Effect of gauge boson mass on chiral symmetry breaking in three-dimensional QED

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In three-dimensional quantum electrodynamics (QED_3) with a massive gauge boson, we investigate the Dyson-Schwinger equation for the fermion self-energy in the Landau gauge and find that chiral symmetry breaking (CSB) occurs when the gauge boson mass ξ is smaller than a finite critical value ξ_{cv} but is suppressed when $\xi > \xi_{cv}$. We further show that the critical value ξ_{cv} does not qualitatively change after considering higher order corrections from the wave function renormalization and vertex function. Based on the relation between CSB and the gauge boson mass ξ, we give a field theoretical description of the competing antiferromagnetic and superconducting orders and, in particular, the coexistence of these two orders in high temperature superconductors. When the gauge boson mass ξ is generated via the instanton effect in a compact QED_3 of massless fermions, our result shows that CSB coexists with the instanton effect in a wide region of ξ, which can be used to study the confinement-deconfinement phase transition.

I. INTRODUCTION

Chiral symmetry breaking (CSB) has been an active research field in particle physics for over 40 years since Nambu and Jona-Lasinio [1] used this idea to generate a fermion mass in a four-fermion model. One fascinating characteristic of CSB is that it can generate fermion mass via fermion-antifermion condensation mediated by a strong gauge field without introducing additional Higgs particles which until now have not been found. The most conclusive evidence for the existence of CSB is provided by the phenomenology of strong interaction, and CSB is widely believed to account for the pions. However, despite the vast amount of theoretical work on CSB, it is not yet clear whether it can be derived from quantum chromodynamics (QCD), primarily due to the complex structure of the SU(3) gauge field. To gain valuable insight into CSB before we can treat it completely, it is very suggestive to study some model that is similar to QCD while being simpler. Three-dimensional quantum electrodynamics (QED_3) is just such a model, and it has attracted intense investigation [2–9] in the past 20 years. QED_3 was shown to exhibit CSB [5–9] and confinement [9], while at the same it is simple enough to be treated with high accuracy. In addition, it has been used to model the physics of many planar condensed matter systems such as high temperature superconductors [10–18] and fractional quantum Hall systems [19].

The breakthrough in research into CSB in QED_3 was caused by a paper of Appelquist *et al.* [5] who found that CSB occurs when the flavor of massless fermions is less than a critical number N_c. They arrived at this conclusion by analytically and numerically solving the Dyson-Schwinger (DS) equation for the fermion self-energy to the lowest order of the $1/N$ expansion. Later, extensive analytical and numerical investigations [6–8] showed that the nature of CSB in

QED_3 remains the same after including higher order corrections to the DS equation.

The above result holds when the gauge boson is massless and but is expected to change when the gauge boson has a finite mass. CSB is a low energy phenomenon because (2+1)-dimensional U(1) gauge field theory is asymptotically free [20] and only in the infrared region is the gauge interaction strong enough to cause fermion condensation. This requires the fermions to be apart from each other. However, when the gauge boson has a finite mass it cannot mediate a long-range interaction. Intuitively, a finite gauge boson mass is repulsive to CSB which is achieved by the formation of fermion-antifermion pairs. Thus it is very interesting to study whether CSB can occur in the presence of a finite gauge boson mass.

CSB is believed to be a nonperturbative phenomenon and hence calculations based on perturbative expansions are incapable of establishing its existence. We will study CSB by means of solving the nonlinear DS equation for the fermion self-energy. Assuming that $A(p^2) = 1$ based on a naive $1/N$ expansion, we get a single integral equation of the gap function $\Sigma(p^2)$. CSB is signalled by the appearance of a squarely integrable nontrivial solution. To solve the DS equation, we will use bifurcation theory and the parameter imbedding method, which not only avoids the convergency problem that usually appears in iteration methods but also can help us distinguish the different bifurcation points. After solving the DS equation, we find that the massless fermions can acquire a finite dynamically generated mass when the gauge boson mass ξ is smaller than a critical value ξ_c. To testify to the robustness of our result against the effect of $A(p^2)$, we will work in a nonlocal gauge in which $A(p^2) \equiv 1$ and the vertex function can be safely replaced by the gamma matrices. We will show that, when the wave function renormalization $A(p^2)$ is included, the result we derive in the Landau gauge remains qualitatively unchanged.

Our study of the influence of the gauge boson mass on the fate of CSB not only is of theoretical interests but can be used to understand important physical phenomena. Actually, starting from the concept of spin-charge separation proposed by Anderson [21], the effective low energy theory of high temperature superconductors is a U(1) gauge theory [10–13]. Superconductivity is achieved when the charge carrying holons Bose condense into a macroscopic quantum state, which generates a finite mass for the gauge boson via the Anderson-Higgs mechanism. The low energy spin fluctuations are captured by the two-component fermions, which are originally massless since they are excited from the *d*-wave gap nodes [11]. On the other hand, CSB is known to correspond to long-range antiferromagnetic (AF) order, which can be seen from the behavior of the AF spin correlation function at low momentum. If we use CSB to describe the AF order and use the gauge boson mass to describe the superconducting (SC) order, our result then leads to a competition between the long-range AF order and the long-range SC order, which is one of the most fundamental issues in modern condensed matter physics. As a compromise in this competition, when the mass of the gauge boson is less than its critical value ξ_c but is finite, there is a coexistence of these two orders in the bulk superconductors.

If the U(1) gauge field is compact in the meaning that the vector potential has a periodicity, then it acquires a finite mass via the instanton effect. Furthermore, permanent confinement of static charges is present when the instanton effect is important. The influence of additional matter fields, especially massless fermions, on the permanent confinement is an unsolved problem. The relation between CSB and the gauge boson mass obtained in this paper is very helpful in studying the confinement to deconfinement phase transition driven by the cooling of massless fermions.

The physical applications of CSB in the presence of a gauge boson mass to high temperature superconductors have been reported in [16]. In this paper, we provide the related field theoretical technique in details. In Sec. II, we derive the DS equation in the presence of a finite mass of the gauge boson in the Landau gauge. We then choose to solve the nonlinear DS equation by means of bifurcation theory and the parameter imbedding method. Section III is devoted to the elementary knowledge of bifurcation theory that will be used in this paper and the detailed calculation steps of the parameter imbedding method. In Sec. IV, we consider the higher order corrections to the wave function renormalization and show that these corrections do not change our result noticeably. In Sec. V, we give a thorough discussion of the competing orders in high temperature superconductors from a field theoretical point of view. In particular, we emphasize the necessity for nonperturbative effects in getting an AF spin correlation that is consistent with experiments. In Sec. VI, we discuss the instanton effect on CSB in compact QED$_3$. The calculation of the AF spin correlation function in the CSB phase is given in the Appendix.

II. DYSON-SCHWINGER EQUATION IN THE LANDAU GAUGE

The three-dimensional U(1) gauge theory of massless fermions is

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 + \sum_{\sigma=1}^{N} \bar{\psi}_\sigma (\partial_\mu - i a_\mu) \gamma_\mu \psi_\sigma, \qquad (1)$$

where the Fermi field ψ_σ is a 4×1 spinor. The $4 \times 4 \gamma_\mu$ matrices obey the algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

The full fermion propagator is

$$G^{-1}(p) = i\gamma \cdot p A(p^2) + \Sigma(p^2), \qquad (2)$$

where $A(p^2)$ is the wave-function renormalization and $\Sigma(p^2)$ the fermion self-energy. The DS equation for the full fermion propagator in momentum space is given by

$$G^{-1}(p) = G_0^{-1}(p) - \int \frac{d^3k}{(2\pi)^3} \gamma_\mu G(k) \Gamma_\nu(p,k) D_{\mu\nu}(p-k), \qquad (3)$$

where $\Gamma_\nu(p,k)$ is the full vertex function and $D_{\mu\nu}(p-k)$ is the full photon propagator. $G_0^{-1}(p)$ is the bare propagator of the massless fermions. Substituting the propagator (2) into Eq. (3) and taking the trace on both sides, we obtain the equation for $\Sigma(p^2)$,

$$\Sigma(p^2) = -\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \text{Tr}[\gamma_\mu G(k) \Gamma_\nu(p,k) D_{\mu\nu}(p-k)]. \qquad (4)$$

Multiplying both sides of Eq. (3) by $\gamma \cdot p$ and then taking the trace on both sides, we obtain the equation for $A(p^2)$,

$$A(p^2) = 1 + \frac{1}{4p^2} \int \frac{d^3k}{(2\pi)^3} \text{Tr}[(i\gamma \cdot k)\gamma_\mu G(k) \Gamma_\nu(p,k)$$
$$\times D_{\mu\nu}(p-k)]. \qquad (5)$$

If the DS equation for $\Sigma(p^2)$ has only vanishing solutions, the fermions remain massless and the Lagrangian (1) respects the chiral symmetries $\psi \rightarrow \exp(i\theta\gamma_{3,5})\psi$, with γ_3 and γ_5 two 4×4 matrices that anticommute with γ_μ ($\mu = 0,1,2$). If the DS equation for $\Sigma(p^2)$ develops a squarely integrable nontrivial solution [22–24], then the originally massless fermions acquire a finite dynamically generated mass which breaks the chiral symmetries.

We starts from a general gauge boson propagator

$$D_{\mu\nu}(q) = D_T(q^2)\left(\delta_{\mu\nu} - g(q^2)\frac{q_\mu q_\nu}{q^2}\right) \qquad (6)$$

with

$$D_T^{-1}(q^2) = q^2[1 + \pi(q^2)] + \xi^2 \qquad (7)$$

where $g(q^2)$ is a gauge-fixing parameter that depends on the three-momentum and ξ is the mass of the gauge boson. We use q to denote the gauge boson momentum, and we have $q^2 = (p-k)^2 = p^2 + k^2 - 2pk\cos\theta$. $\pi(q^2)$ is the vacuum polarization of the gauge boson, which was included initially to

overcome the infrared divergence. If we include only the one-loop diagrams for massless fermions, we can write the vacuum polarization as

$$\pi(q^2) = \frac{N}{8|q|}. \tag{8}$$

Then we have

$$D_T^{-1}(q^2) = q^2\pi(q^2) + \xi^2 = \frac{N}{8}(q+\eta), \tag{9}$$

with

$$\eta = 8\xi^2/N, \tag{10}$$

since at low momentum $\pi(q^2) \gg 1$ [5].

As the lowest order approximation, we neglect the wave-function renormalization $A(p^2)$ and adopt a massive gauge boson propagator in the Landau gauge $[g(q^2)=1]$ as follows:

$$D_{\mu\nu}(p-k) = \frac{8}{N(|p-k|+\eta)}\left(\delta_{\mu\nu} - \frac{(p-k)_\mu(p-k)_\nu}{(p-k)^2}\right). \tag{11}$$

Further, we use the bare vertex, i.e., $\Gamma_\mu(p,k) = \gamma_\mu$, which is usually called the quenched planar approximation. Then the DS equation becomes

$$\Sigma(p^2) = \int \frac{d^3k}{(2\pi)^3} \frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k^2)\gamma^\nu}{k^2 + \Sigma^2(k^2)}. \tag{12}$$

Now we can insert the propagator (11) into the DS equation (12); then

$$\Sigma(p^2) = \frac{4}{N\pi^2}\int dk \frac{k\Sigma(k^2)}{k^2+\Sigma^2(k^2)} \int_{-1}^{1} dz \frac{1}{|p-k|+\eta} \tag{13}$$

where $z = \cos\theta$. After performing the integration with respect to z and introducing an ultraviolet cutoff Λ we finally arrive at the following DS equation:

$$\Sigma(p^2) = \lambda \int_0^\Lambda dk \frac{k\Sigma(k^2)}{k^2+\Sigma^2(k^2)} \frac{1}{p}\left[\left|p+k-\left|p\right.\right.\right.$$
$$\left.\left.-k\right| - \eta \ln\left(\frac{p+k+\eta}{|p-k|+\eta}\right)\right], \tag{14}$$

where $\lambda = 4/N\pi^2$ serves as an effective coupling constant.

If we do not introduce an ultraviolet cutoff ($\Lambda \to \infty$), the critical behavior of Eq. (14) is completely independent of η, as can be easily seen by making the scale transformation $p \to p/\eta$, $k \to k/\eta$, and $\Sigma \to \Sigma/\eta$. We can destroy this scale invariance by introducing an ultraviolet cutoff Λ.

Before we go into the techniques of dealing with the DS equation, we would like to discuss one subtle issue. If the DS equation has only trivial solutions, the fermions remain massless and the chiral symmetries are not broken. However,

not all nontrivial solutions lead to CSB. It is well known that the breaking of chiral symmetry is always accompanied by a Goldstone boson, which is a pseudoscalar bound state composed of a fermion and an antifermion. If CSB happens, there should be a nontrivial solution for the Bethe-Salpeter equation of this bound state. In addition, the bound state wave function must satisfy a normalization condition, which can be converted to a sufficient and necessary condition [22–24] for the nontrivial solutions of the DS equation to signal CSB. It gives a constraint on the form of $\Sigma(p^2)/A(p^2)$ as follows [24]:

$$\int_0^\infty dq \frac{q^2\Sigma^2(q^2)}{q^2A^2(q^2)+\Sigma^2(q^2)} = \text{finite.} \tag{15}$$

It is easy to see that in order to satisfy this condition $\Sigma(p^2)/A(p^2)$ must damp more rapidly than $p^{-1/2}$ in the ultraviolet region ($p \to \infty$). The mass function obtained by Appelquist *et al.* [4] satisfies this condition and hence the nontrivial solutions of the DS equation in QED$_3$ correspond to true CSB solutions [24]. On the other hand, when an ultraviolet cutoff is introduced, the solution $\Sigma(p^2)$ automatically satisfies the squarely integrable condition. We should emphasize that, although the nontrivial solutions with an ultraviolet cutoff all satisfy such a condition, only those solutions that satisfy this condition in the continuum limit are physically sensible. In the case of four-dimensional QED, although the nontrivial solutions with explicit ultraviolet cutoff are squarely integrable they do not satisfy the squarely integrable condition when we take the continuum limit [22,23]. Therefore, the CSB solutions obtained in quenched planar QED$_4$ [25] are not physically meaningful solutions because they are not squarely integrable in the continuum limit, or in other words the associated bound state wave functions cannot be normalized. In QED$_3$, it was found that $\Sigma(p^2)/A(p^2)$ behaves like p^{-2} at $p \to \infty$, which surely satisfies the condition. Since the nontrivial solutions in QED$_3$ are true CSB solutions, we can safely introduce an ultraviolet cutoff Λ without bringing unphysical nontrivial solutions.

A theoretical analysis implies that the critical fermion number N_c of Eq. (14) should depend on Λ/η. To determine when CSB occurs, the DS equation should be solved implicitly. The DS equation is a nonlinear integral equation and hence is very hard to investigate. However, based on general bifurcation theory and the parameter imbedding method, we can find the critical fermion number and the mass function exactly. The detailed program of calculations is the topic of the next section. In the rest of this section, we discuss some qualitative properties extracted from the DS equation (14).

When the gauge boson has a very large mass, for example $\eta \gg \Lambda$, then the DS equation becomes

$$\Sigma(p^2) = \frac{8}{N\pi^2\eta}\int_0^\Sigma dk \frac{k\Sigma(k^2)}{k^2+\Sigma^2(k^2)}. \tag{16}$$

From the momentum dependence of the mass function of fermions, we know that actually it is a constant in this limit. Therefore, the DS integral equation simplifies to the algebraic equation

$$\Sigma \arctan\left(\frac{\Lambda}{8\Sigma}\right) = \frac{N}{8}(1 - \pi^2 \eta). \qquad (17)$$

This equation has no solutions; hence a large enough mass of the gauge boson prevents the occurrence of CSB. We now consider another limit, i.e., when the gauge boson mass is very small. In this limit, the last term in the kernel of Eq. (14) can be dropped safely, leaving a DS equation that is the same as the one studied by Appelquist *et al.* [5]. Then a very small gauge boson mass actually does not affect the critical behavior of QED$_3$. This phenomenon can be understood if it happens that the critical fermion number N_c decreases when the gauge boson mass ξ increases and finally approaches zero for very large ξ. The main purpose of our work is to use QED$_3$ to model condensed matter systems where the physical fermion number is 2, coming from the two components of the spin. Based on the tendency of the critical fermion number in the presence of a finite gauge boson mass, it is quite reasonable to hypothesize that there is a critical value for the gauge boson mass ξ_{cv} above which CSB is inhibited. To make sure that this is actually the case, we should solve the DS equation and find the critical coupling constant λ_c at which the DS equation starts to have nontrivial solutions.

III. SOLVING DS EQUATION USING BIFURCATION THEORY AND PARAMETER IMBEDDING METHOD

Equation (14) is a Hammerstein type nonlinear integral equation. It does not satisfy the conditions of the global eigenfunction theory of nonlinear functional analysis, so its global solutions cannot be obtained directly. However, local bifurcation theory [26–29] can help us to find its complete solutions by first obtaining a local solution near a bifurcation point and then extending its region of validity step by step. This program is most easily achieved by the parameter imbedding method [28,29], which has proved to be a powerful method in studying integral equations.

In order to obtain the bifurcation points we need only find the eigenvalues of the associated Frêchet derivative of the nonlinear DS equation [28,29]. Those eigenvalues that have odd multiplicity are the bifurcation points. Taking the Frêchet derivative of the nonlinear equation (14), we have the following linearized equation:

$$\Sigma(p^2) = \lambda \int_0^{\Lambda/\eta} dk \Sigma(k^2) K(p,k) \qquad (18)$$

with the kernel

$$K(p,k) = \frac{1}{pk}\left[p + k - |p - k| - \ln\left(\frac{p+k+1}{|p-k|+1}\right)\right] \qquad (19)$$

where for calculational convenience we made the transformation $p \to p/\eta$, $k \to k/\eta$, and $\Sigma \to \Sigma/\eta$. The smallest eigenvalue λ_c of this equation is just the bifurcation point from which a nontrivial solution of the DS equation (14) branches off. The complex kernel $K(p,k)$ in the linearized equation (18) makes it very difficult to find an analytical solution.

We now would like to use the parameter imbedding method [28,29] to solve Eq. (18) numerically. To do this, we first analytically continue it in the complex plane of λ; correspondingly, $\Sigma(p^2)$ also becomes a complex function. It can be shown that

$$\int \int |K(x,y)|^2 dx dy < \infty. \qquad (20)$$

Here we use x to denote p^2, and y to denote k^2. From the Fredholm integral equation theory we know that there exists a resolvent function for the kernel $K(x,y)$,

$$R_F(x,y,\lambda) = D_F(x,y,\lambda)/d_F(\lambda), \qquad (21)$$

where the functions $D_F(x,y,\lambda)$ and $d_F(\lambda)$ are analytic with respect to λ. If $d_F(\lambda) \neq 0$, we do not have bifurcation points. The values of λ at which $d_F(\lambda) = 0$ are the bifurcation points. According to the parameter imbedding method, the functions $D_F(x,y,\lambda)$ and $d_F(\lambda)$ are related by the differential-integral equations

$$\frac{d}{d\lambda}d_F(\lambda) = -\int_0^{\Lambda^2/\eta} D_F(x,y,\lambda)dx, \qquad (22)$$

$$\frac{\partial}{\partial\lambda}D_F(x,y,\lambda) = \frac{1}{d_F(\lambda)}\left[D_F(x,y,\lambda)\frac{d}{d\lambda}d_F(\lambda)\right.$$
$$\left. + \int_0^{\Lambda^2/\eta} D_F(x,z,\lambda)D_F(z,y,\lambda)dz\right], \qquad (23)$$

with the initial conditions

$$d_F(0) = 1, \qquad (24)$$

$$D_F(x,y,0) = K(x,y). \qquad (25)$$

One remarkable advantage of the parameter imbedding method is to convert the integral equations with variables x and y to a set of equations in the variable λ. Correspondingly, the boundary conditions in the original equations are replaced by two initial conditions, which are easier to treat in performing numerical calculations. Now, the functions $D_F(x,y,\lambda)$ and $d_F(\lambda)$ can be readily obtained by integrating numerically with respect to λ.

We now should choose an appropriate contour C in the complex λ plane which contains the minimum λ on the real axis at which $d(\lambda) = 0$. The number of zero eigenvalues of the linearized equation (18) inside the contour C [i.e., the zeros of the function $d_F(\lambda)$] is

$$N_E = \frac{1}{2\pi i}\oint_C \frac{1}{d_F(\lambda)}\frac{d}{d\lambda}d_F(\lambda)d\lambda. \qquad (26)$$

We can obtain the eigenvalues by solving the equations

$$\sum_{i=1}^{N_E} \lambda_i^l = \frac{1}{2\pi i}\oint_C \frac{\lambda^l}{d_F(\lambda)}\frac{d}{d\lambda}d_F(\lambda)d\lambda, \qquad (27)$$
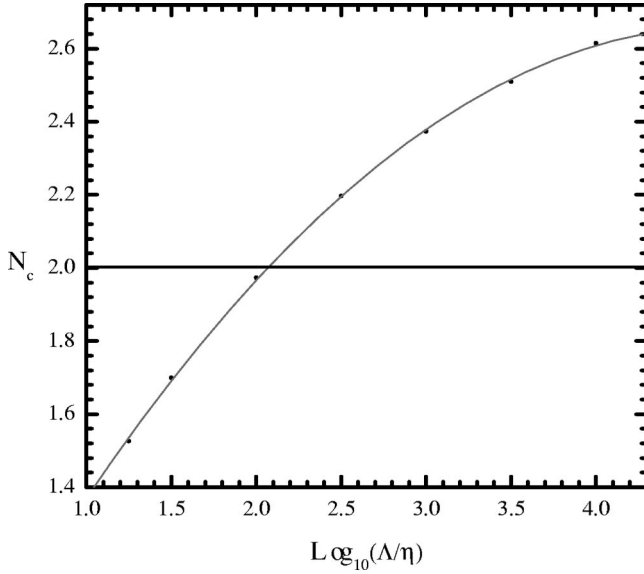
FIG. 1. The dependence of the critical number N_c on $\log_{10}(\Lambda/\eta)$ in the Landau gauge.

with $l=1,\ldots,N_E$. For the present purpose, we only need to know the first bifurcation point, hence we let $N_E=1$.

For $\lambda>\lambda_c$, the DS equation has nontrivial solutions and the massless fermions become massive. The ultraviolet cutoff Λ is provided by the lattice constant and hence is kept fixed. We can obtain the relation of N_c and η by calculating the critical coupling λ_c for different values of Λ/η.

Our numerical result is presented in Fig. 1. The critical fermion number N_c is a monotonically increasing function of Λ/η. For small Λ/η, N_c is smaller than the physical number 2, so CSB does not occur. When Λ/η increases, the critical number N_c increases accordingly and finally becomes larger than 2 at about $\Lambda/\eta_{cv}=100$. Thus we see that there is a critical value of the gauge boson mass ξ_{cv}, below which a finite mass is generated for massless fermions, while beyond it CSB is suppressed.

IV. DYSON-SCHWINGER EQUATION WITH HIGHER ORDER CORRECTIONS

In the last two sections, we investigated the DS equation in the Landau gauge after assuming that $A(p^2)=1$ to simplify calculations. Although this assumption is qualitatively correct, higher order corrections from the wave-function renormalization $A(p^2)$ will alter the critical fermion number N_c quantitatively. However, including $A(p^2)$ makes the DS equations very complicated and we should solve consistently two pairs of nonlinear integral equations. Furthermore, according to the Ward-Takahashi identity, we cannot choose γ_μ as the vertex function in the presence of wave-function renormalization $A(p^2)$. At present, there is no theoretical guidance in determining the vertex function $\Gamma_\mu(p,k)$, and hence one cannot give a guarantee of the legitimacy of a specific choice of vertex function. Here, to simply calculations and partly overcome the embarrassment in choosing the vertex function, we introduce a so-called nonlocal gauge [30–33] in which the wave-function renormalization $A(p^2)$

$\equiv 1$ and the vertex function can be chosen as

$$\Gamma_\mu(p,k)=\gamma_\mu f(p^2,k^2) \tag{28}$$

with f a function of the fermion momentum p^2,k^2. The nonlocal gauge is obtained by solving a differential equation. In this gauge, we need only investigate a single equation for $\Sigma(p^2)$ in studying the chiral phase transition.

Let us go back to the general massive gauge boson propagator (6). If we consider the quenched planar approximation of QED$_3$, i.e., taking $\Pi(p^2)=0$, then the wave function renormalization $A(p^2)\equiv 1$ in the Landau gauge. This result is well known to be exact in QED of dimensions higher than 2. In the case of QED$_3$, the one-loop vacuum polarization is usually introduced explicitly to overcome the severe infrared divergence. In the presence of $\Pi(p^2)$, wave-function renormalization $A(p^2)$ does not equal the identity. It should be obtained by solving two consistent integral equations of $A(p^2)$ and $\Sigma(p^2)$. However, taking advantage of the gauge degrees of freedom of the system, we can simplify the DS equations by choosing an appropriate gauge. In particular, if we can obtain a gauge parameter $g(q^2)$ that satisfies the equation

$$g(q^2)=\frac{2}{q^4 D_T(q^2)}\int_0^{q^2} D_T(z)z\,dz-1, \tag{29}$$

then we find a gauge in which $A(p^2)\equiv 1$ (see the paper of Georgi et al. in Refs. [31] and Ref. [33]). Further, according to the Ward-Takahashi (WT) identity, the vertex function can be chosen as [34]

$$\Gamma_\mu(p,k)=\gamma_\mu. \tag{30}$$

Now the formidable task of solving a pair of integral equations for the wave-function renormalization $A(p^2)$ and the mass function $\Sigma(p^2)$ is simplified to solving a single equation for $\Sigma(p^2)$:

$$\Sigma(p^2)=\int\frac{d^3k}{(2\pi)^3}\frac{\Sigma(k^2)}{k^2+\Sigma^2(k^2)}[3-g(q^2)]D_T(q^2). \tag{31}$$

From $D_T(q^2)$ and Eq. (29), the integral over z can be calculated:

$$\int_0^{q^2} D_T(z)z\,dz=\frac{8}{N}\int_0^{q^2}\frac{1}{z^{1/2}+\eta}z\,dz$$

$$=\frac{16}{N}\left[\frac{1}{3}q^3-\frac{1}{2}\eta q^2+\eta^2 q-\eta^3\ln\left(\frac{q+\eta}{\eta}\right)\right]. \tag{32}$$

Then we obtain a nonlocal gauge parameter

$$g(q^2)=\frac{4}{q^4}(q+\eta)\left[\frac{1}{3}q^3-\frac{\eta}{2}q^2+\eta^2 q-\eta^3\ln\left(1+\frac{q}{\eta}\right)\right]-1. \tag{33}$$

Substituting this $g(q^2)$ into the DS equation (31), after angular integration we have

$$\Sigma(p^2) = \frac{8}{N\pi^2 p} \int_0^\Lambda dk \frac{k\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} \int_{|p-k|}^{p+k} dq \left[\frac{2}{3} - \frac{\eta}{q+\eta} + \frac{\eta}{2q} \right.$$
$$\left. - \frac{\eta^2}{q^2} + \frac{\eta^3}{q^3} \ln\left(1 + \frac{q}{\eta}\right) \right]. \tag{34}$$

In deriving this result, we have used the following formula:

$$\int_0^\pi d\theta \sin\theta f(q^2) = \frac{1}{pk} \int_{|p-k|}^{p+k} q\, dq\, f(q^2). \tag{35}$$

After integrating Eq. (34), we have

$$\Sigma(p^2) = \lambda \int_0^\Lambda dk \frac{k\Sigma(k^2)}{k^2 + \Sigma^2(k^2)} \frac{2}{p} K(p,k,\eta), \tag{36}$$

with

$$K(p,k,\eta) = \frac{2}{3}(p+k-|p-k|) - \frac{\eta}{2} \ln\left(\frac{p+k+\eta}{|p-k|+\eta}\right)$$
$$+ \frac{\eta^2}{2}\left(\frac{1}{p+k} - \frac{1}{|p-k|}\right) + \frac{\eta^3}{2|p-k|^2}$$
$$\times \ln\left(1 + \frac{|p-k|}{\eta}\right) - \frac{\eta^3}{2(p+k)^2} \ln\left(1 + \frac{p+k}{\eta}\right). \tag{37}$$

Here, $\lambda = 4/N\pi^2$ is the effective coupling constant. At first glance, both the third and fourth terms of $K(p,k,\eta)$ have singular behaviors like $1/|p-k|$ which would cause divergence if k approaches p. However, when $|p-k| \to 0$, we can make the expansion

$$\frac{\eta^3}{2|p-k|^2} \ln\left(1 + \frac{|p-k|}{\eta}\right)$$
$$= \frac{\eta^3}{2|p-k|^2}\left(\frac{|p-k|}{\eta} - \frac{(p-k)^2}{2\eta^2} + O(|p-k|^3)\right)$$
$$= \frac{\eta^2}{2|p-k|} - \frac{\eta}{4} + O(|p-k|). \tag{38}$$

Thus the singular terms are exactly cancelled. The same step can be used to show that the singular term $1/(p+q)$ can also be cancelled exactly. Therefore, the kernel $K(p,k,\eta)$ is a smooth function in the whole integration region.

Taking the Fréchet derivative of the nonlinear equation (36), we obtain the linearized equation

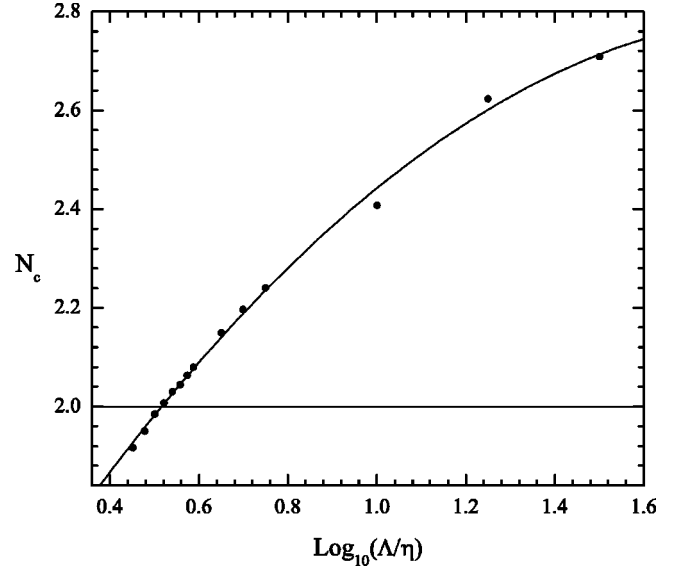$$\Sigma(p^2) = \lambda \int_0^{\Lambda/\eta} dk\, \Sigma(k^2) \frac{2}{pk} K(p,k,\eta) \tag{39}$$

with

FIG. 2. The dependence of the critical number N_c on $\log_{10}(\Lambda/\eta)$ in the nonlocal gauge.

$$K(p,k,\eta) = \frac{2}{3}(p+k-|p-k|) - \frac{1}{2} \ln\left(\frac{p+k+1}{|p-k|+1}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{p+k} - \frac{1}{|p-k|}\right) + \frac{1}{2|p-k|^2} \ln(1+|p-k|)$$
$$- \frac{1}{2(p+k)^2} \ln(1+p+k) \tag{40}$$

where for calculational convenience we made the transformation $p \to p/\eta$, $k \to k/\eta$, and $\Sigma \to \Sigma/\eta$.

Using the steps we presented in the last section, we can solve the linearized equation (39) to obtain the relation between the critical fermion number N_c and the mass ξ of the gauge boson mass. The numerical result is presented in Fig. 2, from which we know that the critical value of the gauge boson mass is about $\Lambda/\eta_{cv} = 3.3$. Although there is a significant change in the critical value ξ_{cv}, the result we obtained in the Landau gauge remains qualitatively correct.

V. COMPETING ORDERS IN HIGH TEMPERATURE SUPERCONDUCTORS

Understanding the competing orders in high temperature cuprate superconductors is one of the most important issues in condensed matter physics. In the presence of competing orders, one order parameter prevails when other orders are suppressed by some external variables. At half filling, the cuprate superconductor is a Mott insulator with long-range antiferromagnetic order. When holes are doped into the Cu-O planes, the material becomes a superconductor at low temperatures and the long-range AF order disappears. Hence there is a competition between the AF order and the SC order, and as a result of this competition the AF order dominates at zero and low doping while the SC order dominates at higher doping. However, even at higher doping the AF order

also has a chance to appear locally where the superconductivity is suppressed by strong external magnetic fields. Recently, elaborate neutron scattering [34] and scanning tunnelling microscopy (STM) [35] experiments found that the AF correlation is significantly enhanced in regions surrounding the vortex cores. In this paper, we will use spin-charge separation and CSB to understand the competing orders.

It has been shown [10–13] that the Lagrangian (1) is the effective low energy theory of undoped cuprates, which have only fermionic excitations because of the presence of a large charge gap. In underdoped cuprates, the electrons fractionalize into spin carrying spinons and charge carrying holons. It has been pointed out [15,16] that the physics of underdoped cuprates is captured by an effective U(1) gauge theory of massless fermions and charged scalar fields

$$
\mathcal{L}_F = \sum_{\sigma=1}^{N} \bar{\psi}_\sigma v_{\sigma,\mu} (\partial_\mu - i a_\mu) \gamma_\mu \psi_\sigma
$$
$$
+ |(\partial_\mu - i a_\mu) b|^2 + V(|b|^2). \tag{41}
$$

Here $b = (b_1, b_2)$ is a doublet of scalar fields representing the holons [15]. $v_{\sigma,0} = 1$ and generally $v_{\sigma,1} \neq v_{\sigma,2}$ as a result of the velocity anisotropy; however, for simplicity we can let $v_{\sigma,1} = v_{\sigma,2} = 1$. Since the spin and charge degrees of freedom are assumed to be separated, there is no Yukawa-type coupling between the fermion field and the scalar field. In the superconducting state, the boson b acquires a nonzero vacuum expectation value, i.e., $\langle b \rangle \neq 0$. This nonzero $\langle b \rangle$ spontaneously breaks the gauge symmetry of the theory and the gauge boson acquires a finite mass ξ via the Anderson-Higgs mechanism.

In the context of high temperature superconductors the U(1) gauge field is introduced as a Lagrangian multiplier to impose a local no-double-occupancy constraint. It has no kinetic term $\sim F_{\mu\nu}^2$ and its dynamics is obtained by integrating out the matter fields. If we only include the one-loop diagram in the vacuum polarization, we get $D_T^{-1}(q^2) = q^2 \pi(q^2) + \xi$. As we have shown previously, the effect of an additional scalar doublet is to shift N in the gauge boson vacuum polarization $\pi(q^2)$ to $N+1$, i.e., $\pi(q^2) = (N+1)/8|q|$. Then the propagator for the gauge boson is

$$
D_{\mu\nu}(p-k) = \frac{8}{(N+1)(|p-k| + \eta)}
$$
$$
\times \left(\delta_{\mu\nu} - \frac{(p-k)_\mu (p-k)_\nu}{(p-k)^2} \right). \tag{42}
$$

From the corresponding DS equation in the nonlocal gauge obtained above, we found a critical gauge boson mass at $\Lambda / \xi_{cv} = 100$. For small ξ, CSB occurs, while for $\xi > \xi_{cv}$, CSB is suppressed.

We now would like to discuss the long-range behavior of the AF correlation function. The AF spin correlation is defined as

$$
\langle S^+ S^- \rangle_0 = -\frac{1}{4} \int \frac{d^3 k}{(2\pi)^3} \mathrm{Tr}[G_0(k) G_0(k+p)], \tag{43}
$$

where $G_0(k)$ is the fermion propagator. If the fermions are massless, then

$$
G_0(k) = -\frac{\gamma \cdot k}{k^2}, \tag{44}
$$

and we have

$$
\langle S^+ S^- \rangle_0 = -\frac{|p|}{16}. \tag{45}
$$

At $p \to 0$, $\langle S^+ S^- \rangle_0 \to 0$, and the AF correlation is heavily lost. This is not a surprising result since our starting point is the resonating valence bond (RVB) picture proposed by Anderson [21], which is just a liquid of spin singlets and hence has only short-range AF correlation. This is not a satisfying situation because a long-range Néel order was observed in experiments shortly after the discovery of cuprate superconductors.

However, even if we start from a RVB ground state, it is still possible to obtain the long-range AF correlation because of the strong correlation nature of Mott insulators. The strong correlation is reflected in the fact that double occupancy on a single lattice is completely inhibited due to the strong Coulomb repulsive force. After this local constraint and quantum fluctuations are taken into account, a strong U(1) gauge field emerges in the effective theory. This gauge field has an important effect on physical properties since it can cause fermion condensation and give the originally massless fermions a finite mass. The AF spin correlation is expected to be significantly enhanced once the fermions become massive. To show that this actually happens, we will calculate the spin correlation function in the CSB phase (see the Appendix for details). Although the dynamically generated fermion mass depends on the three-momentum, here, for simplicity, we assume a constant mass m for the fermions. This approximation is valid because we only care about the low energy properties and $\Sigma(p^2)$ is actually a constant at $p \to 0$. The propagator for the massive fermion is

$$
G(k) = \frac{-(\gamma \cdot k + im)}{k^2 + m^2}, \tag{46}
$$

which leads to

$$
\langle S^+ S^- \rangle_0 = -\frac{1}{4\pi} \left[m + \frac{p^2 + 4m^2}{2|p|} \arcsin\left(\frac{p^2}{p^2 + 4m^2} \right)^{1/2} \right]. \tag{47}
$$

This spin correlation behaves like $-m/2\pi$ as $p \to 0$ and we have a long-range AF correlation when CSB takes place. Therefore, strong fluctuations around the RVB ground state enhance the long-range AF spin correlations.

We should emphasize that calculations based on perturbative expansions cannot be used to obtain the long-range AF

order. It might be argued that including higher order diagrams can enhance the AF spin correlation. However, this argument is not right. If we include the gauge field while keeping the fermions massless, then the spin correlation is [36]

$$\langle S^+ S^- \rangle_{GF} = -\frac{8}{12\pi^2(N+1)} |p| \ln\left(\frac{\Lambda^2}{p^2}\right) \qquad (48)$$

which damps at low momentum $p \rightarrow 0$. Rantner and Wen [36] claimed that the long-range AF correlation can be obtained by reexponentiating the spin correlation function [37]. This scenario is based on their previous statement [14] that the U(1) gauge field cannot generate a finite mass for fermions and hence is a marginal perturbation. This result is derived by considering only the one-loop correction of the gauge field to the fermion self-energy. However, CSB is a nonperturbative phenomenon and whether the gauge field generates a finite mass for the massless fermions can only be settled by investigating the self-consistent DS equation for the fermion self-energy. If Eq. (12) does not have a nonlinear term in the denominator of the kernel, it is a linear equation and *cannot* develop any genuine nontrivial solution. From the point of view of bifurcation theory, a linear operator has no bifurcation points; those are necessary for a phase transition to take place. Once the nonperturbative effect is taken into account, the strong gauge field generates a finite fermion mass which breaks the chiral symmetry and gives rise to long-range AF order (Ref. [38] discussed the correspondence of CSB to AF order in another way). Actually, the formation of long-range AF order spontaneously breaks the rational symmetry of the system and generates a gapless spin wave excitation which corresponds to the massless Goldstone boson. These are hard to understand if we only include the gauge fluctuations without breaking any symmetry. Furthermore, the strong interaction of the gauge field with massless fermions of flavor 2 will unavoidably generate a finite fermion mass.

Now we would like to discuss the application of our result to the interplay of various ground states in high temperature cuprate superconductors. It is well known that the gauge boson mass ξ is proportional to the superfluid density ρ; thus we can use ξ to describe the superconducting order. Otherwise, we use CSB to describe the long-range AF order. Based on the fact that the superfluid density is proportional to the doping concentration, we obtain a clear picture of the evolution of different orders upon increasing the doping concentration. At zero and low doping the gauge boson mass is zero or very small, so CSB and hence AF order are present. When the doping concentration is larger than a critical value δ_{cv} the gauge bosons acquire a mass that is large enough to suppress the CSB and the AF order. Note that superconductivity begins to appear as the ground state of cuprate superconductors at a δ_{sc} that is less than δ_{cv}. Therefore, for $\delta_{sc} < \delta < \delta_{cv}$ there is a coexistence of the AF order and the SC order in the bulk materials. Due to this coexistence, the length scale for AF order to appear should be larger than the vortex scale, which is consistent with the STM experiments of Hoffman *et al.* [35].

When the external magnetic field is stronger than H_{c2}, the superfluid is completely suppressed, and, correspondingly, the gauge bosons become massless. Then CSB reappears in the bulk material and gives a mass to the massless fermions. This mass provides a finite gap for the low energy fermions to be excited; thus at low temperature no fermionic excitations can be observed [15]. This causes the breakdown of the Wiedemann-Franz (WF) law in the normal ground state of cuprate superconductors [15,39]. Thus, based on spin-charge separation and CSB, we give a unified description for both the behavior of the AF spin correlation and the transport properties from a field theoretical point of view. This is the most noticeable advantage of our scenario compared with so many other scenarios [40–46] that also address the problem of local AF order in vortex cores.

VI. INSTANTON EFFECT ON CSB

Confinement is one unresolved problem in modern particle physics. A seminal paper written by Polyakov [47] has shed some light on this problem by studying a three-dimensional compact pure U(1) gauge theory (compact pure QED$_3$). In general, one can define an Abelian gauge theory on a two-dimensional lattice which has the following action:

$$S = \frac{1}{2e^2} \sum_{i,\alpha\beta} (1 - \cos F_{i,\alpha\beta}), \qquad (49)$$

with the field strength

$$F_{i,\alpha\beta} = A_{i,\alpha} + A_{i+\alpha,\beta} - A_{i+\beta,\alpha} - A_{i,\beta}. \qquad (50)$$

Here, the pairs (i,α) are used to denote the links between lattices, with i the beginning of a link and α its direction. If the vector potential $A_{i,\alpha}$ is defined to be a real number on its whole region, i.e., $-\infty \leq A_{i,\alpha} \leq +\infty$, the continuum limit of this action is just that of the usual U(1) gauge as presented in Eq. (1). However, a highly nontrivial physical effect emerges if the vector potential $A_{i,\alpha}$ has angular properties and hence is defined on a circle as $-\pi \leq A_{i,\alpha} \leq \pi$. Due to the periodicity of its action, such a field theory is called compact QED.

Polyakov first considered pure compact QED$_3$ without coupling matter fields to the gauge field. He found that instantons appear in this model as topological solutions of the Euclidean gauge field equations and lead to permanent confinement of static charges which is reflected by the area law for the Wilson integral. Compact QED$_3$ has attracted intense investigation in the past 20 years, initially as a simpler model to study quark confinement. Recently, compact QED$_3$ with matter fields has been used to model the physics of many strongly correlated electron systems [19,48]. However, although it is widely accepted that confinement is present in pure compact QED$_3$, there is no consensus on the fate of permanent confinement when matter fields are included [49,50]. Compared with compact QED$_3$ of scalar fields [49], the situation for compact QED$_3$ of massless fermions is particularly complicated because of the possibility of dynamical mass generation for the fermions.

Since compact QED$_3$ was originally defined on lattices, Monte Carlo numerical simulations are expected to provide important information on CSB, but they suffer from the notorious fermion sign problem. In this paper, we would like to analyze the chiral behavior using the DS equation method. To do this, we map the compact QED$_3$ onto a continuum theory and introduce the ultraviolet cutoff Λ keeping track of its lattice origin. As shown by Polyakov, the gauge field acquires a finite mass due to Debye screening caused by the instantons. We can use the mass ξ of the gauge field to describe the instanton effect and investigate the relation between CSB and the instanton effect by solving the DS equation which consists of a massive gauge boson propagator.

We studied the relation of the gauge boson mass and CSB in the context of high temperature superconductors in the last section. However, the critical gauge boson mass ξ_{cv} is very small in the presence of additional scalar fields, due to the shift from N to $N+1$ by the scalar doublet, and hence CSB can exist only for a small region of ξ. But the critical value ξ_{cv} in the present case [compact U(1) gauge field coupled only to massless fermions] is much larger and there is a wide region of ξ for CSB to take place. In previous papers [5,6] addressing CSB in QED$_3$ the ultraviolet cutoff is provided by $\alpha = N/8$, which is kept fixed when the fermion flavor N is taken to infinity, because for momentum $p > \alpha$ the self-energy function damps rapidly. From $\eta = \xi^2/\alpha$ we know that the critical gauge boson mass is about $\xi_c = \alpha/2$. The instanton effect can coexist with CSB for $\xi < \alpha/2$. If we couple a fermion of one flavor to a compact gauge field, then CSB can coexist with the instanton effect in a much wider region of ξ.

The above result can be used to investigate the possible confinement to deconfinement transition in compact QED$_3$ because whether the fermions have a finite mass is expected to affect the fate of permanent confinement [51]. Such a transition is no doubt of great importance in both particle physics and condensed matter physics, but beyond the scope of this paper.

VII. SUMMARY AND DISCUSSION

In this paper, we have discussed the effect of a finite mass of U(1) gauge bosons on CSB and its physical implications. The gauge boson mass ξ is reflected in the modification of the gauge field propagator, which appears in the DS equation of the fermion self-energy. The DS equation is nonlinear and hence hard to solve. An iteration procedure is the most frequently used numerical calculation method, but it is not clear whether the iteration procedure leads to a convergent result or not. To avoid the problem caused by convergency of the iteration, we make use of bifurcation theory and the parameter imbedding method to numerically investigate the DS equation. Adopting the Landau gauge and neglecting the wave-function renormalization, we found a critical value ξ_{cv} for the gauge boson mass that separates the CSB phase, for $\xi < \xi_{cv}$, and the chiral symmetric phase, for $\xi > \xi_{cv}$. We then showed that including higher order corrections of the wave-function renormalization does not qualitatively change the critical value ξ_{cv}.

We then used our result in two physical systems, high temperature cuprate superconductors and compact QED$_3$. If the gauge boson mass is generated via the Anderson-Higgs mechanism in the superconducting state, the combination of spin-charge separation and CSB provides a field theoretical description of the competition between the AF order and the SC order. As a compromise in this competition, there is microscopic coexistence of these two orders in the bulk materials, which plays an essential role in explaining the local AF order in vortex states observed in neutron scattering and STM experiments. When the periodicity of the gauge field is taken into account, the gauge boson acquires a finite mass via the instanton effect. Since whether the permanent confinement still exists in the presence of fermions depends on the fermion mass, our result can help us to investigate the confinement to deconfinement phase transition, which will be the subject of future study.

APPENDIX

In this appendix we give the details of calculating the spin correlation using the propagator of the massive fermions. When CSB occurs the fermion propagator is

$$G(k) = \frac{-(\gamma \cdot k + im)}{k^2 + m^2}. \tag{A1}$$

Then

$$\langle S^+ S^- \rangle_0 = -\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}[G(k)G(k+p)]$$

$$= -\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}\left[\frac{\gamma \cdot k + im}{k^2 + m^2} \frac{\gamma \cdot (k+p) + im}{(k+p)^2 + m^2} \right]$$

$$= -i \int_0^1 dt \int \frac{d^3k}{(2\pi)^3}$$

$$\times \frac{k \cdot (k+p) - m^2}{\{(k^2+m^2)(1-t) + [(k+p)^2 + m^2]t\}^2}, \tag{A2}$$

where we used the Feynman parametrization formula

$$\frac{1}{ab} = \int_0^1 dt \frac{1}{[at + b(1-t)^2]^2}. \tag{A3}$$

After replacing k by $k - pt$ and making a Wick rotation, we have

$$\langle S^+ S^- \rangle_0 = \int_0^1 dt \int \frac{d^3 k}{(2\pi)^3} \frac{k^2 - m^2 - p^2 t(1-t)}{[k^2 + m^2 + p^2 t(1-t)]^2}. \tag{A4}$$

Using the properties of the Γ function, we can integrate over the momentum k and get

$$\langle S^+ S^- \rangle_0 = \frac{1}{(4\pi)^{3/2}} \frac{\frac{3}{2}\Gamma\left(-\frac{1}{2}\right) - \Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$$

$$\times \int_0^1 dt \frac{1}{[m^2 + p^2 t(1-t)]^{-1/2}}. \tag{A5}$$

Since

$$\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(2)} = -2\pi^{1/2}, \tag{A6}$$

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \pi^{1/2}, \tag{A7}$$

we get

$$\langle S^+ S^- \rangle_0 = -\frac{1}{2\pi} \int_0^1 dt [m^2 + p^2 t(1-t)]^{1/2}$$

$$= -\frac{1}{4\pi}\left[m + \frac{p^2 + 4m^2}{2|p|} \arcsin\left(\frac{p^2}{p^2 + 4m^2}\right)^{1/2} \right]. \tag{A8}$$

This is the AF spin correlation in the CSB phase.

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