p_5+p_6)e^{-t})}{(p_4+p_5+p_6)^2+m^2}+279\text{ permutations}\right). \tag{55}$$

We can obtain these by solving the tree-level integral equations

$$\begin{aligned} e^{y_{2n}t}\mathcal{V}_{2n}(-t;p_1e^{-t},\ldots,p_{2n}e^{-t}) &=\int_0^\infty dt' \Bigg\{ \sum_{k=0}^{[(n-1)/2]} \sum_{\substack{\text{partitions:}\\ I+J=\{2n\}}} e^{y_{2(k+1)}(t+t')} \\ &\times \mathcal{V}_{2(k+1)}(-t-t';p_I e^{-t-t'}) \frac{\Delta(p_I e^{-t-t'})}{p_I^2+m^2} e^{y_{2(n-k)}(t+t')} \\ &\times \mathcal{V}_{2(n-k)}(-t-t';p_J e^{-t-t'}) \Bigg\} \end{aligned} \tag{56}$$

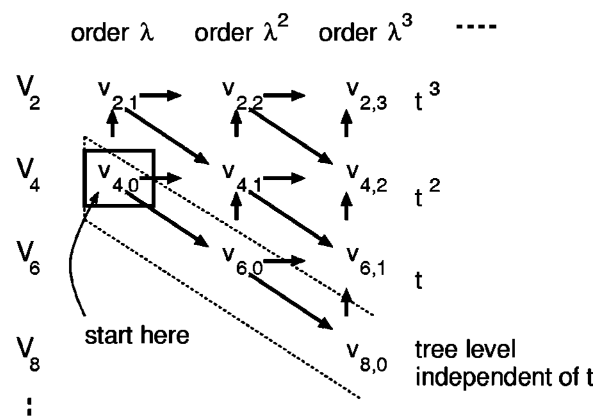

FIG. 1. Flow of perturbative calculations: every vertex upstream is necessary to determine a vertex.

where there is no loop integral over q on the right-hand side. The dependency of the vertices is such that we need only the tree-level vertices $\mathcal{V}_4,\ldots,\mathcal{V}_{2n}$ to construct $\mathcal{V}_{2(n+1)}$ at the tree level.

The next vertex to compute at order λ^n is $\mathcal{V}_{2n}$. The right-hand side of the integral equation (46) has two parts. To get an order λ^n contribution from the first part, we need $\mathcal{V}_2$ to order λ, $\mathcal{V}_4$ to λ^2, $\mathcal{V}_6$ to $\lambda^3,\ldots,\mathcal{V}_{2(n-1)}$ to λ^{n-1}, and $\mathcal{V}_{2n}$ to order λ^{n-1}. All these suffice to be lower order in λ than λ^n since the first part consists of a product of two vertices. To get an order λ^n contribution from the second part, we need $\mathcal{V}_{2(n+1)}$ to order λ^n, which is obtained by the previous step. Proceeding analogously we can calculate $\mathcal{V}_{2(n-1)},\ldots,\mathcal{V}_2$ up to order λ^n.

The flow of perturbative calculations sketched above is shown in Fig. 1. We will elaborate on this further in the next subsection.

B. Perturbative proof of the λ,t dependence of the vertices

The purpose of this subsection is to prove the existence of a perturbative solution to the integral equations (44)–(46) by proving the following λ,t dependence of the vertex functions:

$$\mathcal{V}_{2n}(-t;p_1,\ldots,p_{2n}) = \sum_{k=0}^{\infty} (-\lambda)^{n-1+k} v_{2n,k}(-t;p_1,\ldots,p_{2n};m^2e^{-2t}) \tag{57}$$

where $v_{2n,k}(-t;p_1,\ldots,p_{2n};m^2e^{-2t})$ is an order k polynomial of t. $v_{2n,k}$ corresponds to the k-loop contribution to the vertex. The only exception to Eq. (57) is for $n=1$, for which we take the λ independent part vanishing:

$$v_{2,0}=0. \tag{58}$$

Hence, for $\lambda=0$, all the vertices $\mathcal{V}_{2n}$ vanish. The starting point of the perturbative solution is given by

$$v_{4,0}=1 \tag{59}$$

which is independent of the mass and momenta.

We note that by proving the above λ,t dependence we also prove the assumption on the polynomial behavior of the asymptotic forms (18). Equation (57) gives

$$A_{2n}(-t;p_1,\ldots,p_{2n}) = \sum_{k=0}^{\infty} (-\lambda)^{n-1+k} v_{2n,k}(-t;p_1,\ldots,p_{2n};0) \tag{60}$$

$$B_{2n}(-t;p_1,\ldots,p_{2n}) = \sum_{k=0}^{\infty} (-\lambda)^{n-1+k} \frac{\partial}{\partial m^2} v_{2n,k}(-t;p_1,\ldots,p_{2n};m^2)\bigg|_{m^2=0}. \tag{61}$$

From Eqs. (19),(20) we also obtain

$$A_2(-t)=\sum_{k=1}^{\infty} (-\lambda)^k v_{2,k}(-t;0,0;0) \tag{62}$$

$$B_2(-t)=\sum_{k=1}^{\infty} (-\lambda)^k \frac{\partial}{\partial m^2} v_{2,k}(-t;0,0;m^2)\bigg|_{m^2=0} \tag{63}$$

$$C_2(-t)=\sum_{k=1}^{\infty} (-\lambda)^k \frac{\partial}{\partial p^2} v_{2,k}(-t;p,-p;0)\bigg|_{p^2=0} \tag{64}$$

$$A_4(-t)=-\lambda+\sum_{k=1}^{\infty} (-\lambda)^{1+k} v_{4,k}(-t;0,0,0,0;0). \tag{65}$$

The inductive proof of the λ,t dependence (57) is straightforward. The dependence is valid for the starting point (59) of induction. We wish to prove the validity of the λ,t dependence (57) for $v_{2n,k}$ assuming its validity for all $v_{2n',k'}$ upstream in Fig. 1 where either

$$n'+k'<n+k \tag{66}$$

or

$$n'+k'=n+k \quad \text{and} \quad k'<k. \tag{67}$$

(In Fig. 1, each column has the same $n+k$. As we go toward right, $n+k$ increases. As we go up, k increases, and n decreases.) There are three cases we must consider separately: $n>2$, $n=2$, and $n=1$. First we consider the case $n>2$. By substituting the assumed results into the right-hand side of the integral equation (46) for $\mathcal{V}_{2n}$, we obtain

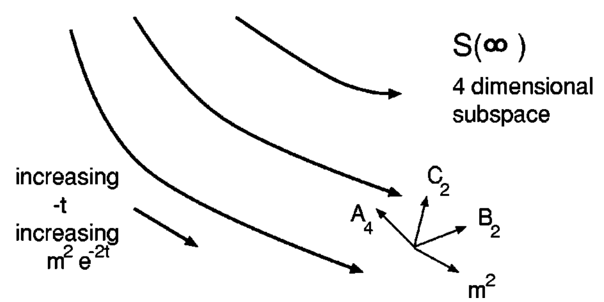

FIG. 2. Flows of ERG: the perfect actions make a 4-dimensional subspace with coordinates m^2, $B_2(0)$, $C_2(0)$, $A_4(0)$.

$$v_{2n,k}(-t;p_1,\ldots,p_{2n};m^2) = \int_0^\infty dt' \Bigg[e^{(y_{2n}+2)t'} \sum_{j=0}^{[(n-1)/2]} \sum_{l=0}^{k} \sum_{\substack{\text{partitions:}\\ I+J=\{2n}} v_{2(j+1),l}(-(t+t');p_I e^{-t'};m^2 e^{-2t'}) \times \frac{\Delta(p_I e^{-t'})}{p_I^2+m^2} v_{2(n-j),k-l}(-(t+t');p_J e^{-t'};m^2 e^{-2t'}) + \frac{1}{2}\int_q \frac{\Delta(q)}{q^2+m^2 e^{-2t'}} e^{y_{2n}t'} v_{2(n+1),k-1}(-(t+t');q,-q,p_1 e^{-t'},\ldots;m^2 e^{-2t'}) \Bigg]. \tag{68}$$

The right-hand side contains only the lower order vertices for which the induction hypothesis is assumed valid. The first sum gives at most order t^k, and the second loop integral gives only t^{k-1}. Hence, $v_{2n,k}$ is an order k polynomial of t.

Next we look at the special case $n=2$. The integral equation (45) gives

$$v_{4,k}(-t;p_1,\ldots,p_4;m^2) = \int_0^\infty dt' \Bigg[e^{2t'} \sum_{i=1}^{4} \sum_{l=1}^{k} v_{2,l}(-(t+t');p_i e^{-t'},-p_i e^{-t'};m^2 e^{-2t'}) \frac{\Delta(p_i e^{-t'})}{p_i^2+m^2} v_{4,k-l}(-(t+t');p_1 e^{-t'},\ldots,p_4 e^{-t'};m^2 e^{-2t'}) + \frac{1}{2}\int_q \Bigg\{ \frac{\Delta(q)}{q^2+m^2 e^{-2t'}} v_{6,k-1}(-(t+t');q,-q,p_1 e^{-t'},\ldots,p_4 e^{-t'};m^2 e^{-2t'}) - \frac{\Delta(q)}{q^2} v_{6,k-1}(-(t+t');q,-q,0,0,0,0;0) \Bigg\} \Bigg] + v_{4,k}(-t;0,0,0,0;0). \tag{69}$$

The first term in the integral gives at most order t^k, and the second loop integral at most order t^{k-1}. The last term is obtained from Eqs. (39),(49) as

$$v_{4,k}(-t;0,0,0,0;0) = z_\lambda^{(k)} - \int_0^t dt' \frac{1}{2}\int_q \frac{\Delta(q)}{q^2} v_{6,k-1}(-t';q,-q,0,0,0,0;0). \tag{70}$$

Since $v_{6,k-1}(-t;q,-q,0,0,0,0;0)$ is a polynomial of order $k-1$ by the induction hypothesis, the above equation implies that $v_{4,k}(-t;0,0,0,0;0)$ is an order k polynomial.

Finally we consider the case $n=1$. The integral equation (44) gives

$$v_{2,k}(-t;p,-p;m^2) = \int_0^\infty dt' \Bigg[e^{4t'} \sum_{l=1}^{k-1} v_{2,l}(-(t+t');pe^{-t'},-pe^{-t'};m^2 e^{-2t'}) \frac{\Delta(pe^{-t'})}{p^2+m^2} \times v_{2,k-l}(-(t+t');pe^{-t'},-pe^{-t'};m^2 e^{-2t'}) + \frac{1}{2}\int_q \Delta(q) \Bigg\{ \frac{1}{q^2+m^2 e^{-2t'}} e^{2t'} \times v_{4,k-1}(-(t+t');q,-q,pe^{-t'},-pe^{-t'};m^2 e^{-2t'}) - \frac{1}{q^2} e^{2t'} v_{4,k-1}(-(t+t');q,-q,0,0;0) - \frac{1}{q^2} p^2 \times \frac{\partial}{\partial p^2} v_{4,k-1}(-(t+t');q,-q,p,-p;0)\bigg|_{p^2=0} - \frac{1}{q^2} m^2 \frac{\partial}{\partial m^2} v_{4,k-1}(-(t+t');q,-q,0,0;m^2)\bigg|_{m^2=0} + \frac{1}{q^4} m^2 v_{4,k-1}(-(t+t');q,-q,0,0;0) \Bigg\} \Bigg] + v_{2,k}(-t;0,0;0) + p^2 \frac{\partial}{\partial p^2} v_{2,k}(-t;p,-p;0)\bigg|_{p^2=0} + m^2 \frac{\partial}{\partial m^2} v_{2,k}(-t;0,0;m^2)\bigg|_{m^2=0}. \tag{71}$$

The first sum gives at most order t^k, and the second loop integral over q gives at most order t^{k-1}. The last line is obtained from Eqs. (36)–(38) and Eqs. (47),(48) as

$$e^{2t} v_{2,k}(-t;0,0;0) = -\int^t dt' e^{2t'} \frac{1}{2}\int_q \frac{\Delta(q)}{q^2} v_{4,k-1}(-t';q,-q,0,0;0) \tag{72}$$

$$\left.\frac{\partial}{\partial p^2} v_{2,k}(-t;p,-p;0)\right|_{p^2=0} = z_\phi^{(k)} - \int_0^t dt' \frac{1}{2}\int_q \frac{\Delta(q)}{q^2} \left.\frac{\partial}{\partial p^2} v_{4,k-1}(-t';q,-q,p,-p;0)\right|_{p^2=0} \tag{73}$$

$$\left.\frac{\partial}{\partial m^2} v_{2,k}(-t;0,0;m^2)\right|_{m^2=0} = z_m^{(k)} - \int_0^t dt' \frac{1}{2}\int_q \Delta(q) \left(\frac{1}{q^2} \left.\frac{\partial}{\partial m^2} v_{4,k-1}(-t';q,-q,0,0;m^2)\right|_{m^2=0} - \frac{1}{q^4} v_{4,k-1}(-t';q,-q,0,0;0) \right). \tag{74}$$

The precise meaning of the integral on the right-hand side of Eq. (72) has been given in the paragraph leading to Eq. (43): the integral converts t^j into an order j polynomial $T_j(t)$. The induction hypothesis implies that the left-hand sides in the above are all at most order t^k. Hence, we have proven that $v_{2,k}$ is at most order t^k.

This concludes the inductive proof of the λ, t dependence given by (57). We have thus proven the existence of a perturbative solution to the integral equations (44)–(46). Since the integral equations define a continuum limit directly, we have proven the perturbative renormalizability of the ϕ^4 theory at the same time.

V. UNIVERSALITY

In the previous section we have shown the existence of a perturbative solution of the ERG integral equations (44)–(46). In this section we consider two issues related to universality: first we will count the independent degrees of freedom of the continuum limit, and second we will consider how the Green functions depend (or not depend) on the choice of a momentum cutoff function $K(p)$.

We first recall that each solution of the integral equations (44)–(46) gives an entire trajectory of the ERG flow in the space $S(\infty)$ of the continuum limit. Each trajectory is parametrized by $-t$ which ranges from $-\infty$ to 0, and it is specified by a squared mass m^2 and three input parameters $B_2(0)$, $C_2(0)$, and $A_4(0)$. We can regard m^2, and $B_2(0)$, $C_2(0)$, $A_4(0)$ as the four coordinates of the end point of the ERG trajectory. Hence, the space $S(\infty)$ is four dimensional (see Fig. 2). According to the usual understanding of the ϕ^4 theory, however, the continuum limit has only two parameters: a squared mass m^2 and a self-coupling constant λ. We wish to reconcile this discrepancy.

Clearly the parameter $A_4(0)$ corresponds to the self-coupling constant λ. The other two parameters $B_2(0)$ and $C_2(0)$, which we can take as zero in the minimal subtraction scheme (50), are related to finite renormalization of the squared mass and wave function, respectively.

Since the space of the continuum limit $S(\infty)$ is physically two-dimensional, there should be a two dimensional group of transformations which relate physically equivalent theories. More concretely, we should be able to find an infinitesimal change of the parameters m^2, $B_2(0)$, $C_2(0)$, and $A_4(0)$ so that the Green functions remain unchanged up to normalization. Such a transformation should map an entire ERG flow to another physically equivalent ERG flow. Without derivation, we write down the infinitesimal transformation $\mathcal{V}_{2n} \to \mathcal{V}_{2n} + \delta\mathcal{V}_{2n}$ with the expected properties:

$$\begin{aligned} e^{2t}\delta\mathcal{V}_2(-t;pe^{-t}) &= \eta(p^2+m^2) + \epsilon m^2 + e^{2t}\mathcal{V}_2(-t;pe^{-t}) \left\{ -\eta + 2(1-K(pe^{-t}))\left(\eta + \frac{\epsilon m^2}{p^2+m^2}\right)\right\} \\ &\quad - (e^{2t}\mathcal{V}_2(-t;pe^{-t}))^2 \frac{K(pe^{-t})(1-K(pe^{-t}))}{p^2+m^2}\left(\eta + \frac{\epsilon m^2}{p^2+m^2}\right) \\ &\quad - \frac{1}{2}\int_q \frac{K(qe^{-t})(1-K(qe^{-t}))}{q^2+m^2}\left(\eta + \frac{\epsilon m^2}{q^2+m^2}\right)\mathcal{V}_4(-t;qe^{-t},-qe^{-t},pe^{-t},-pe^{-t}) \end{aligned} \tag{75}$$

and, for $2n \geqslant 4$,

$$e^{y_{2n}t}\delta V_{2n}(-t;p_1e^{-t},\dots,p_{2n}e^{-t})=\sum_{i=1}^{2n}\left\{-\frac{\eta}{2}+(1-K(p_i\mathrm{e}^{-t}))\left(\eta+\frac{\epsilon m^2}{p_i^2+m^2}\right)\right\}e^{y_{2n}t}\mathcal{V}_{2n}(-t;p_1e^{-t},\dots,p_{2n}e^{-t})$$
$$-\sum_{k=0}^{[(n-1)/2]}\sum_{\substack{\text{partitions:}\\ I+J=\{2n\}}}e^{y_{2(k+1)}t}\mathcal{V}_{2(k+1)}(-t;p_Ie^{-t})\frac{K(p_Ie^{-t})(1-K(p_Ie^{-t}))}{p_I^2+m^2}$$
$$\times\left\{\eta+\frac{\epsilon m^2}{p_I^2+m^2}\right\}e^{y_{2(n-k)}t}\mathcal{V}_{2(n-k)}(-t;p_Je^{-t})-\frac{1}{2}\int_q\frac{K(qe^{-t})(1-K(qe^{-t}))}{q^2+m^2}$$
$$\times\left\{\eta+\frac{\epsilon m^2}{q^2+m^2}\right\}e^{y_{2(n+1)}t}\mathcal{V}_{2(n+1)}(-t;qe^{-t},-qe^{-t},p_1e^{-t},\dots,p_{2n}e^{-t})\qquad(76)$$

where ϵ,η are infinitesimal constants.

The above transformation satisfies the following two properties:

(1) The Green functions change only by normalization:

$$\langle\phi(p_1)\cdots\phi(p_{2n-1})\phi\rangle_{m^2e^{-2t},\mathcal{V}(-t)}$$
$$=(1-n\eta)$$
$$\times\langle\phi(p_1)\cdots\phi(p_{2n-1})\phi\rangle_{m^2(1+\epsilon)e^{-2t},(\mathcal{V}+\delta\mathcal{V})(-t)}.\qquad(77)$$

(2) The transformed vertices $(\mathcal{V}_{2n}+\delta\mathcal{V}_{2n})(-t)$ satisfy the ERG equations (11) for the squared mass $m^2(1+\epsilon)e^{-2t}$.

(For a proof of the above properties, please refer to Ref. [6].)

The infinitesimal transformation defined by Eqs. (75),(76) corresponds to the following infinitesimal change of the parameters:

$$m^2\to m^2(1+\epsilon)\qquad(78)$$

$$B_2(0)\to(1-\eta)B_2(0)+\epsilon+\eta-\frac{1}{2}\int_q K(q)(1-K(q))$$
$$\times\left((-\eta+\epsilon)\frac{1}{q^4}A_4(0;q,-q,0,0)\right.$$
$$\left.+\eta\frac{1}{q^2}B_4(0;q,-q,0,0)\right)\qquad(79)$$

$$C_2(0)\to(1-\eta)C_2(0)+\eta-\eta\frac{1}{2}\frac{\partial}{\partial p^2}$$
$$\times\int_q\frac{K(q)(1-K(q))}{q^2}A_4(0;q,-q,p,-p)\Big|_{p^2=0}\qquad(80)$$

$$A_4(0)\to(1-2\eta)A_4(0)$$
$$-\eta\frac{1}{2}\int_q\frac{K(q)(1-K(q))}{q^2}A_6(0;q,-q,0,0,0,0).\qquad(81)$$

These infinitesimal transformations generate equivalence classes of theories, and the space of the equivalence classes is two-dimensional.

In Ref. [6] we will modify the ERG equations by introducing a running squared mass and an anomalous scale dimension of the field ϕ. With the modification we can no longer take $B_2(0)$ and $C_2(0)$ as arbitrary, and the space $S(\infty)$ becomes two-dimensional.

We now proceed to the next issue. We recall that universality usually means that the Green functions of the scalar field ϕ is unique up to normalization of the field. In other words the Green functions in the continuum limit do not depend on how the continuum limit is taken. In the present context universality demands that we get the same Green functions no matter what momentum cutoff function $K(p)$ we use, as long as $K(p)$ is 1 for small $|p|$ and 0 for large $|p|$. Under a change of K, the Green functions should change in such a way that the differences can be compensated by appropriate finite change of the parameters and normalization of the field.

Let us consider the Green functions computed with the vertices $\{\mathcal{V}_{2n}(-t)\}$ using a modified propagator $(K+\delta K)(p)/(p^2+m^2e^{-2t})$, where the infinitesimal change $\delta K(p)$ vanishes for $|p|<1$ and for large $|p|$. The change of the Green functions due to the modified propagator can be reproduced using the original cutoff function $K(p)$ but using a different set of vertices $\{(\mathcal{V}_{2n}+\delta\mathcal{V}_{2n})(-t)\}$:

$$\langle\phi(p_1)\dots\phi(p_{2n-1})\phi\rangle_{K,m^2e^{-2t},(\mathcal{V}+\delta\mathcal{V})(-t)}$$
$$=\langle\phi(p_1)\dots\phi(p_{2n-1})\phi\rangle_{K+\delta K,m^2e^{-2t},\mathcal{V}(-t)}.\qquad(82)$$

The change $\{\delta\mathcal{V}_{2n}(-t)\}$ of the vertices necessary for the above equality is most easily obtained by a diagrammatic consideration. By interpreting the δK not as part of a propagator but as part of a vertex, we find that the appropriate infinitesimal change of the vertices is given by

$$
\begin{aligned}
&e^{y_{2n}t}\delta\mathcal{V}_{2n}(-t;p_1e^{-t},\ldots,p_{2n}e^{-t})\\
&\quad=-\sum_{k=0}^{[(n-1)/2]}\sum_{\substack{\text{partitions:}\\ I+J=\{2n\}}} e^{y_{2(k+1)}t}\mathcal{V}_{2(k+1)}(-t;p_Ie^{-t})\\
&\qquad\times\frac{\delta K(p_Ie^{-t})}{p_I^2+m^2}e^{y_{2(n-k)}t}\mathcal{V}_{2(n-k)}(-t;p_Je^{-t})\\
&\qquad-\frac{1}{2}\int_q\frac{\delta K(qe^{-t})}{q^2+m^2}e^{y_{2(n+1)}t}\mathcal{V}_{2(n+1)}\\
&\qquad\times(-t;qe^{-t},-qe^{-t},p_1e^{-t},\ldots,p_{2n}e^{-t}). \qquad (83)
\end{aligned}
$$

It is straightforward to check that the vertices $\{(\mathcal{V}_{2n}+\delta\mathcal{V}_{2n})(-t)\}$ satisfy the ERG equations (11) with the squared mass m^2e^{-2t}.

The above change of the vertices corresponds to the following change of the input parameters to the integral ERG equation:

$$
\delta B_2(0)=-\frac{1}{2}\int_q\delta K(q)\left(\frac{1}{q^2}B_4(0;q,-q,0,0)-\frac{1}{q^4}A_4(0;q,-q,0,0)\right) \qquad (84)
$$

$$
\delta C_2(0)=-\frac{1}{2}\frac{\partial}{\partial p^2}\int_q\frac{\delta K(q)}{q^2}A_4(0;q,-q,p,-p)\bigg|_{p^2=0} \qquad (85)
$$

$$
\delta A_4(0)=-\frac{1}{2}\int_q\frac{\delta K(q)}{q^2}A_6(0;q,-q,0,0,0,0). \qquad (86)
$$

Hence, the ERG trajectory specified by m^2, $B_2(0)$, $C_2(0)$, and $A_4(0)$ in $S(\infty)$ with the cutoff $K+\delta K$ is equivalent to the ERG trajectory specified by m^2, $(B_2+\delta B_2)(0)$, $(C_2+\delta C_2)(0)$, and $(A_4+\delta A_4)(0)$ in $S(\infty)$ with the cutoff K. Thus, with this equivalence, the space of theories in the continuum limit is independent of the choice of a momentum cutoff function K. In other words the continuum limit is universal.

VI. CONCLUSION

In this paper we have reformulated the exact renormalization group equation of Wilson in terms of integral equations. The advantage of the integral equations is that they define the continuum limit of a theory directly. So far the exact renormalization group has been studied as differential (or difference) equations, and for perturbation theory it has been used mainly as a method of regularization which is particularly convenient for formal studies. The continuum limit has to be constructed by first introducing a bare theory and then taking the bare theory to a critical point. In comparison the integral equation approach has two advantages: first we can construct the continuum limit directly, and second the integral equation naturally provides a self-determining perturbative procedure.

The integral equations are somewhat cumbersome to write down due to the subtractions necessary for the two- and four-point vertices. However, the analysis of the structure of the perturbative solution is straightforward, and the proof of the existence of a perturbative solution given in Sec. IV is one of the simplest proofs (if not the simplest) of renormalizability of ϕ^4 theory in the literature.

Some questions left unanswered in this paper will be answered in a forthcoming paper [6]. In particular it should be interesting to relate the ordinary renormalization group equations of the renormalized parameters and fields to the exact renormalization group equations. The lowest order results given in Ref. [7] will be extended to all orders in perturbation theory in Ref. [6] by modifying the exact renormalization group equation.

The exact renormalization group has been applied to a wide variety of theories such as gauge theories, chiral theories, theories with spontaneous symmetry breaking, supersymmetric theories, and theories with a real ultraviolet fixed point. (For example, see Refs. [8,9] for applications to gauge, chiral, and supersymmetric theories.) We expect that the integral equation approach introduced in this paper will further simplify the perturbative studies of those theories.

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APPENDIX A: LOWEST ORDER CALCULATIONS

We choose the minimal subtraction (MS) scheme:

$$
B_2(0)=C_2(0)=0, \quad A_4(0)=-\lambda. \qquad (A1)
$$

1. Order λ

At order λ we find

$$
\mathcal{V}_4(-t;p_1e^{-t},\ldots,p_4e^{-t})=(-\lambda)v_{4,0},
$$

$$
\mathcal{V}_2(-t;pe^{-t})=(-\lambda)v_{2,1}(-t) \qquad (A2)
$$

where

$$
v_{4,0}=1 \qquad (A3)
$$

$$
\begin{aligned}
e^{2t}v_{2,1}(-t)
&= \int_0^\infty dt' \frac{1}{2}\int_q \left[\frac{\Delta(qe^{-(t+t')})}{q^2+m^2} - \frac{\Delta(qe^{-(t+t')})}{q^2} \right.\\
&\quad \left. + m^2 \frac{\Delta(qe^{-(t+t')})}{q^4} \right] - \frac{1}{2} e^{2t} T_0 \int_q \frac{\Delta(q)}{q^2}\\
&\quad + t m^2 \frac{1}{2}\int_q \frac{\Delta(q)}{q^4}\\
&= \frac{1}{2}\int_q \left[\frac{1-K(qe^{-t})}{q^2+m^2} - \frac{1}{q^2} + (1-K(qe^{-t}))\frac{m^2}{q^4} \right]\\
&\quad + t m^2 \frac{1}{2}\int_q \frac{\Delta(q)}{q^4}
\end{aligned} \tag{A4}
$$

where we used

$$
\frac{d}{dt}K(pe^{-t}) = \Delta(pe^{-t}) \tag{A5}
$$

and $T_0 = 1/2$ is defined by

$$
\frac{d}{dt}(e^{2t}T_0) = e^{2t}. \tag{A6}
$$

2. Order λ^2

Up to order λ^2 we find

$$
\begin{aligned}
\mathcal{V}_6(-t; p_1 e^{-t}, \dots, p_6 e^{-t})
&= (-\lambda)^2 v_{6,0}(p_1 e^{-t}, \dots, p_6 e^{-t}; m^2 e^{-2t})
\end{aligned} \tag{A7}
$$

$$
\begin{aligned}
\mathcal{V}_4(-t; p_1 e^{-t}, \dots, p_4 e^{-t})
&= (-\lambda) v_{4,0} + (-\lambda)^2 v_{4,1}(-t; p_1 e^{-t}, \cdots, p_4 e^{-t}; m^2 e^{-2t})
\end{aligned} \tag{A8}
$$

$$
\begin{aligned}
\mathcal{V}_2(-t; pe^{-t})
&= (-\lambda) v_{2,1}(-t) + (-\lambda)^2\\
&\quad \times v_{2,2}(-t; pe^{-t}, -pe^{-t}; m^2 e^{-2t}).
\end{aligned} \tag{A9}
$$

We must start from the six-point function:

$$
\begin{aligned}
e^{-2t} v_{6,0}(p_1 e^{-t}, \dots, p_6 e^{-t})
&= \frac{1-K((p_1+p_2+p_3)e^{-t})}{(p_1+p_2+p_3)^2+m^2} + 5 \text{ permutations.}
\end{aligned} \tag{A10}
$$

This implies the asymptotic form

$$
A_6(-t; q, -q, 0,0,0,0) = (-\lambda)^2 \frac{6(1-K(q))}{q^2}. \tag{A11}
$$

Hence, we obtain

$$
\begin{aligned}
v_{4,1}(-t; p_1 e^{-t}, \dots, p_4 e^{-t}; m^2 e^{-2t})
&= \int_0^\infty dt' \sum_{i=1}^4 \frac{\Delta(p_i e^{-(t+t')})}{p_i^2+m^2} e^{2(t+t')} v_{2,1}(-(t+t')) + \int_0^\infty dt' \frac{1}{2}\int_q \left[\frac{\Delta(q^{-(t+t')})}{q^2+m^2} e^{-2(t+t')} \right.\\
&\quad \left. \times v_{6,0}(qe^{-(t+t')}, -qe^{-(t+t')}, p_1 e^{-(t+t')}, \dots, p_4 e^{-(t+t')}) - \frac{\Delta(qe^{-(t+t')})}{q^2} \frac{6(1-K(qe^{-(t+t')}))}{q^2} \right]\\
&\quad - t \frac{1}{2}\int_q \frac{\Delta(q)}{q^2} \frac{6(1-K(q))}{q^2}\\
&= \sum_{i=1}^4 \frac{1-K(p_i e^{-t})}{p_i^2+m^2} e^{2t} v_{2,1}(-t) + \frac{1}{2}\int_q \left[\frac{1-K(qe^{-t})}{q^2+m^2} \left(\frac{1-K((p_1+p_2+q)e^{-t})}{(p_1+p_2+q)^2+m^2} + 2 \text{ permutations} \right) \right.\\
&\quad \left. - 3\frac{(1-K(qe^{-t}))^2}{q^4} \right] - 3t \int_q \frac{\Delta(q)(1-K(q))}{q^4}
\end{aligned} \tag{A12}
$$

where we used

$$
\Delta(qe^{-t})(1-K(qe^{-t})) = -\frac{d}{dt}\frac{1}{2}(1-K(qe^{-t}))^2. \tag{A13}
$$

The expression for $v_{2,2}$ is omitted.

APPENDIX B: CONSTRUCTION OF $T_k(t)$

The k-th order polynomial $T_k(t)$ is defined by

$$\frac{d}{dt}(e^{2t}T_k(t))=e^{2t}t^k. \tag{B1}$$

By substituting

$$T_k(t)=\sum_{l=0}^{k} c_l t^{k-l} \tag{B2}$$

into the definition, we obtain a recursion relation for c_l whose solution is

$$c_l=(-)^l \frac{k(k-1)\cdots(k-l+1)}{2^{l+1}} \Leftrightarrow c_0=\frac{1}{2}, \quad c_1=-\frac{k}{4},$$

$$c_2=\frac{k(k-1)}{8}, \ldots, \quad c_k=(-)^k \frac{k!}{2^{k+1}}. \tag{B3}$$

Using $T_k(t)$, we can construct a map from an n-th order polynomial $P_n(t)$ to another n-th order polynomial:

$$P_n(t)=\sum_{k=0}^{n} P_{n,k}t^k \rightarrow Q_n(t)=\sum_{k=0}^{n} P_{n,k}T_k(t). \tag{B4}$$

By definition of $T_k(t)$, this has the obvious consequence

$$\frac{d}{dt}(e^{2t}Q_n(t))=e^{2t}P_n(t). \tag{B5}$$

An important property of the above map is its invariance under translation. Namely, if the polynomial $P_n(t)$ maps to $Q_n(t)$, then the shifted polynomial $P_n(t-\Delta t)$, where Δt is a constant, maps to the shifted polynomial $Q_n(t-\Delta t)$. This implies that the map from $P_n(t)$ to $Q_n(t)$ is defined independent of the choice of the origin of the variable t.

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[10] At each order of perturbation theory, the approach is modified by a finite integral power of t.

[11] For $m^2=0$ to correspond to a physically massless theory, it is important to take the momentum cutoff function $K(p)$ independent of m^2. It is the well-known fine tuning problem of the ϕ^4 theory that the m^2 term is exponentially small compared to the $A_2(-t)$ term.

[12] The problem of the minimal subtraction scheme (50) is that it is not RG invariant. $B_2(0)=C_2(0)=0$ does not imply $B_2(-t)=C_2(-t)=0$ for arbitrary t. To introduce an RG invariant MS scheme, we must modify our ERG equation.