Gravitational radiation in *D*-dimensional spacetimes

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Gravitational wave solutions to Einstein's equations and their generation are examined in *D*-dimensional flat spacetimes. First the plane wave solutions are analyzed; then the wave generation is studied with the solution for the metric tensor being obtained with the help of retarded *D*-dimensional Green's functions. Because of the difficulties in handling the wave tails in odd dimensions we concentrate our study on even dimensions. We compute the metric quantities in the wave zone in terms of the energy-momentum tensor at retarded time. Some special cases of interest are studied. First we study the slow motion approximation, where the *D*-dimensional quadrupole formula is deduced. Within the quadrupole approximation, we consider two cases of interest: a particle in circular orbit and a particle falling radially into a higher dimensional Schwarzschild black hole. Then we turn our attention to the gravitational radiation emitted during collisions lasting zero seconds, i.e., hard collisions. We compute the gravitational energy radiated during the collision of two point particles, in terms of a cutoff frequency. In the case in which at least one of the particles is a black hole, we argue that this cutoff frequency should be close to the lowest gravitational quasinormal frequency. In this context, we compute the scalar quasinormal frequencies of higher dimensional Schwarzschild black holes. Finally, as an interesting new application of this formalism, we compute the gravitational energy release during the quantum process of black hole pair creation. These results might be important in light of the recent proposal that there may exist extra dimensions in the Universe, one consequence of which may be black hole creation at the Large Hadron Collider at CERN.

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I. INTRODUCTION

One expects to finally detect gravitational waves in the forthcoming years. If this happens, and if the observed waveforms match the predicted templates, general relativity will have passed a crucial test. Moreover, if one manages to cleanly separate gravitational waveforms, we will open a new and exciting window on the Universe, a window from which one can look directly into the heart of matter, as gravitational waves are weakly scattered by matter. A lot of effort has been spent in recent years on trying to build gravitational wave detectors, and a new era will begin with gravitational wave astronomy [1,2]. The property that makes gravitational wave astronomy attractive, the weakness with which gravitational waves are scattered by matter, is also the major source of technical difficulties when trying to develop an apparatus that interacts with them. Nevertheless, some of these highly nontrivial technical difficulties have been surmounted, and we have detectors already operating [3–5]. Additional effort is being dedicated by theoreticians in trying to obtain accurate templates for the various physical processes that may give rise to the waves impinging on the detector. We now have a well established theory of wave generation and propagation, which started with Einstein and his quadrupole formula. The quadrupole formula expresses the energy lost to gravitational waves by a system moving at low velocities, in terms of its energy content. The quadrupole formalism is the most famous example of slow motion techniques to compute wave generation. All these techniques break the nonlinearity of Einstein's equations by imposing a power series in some small quantity and keeping only the lowest or the lowest few order terms. The quadrupole formalism starts from a flat background and expands the relevant quantities in R/λ, where R is the size of the source and λ the wavelength of the waves. Perturbation formalisms, on the other hand, start from some nonradiative background, whose metric is known exactly, for example, the Schwarzschild metric, and expand in deviations from that background metric. For a catalog of the various methods and their description we refer the reader to the review works by Thorne [6] and Damour [7]. The necessity of developing all such methods was driven, of course, by the lack of exact radiative solutions to Einstein's equations (although there are some worthy exceptions, such as the *C* metric [8]), and by the fact that even nowadays solving the full set of Einstein's equations numerically is a monumental task, and has been done only for the more tractable physical situations. All the existing methods seem to agree with each other when it comes down to the computation of waveforms and energies radiated during physical situations, and also agree with the few available results from a fully numerical evolution of Einstein's equations.

In this work we extend some of these results to higher dimensional spacetimes. There are several reasons why one should now try to do it. It seems impossible to formulate in four dimensions a consistent theory that unifies gravity with the other forces in nature. Thus, most efforts in this direction have considered a higher dimensional arena for our Universe, one example being string theories which have recently

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made some remarkable achievements. Moreover, recent investigations [9] propose the existence of extra dimensions in our Universe in order to solve the hierarchy problem, i.e., the huge difference between the electroweak and the Planck scales, $m_{\rm EW}/M_{\rm Pl}\sim 10^{-17}$. The fields of the standard model would inhabit a four-dimensional submanifold, the brane, whereas the gravitational degrees of freedom would propagate throughout all dimensions. One of the most spectacular consequences of this scenario would be the production of black holes at the Large Hadron Collider (LHC) at CERN [10] (for recent relevant work related to this topic we refer the reader to [11–13]). Now, one of the experimental signatures of black hole production will be a missing energy, perhaps a large fraction of the center of mass energy [12]. This will happen because, when the partons collide to form a black hole, some of the initial energy will be converted to gravitational waves, and due to the small amplitudes involved there is no gravitational wave detector capable of detecting them, so they will appear as missing. Thus, the collider could in fact indirectly serve as a gravitational wave detector. This calls for the calculation of the energy given away as gravitational waves when two high energy particles collide to form a black hole, which lives in all the dimensions. The work done so far on this subject [14,15] in higher dimensions is mostly geometric, and generalizes a construction by Penrose to find trapped surfaces on the union of two shock waves, describing boosted Schwarzschild black holes. On the other hand, there are clues [12,13] indicating that a formalism described by Weinberg [16] to compute the gravitational energy radiated in the collision of two point particles gives results correct to an order of magnitude when applied to the collision of two black holes. This formalism assumes a hard collision, i.e., a collision lasting zero seconds. It is important to apply this formalism in higher dimensions, trying to see if there is agreement between the two results. This is one of the topics discussed in this paper. The other topic we study in this paper is the quadrupole formula in higher dimensions. Due to the difficulties in handling the wave tails in odd dimensions, we concentrate our study on even dimensions.

This paper is organized as follows. In Sec. II we linearize Einstein's equations in a flat D-dimensional background and arrive at an inhomogeneous wave equation for the metric perturbations. The source-free equations are analyzed in terms of plane waves, and then the general solution to the homogeneous equation is deduced in terms of the D-dimensional retarded Green's function. In Sec. III we compute the D-dimensional quadrupole formula (assuming slowly moving sources), expressing the metric and the radiated energy in terms of the time-time component of the energy-momentum tensor. We then apply the quadrupole formula to two cases: a particle in circular motion in a generic background, and a particle falling into a D-dimensional Schwarzschild black hole. In Sec. IV we consider a hard collision between two particles, i.e., the collision takes zero seconds, and introduce a cutoff frequency necessary to have meaningful results. We then apply it to the case where one of the colliding particles is a black hole. We propose that this cutoff should be related to the gravitational quasinormal frequency of the black hole, and compute some values of the scalar quasinormal frequencies for higher dimensional Schwarzschild black holes, expecting that the gravitational quasinormal frequencies will behave in the same manner. Finally, we apply this formalism to compute the generation of gravitational radiation during black hole pair creation in four and higher dimensions, a result that has never been worked out, even for $D=4$. In our presentation we shall mostly follow Weinberg's [16] exposition.

II. LINEARIZED D-DIMENSIONAL EINSTEIN'S EQUATIONS

Because of the nonlinearity of Einstein's equations, the treatment of the gravitational radiation problem is not an easy one since the energy-momentum tensor of the gravitational wave contributes to its own gravitational field. To overcome this difficulty it is a standard procedure to work only with the weak radiative solution, in the sense that the energy-momentum content of the gravitational wave is small enough in order to allow us to neglect its contribution to its own propagation. This approach is justified in practice since we expect the detected gravitational radiation to be of low intensity.

A. The inhomogeneous wave equation

We begin this subsection by introducing the general background formalism (whose details can be found, e.g., in [16]) that will be needed in later sections. Then we obtain the linearized inhomogeneous wave equation.

Greek indices vary as $0,1,\ldots,D-1$ and latin indices as $1,\ldots,D-1$ and our units are such that $c\equiv 1$. We work on a D-dimensional spacetime described by a metric $g_{\mu\nu}$ that approaches asymptotically the D-dimensional Minkowski metric $\eta_{\mu\nu}=\mathrm{diag}(-1,+1,\ldots,+1)$, and thus we can write

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},\quad \mu,\nu=0,1,\ldots,D-1, \tag{1}$$

where $h_{\mu\nu}$ is small, i.e., $|h_{\mu\nu}|\ll 1$, so that it represents small corrections to the flat background. The exact Einstein field equations $G_{\mu\nu}=8\pi\mathcal{G}T_{\mu\nu}$ (with $\mathcal{G}$ being the usual Newton constant) can then be written as

$$R^{(1)}{}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}R^{(1)\alpha}{}_{\alpha}=8\pi\mathcal{G}\tau_{\mu\nu}, \tag{2}$$

with

$$\tau^{\mu\nu}\equiv\eta^{\mu\alpha}\eta^{\nu\beta}(T_{\alpha\beta}+t_{\alpha\beta}). \tag{3}$$

Here $R^{(1)}{}_{\mu\nu}$ is the part of the Ricci tensor linear in $h_{\mu\nu}$, $R^{(1)\alpha}{}_{\alpha}=\eta^{\alpha\beta}R^{(1)}{}_{\beta\alpha}$, and $\tau_{\mu\nu}$ is the effective energy-momentum tensor, containing contributions from $T_{\mu\nu}$, the energy-momentum tensor of the matter source, and $t_{\mu\nu}$ which represents the gravitational contribution. The pseudotensor $t_{\mu\nu}$ contains the difference between the exact Ricci terms and the Ricci terms linear in $h_{\mu\nu}$,

$$t_{\mu\nu}=\frac{1}{8\pi\mathcal{G}}\left[R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R^{\alpha}{}_{\alpha}-R^{(1)}{}_{\mu\nu}+\frac{1}{2}\eta_{\mu\nu}R^{(1)\alpha}{}_{\alpha}\right]. \tag{4}$$

The Bianchi identities imply that $\tau_{\mu\nu}$ is locally conserved,

$$\partial_{\mu}\tau^{\mu\nu}=0. \tag{5}$$

Introducing the Cartesian coordinates $x^{\alpha}=(t,\mathbf{x})$ with $\mathbf{x}=x^{i}$, and considering a $D-1$ volume V with a boundary spacelike surface S with dimension $D-2$ whose unit exterior normal is $\mathbf{n}$, Eq. (5) yields

$$\frac{d}{dt}\int_{V}d^{D-1}\mathbf{x}\ \tau^{0\nu}=-\int_{S}d^{D-2}\mathbf{x}\ n_{i}\tau^{i\nu}. \tag{6}$$

This means that one may interpret

$$p^{\nu}\equiv\int_{V}d^{D-1}\mathbf{x}\ \tau^{0\nu} \tag{7}$$

as the total energy-momentum (pseudo)vector of the system, including matter and gravitation, and $\tau^{i\nu}$ as the corresponding flux. Since the matter contribution is contained in $t^{\mu\nu}$, the flux of gravitational radiation is

$$\text{Flux}=\int_{S}d^{D-2}\mathbf{x}\ n_{i}t^{i\nu}. \tag{8}$$

In this context of linearized general relativity, we neglect terms of order higher than the first in $h_{\mu\nu}$ and all the indices are raised and lowered using $\eta^{\mu\nu}$. We also neglect the contribution of the gravitational energy-momentum tensor $t_{\mu\nu}$ (i.e., $|t_{\mu\nu}|\ll|T_{\mu\nu}|$) since from Eq. (4) we see that $t_{\mu\nu}$ is of higher order in $h_{\mu\nu}$. Then the conservation equations (5) yield

$$\partial_{\mu}T^{\mu\nu}=0. \tag{9}$$

In this setting and choosing a convenient coordinate system that obeys the harmonic (also called Lorentz) gauge conditions,

$$2\partial_{\mu}h^{\mu}{}_{\nu}=\partial_{\nu}h^{\alpha}{}_{\alpha} \tag{10}$$

(where $\partial_{\mu}=\partial/\partial x^{\mu}$), the first order Einstein field equations (2) yield

$$\Box h_{\mu\nu}=-16\pi\mathcal{G}S_{\mu\nu}, \tag{11}$$

$$S_{\mu\nu}=T_{\mu\nu}-\frac{1}{D-2}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}, \tag{12}$$

where $\Box=\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$ is the D-dimensional Laplacian, and $S_{\mu\nu}$ will be called the modified energy-momentum tensor of the matter source. Equations (11) and (12) subject to Eq. (10) allow us to find the gravitational radiation produced by a matter source $S_{\mu\nu}$.

B. The plane wave solutions

In vacuum, the linearized equations for the gravitational field are $R^{(1)}{}_{\mu\nu}=0$ or, equivalently, the homogeneous equations $\Box h_{\mu\nu}=0$, subjected to the harmonic gauge conditions (10). The solutions of these equations, the plane wave solutions, are important since the general solutions of the inhomogeneous equations (10) and (11) approach the plane wave solutions at large distances from the source. Setting $k_{\alpha}=(-\omega,\mathbf{k})$ with ω and $\mathbf{k}$ being, respectively, the frequency and wave vector, the plane wave solutions can be written as a linear superposition of solutions of the kind

$$h_{\mu\nu}(t,\mathbf{x})=e_{\mu\nu}e^{ik_{\alpha}x^{\alpha}}+e^{*}_{\mu\nu}e^{-ik_{\alpha}x^{\alpha}}, \tag{13}$$

where $e_{\mu\nu}=e_{\nu\mu}$ is called the polarization tensor and $*$ means the complex conjugate. These solutions satisfy Eq. (11) with $S_{\mu\nu}=0$ if $k_{\alpha}k^{\alpha}=0$, and obey the harmonic gauge conditions (10) if $2k_{\mu}e^{\mu}{}_{\nu}=k_{\nu}e^{\mu}{}_{\mu}$.

An important issue that must be addressed is the number of different polarizations that a gravitational wave in D dimensions can have. The polarization tensor $e_{\mu\nu}$, being symmetric, has in general $D(D+1)/2$ independent components. However, these components are subject to the D harmonic gauge conditions that reduce by D the number of independent components. In addition, under the infinitesimal change of coordinates $x'^{\mu}=x^{\mu}+\xi^{\mu}(x)$, the polarization tensor transforms into $e'_{\mu\nu}=e_{\mu\nu}-\partial_{\nu}\xi_{\mu}-\partial_{\mu}\xi_{\nu}$. Now, $e'_{\mu\nu}$ and $e_{\mu\nu}$ describe the same physical system for arbitrary values of the D parameters $\xi^{\mu}(x)$. Therefore, the number of independent components of $e_{\mu\nu}$, i.e., the number of polarization states of a gravitational wave in D dimensions, is $D(D+1)/2-D-D=D(D-3)/2$. From this computation we can also see that gravitational waves are present only when $D>3$. Therefore, from now on we assume $D>3$ whenever we refer to D. In what concerns the helicity of the gravitational waves, for arbitrary D the gravitons are always spin 2 particles.

To end this subsection on gravitational plane wave solutions, we present the average gravitational energy-momentum tensor of a plane wave, a quantity that will be needed later. Notice that in vacuum, since the matter contribution is zero ($T_{\mu\nu}=0$), we cannot neglect the contribution of the gravitational energy-momentum tensor $t_{\mu\nu}$. From Eq. (4), and neglecting terms of order higher than h^{2}, the gravitational energy-momentum tensor of a plane wave is given by

$$t_{\mu\nu}\simeq\frac{1}{8\pi\mathcal{G}}\left[R^{(2)}{}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}R^{(2)\alpha}{}_{\alpha}\right], \tag{14}$$

and through a straightforward calculation (see, e.g., [16] for details) we get the average gravitational energy-momentum tensor of a plane wave,

$$\langle t_{\mu\nu}\rangle=\frac{k_{\mu}k_{\nu}}{16\pi\mathcal{G}}\left[e^{\alpha\beta}e^{*}_{\alpha\beta}-\frac{1}{2}|e^{\alpha}{}_{\alpha}|^{2}\right]. \tag{15}$$

C. The *D*-dimensional retarded Green's function

The general solution to the inhomogeneous differential equation (11) may be found in the usual way in terms of a Green's function as

$$h_{\mu\nu}(t,\mathbf{x}) = -16\pi\mathcal{G}\int dt' \int d^{D-1}\mathbf{x}' S_{\mu\nu}(t',\mathbf{x}') \times G(t-t',\mathbf{x}-\mathbf{x}') + \text{homogeneous solutions}, \tag{16}$$

where the Green's function $G(t-t',\mathbf{x}-\mathbf{x}')$ satisfies

$$\eta^{\mu\nu}\partial_\mu\partial_\nu G(t-t',\mathbf{x}-\mathbf{x}') = \delta(t-t')\,\delta(\mathbf{x}-\mathbf{x}'), \tag{17}$$

where $\delta(z)$ is the Dirac delta function. In the momentum representation this reads

$$G(t,\mathbf{x}) = -\frac{1}{(2\pi)^D}\int d^{D-1}\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\int d\omega\,\frac{e^{-i\omega t}}{\omega^2-k^2}, \tag{18}$$

where $k^2 = k_1^2 + k_2^2 + \cdots + k_{D-1}^2$. To evaluate this, it is convenient to perform the k integral by using spherical coordinates in the $(D-1)$-dimensional k space. The required transformation, along with some useful formulas which shall be used later on, is given in the Appendix. The result for the retarded Green's function in those spherical coordinates is

$$G^{\text{ret}}(t,\mathbf{x}) = -\frac{\Theta(t)}{(2\pi)^{(D-1)/2}}\times\frac{1}{r^{(D-3)/2}} \times\int k^{(D-3)/2}J_{(D-3)/2}(kr)\sin(kt)\,dk, \tag{19}$$

where $r^2 = x_1^2 + x_2^2 + \cdots + x_{D-1}^2$, and $\Theta(t)$ is the Heaviside function defined as

$$\Theta(t) = \begin{cases} 1 & \text{if} \quad t>0, \\ 0 & \text{if} \quad t<0. \end{cases} \tag{20}$$

The function $J_{(D-3)/2}(kr)$ is a Bessel function [17,18]. The structure of the retarded Green's function will depend on the parity of D, as we shall see. This dependence on the parity, which implies major differences between even and odd spacetime dimensions, is connected to the structure of the Bessel function. For even D, the index of the Bessel function is semi-integral and then the Bessel function is expressible in terms of elementary functions, while for odd D this does not happen. A concise explanation of the difference between retarded Green's functions in even and odd D, and the physical consequences that it entails, is presented in [19] (see also [20–22]). A complete derivation of the Green's function in higher dimensional spaces may be found in Hassani [23]. The result is

$$G^{\text{ret}}(t,\mathbf{x}) = \frac{1}{4\pi}\left[-\frac{\partial}{2\pi r\,\partial r}\right]^{(D-4)/2}\left[\frac{\delta(t-r)}{r}\right], \quad D \text{ even}, \tag{21}$$

$$G^{\text{ret}}(t,\mathbf{x}) = \frac{\Theta(t)}{2\pi}\left[-\frac{\partial}{2\pi r\,\partial r}\right]^{(D-3)/2}\left[\frac{1}{\sqrt{t^2-r^2}}\right], \quad D \text{ odd}. \tag{22}$$

It is sometimes convenient to work with the Fourier transform (in the time coordinate) of the Green's function. One finds [23] an analytical result independent of the parity of D:

$$G^{\text{ret}}(\omega,\mathbf{x}) = \frac{i^D\pi}{2(2\pi)^{(D-1)/2}}\left(\frac{\omega}{r}\right)^{(D-3)/2} H^1_{(D-3)/2}(\omega r), \tag{23}$$

where $H^1_\nu(z)$ is a modified Bessel function [17,18]. Of course, the different structure of the Green's function for different D is again embodied in these Bessel functions. Equations (21) and (23) are one of the most important results we shall use in this paper. For $D=4$ Eq. (21) obviously reproduces well known results [23]. Now, one sees from Eq. (21) that, although there are delta function derivatives on the even D Green's function, the localization of the Green's function on the light cone is preserved. However, Eq. (22) tells us that the retarded Green's function for odd dimensions is nonzero inside the light cone. The consequence, as has been emphasized by different authors [19,22,24], is that for odd D the Huygens principle does not hold: the fact that the retarded Green's function support extends to the interior of the light cone implies the appearance of radiative tails in Eq. (16). In other words, we still have a propagation phenomenon for the wave equation in odd dimensional spacetimes, in so far as a localized initial state requires a certain time to reach a point in space. Huygens principle no longer holds, because the effect of the initial state is not sharply limited in time: once the signal has reached a point in space, it persists there indefinitely as a reverberation.

This fact coupled to the analytic structure of the Green's function in odd dimensions make it hard to get a grip on radiation generation in odd dimensional spacetimes. Therefore, from now on we shall focus on even dimensions, for which the retarded Green's function is given by Eq. (21).

D. The even *D*-dimensional retarded solution in the wave zone

The retarded solution for the metric perturbation $h_{\mu\nu}$, obtained by using the retarded Green's function (21) and discarding the homogeneous solution in Eq. (16), is given by

$$h_{\mu\nu}(t,\mathbf{x}) = 16\pi\mathcal{G}\int dt' \int d^{D-1}\mathbf{x}' S_{\mu\nu}(t',\mathbf{x}') \times G^{\text{ret}}(t-t',\mathbf{x}-\mathbf{x}'), \tag{24}$$

with $G^{\text{ret}}(t-t',\mathbf{x}-\mathbf{x}')$ as in Eq. (21). For $D=4$, for example, one has

$$G^{\text{ret}}(t,\mathbf{x}) = \frac{1}{4\pi}\,\frac{\delta(t-r)}{r}, \quad D=4, \tag{25}$$

which is the well known result. For $D=6$, we have

$$G^{\rm ret}(t,\mathbf{x})=\frac{1}{8\pi^2}\left(\frac{\delta'(t-r)}{r^2}+\frac{\delta(t-r)}{r^3}\right),\quad D=6, \tag{26}$$

where $\delta'(t-r)$ means the derivative of the Dirac delta function with respect to its argument. For $D=8$, we have

$$G^{\rm ret}(t,\mathbf{x})=\frac{1}{16\pi^3}\left(\frac{\delta''(t-r)}{r^3}+3\frac{\delta'(t-r)}{r^4}+3\frac{\delta(t-r)}{r^5}\right),$$

$$D=8. \tag{27}$$

We see that in general even D dimensions the Green's function consists of inverse integer powers in r, spanning all values between $1/r^{(D-2)/2}$ and $1/r^{D-3}$, including these. Now, the retarded solution is given by Eq. (24) as a product of the Green's function times the modified energy-momentum tensor $S_{\mu\nu}$. The net result of having derivatives on the delta functions is to transfer these derivatives to the energy-momentum tensor as time derivatives [this can be seen by integrating Eq. (24) by parts in the t integral].

A close inspection then shows that the retarded field possesses a kind of peeling property in that it consists of terms with different falloff at infinity. Explicitly, this means that the retarded field will consist of a sum of terms possessing all integer inverse powers in r between $(D-2)/2$ and $D-3$. The term that dies off more quickly at infinity is $1/r^{D-3}$, typically a static term, since it comes from the Laplacian. As a matter of fact this term was already observed in a higher dimensional black hole by Tangherlini [25] (see also Myers and Perry [26]). We will see that the term falling more slowly, the one that varies as $1/r^{(D-2)/2}$, gives rise to gravitational radiation. It is well defined, in the sense that the power crossing sufficiently large hyperspheres with different radii is the same, because the volume element varies as r^{D-2} and the energy as $|h|^2\sim 1/r^{D-2}$.

In radiation problems, one is interested in finding the field at large distances from the source, $r\gg\lambda$, where λ is the wavelength of the waves, and also much larger than the source's dimensions R. This is defined as the wave zone. In the wave zone, one may neglect all terms in the Green's function that decay faster than $1/r^{(D-2)/2}$. So, in the wave zone, we find

$$h_{\mu\nu}(t,\mathbf{x})=-8\pi\mathcal{G}\frac{1}{(2\pi r)^{(D-2)/2}}\partial_t^{(D-4)/2}\times\left[\int d^{D-1}\mathbf{x}'S_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')\right], \tag{28}$$

where $\partial_t^{(D-4/2)}$ stands for the $[(D-4)/2]$th derivative with respect to time. For $D=4$ Eq. (28) yields the standard result [16]:

$$h_{\mu\nu}(t,\mathbf{x})=-\frac{4\mathcal{G}}{r}\int d^{D-1}\mathbf{x}'S_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}'),\quad D=4. \tag{29}$$

To find the Fourier transform of the metric, one uses the representation (23) for the Green's function. Now, in the wave zone, the Green's function may be simplified using the asymptotic expansion for the Bessel function [18]:

$$H^1_{(D-3)/2}(\omega r)\sim\sqrt{\frac{2}{\pi(\omega r)}}e^{i[\omega r-(\pi/4)(D-2)]},\quad \omega r\to\infty. \tag{30}$$

This yields

$$h_{\mu\nu}(\omega,\mathbf{x})=-\frac{8\pi\mathcal{G}}{(2\pi r)^{(D-2)/2}}\omega^{(D-4)/2}e^{i\omega r}\times\int d^{D-1}\mathbf{x}'S_{\mu\nu}(\omega,\mathbf{x}'). \tag{31}$$

This could also have been arrived at directly from Eq. (28), using the rule time derivative $\to -i\omega$ for Fourier transforms. Equations (28) and (31) are one of the most important results derived in this paper, and will be the basis for the whole subsequent section. Similar equations, but not as general as the ones presented here, were given by Chen, Li, and Lin [27] in the context of gravitational radiation by a rolling tachyon.

To get the energy spectrum, we use Eq. (12), yielding

$$\frac{d^2E}{d\omega d\Omega}=2\mathcal{G}\frac{\omega^{D-2}}{(2\pi)^{D-4}}\times\left(T^{\mu\nu}(\omega,\mathbf{k})T^*_{\mu\nu}(\omega,\mathbf{k})-\frac{1}{D-2}|T^\lambda_\lambda(\omega,\mathbf{k})|^2\right). \tag{32}$$

III. THE EVEN D-DIMENSIONAL QUADRUPOLE FORMULA

A. Derivation of the even D-dimensional quadrupole formula

When the velocities of the sources that generate the gravitational waves are small, it is sufficient to know the T^{00} component of the gravitational energy-momentum tensor in order to have a good estimate of the energy they radiate. In this subsection, we will deduce the D-dimensional quadrupole formula and in the next subsection we will apply it to (1) a particle in circular orbit and (2) a particle in free fall into a D-dimensional Schwarzschild black hole.

We start by recalling that the Fourier transform of the energy-momentum tensor is

$$T_{\mu\nu}(\omega,\mathbf{k})=\int d^{D-1}\mathbf{x}'e^{-i\mathbf{k}\cdot\mathbf{x}'}\int dt\ e^{i\omega t}T^{\mu\nu}(t,\mathbf{x})+\text{c.c.}, \tag{33}$$

where c.c. means the complex conjugate of the preceding term. Then, the conservation equations (9) for $T^{\mu\nu}(t,\mathbf{x})$ applied to Eq. (33) yield $k^\mu T_{\mu\nu}(\omega,\mathbf{k})=0$. Using this last result we obtain $T_{00}(\omega,\mathbf{k})=\hat{k}^j\hat{k}^iT_{ji}(\omega,\mathbf{k})$ and $T_{0i}(\omega,\mathbf{k})=$

$-\hat{k}^j T_{ji}(\omega,\mathbf{k})$, where $\hat{\mathbf{k}}=\mathbf{k}/\omega$. We can then write the energy spectrum, Eq. (32), as a function only of the spacelike components of $T^{\mu\nu}(\omega,\mathbf{k})$,

$$\frac{d^2E}{d\omega d\Omega}=2\mathcal{G}\frac{\omega^{D-2}}{(2\pi)^{D-4}}\Lambda_{ij,lm}(\hat{k})T^{*ij}(\omega,\mathbf{k})T^{ij}(\omega,\mathbf{k}), \tag{34}$$

where

$$\Lambda_{ij,lm}(\hat{k})=\delta_{il}\delta_{jm}-2\hat{k}_j\hat{k}_m\delta_{il}+\frac{1}{D-2}(-\delta_{ij}\delta_{lm}+\hat{k}_l\hat{k}_m\delta_{ij}+\hat{k}_i\hat{k}_j\delta_{lm})+\frac{D-3}{D-2}\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m. \tag{35}$$

At this point, we make a new approximation (in addition to the wave zone approximation) and assume that $\omega R\ll 1$, where R is the source's radius. In other words, we assume that the internal velocities of the sources are small and thus the source's radius is much smaller than the characteristic wavelength $\sim 1/\omega$ of the emitted gravitational waves. Within this approximation, one can set $e^{-i\mathbf{k}\cdot\mathbf{x}'}\sim 1$ in Eq. (33) (since $R=|\mathbf{x}'|_{\max}$). Moreover, after a straightforward calculation, one can also use in Eq. (34) the approximation $T^{ij}(\omega,\mathbf{k})\simeq -(\omega^2/2)D_{ij}(\omega)$, where

$$D_{ij}(\omega)=\int d^{D-1}\mathbf{x}\; x^i x^j T^{00}(\omega,\mathbf{x}). \tag{36}$$

Finally, using

$$\int d\Omega_{D-2}\;\hat{k}_i\hat{k}_j=\frac{\Omega_{D-2}}{D-1}\delta_{ij},$$

$$\int d\Omega_{D-2}\;\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m=\frac{3\Omega_{D-2}}{D^2-1}(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}), \tag{37}$$

where Ω_{D-2} is the $(D-2)$-dimensional solid angle defined in Eq. (A5), we obtain the D-dimensional quadrupole formula

$$\frac{dE}{d\omega}=\frac{2^{2-D}\pi^{-(D-5)/2}\mathcal{G}(D-3)D}{\Gamma[(D-1)/2](D^2-1)(D-2)}\omega^{D+2}\times[(D-1)D^*_{ij}(\omega)D_{ij}(\omega)-|D_{ii}(\omega)|^2], \tag{38}$$

where the Gamma function $\Gamma[z]$ is defined in the Appendix. As the dimension D grows it is seen that the rate of gravitational energy radiated increases as ω^{D+2}. Sometimes it will be more useful to have the time rate of emitted energy:

$$\frac{dE}{dt}=\frac{2^{2-D}\pi^{-(D-5)/2}\mathcal{G}(D-3)D}{\Gamma[(D-1)/2](D^2-1)(D-2)}\times[(D-1)\partial_t^{(D+2)/2}D^*_{ij}(t)\partial_t^{(D+2)/2}D_{ij}(t)-|\partial_t^{(D+2)/2}D_{ii}(t)|^2]. \tag{39}$$

For $D=4$, Eq. (39) yields the well known result [16]

$$\frac{dE}{dt}=\frac{\mathcal{G}}{5}\left[\partial_t^3 D^*_{ij}(t)\partial_t^3 D_{ij}(t)-\frac{1}{3}|\partial_t^3 D_{ii}(t)|^2\right]. \tag{40}$$

B. Applications of the quadrupole formula: Test particles in a background geometry

The quadrupole formula has been used successfully in almost all kinds of problems involving gravitational wave generation. By "successful," we mean that it agrees with other more accurate methods. Its simplicity and the fact that it gives results correct to within a few percent make it an invaluable tool in estimating gravitational radiation emission. We shall in the following present two important examples of the application of the quadrupole formula.

1. A particle in circular orbit

The radiation generated by particles in circular motion was perhaps the first situation to be considered in the analysis of gravitational wave generation. For orbits with low frequency, the quadrupole formula yields excellent results. As expected it is difficult to find in nature a system with perfect circular orbits; they will in general be elliptic. In this case the agreement is also remarkable, and one finds that the quadrupole formalism can account with precision for the increase in period of the pulsar PSR 1913+16, due to gravitational wave emission [28]. In four dimensions the full treatment of the elliptic orbital motion is discussed by Peters [29]. In dimensions higher than four, it has been shown [25] that there are no stable geodesic circular orbits, and so geodesic circular motion is not as interesting for higher D. For this reason, and also because we only want to put in evidence the differences that arise in gravitational wave emission as one varies the spacetime dimension D, we will just analyze the simple circular, not necessarily geodesic motion, to see whether the results are nontrivially changed as one increases D. Consider then two bodies of equal mass m in circular orbits a distance l apart. Suppose they revolve around the center of mass, which is at $l/2$ from both masses, and that they orbit with frequency ω in the x-y plane. A simple calculation [29,30] yields

$$D_{xx}=\frac{ml^2}{4}\cos(2\omega t)+\text{const}, \tag{41}$$

$$D_{yy}=-D_{xx}, \tag{42}$$

$$D_{xy}=\frac{ml^2}{4}\sin(2\omega t)+\text{const}, \tag{43}$$

independently of the dimension in which they are embedded and with all other components being zero. We therefore get, from Eq. (39),

$$\frac{dE}{dt}=\frac{2\mathcal{G}D(D-3)}{\pi^{(D-5)/2}\Gamma[(D-1)/2](D+1)(D-2)}m^2l^4\omega^{D+2}. \tag{44}$$

For $D=4$ one gets

$$\frac{dE}{dt}=\frac{8\mathcal{G}}{5}m^2l^4\omega^6, \tag{45}$$

which agrees with known results [29,30]. Equation (44) is telling us that as one climbs up in dimension number D the frequency effect gets more pronounced.

2. A particle falling radially into a higher dimensional Schwarzschild black hole

As yet another example of the use of the quadrupole formula Eq. (39) we now calculate the energy given away as gravitational waves when a point particle with mass m falls into a D-dimensional Schwarzschild black hole, a metric first given in [25]. Historically, the case of a particle falling into a $D=4$ Schwarzschild black hole was one of the first to be studied [31,32] in connection with gravitational wave generation, and later served as a model calculation when one wanted to evolve Einstein's equations fully numerically [33,34]. This process was first studied [32] by numerically solving Zerilli's [31] wave equation for a particle at rest at infinity and then falling into a Schwarzschild black hole. Davis *et al.* [32] found numerically that the amount of energy radiated to infinity as gravitational waves was $\Delta E_{\rm num}=0.01m^2/M$, where m is the mass of the particle falling in and M is the mass of the black hole. It is found that the $D=4$ quadrupole formula yields [35] $\Delta E_{\rm quad}=0.019m^2/M$, so it is of the same order of magnitude as that given by fully relativistic numerical results. Despite the fact that the quadrupole formula fails somewhere near the black hole (the motion is not slow, and the background is certainly not flat), it looks as if one can get an idea of how much radiation will be released with the help of this formula. Based on this good agreement, we shall now consider this process but for higher dimensional spacetimes. The metric for the D-dimensional Schwarzschild black hole in $(t,r,\theta_1,\theta_2,\ldots,\theta_{D-2})$ coordinates (see the Appendix) is

$$ds^2=-\left(1-\frac{16\pi\mathcal{G}M}{(D-2)\Omega_{D-2}}\frac{1}{r^{D-3}}\right)dt^2+\left(1-\frac{16\pi\mathcal{G}M}{(D-2)\Omega_{D-2}}\frac{1}{r^{D-3}}\right)^{-1}dr^2+r^{D-2}d\Omega_{D-2}^2. \tag{46}$$

Consider a particle falling along a radial geodesic, and at rest at infinity. Then the geodesic equations give

$$\frac{dr}{dt}\sim\frac{16\pi\mathcal{G}M}{(D-2)\Omega_{D-2}}\frac{1}{r^{D-3}}, \tag{47}$$

where we make the flat space approximation $t=\tau$. We then have, in these coordinates, $D_{11}=r^2$, and all other components vanish. From Eq. (39) we get the energy radiated per second, which yields

$$\frac{dE}{dt}=\frac{2^{2-D}\pi^{-(D-5)/2}\mathcal{G}(D-3)}{\Gamma[(D-1)/2](D^2-1)}D|\partial_t^{(D+2)/2}D_{11}|^2. \tag{48}$$

We can perform the derivatives and integrate to get the total energy radiated. There is a slight problem though: where do we stop the integration? The expression for the energy diverges at $r=0$ but this is no problem, as we know that as the particle approaches the horizon the radiation will be infinitely redshifted. Moreover, the standard picture [35] is that of a particle falling in, and in the last stages being frozen near the horizon. With this in mind we integrate from $r=\infty$ to some point near the horizon, say $r=b\times r_+$, where r_+ is the horizon radius and b is some number larger than unity, and we get

$$\Delta E=A\frac{D(D-2)\pi}{2^{2D-4}}\times b^{(9-D^2)/2}\times\frac{m^2}{M}, \tag{49}$$

where

$$A=\frac{(3-D)^2(5-D)^2(7-3D)^2(8-4D)^2(9-5D)^2\cdots(D/2+4-D^2/2)^2}{\Gamma[(D-1)/2]^2(D-1)(D+1)(D+3)}. \tag{50}$$

To understand the effect of both the dimension number D and the parameter b on the total energy radiated according to the quadrupole formula, we list in Table I some values ΔE for different dimensions, and b between 1 and 1.3.

The parameter b is in fact a measure of our ignorance of what goes on near the black hole horizon, so if the energy radiated does not vary much with b it means that our lack of knowledge does not affect the results very much. For $D=4$ that indeed happens. Putting $b=1$ gives an energy only 2.6 times larger than with $b=1.3$, and still very close to the fully relativistic numerical result for $0.01m^2/M$. However, as we increase D, the effect of b increases dramatically. For $D=12$, for example, we can see that a change in b from 1 to 1.3 gives a corresponding change in ΔE of 3×10^6 to 0.0665. This is eight orders of magnitude lower. Since there is as yet no Regge-Wheeler-Zerilli [31,36] wave function for higher dimensional Schwarzschild black holes, there are no fully relativistic numerical results to compare our results with. Thus $D=4$ is just the perfect dimension to predict, through the quadrupole formula, the gravitational energy coming from collisions between particles and black holes, or between small and massive black holes. It is not a problem related to the quadrupole formalism, but rather one related to D. A small change in parameters translates itself, for high D,

TABLE I. The energy radiated by a particle falling from rest into a higher dimensional Schwarzschild black hole, as a function of dimension. The integration is stopped at $b\times r_+$, where r_+ is the horizon radius.

	$\Delta E\times M/m^2$		
D	$b=1$	$b=1.2$	$b=1.3$
4	0.019	0.01	0.0076
6	0.576	0.05	0.0167
8	180	1.19	0.13
10	24567	6.13	0.16
12	3.3×10^6	14.77	0.0665

into a large variation in the final result. Thus, as the dimension D grows, the knowledge of the cutoff radius $b\times r_+$ becomes essential for accurate computation of the energy released.

IV. INSTANTANEOUS COLLISIONS IN EVEN *D* DIMENSIONS

In general, whenever two bodies collide or scatter there will be gravitational energy released due to the changes in momentum involved in the process. If the collision is hard, meaning that the incoming and outgoing trajectories have constant velocities, there is a method first envisaged by Weinberg [16,37], and later explored in [38] by Smarr to compute exactly the metric perturbation and energy released. The method is valid for arbitrary velocities (one will still be working in the linear approximation, so energies have to be low). Basically, it assumes a collision lasting for zero seconds. It was found that in this case the resulting spectra were flat, precisely what one would expect based on one's experience with electromagnetism [39], and so to give a meaning to the total energy a cutoff frequency is needed. This cutoff frequency depends upon some physical cutoff in the particular problem. We shall now generalize this construction for arbitrary dimensions.

A. Derivation of the radiation formula in terms of a cutoff for a head-on collision

Consider therefore a system of freely moving particles with D-momenta P_i^μ, energies E_i, and $(D-1)$-velocities $\mathbf{v}$, which due to the collision change abruptly at $t=0$ to corresponding primed quantities. For such a system, the energy-momentum tensor is

$$T^{\mu\nu}(t,\mathbf{v})=\sum \frac{P_i^\mu P_i^\nu}{E_i}\delta^{D-1}(\mathbf{x}-\mathbf{v}t)\Theta(-t) + \frac{P'^\mu_i P'^\nu_i}{E'_i}\delta^{D-1}(\mathbf{x}'-\mathbf{v}'t)\Theta(t), \tag{51}$$

from which, using Eqs. (31) and (32), one can get the quantities $h_{\mu\nu}$ and also the radiation emitted. Let us consider the particular case in which one has a head-on collision of two particles, particle 1 with mass m_1 and Lorentz factor γ_1 and particle 2 with mass $m=m_2$ with Lorentz factor γ_2, colliding to form a particle at rest. Without loss of generality, one may orient the axis so that the motion is in the (x_{D-1},x_D) plane, and the x_D axis is the radiation direction (see the Appendix). We then have

$$P_1=\gamma_1 m_1(1,0,0,\ldots,v_1\sin\theta_1,v_1\cos\theta_1),$$
$$P'_1=(E'_1,0,0,\ldots,0,0), \tag{52}$$
$$P_2=\gamma_2 m_2(1,0,0,\ldots,-v_2\sin\theta_1,-v_2\cos\theta_1),$$
$$P'_2=(E'_2,0,0,\ldots,0,0). \tag{53}$$

Momentum conservation leads to the additional relation $\gamma_1 m_1 v_1=\gamma_2 m_2 v_2$. Replacing Eqs. (52) and (53) in the energy-momentum tensor (51) and using Eq. (32) we find

$$\frac{d^2E}{d\omega d\Omega}=\frac{2\mathcal{G}}{(2\pi)^{D-2}}\frac{D-3}{D-2}\times\frac{\gamma_1^2 m_1^2 v_1^2(v_1+v_2)^2\sin\theta_1^4}{(1-v_1\cos\theta_1)^2(1+v_2\cos\theta_1)^2}\times\omega^{D-4}. \tag{54}$$

We see that for the arbitrary (even) D the spectrum is not flat. Flatness happens only for $D=4$. For any D, the total energy integrated over all frequencies would diverge so one needs a cutoff frequency which will depend on the particular problem under consideration. Integrating Eq. (54) from $\omega=0$ to the cutoff frequency ω_c, we have

$$\frac{dE}{d\Omega}=\frac{2\mathcal{G}}{(2\pi)^{D-2}}\frac{1}{D-2}\times\frac{\gamma_1^2 m_1^2 v_1^2(v_1+v_2)^2\sin\theta_1^4}{(1-v_1\cos\theta_1)^2(1+v_2\cos\theta_1)^2}\times\omega_c^{D-3}. \tag{55}$$

Two limiting cases are of interest here, namely, (i) the collision between identical particles and (ii) the collision between a light particle and a very massive one. In case (i), replacing $m_1=m_2=m$, $v_1=v_2=v$, Eq. (55) gives

$$\frac{dE}{d\Omega}=\frac{8\mathcal{G}}{(2\pi)^{D-2}}\frac{1}{D-2}\frac{\gamma^2 m^2 v^4\sin\theta_1^4}{(1-v^2\cos^2\theta_1)^2}\times\omega_c^{D-3}. \tag{56}$$

In case (ii) considering $m_1\gamma_1\equiv m\gamma\ll m_2\gamma_2$, $v_1\equiv v\gg v_2$, Eq. (55) yields

$$\frac{dE}{d\Omega}=\frac{2\mathcal{G}}{(2\pi)^{D-2}}\frac{1}{D-2}\frac{\gamma^2 m^2 v^4\sin\theta_1^4}{(1-v\cos\theta_1)^2}\times\omega_c^{D-3}. \tag{57}$$

Notice that the technique just described is expected to break down if the velocities involved are very low, since then the collision would not be instantaneous. In fact, a condition for this method to work can be stated.

Indeed, one can see from Eq. (55) that, if $v \to 0$, $dE/d\omega \to 0$, even though we know (see Sec. IV A) that $\Delta E \neq 0$. In any case, if the velocities are small one can use the quadrupole formula instead.

B. Applications: The cutoff frequency when one of the particles is a black hole and radiation from black hole pair creation

1. The cutoff frequency when one of the head-on colliding particles is a black hole

We shall now restrict ourselves to the case (ii) of the last subsection, in which at least one of the particles participating in the collision is a massive black hole, with mass $M \gg m$ (where we have put $m_1 = m$ and $m_2 = M$). Formulas (55)–(57) are useless unless one is able to determine the cutoff frequency ω_c present in the particular problem under consideration. In the situation where one has a small particle colliding at high velocities with a black hole, it has been suggested by Smarr [38] that the cutoff frequency should be $\omega_c \sim 1/2M$, presumably because the characteristic collision time is dictated by the large black hole whose radius is $2M$. Using this cutoff he finds

$$\Delta E_{\text{Smarr}} \sim 0.2\gamma^2 \frac{m^2}{M}. \tag{58}$$

The exact result, using a relativistic perturbation approach which reduces to the numerical integration of a second order differential equation (the Zerilli wave function), has been given by Cardoso and Lemos [13] as

$$\Delta E_{\text{exact}} = 0.26\gamma^2 \frac{m^2}{M}. \tag{59}$$

This is equivalent to saying that $\omega_c = 0.613/M \sim 1/1.63M$, and so it looks as if the cutoff is indeed the inverse of the horizon radius. However, in the numerical work by Cardoso and Lemos, it was found that it was not the presence of a horizon that contributed to this cutoff, but the presence of a potential barrier V outside the horizon. By decomposing the field in tensorial spherical harmonics with the index l standing for the angular quantum number, we found that for each l the spectrum is indeed flat [as predicted by Eq. (55) for $D=4$], until a cutoff frequency ω_{c_l}, which was numerically equal to the lowest gravitational quasinormal frequency ω_{QN}. For $\omega > \omega_{c_l}$ the spectrum decays exponentially. This behavior is illustrated in Fig. 1. The quasinormal frequencies [40] are those frequencies that correspond to only outgoing waves at infinity and only ingoing waves near the horizon. As such, the gravitational quasinormal frequencies will in general have a real and an imaginary part, the latter denoting gravitational wave emission and therefore a decay in the perturbation. There have been a wealth of works dwelling on quasinormal modes on asymptotically flat spacetimes [40], due to its close connection with gravitational wave emission, and also on nonasymptotically flat spacetimes, like asymptotically anti–de Sitter [41] or asymptotically de Sitter [42] spacetimes, mainly due to the AdS conformal field theory

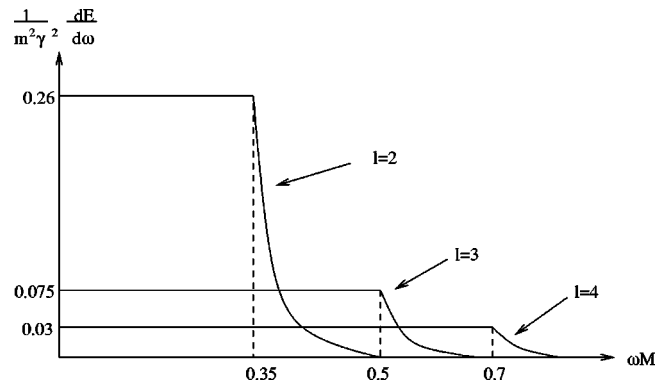

FIG. 1. The energy spectra as a function of the angular number l, for a highly relativistic particle falling into a $D=4$ Schwarzschild black hole [13]. The particle begins to fall with a Lorentz factor γ. Notice that for each l there is a cutoff frequency ω_{c_l} which is equal to the quasinormal frequency ω_{QN} after which the spectrum decays exponentially. So it is clearly seen that ω_{QN} works as a cutoff frequency. The total energy radiated is given by a sum over l, which is the same as saying that the effective cutoff frequency is given by a weighted average of the various ω_{c_l}.

(CFT) and dS/CFT [43] correspondence conjecture. We argue here that it is indeed the quasinormal frequency that dictates the cutoff, and not the horizon radius. For $D=4$ it so happens that the weighted average of ω_{c_l} is $0.613/M$, which, as we said, is quite similar to $r_+ = 1/2M$. The reason for the cutoff being dictated by the quasinormal frequency can be understood using some WKB intuition. The presence of a potential barrier outside the horizon means that waves with some frequencies get reflected back on the barrier while others can cross. Frequencies such that ω^2 is lower than the maximum barrier height $V_{\max}$ will be reflected back to infinity where they will be detected. However, frequencies ω^2 larger than the maximum barrier height cross the barrier and enter the black hole, thereby being absorbed and not contributing to the energy detected at infinity. So only frequencies ω^2 lower than this maximum barrier height are detected at infinity. It has been shown [44] that the gravitational quasinormal frequencies are to first order equal to the square root of the maximum barrier height. In view of this picture, and considering the physical meaning of the cutoff frequency, it seems quite natural to say that the cutoff frequency is equal to the quasinormal frequency. If the frequencies are higher than the barrier height, they do not get reflected back to infinity. This discussion is very important to understanding how the total energy varies with the number D of dimensions. In fact, if we set $\omega_c \sim 1/r_+$, we find that the total energy radiated decreases rapidly with increasing dimension number, because r_+ increases rapidly with the dimension. This conflicts with recent results [14,15], which, using shock waves that describe boosted Schwarzschild black holes and searching for apparent horizons, indicate an increase with D. So we need the gravitational quasinormal frequencies for higher dimensional Schwarzschild black holes. To arrive at a wave equation for gravitational perturbations of higher dimensional Schwarzschild black holes, and therefore to compute its gravitational quasinormal frequencies, one needs to

TABLE II. The lowest scalar quasinormal frequencies for spherically symmetric ($l=0$) scalar perturbations of higher dimensional Schwarzschild black holes, obtained using a WKB method [44]. Notice that the real part of the quasinormal frequency is always of the same order of magnitude as the square root of the maximum barrier height. We show also the maximum barrier height as well as the horizon radius as a function of dimension D. The mass M of the black hole has been set to 1.

D	$\mathrm{Re}[\omega_{\mathrm{QN}}]$	$\mathrm{Im}[\omega_{\mathrm{QN}}]$	$\sqrt{\mathrm{V}_{\max}}$	$1/r_+$
4	0.10	−0.12	0.16	0.5
6	1.033	−0.713	1.441	1.28
8	1.969	−1.023	2.637	1.32
10	2.779	−1.158	3.64	1.25
12	3.49	−1.202	4.503	1.17

decompose Einstein's equations in D-dimensional tensorial harmonics, which would lead to some quite complex expressions. It is not necessary to go that far though, because one can get an idea of how the gravitational quasinormal frequencies vary by searching for the quasinormal frequencies of scalar perturbations, and scalar quasinormal frequencies are a lot easier to find. One hopes that the scalar frequencies will behave with D in the same manner as do the gravitational ones. Scalar perturbations in D-dimensional Schwarzschild spacetimes obey the wave equation (consult [45] for details)

$$\frac{\partial^2\phi(\omega,r)}{\partial r_*^2}+[\omega^2-V(r)]\phi(\omega,r)=0. \tag{60}$$

The potential $V(r)$ appearing in Eq. (60) is given by

$$V(r)=f(r)\left[\frac{a}{r^2}+\frac{(D-2)(D-4)f(r)}{4r^2}+\frac{(D-2)f'(r)}{2r}\right], \tag{61}$$

where $a=l(l+D-3)$ is the eigenvalue of the Laplacian on the hypersphere S^{D-2}, the tortoise coordinate r_* is defined as $\partial r/\partial r_*=f(r)=\{1-[16\pi\mathcal{G}M/(D-2)\Omega_{D-2}](1/r^{D-3})\}$, and $f'(r)=df(r)/dr$. We have found the quasinormal frequencies of spherically symmetric ($l=0$) scalar perturbations, by using a WKB approach developed by Schutz, and collaborators [44]. The results are presented in Table II, where we also show the maximum barrier height of the potential in Eq. (61), as well as the horizon radius.

The first thing worth noticing is that the real part of the scalar quasinormal frequency is to first order reasonably close to the square root of the maximum barrier height $\sqrt{V_{\max}}$, supporting the previous discussion. Furthermore, the scalar quasinormal frequency grows more rapidly than the inverse of the horizon radius $1/r_+$ as one increases D. In fact, the scalar quasinormal frequency grows with D while the horizon radius r_+ gets smaller. Note that, from purely dimensional arguments, for fixed D, $\omega\propto 1/r_+$. The statement here is that the constant of proportionality depends on the dimension D—more explicitly it grows with D—and can be found from Table II. Assuming that the gravitational quasinormal frequencies will have the same behavior (and some very recent studies [46] relating black hole entropy and damped quasinormal frequencies seem to point that way), the total energy radiated during high energy collisions does indeed increase with D, as some studies [14,15] seem to indicate.

2. The gravitational energy radiated during black hole pair creation

As a new application of this instantaneous collision formalism, we now consider the gravitational energy released during the quantum creation of pairs of black holes, a process that as far as we know has not been analyzed in the context of gravitational wave emission, even for $D=4$. It is well known that vacuum quantum fluctuations produce virtual electron-positron pairs. These pairs can become real [47] if they are pulled apart by an external electric field, in which case the energy for the pair materialization and acceleration comes from the external electric field energy. Likewise, a black hole pair can be created in the presence of an external field whenever the energy pumped into the system is enough to make the pair of virtual black holes real (see Dias [48] for a review on black hole pair creation). If one tries to predict the spectrum of radiation coming from pair creation, one expects, of course, a spectrum characteristic of accelerated masses, but one also expects that this follows some kind of signal indicating pair creation. In other words, the process of pair creation itself, which involves the sudden creation of particles, must imply emission of radiation. It is this phase we shall focus on, forgetting the subsequent emission of radiation caused by the acceleration.

Pair creation is a purely quantum-mechanical process in nature, with no classical explanation. But, given that the process does occur, one may ask about the spectrum and intensity of the radiation accompanying it. The sudden creation of pairs can be viewed for our purposes as an instantaneous creation of particles (i.e., the time reversed process of instantaneous collisions), the violent acceleration of particles initially at rest to some final velocity in a very short time, and the technique described at the beginning of this section applies. This is quite similar to another purely quantum-mechanical process, the beta decay. The electromagnetic radiation emitted during beta decay has been computed classically by Chang and Falkoff [49] and is also presented in Jackson [39]. The classical calculation is similar in all aspects to the one described in this section (the instantaneous collision formalism) assuming the sudden acceleration to energies E of a charge initially at rest, and also requires a cutoff in the frequency, which has been assumed to be given by the uncertainty principle $\omega_c\sim E/\hbar$. Assuming this cutoff, one finds that the agreement between the classical calculation and the quantum calculation [49] is extremely good (especially in the low frequency regime), and, more important, was verified experimentally. Summarizing, formula (56) also describes the gravitational energy radiated when two black holes, each with mass m and energy E, form through quantum pair creation. The typical pair creation time can be estimated by the uncertainty principle $\tau_{\mathrm{creation}}\sim\hbar/E\sim\hbar/m\gamma$, and thus we find the cutoff frequency as

$$\omega_c \sim \frac{1}{\tau_{\text{creation}}} \sim \frac{m\gamma}{\hbar}. \tag{62}$$

Here we would like to draw the reader's attention to the fact that the units of Planck's constant $\hbar$ change with dimension number D: according to our convention of setting $c=1$ the units of $\hbar$ are $[M]^{(D-2)/(D-3)}$. With this cutoff, we find the spectrum of the gravitational radiation emitted during pair creation to be given by Eq. (54) with $m_1=m_2$ and $v_1=v_2$ (we are considering the pair creation of two identical black holes):

$$\frac{d^2E}{d\omega d\Omega} = \frac{8\mathcal{G}}{(2\pi)^{D-2}} \frac{D-3}{D-2} \frac{\gamma^2 m^2 v^4 \sin\theta_1^4}{(1-v^2\cos^2\theta_1)^2} \times \omega^{D-4}, \tag{63}$$

and the total frequency integrated energy per solid angle is

$$\frac{dE}{d\Omega} = \frac{8\mathcal{G}}{(2\pi)^{D-2}(D-2)} \frac{v^4 \sin\theta_1^4}{(1-v^2\cos^2\theta_1)^2} \times \frac{(m\gamma)^{D-1}}{\hbar^{D-3}}. \tag{64}$$

For example, in four dimensions and for pairs with $v\sim 1$ one obtains

$$\frac{dE}{d\omega} = \frac{4\mathcal{G}}{\pi}\gamma^2 m^2, \tag{65}$$

and will have for the total energy radiated during production itself, using the cutoff frequency (62),

$$\Delta E = \frac{4\mathcal{G}}{\pi}\frac{\gamma^3 m^3}{\hbar}. \tag{66}$$

This could lead under appropriate numbers of m and γ to huge quantities. Although one cannot be sure as to the cutoff frequency, and therefore the total energy (66), it is extremely likely that, as was verified experimentally in beta decay, the zero frequency limit, Eq. (65), is exact.

V. SUMMARY AND DISCUSSION

We have developed the formalism to compute gravitational wave generation in higher D dimensional spacetimes, with D even. Several examples have been worked out, and one cannot help the feeling that our apparently four-dimensional world is the best one to make predictions about the intensity of gravitational waves in concrete situations, in the sense that a small variation of parameters leads in high D to a huge variation of the energy radiated. Much more work is still needed if one wants to make precise predictions about gravitational wave generation in D-dimensional spacetimes. For example, it would be important to find a way to treat gravitational perturbations of higher dimensional Schwarzschild black holes. One of the examples worked out, the gravitational radiation emitted during black hole pair creation, was not previously considered in the literature, and it seems to be a good candidate, even in $D=4$, to radiate intensely through gravitational waves.

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APPENDIX: SPHERICAL COORDINATES IN $D-1$ DIMENSIONS

In this appendix we list some important formulas and results used throughout this paper. We shall first present the transformation mapping $(D-1)$ Cartesian coordinates $(x_1,x_2,x_3,\ldots,x_{D-1})$ onto $(D-1)$ spherical coordinates $(r,\theta_1,\theta_2,\ldots,\theta_{D-2})$. The transformation reads

$$\begin{aligned}
x_1 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2},\\
x_2 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-3}\cos\theta_{D-2},\\
&\vdots\\
x_i &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-i-1}\cos\theta_{D-i},\\
&\vdots\\
x_{D-1} &= r\cos\theta_1.
\end{aligned}$$

The Jacobian of this transformation is

$$J = r^{D-2}\sin\theta_1^{D-3}\sin\theta_2^{D-4}\cdots\sin\theta_i^{D-i-2}\cdots\sin\theta_{D-3}, \tag{A1}$$

and the volume element becomes

$$d^{D-1}\mathbf{x} = J dr d\theta_1 d\theta_2\cdots d\theta_{D-2} = r^{D-2} dr d\Omega_{D-2}, \tag{A2}$$

where

$$d\Omega_{D-2} = \sin\theta_1^{D-3}\sin\theta_2^{D-4}\cdots\sin\theta_{D-3}\, d\theta_1 d\theta_2\cdots d\theta_{D-2}, \tag{A3}$$

is the element of the $(D-1)$-dimensional solid angle. Finally, using [50]

$$\int_0^\pi \sin\theta^n = \sqrt{\pi}\,\frac{\Gamma[(n+1)/2]}{\Gamma[(n+2)/2]}, \tag{A4}$$

this yields

$$\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}. \tag{A5}$$

Here, $\Gamma[z]$ is the Gamma function, whose definition and properties are listed in [18]. In this work the main properties of the Gamma function that were used are $\Gamma[z+1]=z\Gamma[z]$ and $\Gamma[1/2]=\sqrt{\pi}$.

[1] B. F. Schutz, Class. Quantum Grav. **16**, A131 (1999); B. F. Schutz and F. Ricci, in *Gravitational Waves*, edited by I. Ciufolini *et al.* (Institute of Physics Publishing, Bristol, 2001).

[2] S. A. Hughes, astro-ph/0210481.

[3] K. Danzmann *et al.*, in *Gravitational Wave Experiments*, edited by E. Coccia, G. Pizzella, and F. Ronga (World Scientific, Singapore, 1995).

[4] A. Abramovici *et al.*, Science **256**, 325 (1992).

[5] C. Bradaschia *et al.*, in *Gravitation* 1990, Proceedings of the Banff Summer Institute, Banff, Alberta, 1990, edited by R. Mann and P. Wesson (World Scientific, Singapore, 1991).

[6] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980); in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1989).

[7] T. Damour, in *Gravitational Radiation*, edited by Nathalie Deruelle and Tsvi Piran (North-Holland Publishing Company, New York, 1983).

[8] W. Kinnersley and M. Walker, Phys. Rev. D **2**, 1359 (1970); O. J. C. Dias and J. P. S. Lemos, *ibid.* **67,** 064001 (2003).

[9] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); Phys. Rev. D **59**, 086004 (1999); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998).

[10] P. C. Argyres, S. Dimopoulos, and J. March-Russell, Phys. Lett. B **441**, 96 (1998); S. Dimopoulos and G. Landsberg, Phys. Rev. Lett. **87**, 161602 (2001); S. B. Giddings and S. Thomas, Phys. Rev. D **65**, 056010 (2002); S. D. H. Hsu, hep-ph/0203154; H. Tu, hep-ph/0205024; A. Jevicki and J. Thaler, Phys. Rev. D **66**, 024041 (2002); Y. Uehara, hep-ph/0205199.

[11] P. Kanti and J. March-Russell, Phys. Rev. D **66**, 024023 (2002); hep-ph/0212199; D. Ida, Kin-ya Oda, and S. C. Park, PE: 027306 Phys. Rev. D (to be published), hep-th/0212108; V. Frolov and D. Stojkovic, Phys. Rev. Lett. **89**, 151302 (2002); gr-qc/0211055.

[12] V. Cardoso and J. P. S. Lemos, ''Gravitational Radiation from the Radial Infall of Highly Relativistic Point Particles into Kerr Black Holes,'' gr-qc/0211094.

[13] V. Cardoso and J. P. S. Lemos, Phys. Lett. B **538**, 1 (2002); gr-qc/0207009.

[14] D. M. Eardley and S. B. Giddings, Phys. Rev. D **66**, 044011 (2002).

[15] H. Yoshino and Y. Nambu, Phys. Rev. D **67**, 024009 (2003).

[16] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[17] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1995).

[18] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1970).

[19] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1962), Chap. VI.

[20] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, CT, 1923).

[21] J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle* (Oxford University Press, Oxford, 1986).

[22] D. V. Gal'tsov, Phys. Rev. D **66**, 025016 (2002).

[23] S. Hassani, *Mathematical Physics* (Springer-Verlag, New York, 1998).

[24] P. O. Kazinski, S. L. Lyakhovich, and A. A. Sharapov, Phys. Rev. D **66**, 025017 (2002); B. P. Kosyakov, Theor. Math. Phys. **119**, 493 (1999).

[25] F. R. Tangherlini, Nuovo Cimento **27**, 636 (1963).

[26] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).

[27] B. Chen, M. Li, and Feng-Li Lin, J. High Energy Phys. **11**, 050 (2002).

[28] J. M. Weisberg and J. H. Taylor, astro-ph/0211217.

[29] P. C. Peters, Phys. Rev. **136**, B1224 (1964); P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).

[30] B. F. Schutz, *A First Course in General Relativity* (Cambridge University Press, Cambridge, England, 1985).

[31] F. Zerilli, Phys. Rev. Lett. **24**, 737 (1970); Phys. Rev. D **2**, 2141 (1970).

[32] M. Davis, R. Ruffini, W. H. Press, and R. H. Price, Phys. Rev. Lett. **27**, 1466 (1971).

[33] *Sources of Gravitational Radiation*, edited by S. L. Smarr (Cambridge University Press, Cambridge, England, 1979).

[34] P. Anninos, D. Hobill, E. Seidel, L. Smarr, and W. M. Suen, Phys. Rev. Lett. **71**, 2851 (1993); R. J. Gleiser, C. O. Nicasio, R. H. Price, and J. Pullin, *ibid.* **77**, 4483 (1996).

[35] R. Ruffini, Phys. Rev. D **7**, 972 (1973); M. J. Fitchett, Ph.D. thesis, University of Cambridge, Cambridge, England, 1984.

[36] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[37] S. Weinberg, Phys. Lett. **9**, 357 (1964); Phys. Rev. **135**, B1049 (1964).

[38] L. Smarr, Phys. Rev. D **15**, 2069 (1977); R. J. Adler and B. Zeks, *ibid.* **12**, 3007 (1975).

[39] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).

[40] K. D. Kokkotas and B. G. Schmidt, Living Rev. Relativ. **2**, 2 (1999).

[41] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D **62**, 024027 (2000); V. Cardoso and J. P. S. Lemos, *ibid.* **63**, 124015 (2001); **64**, 084017 (2001); Class. Quantum Grav. **18**, 5257 (2001); D. Birmingham, I. Sachs, and S. N. Solodukhin, Phys. Rev. Lett. **88**, 151301 (2002); R. A. Konoplya, Phys. Rev. D **66**, 044009 (2002); S. F. J. Chan and R. B. Mann, *ibid.* **55**, 7546 (1997); I. G. Moss and J. P. Norman, Class. Quantum Grav. **19**, 2323 (2002); R. Aros, C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **67**, 044014 (2003); D. T. Son and A. O. Starinets, J. High Energy Phys. **09**, 042 (2002).

[42] F. Mellor and I. Moss, Phys. Rev. D **41**, 403 (1990); E. Abdalla, B. Wang, A. Lima-Santos, and W. G. Qiu, Phys. Lett. B **538**, 435 (2002).

[43] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 253 (1998).

[44] B. F. Schutz and C. M. Will, Astrophys. J. Lett. **291**, L33 (1985); S. Iyer and C. M. Will, Phys. Rev. D **35**, 3621 (1987); S. Iyer, *ibid.* **35**, 3632 (1987).

[45] V. Cardoso and J. P. S. Lemos, Phys. Rev. D **66**, 064006 (2002).

[46] O. Dreyer, gr-qc/0211076; G. Kunstatter, gr-qc/0211076.

[47] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[48] Ó. J. C. Dias, in *Proceedings of the Xth Portuguese Meeting on Astronomy and Astrophysics*, Lisbon, 2000, edited by J. P. S. Lemos *et al.* (World Scientific, Singapore, 2001), p. 109, gr-qc/0106081.

[49] C. S. Wang Chang and D. L. Falkoff, Phys. Rev. **76**, 365 (1949).

[50] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).