Optimization of the derivative expansion in the nonperturbative renormalization group

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We study the optimization of nonperturbative renormalization group equations truncated both in fields and derivatives. On the example of the Ising model in three dimensions, we show that the principle of minimal sensitivity can be unambiguously implemented at order ∂^2 of the derivative expansion. This approach allows us to select optimized cutoff functions and to improve the accuracy of the critical exponents ν and η. The convergence of the field expansion is also analyzed. We show in particular that its optimization does *not* coincide with optimization of the accuracy of the critical exponents.

I. INTRODUCTION

During the last ten years the Wilson-Kadanoff approach [1,2] to the renormalization group (RG), based on the block spin concept, has been the subject of a revival in both statistical physics and field theory. This originates in recent developments [3–5] which have now turned it into an efficient tool, the effective average action method [3] allowing one to investigate nonperturbative phenomena. This method implements on the effective action Γ—the Gibbs free energy—the idea of integration of high-energy modes that underlies any RG approach. The whole method consists in building an effective free energy Γ_k at a scale k for the high-energy modes that have been integrated out and in following its evolution with the scale k through an exact equation [3]. The main drawback of this equation is that it cannot be handled in actual calculations without truncations of Γ_k. It is thus of the utmost importance to know whether the truncations used provide converged and accurate results. As is well known, the problem of convergence is also crucial in perturbation theory where it requires the resummation of series. Let us emphasize that this problem is far from being solved in general since Borel summability, which is the key point to resum perturbative series, is not generically proven and may even turn out not to hold (see [6,7] for a review). It is then important to dispose of an alternative method, not relying on an expansion in a coupling constant and thus not requiring *a priori* resummation. Good indications on the convergence properties of the effective average action method have been already provided by its ability to tackle with highly nontrivial problems, such as low-energy QCD [8], the Abelian Higgs model relevant for superconductivity [9], the phase diagram of He_3 [10], frustrated magnets [11–14], the Gross-

Neveu model in three dimensions [15,16], the randomly dilute Ising model [17], the Kosterlitz-Thouless transition [18,19], etc. (see [20] for a review and [21] for an exhaustive bibliography). A systematic investigation of the convergence and accuracy issues is, however, still lacking.

We propose here, on the example of the three-dimensional Ising model, to study the convergence and optimization of the accuracy of the effective average action method truncated both in derivatives, at order ∂^2, and in fields. We study, in particular, the role of the cutoff function, used to separate the low- and high-energy modes, on the determination of the critical exponents ν and η.

In Sec. II, we briefly introduce the basic ideas underlying the effective average action method. We then discuss in Sec. III the truncations necessary to deal with concrete calculations. We motivate, in Sec. IV, the use of the principle of minimal sensitivity (PMS) to optimize the results. Then, we apply this technique successively within the local potential approximation (LPA), Sec. V and at order ∂^2 of the derivative expansion, Sec. VI.

II. THE EFFECTIVE AVERAGE ACTION METHOD

Historically, the block spin concept was first implemented, in the continuum, on the Hamiltonian. This procedure consists in separating, within the partition function, the microscopic fields into a high- and a low-energy part and in integrating out the high-energy part to get an effective Hamiltonian for the remaining low-energy modes. The iteration of this procedure generates a sequence, a flow, of scale-dependent Hamiltonians, parametrized by a running scale k, and describing the same long distance physics. The critical properties are then determined by the behavior of the system around the fixed point of the flow of Hamiltonians [2]. However, due to technical difficulties [5,22,23,21] this nonperturbative renormalization procedure has been mainly used as a conceptual basis for perturbative calculations rather than as a practical tool to investigate nonperturbative aspects of field theories and critical phenomena. The situation has changed

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when it has been realized, mainly by Ellwanger [24–27], Morris [5,22,28–31,23], and Wetterich [32,33,3,34–36] that, rather than the Hamiltonian H, one should consider the effective action Γ—the Gibbs free energy—as the central quantity to be renormalized. In the spirit of the original Wilsonian formulation one builds a *running* effective action Γ_k that only includes high-energy fluctuations with momenta $q^2 > k^2$. This implies that, on the one hand, at the underlying microscopic scale $k = \Lambda$, Γ_k coincides with the classical Hamiltonian H since no fluctuation has yet been taken into account. On the other hand, when the running scale is lowered to $k = 0$, i.e., when *all* fluctuations have been integrated out, the standard effective action Γ is recovered. To summarize, Γ_k continuously interpolates between the microscopic Hamiltonian H and the free energy:

$$\Gamma_{k=\Lambda} = H,$$

$$\Gamma_{k=0} = \Gamma. \tag{1}$$

Since, by definition, Γ_k is built up from the high-energy fluctuations of the microscopic system, the low-energy modes—with $q^2 < k^2$—must be removed from the running partition function. This is most easily achieved by adding to the original Hamiltonian a scale-dependent mass term ΔH_k. Then, the running partition function with a source term writes [20]:

$$\mathcal{Z}_k[J] = \int D\chi \, e^{-H[\chi] - \Delta H_k[\chi] + J \cdot \chi} \tag{2}$$

with $J \cdot \chi = \int d^d q \, J(q) \chi(-q)$ and

$$\Delta H_k[\chi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q) \chi(q) \chi(-q), \tag{3}$$

where $\chi(q)$ is the microscopic field. In Eq. (3), $R_k(q)$ is chosen in such a way that it acts as a cut-off function that decouples the low- and high-energy modes. This imposes several constraints:

$$R_k(q) \sim k^2 \quad \text{for} \quad q^2 \ll k^2, \tag{4}$$

$$R_k(q) \to 0 \quad \text{for} \quad q^2 \gg k^2. \tag{5}$$

Equation (4) means that, at low momentum with respect to k, $R_k(q)$ essentially acts as a mass, i.e., an infrared cutoff, which prevents the propagation of the low-energy modes. This ensures that these modes do not contribute to Γ_k [59]. Equation (5) implies that $R_k(q)$ does not affect the propagation of high-energy modes. They are thus almost fully taken into account in $\mathcal{Z}_k$ and, consequently, in Γ_k.

In order to recover the limits (1), $R_k(q)$ must also satisfy

$$R_k(q) \to \infty \quad \text{when} \quad k \to \Lambda \quad \text{at fixed} \quad q, \tag{6}$$

which ensures that Γ_k coincides with the microscopic Hamiltonian H when $k \to \Lambda$, and

$$R_k(q) \to 0 \quad \text{identically when} \quad k \to 0, \tag{7}$$

which ensures that, in the limit of vanishing k, one recovers the standard effective action Γ. Note that since we are only interested here in the universal long distance behavior and not in quantities depending on microscopic details, we send Λ to ∞.

The effective average action Γ_k is then defined as

$$\Gamma_k[\phi] = -\ln \mathcal{Z}_k[J] + J \cdot \phi - \Delta H_k[\phi], \tag{8}$$

where ϕ stands for the running order parameter $\phi_k(q)$:

$$\phi_k(q) = \langle \chi(q) \rangle_k = \left. \frac{\delta \ln \mathcal{Z}_k[J]}{\delta J(q)} \right|_{J=0}. \tag{9}$$

It follows from the definition (8) that $\Gamma_k[\phi]$ essentially corresponds to the Legendre transform of $\ln \mathcal{Z}_k[J]$, up to the mass term ΔH_k which allows us to recover the limits (1) [37].

The effective average action Γ_k follows an exact equation which controls its evolution with the running scale k [3]:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_t R_k(q) \{\Gamma_k^{(2)}[\phi(q)] + R_k(q)\}^{-1}, \tag{10}$$

where $t = \ln(k/\Lambda)$ and $\Gamma_k^{(2)}[\phi]$ is the second functional derivative of Γ_k with respect to the field ϕ. We emphasize that Eq. (10) is exact and thus contains all perturbative and nonperturbative features of the underlying theory (see [37] for technical details and [20] for a review of the applications of this equation to concrete physical issues).

III. TRUNCATIONS OF THE EFFECTIVE AVERAGE ACTION

Equation (10) is a functional partial integro-differential equation that has obviously no known solution in the general case. Therefore, to render it tractable, one has to truncate the effective action Γ_k. The most natural truncation, well suited to the study of the long distance physics, is the derivative expansion. It consists in writing an *ansatz* for Γ_k as a power series in $\partial\phi$. Let us first consider the case of an $O(N)$ invariant theory for which the *ansatz* at the order ∂^2 writes [37]

$$\Gamma_k[\phi] = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} Z_k(\rho)(\partial_\mu \vec{\phi})^2 + \frac{1}{4} Y_k(\rho)(\partial_\mu \rho)^2 \right. $$
$$\left. + O(\partial^4) \right\}, \tag{11}$$

where $\vec{\phi}$ is an N-component vector and $\rho = \vec{\phi}^2/2$ is the $O(N)$ invariant. In Eq. (11), $U_k(\rho)$ corresponds to the potential part of Γ_k while $Z_k(\rho)$ and $Y_k(\rho)$ correspond to the field renormalization functions. Thus, with $Z_k(\rho) = 1$ and $Y_k(\rho) = 0$, Eq. (11) provides the *ansatz* for the so-called local potential approximation (LPA) where the anomalous dimension vanishes. This kind of *ansatz* has been successfully used in several cases among which are the $O(N)$ [20] and Gross-Neveu models [15,16]. However, to deal with more complicated models, e.g., with matrixlike order parameters, a further ap-

proximation is almost unavoidable [11,13,38]. Indeed, when the symmetry is lower than $O(N)$, there are several invariants and the number of independent functions analogous to $Z_k(\rho, \dots)$ and $Y_k(\rho, \dots)$ grows. In this case, the integration of the flow can be very demanding. It is then very convenient to further truncate the functions $U_k(\rho, \dots)$, $Z_k(\rho, \dots)$ in power series of ρ and of all other invariants.

Here, we focus on the Ising model, described by a scalar, $\mathbb{Z}_2$-invariant field theory, considered as a toy model to study the derivative and field expansions. In this case, since the only independent field renormalization function is $Z_k(\rho)$, the function $Y_k(\rho)$ can be set to zero. The field truncation then writes

$$U_k(\rho) = \sum_{i=1}^{n} U_{i,k}(\rho - \rho_0)^i,$$

$$Z_k(\rho) = \sum_{i=0}^{p} Z_{i,k}(\rho - \rho_0)^i, \tag{12}$$

where $\rho_0 = \phi_0^2/2$, ϕ_0 being a particular configuration of the field ϕ. We shall come back to this point later.

The truncation in fields conveys two nice properties. First, with the *ansatz* (11) and (12), the RG flow equation (10) leads to a finite set of ordinary coupled differential equations for the coupling constants $U_{i,k}$'s and $Z_{i,k}$'s that is simpler to solve than the partial differential equations obeyed by the full functions $U_k(\rho)$ and $Z_k(\rho)$. Second, even the lowest order approximations, in which only the first nontrivial terms of $U_k(\rho)$ and $Z_k(\rho)$ are kept, give a fairly good qualitative picture of the physics [37,20].

However, the study of the truncated version of Eq. (10) raises several important questions.

(i) Does the derivative expansion converge and does it provide a satisfying accuracy at low orders? The question of the convergence of the derivative expansion, in its full generality, has not yet been considered and appears to be a major and open challenge. In practice, one is less interested in this delicate question than in the quality of the results and their improvement as the order of the derivative expansion is increased. In the case of $O(N)$ models, very accurate results have been obtained at second order in the derivative expansion. For instance, Wetterich *et al.* have shown that handling the full field dependence of the potential $U_k(\rho)$ and of the field renormalization functions $Z_k(\rho)$ and $Y_k(\rho)$ leads to results that can compete with the world best estimates, at least for the critical exponent ν [20]. The value obtained for the anomalous dimension η is less accurate. Its definition being linked to the momentum dependence of the two-point correlation function, an accurate determination of η probably requires higher-order terms in the derivative expansion. This question will be investigated in a forthcoming article [39].

(ii) Does the field expansion of $U_k(\rho)$ and $Z_k(\rho)$ converge and how rapidly? Once again, the general question of this convergence has not yet been investigated. Nevertheless, several works have dealt with field truncations at high order within the LPA [40–44] or with a field-independent field

renormalization [37], i.e., with $Z_k(\rho) = Z_{0,k}$. They suggest that a few orders suffice to obtain reasonably converged values of critical exponents. To our knowledge, their computation using also an expansion of $Z_k(\rho)$ has been only studied in the Ising model and using a power-law cutoff function [44]. In this study we extend this analysis to two other families of cutoff functions, leading to more accurate results.

Questions (i) and (ii) are linked with a corollary issue, which resides in the choice of cutoff function.

(iii) Can the accuracy be improved through the choice of cutoff function R_k? Of course, when no truncation is made, an exact solution for $\Gamma[\phi] = \lim_{k \to 0} \Gamma_k[\phi]$ does not depend on the function R_k used, whereas any kind of truncation induces a spurious dependence on it. One can thus wonder how to optimize the choice of this cutoff function. This question is not as trivial as it seems since one has to decide on an optimization criterion: rapidity of convergence of the expansions in powers of derivatives, fields, or amplitudes [41,42,45,43,46]? Accuracy of the results? Sensitivity of the results with respect to the cutoff? We specifically concentrate on these two latter issues in the following.

IV. OPTIMIZATION AND PRINCIPLE OF MINIMAL SENSITIVITY

Up to now, attempts to optimize nonperturbative RG equations have been mainly worked out in the Polchinski equation [47], in particular at second order in the derivative expansion. For instance, Ball *et al.* [48] and Comellas [49] have tried to suppress the cutoff and normalization dependence of the exponents ν and η by using the principle of minimal sensitivity (PMS) [50]. We shall not pursue this within this framework since it has now been widely recognized that the effective average action method is the most efficient way to deal with the nonperturbative RG. We will thus consider this latter formalism.

In the context of the effective average action method, within the framework of LPA, Litim has proposed to consider the quantity C, defined by [42,45,43,51,46,52]:

$$\min_{q^2 \geq 0} \{\Gamma_k^{(2)}[\phi(q)]|_{\phi = \phi_0} + R_k(q)\} = Ck^2, \tag{13}$$

where $\Gamma_k^{(2)}[\phi(q)] + R_k(q)$ is the inverse of the full regularized propagator and C parametrizes the gap amplitude. According to Litim, the gap is bounded from above and the best cutoff functions are those which maximize this gap [42,45,43,51,46,52]:

$$C_{\mathrm{opt}} = \max(C) \quad \text{when varying } R_k. \tag{14}$$

The idea behind this criterion is that the larger C, the more stable the truncated RG flow. Indeed, it has been shown that the maximum of the gap corresponds to the largest radius of convergence of an amplitude expansion. This suggests that the optimal selected regulators should have nice properties, such as improving the convergence of the field expansion [42,43,51,46,52]. Moreover, in [43] it has been shown that, within the LPA, the criterion (14) is also linked to a PMS.

At this stage, let us shed light on some important features of the "gap criterion." First Eq. (14) does *not* select a unique cutoff function: many R_k maximizing the gap have been exhibited, for instance, in [42]. Also, the various optimized cutoff functions, solutions of Eq. (14), can lead to quantitatively different critical exponents, depending on the specific properties of a given cutoff function, like its asymptotic behavior (see below and compare [46] and [44]). The quality of the results therefore relies on the choice of the type of optimized regulator. Second, beyond the LPA, the implementation of the gap criterion (14) appears to be nontrivial. Indeed, the field renormalization function $Z_k(\rho_0)$ induces an implicit R_k dependence in $\Gamma_k^{(2)}[\phi(q)]$ that complicates the maximization of the gap. Moreover, it is not completely clear whether, beyond the LPA, this criterion would still convey the nice properties it shows at the lowest order of the derivative expansion and, in particular, its link to a PMS. As we are specifically concerned here with the question of the sensitivity of the results with respect to the cutoff function, we favor a method that directly probes the dependence of the critical exponents on the cutoff function. We have decided to base our analysis on the PMS, which can always be simply implemented and has already proven its efficiency.

Let us recall how it works. Suppose, for instance, that we compute a quantity Q in an approximate way. The approximation used may induce a dependence of Q on a parameter—denoted here by α—which is spurious. The PMS consists in choosing for α the value α_{PMS} for which Q is stationary:

$$\left.\frac{dQ(\alpha)}{d\alpha}\right|_{\alpha_{PMS}} = 0. \tag{15}$$

One thus expects that imposing such a constraint, satisfied by Q computed without approximation, improves the approximate determination of this quantity. The obvious drawback of this method is that Eq. (15) can have many solutions. This worsens if several quantities are simultaneously studied, and lead to distinct solutions. An additional criterion is then necessary to select a unique one.

We first study the LPA of the scalar, $\mathbb{Z}_2$-invariant field theory relevant for the Ising model. We show that the PMS allows one to optimize the quality of the results. We then study the $O(\partial^2)$ approximation of the derivative expansion and show that the PMS leads to accurate results provided we add some new inputs to discriminate the solutions.

V. THE LOCAL POTENTIAL APPROXIMATION OF THE ISING MODEL

Let us recall that the LPA consists in approximating Γ_k by

$$\Gamma_k[\phi] = \int d^d x \left\{ U_k(\rho) + \frac{1}{2}(\partial\phi)^2 \right\}, \tag{16}$$

i.e., in neglecting the field renormalization. This *ansatz*, once plugged into Eq. (10), enables us to get the evolution equa-

tion for U_k. Actually, working with dimensionless quantities is necessary to get a fixed point, so that we define

$$r(y) = \frac{R_k(q^2)}{q^2} \quad \text{with} \quad y = \frac{q^2}{k^2},$$

$$u_k = k^{-d} U_k,$$

$$\widetilde{\rho} = k^{2-d} \rho. \tag{17}$$

The RG equation obeyed by u_k writes

$$\frac{\partial u_k}{\partial t} = -d u_k + (d-2)\widetilde{\rho} u_k' - v_d L_0^d(w), \tag{18}$$

where $u_k = u_k(\widetilde{\rho})$, $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$, the prime means derivation with respect to $\widetilde{\rho}$, $w = u_k' + 2\widetilde{\rho} u_k''$, and

$$L_0^d(w) = \int_0^\infty dy\, y^{d/2-1} \frac{2y^2 r'(y)}{y[1+r(y)]+w}. \tag{19}$$

The nonperturbative features of the evolution of the potential are entirely encoded in the integral L_0^d, called threshold function [37].

We now study Eq. (18) within a field truncation

$$u_k(\widetilde{\rho}) = \sum_{i=1}^n u_i (\widetilde{\rho} - \widetilde{\rho}_0)^i, \tag{20}$$

where we have suppressed the index k for the coupling constants. Once $u_k(\widetilde{\rho})$ is truncated at a finite order n of the field expansion, the field configuration $\widetilde{\rho}_0$ around which it is expanded matters. Two configurations have been widely studied: the vanishing field configuration, $\widetilde{\rho}_0 = 0$, and the configuration where $u_k(\widetilde{\rho})$ has a nontrivial minimum [60]:

$$\left.\frac{\partial u_k}{\partial\widetilde{\rho}}\right|_{\widetilde{\rho}_0} = 0. \tag{21}$$

All the studies performed using field truncations show that the convergence properties are improved by expanding around the minimum rather than around the zero field configuration [28,44]. Therefore, we choose the former.

We also need to choose families of cutoff functions R_k to perform calculations. For simplicity, we restrict for now our study to families of cutoff functions R_k depending on a single parameter. We extend this to a two-parameter family in Sec. VI D. We consider two usual cutoff functions. The first one is the exponential cutoff, which has been often used and constitutes an efficient and robust regulator [37]. The other one, the theta cutoff, has been introduced by Litim [45]. It presents the advantage of leading to threshold functions that can be analytically computed. We extend these functions, by multiplying them by a factor α, to two one-parameter families [20,46]:
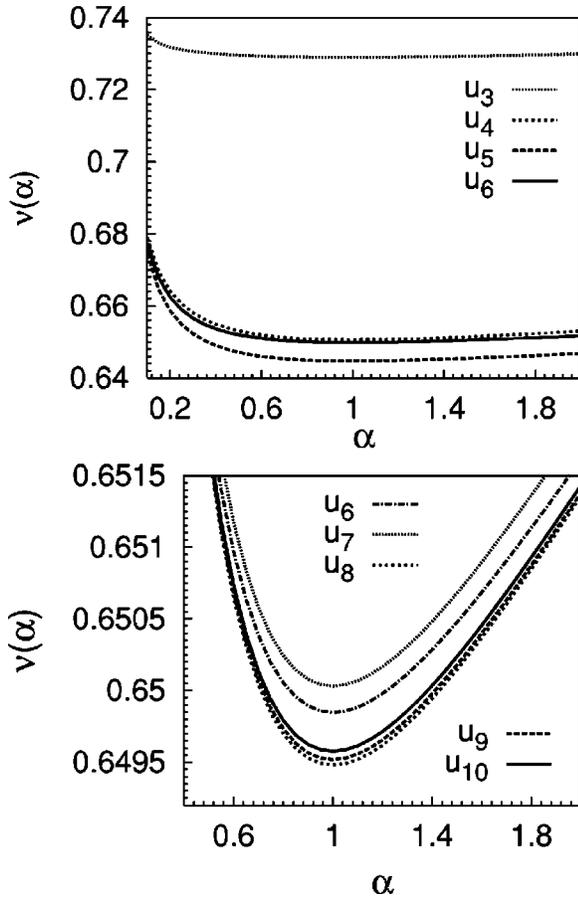
FIG. 1. Curves $\nu(\alpha)$ for the cutoff $r_{\theta,\alpha}$, for different truncations of the potential $u_k(\tilde{\rho})$. Note that for $n \geqslant 6$ (lower figure) the ν axis is magnified.

$$r_{\exp,\alpha}(y) = \alpha \frac{1}{e^y - 1},$$

$$r_{\theta,\alpha}(y) = \alpha \left(\frac{1}{y} - 1 \right) \theta(1-y). \qquad (22)$$

Note that both of these cutoff functions can be optimized according to the gap criterion.

For both families, we investigate the α dependence of the critical exponent ν over a large range of α, for each order n of the field expansion, up to the tenth power of $\tilde{\rho}$. We indeed expect the most relevant operators to be contained in the first terms, and thus the evolution of ν as a function of the order of the truncation to be stabilized at, or before, the tenth order.

We find that, at each order, $\nu(\alpha)$ exhibits a single point of minimal sensitivity for both cutoff functions. For $r_{\theta,\alpha}$ (Fig. 1) the minimum occurs at $\alpha_{PMS}=1$, as already found in [46], with an optimized ν equal to $\nu(\alpha_{PMS})=\nu_{PMS}=0.650$. For $r_{\exp,\alpha}$ (Fig. 2) one has $\alpha_{PMS}=6.03$ and $\nu_{PMS}=0.651$ (see Table I). Both cutoff functions lead to very similar optimal results for ν, differing by less than 0.5% to all orders n, as shown in Fig. 3. The converged values of ν are reached below the percent level in both cases after only a few orders ($n=4$), as expected.
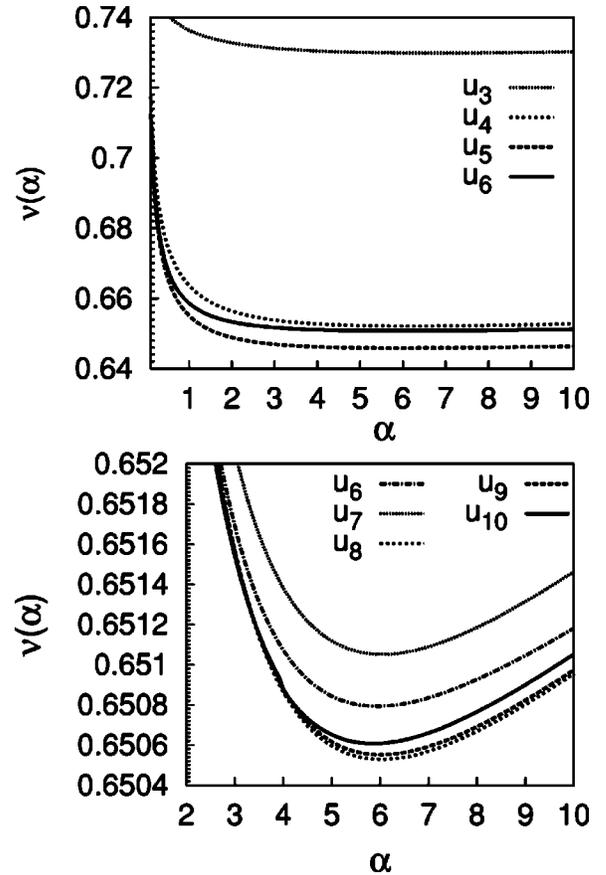

FIG. 2. Curves $\nu(\alpha)$ for the cutoff $r_{\exp,\alpha}$ for different truncations of the potential $u_k(\tilde{\rho})$. Note that for $n \geqslant 6$ (lower figure) the ν axis is magnified.

TABLE I. Critical exponents of the three-dimensional Ising model. (a), (b), (b'), (c), and (d) are computed from the effective average action method: (a) with $r_{\exp,\alpha_{PMS}}$ (present work); (b) with $r_{\theta,\alpha_{PMS}}$ (present work); (b') with $r_{\theta,\alpha=1}$ [46]; (c) with a power-law cutoff [44,53]; (d) with $r_{\exp,\alpha=1}$ without field expansion [54]; (e) from perturbation theory including 7-loop contributions [55]; (f) from Monte Carlo simulations [56]; (g) from experiment in mixing transition [57]; (h) from experiment in liquid-vapor transition (computed from $3\nu=2-\alpha$ [58]).

	Ref.	ν	η
	(a)	0.651	0
LPA	(b),(b')	0.650	0
	(c)	0.660	0
	(d)	0.6307	0.0467
∂^2	(a)	0.6281	0.0443
	(b)	0.6260	0.0470
	(c)	0.6175	0.0542
7-loop	(e)	0.6304(13)	0.0335(25)
MC	(f)	0.6297(5)	0.0362(8)
Exp.	(g)	0.636(31)	0.045(11)
	(h)	0.6298(90)	

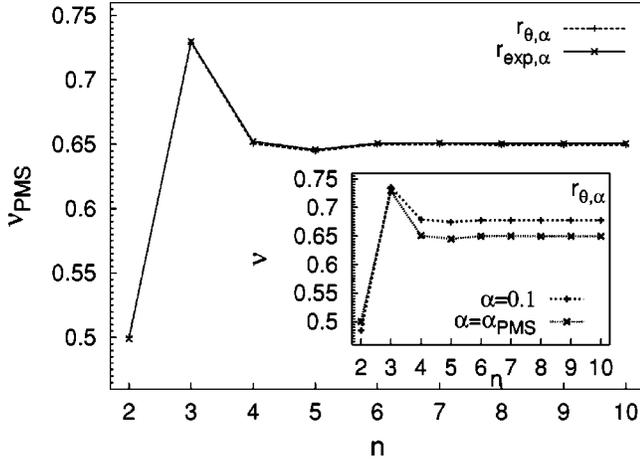FIG. 3. ν_{PMS} for $r_{\theta,\alpha}$ and $r_{\exp,\alpha}$ as a function of the order n of the field truncation of $u_k(\tilde{\rho})$. The two curves almost superimpose for all n. In the inset, ν is displayed for $r_{\theta,\alpha}$ for both α_{PMS} and $\alpha = 0.1$.

The inset of Fig. 3, where the evolution of ν with the order n of the field expansion is compared for $\alpha = 0.1$—chosen for illustrative purpose—and $\alpha = \alpha_{PMS} = 1$, shows that the same convergence level is reached independently of α right from the $n = 4$ order, though the asymptotic values of $\nu(0.1)$ and $\nu(\alpha_{PMS})$ differ significantly. This shows that the rapidity of convergence criterion is helpless here to select a cutoff.

We now compare our results with those obtained through the gap criterion. As displayed in Fig. 1, $\alpha_{PMS} = 1$ exactly with $r_{\theta,\alpha}$ to all orders. For this cutoff function the α_{PMS} value coincides with that given by the gap criterion [46]. For $r_{\exp,\alpha}$, α_{PMS} converges to 6.03 (see lower curves in Fig. 2) whereas the gap criterion selects an optimal parameter $\alpha_0 = 3.92$ [42]. In this case, the two methods seem to differ. However, since the variations of ν, once converged, do not exceed 1% in the whole range $\alpha \in [\alpha_1 \simeq 1.2, \alpha_2 \simeq 74]$, we do not expect the two methods of optimization to lead to drastically different critical exponents. Indeed, $\nu(\alpha_0) = \nu(\alpha_{PMS})$ up to 10^{-4}. Thus for the two families of regulators considered here, the PMS and gap criteria coincide. It has been argued that this property holds, within the LPA, for more general families of regulators [43,46].

Note that for the exponential cutoff, the standard choice $\alpha = 1$ leads to $\nu = 0.658$. This value, which does not correspond to an optimized one, differs by a little bit more than 1% from ν_{PMS}. For completeness, we also mention that the power-law regulator optimized via the gap criterion—$r(y) = y^{-2}$—leads to a less accurate result: $\nu = 0.660$ [44,53].

Finally, let us emphasize that the world best value $\nu = 0.6304(13)$ (see Table I) lies below all curves $\nu(\alpha)$ for both cutoff functions and that the PMS solutions for ν are minima. Thus, $\nu(\alpha_{PMS})$ is the most accurate value achievable within each family of cutoff functions studied here. The PMS therefore constitutes a powerful method to optimize the cutoff function in order to reach the best accuracy on the critical exponents.

VI. ORDER ∂^2 OF THE DERIVATIVE EXPANSION

We now show how the PMS can be consistently implemented at the order ∂^2 of the derivative expansion for which, as far as we know, no optimization procedure has ever been implemented within the effective average action method. We dispose of two physical quantities candidates for a PMS analysis: ν and η. We perform both analyses independently, with each cutoff function. We show in Sec. VI A that the PMS allows one to improve the accuracy on both exponents. We especially highlight that accuracy is not synonymous with rapidity of convergence of the field expansion. In Sec. VI B we bring out a necessary condition for the independent implementation of the two PMS on ν and η to be consistent. We then check that our results meet this condition. In Sec. VI C we exhibit cases where, contrary to what occurs in the LPA, multiple PMS solutions exist. We show that a unique one can be selected thanks to general arguments. We end up by extending the analysis to a two-parameter family of cutoff functions.

A. Accuracy of the PMS solution and convergence of the field expansion

This section is devoted to showing that the PMS is still, at order ∂^2, the appropriate tool to find, within a class of cutoff functions, the one giving the best accuracy. Though it seems counterintuitive, we emphasize that this cutoff function does not coincide with the one providing the fastest convergence of the field expansion of $Z_k(\rho)$. To this purpose, we implement both PMS independently on ν and η, postponing the coherence of this to the next section.

Working with a nontrivial field renormalization function $Z_k(\rho)$, dimensionless *and* renormalized quantities are necessary in order to get a fixed point, so that we define

$$r(y) = \frac{R_k(q^2)}{Z_{0,k} q^2} \quad \text{with} \quad y = \frac{q^2}{k^2},$$

$$u_k(\tilde{\rho}) = k^{-d} U_k(\tilde{\rho}),$$

$$\tilde{\rho} = Z_{0,k} k^{2-d} \rho,$$

$$z_k(\tilde{\rho}) = \frac{Z_k(\tilde{\rho})}{Z_{0,k}}, \tag{23}$$

where $Z_{0,k}$ is defined in Eq. (12). The RG equation obeyed by $z_k(\tilde{\rho})$ writes [37,54]

$$\frac{\partial z_k}{\partial t} = \eta z_k + \tilde{\rho} z_k'(d - 2 + \eta) + v_d(z_k' + 2\tilde{\rho} z_k'') L_1^d(w, z_k, \eta)$$

$$- 4 v_d \tilde{\rho} z_k'(3 u_k'' + 2\tilde{\rho} u_k''') L_2^d(w, z_k, \eta)$$

$$- 2 v_d(2 + 1/d) \tilde{\rho}(z_k')^2 L_2^{d+2}(w, z_k, \eta) + (4/d)$$

$$\times v_d \tilde{\rho}(3 u_k'' + 2\tilde{\rho} u_k''')^2 M_4^d(w, z_k, \eta) + (8/d)$$

$$\times v_d \tilde{\rho} z_k'(3 u_k'' + 2\tilde{\rho} u_k''') M_4^{d+2}(w, z_k, \eta)$$

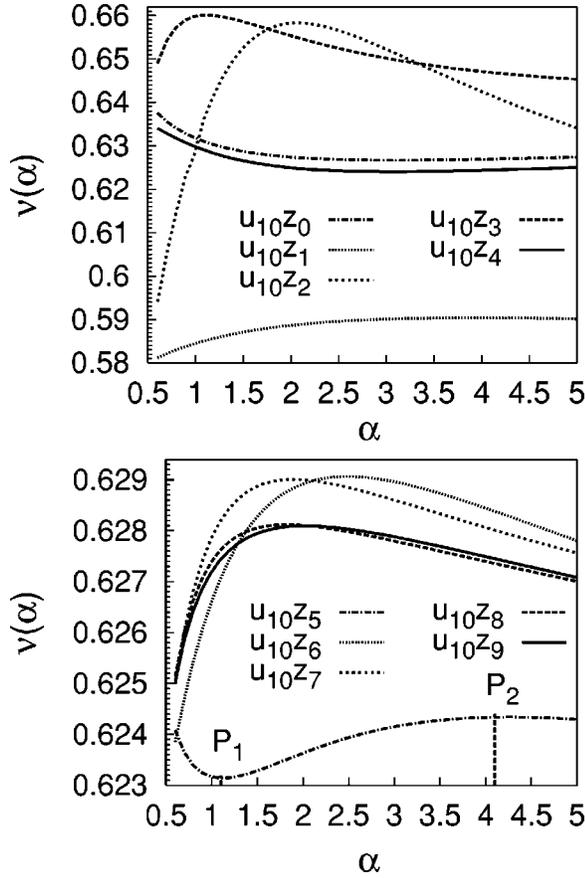$$+ (4/d) v_d \tilde{\rho}(z_k')^2 M_4^{d+4}(w, z_k, \eta), \tag{24}$$

FIG. 4. Curves $\nu(\alpha)$ for $r_{\exp,\alpha}$ for different truncations of the field renormalization $z_k(\tilde{\rho})$. For $p \geq 5$ (lower figure) the ν axis is magnified. Note that the curve $u_{10}z_5$ shows two extrema [62].
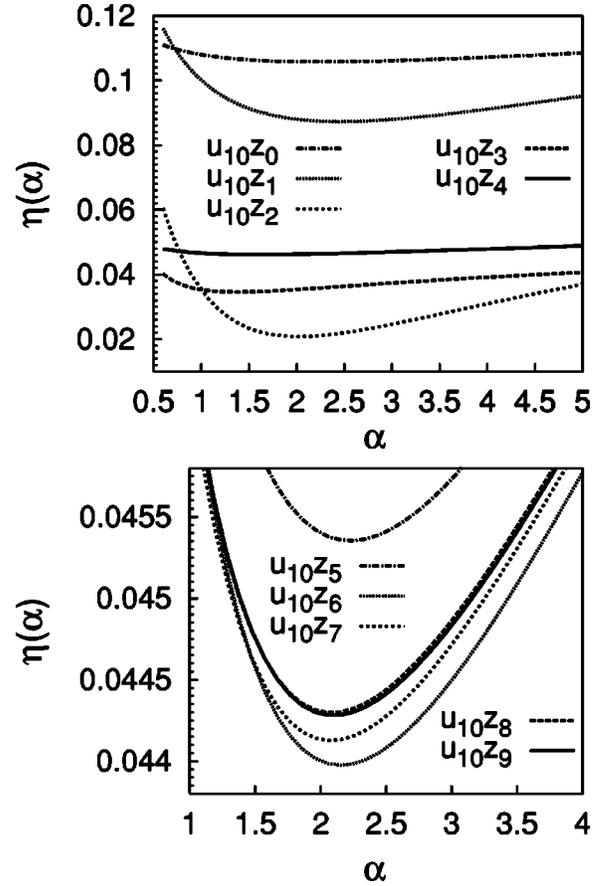

FIG. 5. Curves $\eta(\alpha)$ for $r_{\exp,\alpha}$ for different truncations of the field renormalization $z_k(\tilde{\rho})$. For $p \geq 5$ (lower figure) the η axis is magnified.

where $w = u' + 2\tilde{\rho}u''$, the prime means a derivative with respect to $\tilde{\rho}$, and the threshold functions are defined, for $n \geq 1$, by

$$L_n^d(w, z_k, \eta) = n \int_0^\infty dy\, y^{d/2-1} \frac{2y^2 r'(y) + \eta y r(y)}{(P(y) + w)^{n+1}}, \quad (25)$$

$$
\begin{aligned}
M_n^d(w, z_k, \eta) = \int_0^\infty dy\, y^{d/2} \frac{1 + r(y) + y r'(y)}{[P(y) + w]^n} \Bigg\{ y[1 + r(y)] \\
+ y r'(y)][\eta r(y) + 2 y r'(y)] \left(\frac{n}{P(y) + w} \right) \\
- 2\eta[r(y) + y r'(y)] - 4y[2 r'(y) \\
+ y r''(y)] \Bigg\},
\end{aligned}
\quad (26)
$$

where

$$P(y) = y[z_k + r(y)]. \quad (27)$$

The anomalous dimension η is given by

$$\eta = -\frac{d}{dt} \ln Z_{0,k}. \quad (28)$$

As previously, we truncate the field renormalization function $z_k(\tilde{\rho})$ up to the pth power of $\tilde{\rho}$:

$$z_k(\tilde{\rho}) = \sum_{i=0}^p z_i (\tilde{\rho} - \tilde{\rho}_0)^i. \quad (29)$$

We use, for the potential $u_k(\tilde{\rho})$, the expansion given in Eq. (20), up to the $\tilde{\rho}^{10}$ term, which represents a very accurate approximation of $u_k(\tilde{\rho})$ in the vicinity of its minimum as shown in the preceding section. We expand $z_k(\tilde{\rho})$ up to the ninth power of $\tilde{\rho}$ which turns out to be sufficient to obtain converged results.

At each order p of the field expansion of $z_k(\tilde{\rho})$, we have computed the exponents ν and η as functions of α for both cutoff functions $r_{\theta,\alpha}$ and $r_{\exp,\alpha}$. Figures 4 and 5 gather the curves representing these functions, labelled $u_{10}z_p$, $p = 0, \ldots, 9$, on the example of $r_{\exp,\alpha}$. They are displayed on a range of α around the extremum and separated in two distinct figures since the $p \geq 5$ curves would be superimposed without magnification. This seems to indicate that the
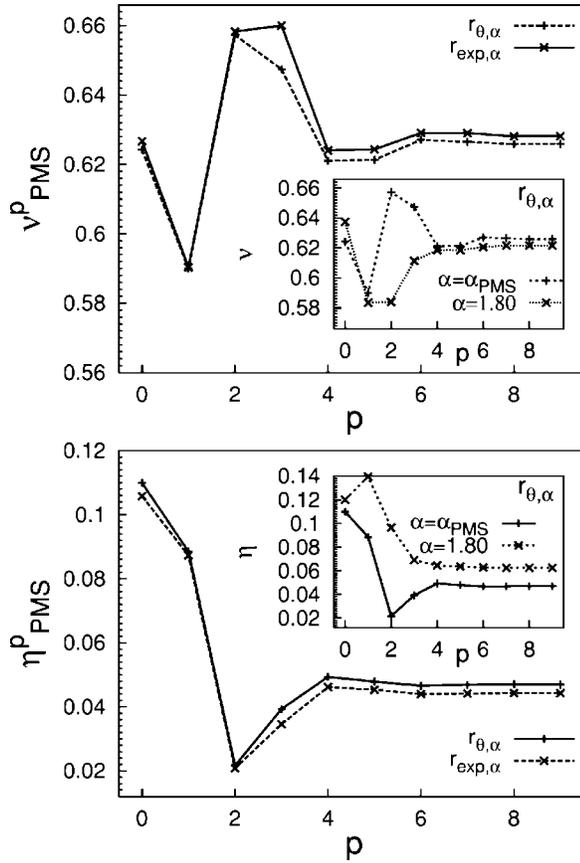
FIG. 6. ν_{PMS} and η_{PMS} for both $r_{\exp,\alpha}$ and $r_{\theta,\alpha}$ as functions of the order p of the field expansion of $z_k(\widetilde{\rho})$. In the insets are displayed, for $r_{\theta,\alpha}$, and for two distinct values of α, $\alpha = \alpha_{PMS}(p)$, and $\alpha = 1.8$, the critical exponents ν (upper inset) and η (lower inset) as functions of p.

field expansion converges, at least on the whole range of α studied. The same conclusion holds for $r_{\theta,\alpha}$, with very similar curves (that we therefore do not show), up to the important subtlety, discussed in Sec. VIC, that two PMS solutions exist in this case and that only one has to be considered. We call $\nu^{\infty}(\alpha)$ and $\eta^{\infty}(\alpha)$ the two limit functions obtained for $p \to \infty$. In practice, we approximate these functions by those at $p = 9$.

Let us emphasize that for both cutoff functions, (i) the rapidity of convergence to $\nu^{\infty}(\alpha)$ and $\eta^{\infty}(\alpha)$ and (ii) the asymptotic values ν^{∞} and η^{∞} both depend on α. One can thus naturally wonder whether the values of α for which the convergence is the fastest coincide with those for which the exponents are the most accurate compared with the world best results. We shall show that this is *not* the case contrary to what is widely believed.

1. Accuracy

We first point out that the PMS exponents are, as in the LPA case, the most accurate ones. We have determined, for each p, the values $\alpha_{PMS}^{\nu}(p)$ and $\alpha_{PMS}^{\eta}(p)$ for which, respectively, ν and η reach their extremum. The corresponding exponents are referred to, in the following, as

$\nu_{PMS}^{p} = \nu(\alpha_{PMS}^{\nu}(p))$ and $\eta_{PMS}^{p} = \eta(\alpha_{PMS}^{\eta}(p))$. The obtained PMS asymptotic values are $\nu_{PMS}^{\infty} = 0.6281$ and $\eta_{PMS}^{\infty} = 0.0443$ for the exponential cutoff, and $\nu_{PMS}^{\infty} = 0.6260$ and $\eta_{PMS}^{\infty} = 0.0470$ for the theta cutoff. These values of the exponents are indeed the best achievable within each class of cutoff functions studied, since the world best value of ν lies above the sets of curves in Fig. 4 and since the extremum is a maximum (and vice versa for η) (see Table I and [61]). The PMS is thus, as in the LPA case, the appropriate tool to find, among a family of cutoff functions, the one providing the best accuracy.

2. Rapidity of convergence

The evolution of ν_{PMS}^{p} and η_{PMS}^{p} with the order p of the field expansion of $z_k(\widetilde{\rho})$ is displayed in Fig. 6 for both cutoff functions. The convergence of ν_{PMS} and η_{PMS}, at the percent level, requires at least $p = 4$ for both cutoff functions. However, there exist values of the parameter α, for instance $\alpha = 1.80$ for $r_{\theta,\alpha}$, for which the convergence is faster than for α_{PMS}. This is illustrated in the insets of Fig. 6. Indeed $\eta(\alpha = 1.8)$ has already converged at the percent level for $p = 3$, but to a different value than η_{PMS}^{∞}. Thus, the PMS exponents, which are the most accurate, are not those converging the fastest.

We conclude that (i) the PMS leads to the most accurate exponents within each class of cutoffs studied, (ii) a criterion based on rapidity of convergence of the field expansion would be misleading here since it would select cutoff functions leading to exponents significantly differing from the PMS ones.

B. Consistency condition for independent PMS implementations

We have implemented and discussed the PMS analyses independently on ν and η along the preceding section. This has naturally led us to two distinct PMS values of α at each order p, $\alpha_{PMS}^{\nu}(p)$, and $\alpha_{PMS}^{\eta}(p)$. One can thus wonder whether it makes sense to compute two different quantities with two different cutoff functions. We now provide a natural condition for the whole procedure to be consistent.

Let us notice that since the field expansion seems to converge (as shown in the preceding section), the two sequences $\alpha_{PMS}^{\nu}(p)$ and $\alpha_{PMS}^{\eta}(p)$ also converge. The asymptotic value $\alpha_{PMS}^{\eta}(p = \infty)$ (resp. $\alpha_{PMS}^{\nu}(p = \infty)$) is the one that achieves the minimum dependence of the exponent η (resp. ν) on the cutoff function at order ∂^2 of the derivative expansion. There is no reason for them to coincide. However, the discrepancy between the α_{PMS}'s does not matter as long as choosing one or the other does not change significantly the value of each exponent. A consistency condition is thus

$$\nu(\alpha_{PMS}^{\eta}(\infty)) \simeq \nu(\alpha_{PMS}^{\nu}(\infty)) \qquad (30)$$

and

$$\eta(\alpha_{PMS}^{\nu}(\infty)) \simeq \eta(\alpha_{PMS}^{\eta}(\infty)). \qquad (31)$$
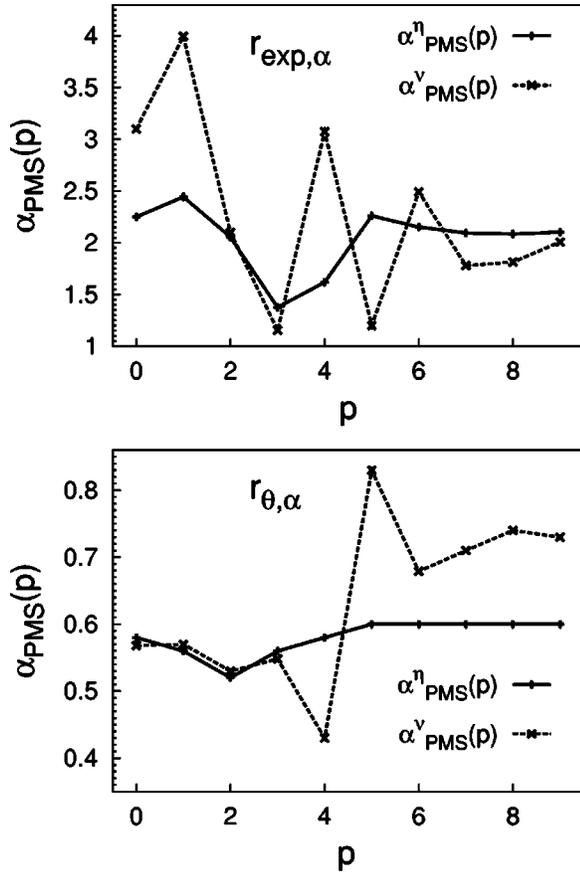
FIG. 7. Plot of α_{PMS}^{η} and α_{PMS}^{ν} as functions of the order p of the field expansion of $z_k(\tilde{\rho})$ for both cutoff functions $r_{\exp,\alpha}$ and $r_{\theta,\alpha}$. α_{PMS}^{η} (resp. α_{PMS}^{ν}) is the value where lies the PMS extremum of η (resp. ν).

FIG. 8. Curves $\nu(\alpha)$ and $\eta(\alpha)$ for both cutoff functions $r_{\theta,\alpha}$ and $r_{\exp,\alpha}$ within LPA (labeled u_{10}) and at $O(\partial^2)$ of the derivative expansion (labeled $u_{10}z_9$) for the maximal truncations of $u_k(\tilde{\rho})$ and $z_k(\tilde{\rho})$ computed here. The two PMS extrema for $r_{\theta,\alpha}$ are shown for both ν and η.

Reciprocally, large discrepancies between the values, at the two α_{PMS}'s, of an exponent would be an indication of a failure of convergence. It could be imputed to either a too low order of expansion, or to an unappropriate choice of cutoff functions family.

In principle, we should check the consistency over the whole set of exponents describing the model. Let us, however, show that once this condition is satisfied by two independent exponents, it is automatically satisfied by all the others, provided the scaling relations hold within the chosen truncation scheme (in fields and derivatives). Let us first emphasize that it has been observed in all instances where it has been studied that the scaling relations remain precisely verified order by order in the field expansion, although the exponents vary much with the order. We thus assume that computing the critical exponents either directly or from the scaling relations is (almost) equivalent. In this case, an exponent, γ for instance, related to ν and η through the scaling relation

$$\gamma(\alpha) = \nu(\alpha)[2 - \eta(\alpha)] \qquad (32)$$
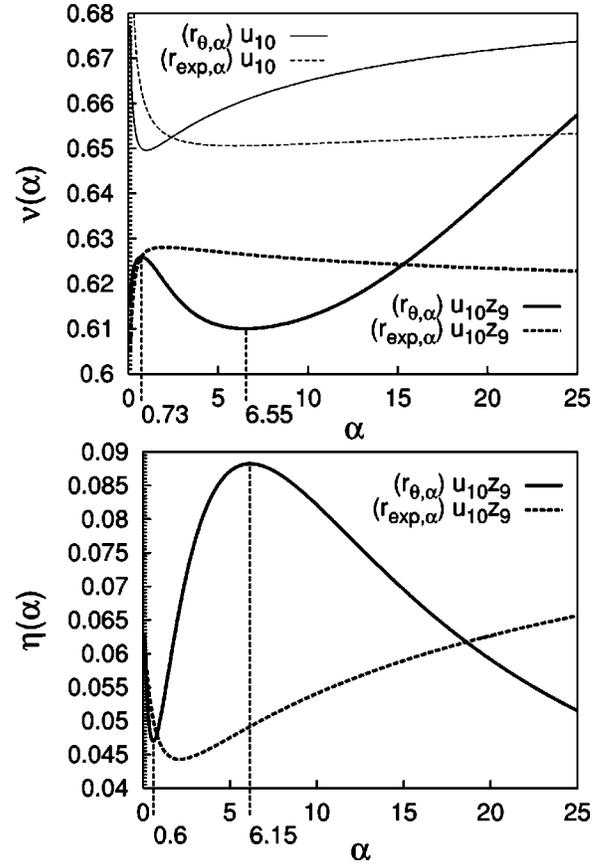
obviously verifies for all α

$$\frac{d\gamma}{d\alpha} = \frac{d\nu}{d\alpha}(2-\eta) - \nu\frac{d\eta}{d\alpha}. \qquad (33)$$

In the simple case where α_{PMS}^{ν} coincides with α_{PMS}^{η}, we deduce from Eq. (33) that $\gamma(\alpha)$ also reaches its extremum for this α_{PMS}. Thus, $\alpha_{PMS}^{\gamma} = \alpha_{PMS}^{\nu} = \alpha_{PMS}^{\eta}$ and the consistency is trivially verified for γ also. In the general case where the α_{PMS}'s are distinct, if they correspond to consistent exponents ν and η according to Eqs. (30) and (31), one is ensured that both exponents are almost stationary between these two α_{PMS}'s, provided the functions $\nu(\alpha)$ and $\eta(\alpha)$ are smooth enough in this range. Hence, it follows from Eq. (33) that $d\gamma/d\alpha$ almost vanishes both at $\alpha = \alpha_{PMS}^{\eta}$ and at $\alpha = \alpha_{PMS}^{\nu}$. This means that $\gamma(\alpha)$ is also stationary around these points, and thus, γ computed from a PMS analysis should verify

$$\gamma(\alpha_{PMS}^{\gamma}(\infty)) \simeq \gamma(\alpha_{PMS}^{\eta}(\infty)) \simeq \gamma(\alpha_{PMS}^{\nu}(\infty)), \qquad (34)$$

i.e., γ meets the consistency condition. Using the same argument for all the other exponents, we deduce that the independent implementations of the PMS on all exponents are consistent once they are for two independent ones.

Let us now examine our results. Figure 7 sketches

$\alpha_{PMS}^{\nu}(p)$ and $\alpha_{PMS}^{\eta}(p)$ as functions of the order p of the field truncation for both cutoff functions $r_{\exp,\alpha}$ and $r_{\theta,\alpha}$. Let us set out a few comments. First, the functions $\alpha_{PMS}^{\nu}(p)$ and $\alpha_{PMS}^{\eta}(p)$ converge as expected. On the one hand, $\alpha_{PMS}^{\eta}(p)$ turns out to be very stable, and roughly converging as fast as η_{PMS}^{p}. This originates in the very peaked shape of the function $\eta(\alpha)$ (lower curves of Fig. 5). On the other hand, α_{PMS}^{ν} shows larger oscillations, due to the flatness of the function $\nu(\alpha)$ (lower curves of Fig. 4). It is worth mentioning that since the exponents have almost converged at $p=4$ (Fig. 6), the fluctuations on the corresponding α_{PMS} values induce negligible variations on them for $p \geqslant 4$.

Let us now show that the independent analyses of η and ν give consistent results with respect to Eqs. (30) and (31). The asymptotic values are approximated by those at $p=9$. The consistency condition is trivially verified for $r_{\exp,\alpha}$ since in this case $\alpha_{PMS}^{\eta}(\infty) \simeq \alpha_{PMS}^{\nu}(\infty)$ (see Fig. 7). For $r_{\theta,\alpha}$, we find

$$|\nu(\alpha_{PMS}^{\eta}(\infty)) - \nu(\alpha_{PMS}^{\nu}(\infty))| \simeq 10^{-4},$$

$$|\eta(\alpha_{PMS}^{\eta}(\infty)) - \eta(\alpha_{PMS}^{\nu}(\infty))| \simeq 6 \times 10^{-4}, \qquad (35)$$

which are both negligible. Thus, in this case also, the consistency condition is fulfilled. We draw the conclusion that the PMS analyses have selected a unique optimal value for each exponent ν and η although the corresponding α_{PMS}'s do not coincide. They enable us to deduce the remaining critical exponents as well.

C. Discrimination of multiple PMS extrema

The results discussed in Sec. VI A are associated with a particular PMS solution while several ones can exist, leading to significantly different exponents [62]. This happens for $r_{\theta,\alpha}$ (see Fig. 8). We now expose the general arguments we used to discriminate between the different PMS solutions.

Suppose that the derivative expansion is studied order by order without field truncation (or equivalently that the field expansion is perfectly converged). If the derivative expansion converges, the corrections on exponents must be smaller and smaller as the order of the expansion is increased, at least at sufficiently large order. On the other hand, as the asymptotic value of any observable is exact, it must be independent of the cutoff function. Thus, for any quantity, all cutoff functions lead to the same asymptotic—exact—value, although not at the same speed. In practice, the aim is to reach it as fast as possible. This means that, at least beyond a certain order, the best cutoff *for the derivative expansion* is the one which leads to the fastest convergence. Note that this is not the case for the field expansion where the rapidity of convergence does not provide a criterion to discriminate between various PMS solutions.

Of course, this asymptotic value could be reached only after large fluctuations occuring at first orders, as in the field expansion (see Fig. 6). However, contrary to this case and provided η is not too large, we expect the first orders of the derivative expansion to already lead to reliable results. Under this hypothesis, we get two natural criteria to select a unique
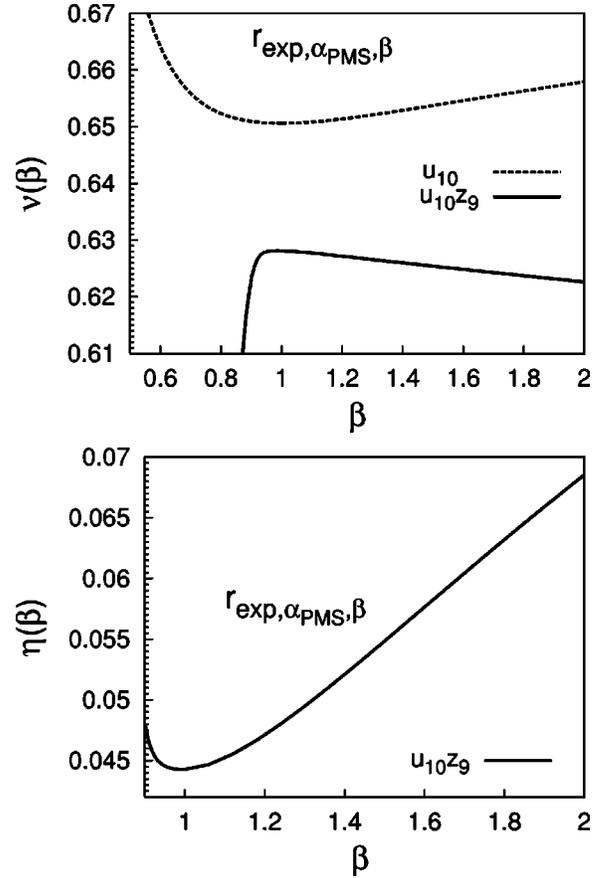

FIG. 9. Curves $\nu(\beta)$ and $\eta(\beta)$ for $r_{\exp,\alpha_{PMS},\beta}$ within LPA (labeled u_{10}) and at $O(\partial^2)$ of the derivative expansion (labeled $u_{10}z_9$) for the maximal truncations of $u_k(\tilde{\rho})$ and $z_k(\tilde{\rho})$ considered here. α_{PMS} is the value obtained in Sec. VI A.

PMS solution when several exist. The first one consists in keeping, for each family of cutoff functions, only the PMS solutions that have a counterpart in the other(s) family(ies), i.e., that lead to (almost) the same critical exponents. This means in our case that we keep only the PMS solutions that verify (in obvious notations)

$$\nu_{PMS}^{\exp} \simeq \nu_{PMS}^{\theta},$$

$$\eta_{PMS}^{\exp} \simeq \eta_{PMS}^{\theta}, \qquad (36)$$

since these exponents are stationary not only inside a family of cutoff functions but also from one family to the other. The second criterion consists in applying our previous hypothesis of rapid convergence already at order ∂^2: we assume that no large fluctuation occurs between the LPA and ∂^2 approximation. We thus select the PMS solution that minimizes, on the exponents, the correction of order ∂^2 to the LPA.

Both criteria allow one to discriminate between the two distinct PMS solutions obtained for ν and η with $r_{\theta,\alpha}$ (see the curve $u_{10}z_9$ in Fig. 8). They happen to pair for both exponents, at roughly $\alpha_{PMS}^{\eta} \simeq \alpha_{PMS}^{\nu} \simeq 0.7$ and $\alpha_{PMS}^{\eta} \simeq \alpha_{PMS}^{\nu} \simeq 6.5$. According to the second criterion, we

exclude the second PMS solutions located at $\alpha_{PMS} \simeq 6.5$, which lead for both ν and η to much larger deviations than the first ones compared with the LPA result: $\eta = 0$ and $\nu = 0.650$ [see Fig. 8, curve $(r_{\theta,\alpha})u_{10}$]. The first criterion leads to the same choice since (i) we have checked that with $r_{\exp,\alpha}$ only one PMS solution exists for ν (resp. for η), and (ii) the corresponding exponent is very similar to the one at the first PMS solution for ν (resp. for η) with $r_{\theta,\alpha}$, see Table I and Fig. 8. Thus, our two criteria to select a unique PMS solution are consistent [63].

D. Influence of a second parameter

In the preceding sections, we have restricted our analyses to the influence of the parameter α, the amplitude of the cutoff functions, on the critical exponents. The optimized results obtained with the two families of cutoff functions are very close together. It is thus natural to test the robustness of this result. In this section we investigate the influence of other deformations of the usual cutoff functions focusing on the exponential cutoff. Two generalizations of $r_{\exp,\alpha}$ come naturally. They consist in changing (i) $\exp y \rightarrow \exp \beta y$ and (ii) $\exp y \rightarrow \exp y^{\beta}$ [41,32,3,42]. The deformation (i) is actually useless since it is equivalent to a rescaling of the running scale k in R_k which is immaterial. We hence study the two-parameter generalization of $r_{\exp}$:

$$r_{\exp,\alpha,\beta}(y) = \alpha \frac{1}{e^{y^{\beta}} - 1}. \tag{37}$$

We perform the full PMS analyses of ν and η over the two-parameter space spanned by α and β, within the LPA and at order ∂^2 of the derivative expansion, for the maximal field truncations of $u_k(\tilde{\rho})$ and $z_k(\tilde{\rho})$ considered here. We find a unique two-dimensional PMS for both exponents, and at both orders. It lies at $\alpha_{PMS}^{\eta} \simeq \alpha_{PMS}^{\nu} = 2.25$, $\beta_{PMS}^{\nu} \simeq \beta_{PMS}^{\eta} = 0.98$ and gives $\eta_{PMS} = 0.04426$ and $\nu_{PMS} = 0.6281$ at order ∂^2. It turns out that our prior choice $\beta = 1$ was very close to β_{PMS}, and thus the α optimization performed in the previous section already enabled us to almost reach this minimum. The two-parameter PMS exponents thus differ by less than a tenth of a percent from those obtained previously (see Table I).

For illustration purposes, we isolate in Fig. 9 the behavior of the β parameter, fixing α to its PMS value determined in Sec. VI A. It displays the $\eta(\beta)$ and $\nu(\beta)$ functions for the converged field truncations. Both exponents exhibit a single PMS solution for β very close to one ($\beta_{PMS} = 1.001$ in LPA and $\beta_{PMS}^{\nu} = \beta_{PMS}^{\eta} = 0.993$ at order ∂^2).

As shown in Fig. 9, the β dependence of ν and η is quite sharp. It raises a natural question: had we fixed β far from β_{PMS} to perform the α PMS analysis, what would we have obtained? In other words, would the α optimization have sufficed to retrieve exponents close to the two-parameter PMS ones? To investigate this question, we have fixed $\beta = 2$, which seems from Fig. 9 to alter much η, and determined α_{PMS}^{ν} and α_{PMS}^{η}. The corresponding exponents are $\eta_{PMS}^{\beta=2} = 0.05573$, $\nu_{PMS}^{\beta=2} = 0.6246$ at order ∂^2. The discrep-

ancy with the two-parameter PMS exponents is quite significant for η, whereas the larger exponents—ν and the others computed from the scaling relations—only undergo a few percent variation. This originates in the difference of nature of both exponents. On the one hand, the exponent ν is related to the behavior of the mass, embodied in the minimum of the effective potential. The weakness of the sensitivity of ν on the cutoff function, at order ∂^2 of the derivative expansion, suggests that the effective potential is already well approximated at this order, and thus provides an accurate determination of ν, close to the exact value. On the other hand, η describes the momentum-dependent part of the two-spin correlation function, for which the order ∂^2 truncation constitutes a very rough *ansatz*. Hence, the determination of η is rather poor at this order and improving it probably requires higher derivative orders. This is directly reflected in the non-negligible dependence of η on the cutoff function underlined above.

The conclusion to be drawn from this is that, as previously, the PMS is the appropriate method to select, among a class of cutoff functions, the one that achieves the best accuracy, in so far as it minimizes the distance to the world best values for both exponents and at both orders. Moreover, the PMS reveals itself all the more crucial that the variations with respect to a given parameter are large.

VII. CONCLUSION

We have implemented the principle of minimal sensitivity to improve critical exponents within the framework of the nonperturbative RG. We have shown that it always allows us to reach the most accurate results achievable in the class of cutoff functions under scrutiny. Within the LPA, the PMS exponents turn out to almost coincide with those obtained through the principle of maximization of the gap, and the method is easily generalizable at order ∂^2.

Two main drawbacks are usually attributed to the implementation of the PMS: (i) several solutions of the PMS can exist and render its implementation ambiguous, (ii) it is not clear whether it indeed improves the results. We have shown on the example of the Ising model, that within the context of the effective average action method, these drawbacks either can be circumvented or do not exist at all. We have indeed brought out that a unique solution of the PMS can always be selected, thanks to very reasonable criteria, and furthermore this solution represents the most accurate determination of the critical exponents. The PMS thus appears as a safe and powerful method to optimize the results obtained in the nonperturbative RG context. An important and rather unexpected aspect of our analysis is that the rapidity of convergence of the field expansion is not optimal where the accuracy is.

Let us also emphasize that, even within a rather modest truncation involving the potential expansion up to order $\tilde{\rho}^5$ and the field renormalization expansion up to order $\tilde{\rho}^4$, the accuracy reached on ν is below the percent level compared with the world best results. This suggests that, with the same

kind of computational complexity, a comparable accuracy can be achieved for more complicated models.

Finally, the determination of η is poorer, which is to be imputed to the roughness of the *ansatz* to describe the full momentum dependence of the two-spin correlation function. Improving it is likely to require the inclusion of terms of order ∂^4. This will be investigated in [39].

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[59] Note that the behavior of $R_k(q)$ given in Eq. (4) is not the most general one. For instance, $R_k(q)$ can be a power law cutoff, i.e., $R_k(q) = q^2 (q^2/k^2)^{-a}$ in which case the constraint becomes $R_k(q) \gg q^2$ (for $q^2 \ll k^2$).

[60] Let us notice that although the magnetization vanishes at the critical temperature, the dimensionless minimum of the fixed point potential u_* does not vanish in general.

[61] ν is defined in our calculation as minus the inverse of the relevant eigenvalue of the stability matrix, linearized around the fixed point. This computation differs from the one implemented in [54]. It generates a small discrepancy (0.5%) between the (d) value in Table I and $\nu(\alpha = 1) = 0.6272$ computed here, see the lower curves in Fig. 4.

[62] Note that several PMS solutions can lead to almost degenerate critical exponents. See for instance the curve $u_{10}z_5$ in Fig. 4 for $r_{\exp,\alpha}$ where two PMS exist at P_1 and P_2. In this case, either point can be selected arbitrarily, since anyway the discrepancy between $\nu_{PMS}(P_1)$ and $\nu_{PMS}(P_2)$ is negligible (it does not exceed a few tenths of percent here).

[63] Actually, for $\alpha \to \infty$, both ν and η approach an asymptotic value for $r_{\exp,\alpha}$ that, by extending the notion of PMS to infinite α could be considered as a second PMS solution. However, the values of both ν and η thus obtained, $\nu(\alpha = \infty) \simeq 0.60$ and $\eta(\alpha = \infty) \simeq 0.124$, are far from those at the second PMS of $r_{\theta,\alpha} - \nu \simeq 0.61$ and $\eta \simeq 0.088$, and therefore cannot be considered as consistent.