Causal Barrett-Crane model: Measure, coupling constant, Wick rotation, symmetries, and observables

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We discuss various features and details of two versions of the Barrett-Crane spin foam model of quantum gravity: first of the *Spin*(4)-symmetric Riemannian model and second of the $SL(2,\mathbb{C})$-symmetric Lorentzian version in which all tetrahedra are spacelike. Recently, Livine and Oriti proposed to introduce a causal structure into the Lorentzian Barrett-Crane model from which one can construct a path integral that corresponds to the causal (Feynman) propagator. We show how to obtain convergent integrals for the $10j$ symbols and how a dimensionless constant can be introduced into the model. We propose a ''Wick rotation'' which turns the rapidly oscillating complex amplitudes of the Feynman path integral into positive real and bounded weights. This construction does not yet have the status of a theorem, but it can be used as an alternative definition of the propagator and makes the causal model accessible by standard numerical simulation algorithms. In addition, we identify the local symmetries of the models and show how their four-simplex amplitudes can be reexpressed in terms of the ordinary relativistic $10j$ symbols. Finally, motivated by possible numerical simulations, we express the matrix elements that are defined by the model, in terms of the continuous connection variables and determine the most general observable in the connection picture. Everything is done on a fixed two complex.

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I. INTRODUCTION

Spin foam models have been proposed as candidates for a quantum theory of gravity; see, for example, the review articles [1,2]. A spin foam [3] whose symmetry group is a suitable Lie group G, is an abstract oriented two-complex consisting of faces, edges and vertices, together with a coloring of the faces with representations of G and a coloring of the edges with compatible intertwiners (representation morphisms) of G. Spin foam models are defined by a path integral in terms of a sum over spin foams, often over all colorings of a fixed two complex or in addition over a class of two complexes.

The most carefully studied model in this context is the Barrett-Crane model [4] which was initially formulated for a Riemannian signature and a local *Spin*(4) symmetry. A version with Lorentzian signature and $SL(2,\mathbb{C})$ symmetry can be constructed along similar lines. Here we are interested in the model [5] in which all tetrahedra are spacelike; i.e., if the model is formulated on the two complex dual to a triangulated four manifold, then the model assigns a geometry to the two complex such that each tetrahedron has a timelike normal vector.

The idea for the construction of the Barrett-Crane model [3,4,6,23] can be sketched as follows. General relativity in four dimensions is reformulated as a topological *BF* theory with a symmetry group *Spin*(4) or $SL(2,\mathbb{C})$, depending on the signature, subject to bivector constraints which break the topological properties and which ensure that the theory is classically equivalent to general relativity, possibly allowing degenerate metrics. Topological *BF* theory is then regularized and quantized on a triangulated four manifold which results in a topological spin foam model. The bivector constraints are finally implemented into this quantum theory. The result is a spin foam model which dynamically assigns geometric data to a purely combinatorial triangulation.

The path integral of the spin foam model can then be used in order to define the matrix elements of some operator between spin network states. There have been different conjectures, for example, that it is some unitary ''time evolution'' operator or that this operator is the projection from some kinematical Hilbert space onto the physical Hilbert space of quantum gravity. The precise role of the fourth direction (''time?'') in this path integral, however, remained obscure. In particular it was observed [7], see also Ref. [8], that the amplitudes of this path integral are positive real so that it does not look similar to a complex oscillating ''real time'' path integral at all. Formally, it could be an Euclidean (''imaginary time'') path integral, but this was not the intention of the construction, and a physical interpretation of this picture is also lacking.

Recently, Livine and Oriti [9] proposed a modification of the amplitudes of the Lorentzian Barrett-Crane model in which they employ, for each pair of triangle and four simplex, only one out of two summands of the amplitude with a particular sign in the exponent. This guarantees that the construction is compatible with a causal structure imposed on the four simplices and that the model resembles a ''real time'' Feynman (i.e., causal) path integral of a quantum field theory with four-dimensional Lorentzian Regge action. In the following, we call this version of the model the causal Lorentzian Barrett-Crane model. Livine and Oriti [9] derive consistency conditions on the relevant signs for the construction of this model.

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A causal version of the Riemannian Barrett-Crane model can be defined by analogy. This, however, is not more than just a toy model because in this case the causal structure has to be imposed completely by hand and is no longer related to the signature of the metric. We do include this possibility in the following because it is occasionally very helpful for technical reasons.

Given the causal model with its definition of the Feynman path integral, there are a number of natural questions to ask. What is the status of the measure? Are the integrals convergent, at least for a fixed triangulation and a fixed assignment of representations to the triangles? Having identified an action for the path integral, is there any coupling constant in the model which can affect the dominant contributions to the path integral? Is there a consistent ''Wick rotation'' in order to render all amplitudes positive real and to obtain an ''imaginary time'' model whose physical interpretation we understand? Which numbers can we extract from the model? In the present article, we address various aspects of these questions.

In particular, we give an explicit construction of suitable sign factors that satisfy the conditions of Livine and Oriti [9]. We rephrase the causal model so that it becomes manifest that a future-pointing increase in the lapse function of the path integral (for details, see Ref. [9]) corresponds to four simplices with positive four volume. We demonstrate how to split measure and e^{iS} amplitude so that all integrals originating from the $10j$ symbols are well defined, and we introduce a dimensionless coupling constant into the model.

As far as the Wick rotation is concerned, we follow ideas from the area of dynamical triangulations [10] and proceed four simplex by four simplex, introducing i or $-i$ into the exponents in order to obtain an Euclidean action, depending on whether the simplex itself or its opposite oriented counterpart appears in the Feynman propagator. It should be pointed out that the relation of this Euclidean theory with the original one does not yet have the status of a theorem comparable to the situation in axiomatic quantum field theory. What is the status of Wick rotation in quantum gravity in general? In a path integral of quantum gravity one has to sum, in one way or another, over all possible four metrics most of which do not admit any global time coordinate which one could use in order to Wick rotate a Lorentzian to a Riemannian manifold. In addition, the choice of a time coordinate is obviously not a diffeomorphism invariant concept. These problems render Wick rotation in a generic quantum gravity setting highly questionable. Why then discuss Wick rotation at all?

The answer is that one should not take Wick rotation literally and not try to substitute $t \mapsto -i\tau$ for some coordinate t. Rather, there exists a generalization [11] of the Osterwalder-Schrader Euclidean reconstruction theorem to background independent theories which relates a path integral theory which is defined on a four manifold of the topology $\Sigma \times \mathbb{R}$ and whose measure is reflection positive, with a canonical quantum theory in terms of Hilbert space and Hamiltonian on Σ. In a spin foam context, such a procedure will be independent of the metric signature because the local frame symmetry is treated as an internal symmetry in the underlying first order formulation of general relativity. We therefore expect that we can independently euclideanize both the Riemannian and the Lorentzian signature models and that their signatures and symmetry groups do not change under such a transformation.

Since up to now, there does not exist any generally accepted way of achieving independence of the Barrett-Crane model from the chosen triangulation, we cannot restore the full diffeomorphism symmetry. Therefore we cannot yet verify all of the generalized Osterwalder-Schrader axioms [11].

The main motivation for searching an Euclidean version of the causal model is the fact that the resulting model with its positive real amplitudes can be tackled by standard simulation algorithms. We call the resulting ''Wick rotated'' model the *Euclidean* (Riemannian or Lorentzian) Barrett-Crane model. Note that the term *Euclidean* refers to the use of ''imaginary time'' as opposed to the term *Riemannian* which denotes the metric signature.[1]

In the resulting Euclidean model, one wishes to calculate interesting quantities. We address the question of which are suitable observables (here meaning numbers we can extract from the model on a fixed two complex) in the *connection picture*, the reformulation which uses continuous variables and which was developed for the Riemannian model in Ref. [12] and calculate the most general function of the connection variables that is compatible with the local symmetries of the model. Finally, starting from the analysis of the local symmetries, we show how both the causal and the Euclidean model can still be reexpressed in terms of relativistic $10j$ symbols which are familiar from the original model.

The present article is organized as follows. In Sec. II, we introduce our notation for oriented two complexes and introduce the Riemannian and Lorentzian Barrett-Crane models in their original formulation. In Sec. III, we review the causal versions of these models, present a construction of all required signs, carefully choose an appropriate measure and identify a dimensionless coupling constant. The transition amplitudes are euclideanized in Sec. IV. We study in Sec. V the most general observables of the model in the connection picture. In Sec. VI, we show how the causal and the Euclidean models can be rephrased in terms of the relativistic $10j$ symbols, similar to the original formulation of the model. Section VII contains some concluding comments.

II. NOTATION AND CONVENTIONS

A. Triangulations and two complexes

We consider the triangulation of an oriented piecewise linear four manifold M and its dual two-complex. This two complex is described by sets V of vertices, E of edges, and F of faces together with maps indicating the source $\partial_-(e) \in V$ and target $\partial_+(e) \in V$ of each edge $e \in E$ as well as all

[1]Unfortunately, the term ''Euclidean'' was historically also used in order to denote the Riemannian signature.

edges $\partial_j(f) \in E$ in the boundary of each face $f \in F$. Here $1 \leqslant j \leqslant N(f)$ enumerates all these edges.

As far as the orientations are concerned, we write $\varepsilon(v) \in \{-1,+1\}$ depending on whether the four simplex dual to $v \in V$ is isomorphic to a simplex in M or whether this is true for its counterpart with opposite orientation, denoted by v^*. For each $v \in V$, $e \in E$, we write $\varepsilon(v,e) \in \{-1,+1\}$ for the orientation of the tetrahedron dual to e in the boundary of the four simplex dual to v.

In the interior of M, each tetrahedron is contained in the boundary of exactly two four simplices so that it appears once with either orientation. Therefore

$$\varepsilon(v_1,e)\varepsilon(v_1) = -\varepsilon(v_2,e)\varepsilon(v_2), \tag{2.1}$$

where the tetrahedron dual to $e \in E$ is contained in the boundary of the two four-simplices dual to $v_1, v_2 \in V$.

Finally, we write $\varepsilon(e,f) \in \{-1,+1\}$ for the orientation of the triangle (dual to the face) $f \in F$ in the boundary of the tetrahedron (dual to the edge) $e \in E$.

In our formulas, we always use the notation of the two complex (V,E,F). If we mean the two complex dual to a triangulation, we often omit the words "dual to" and speak of the four simplex $v \in V$, the tetrahedron $e \in E$, etc. As the motivation of the causal Barrett-Crane model involves results from Regge calculus, we are initially restricted to these two complexes dual to triangulations. In the subsequent sections of this article, however, our formulas will be valid on any oriented two complex (V,E,F).

B. The original models

The partition function of the Riemannian or *Spin*(4)-symmetric[2] Barrett-Crane model [4] can be written [12]

$$Z_R = \left(\prod_{f\in F} \sum_{j_f \in 1/2\mathbb{N}_0} (2j_f+1)\right)\left(\prod_{e\in E} \int_{S^3} dx_e^{(+)} \int_{S^3} dx_e^{(-)}\right) \times \left(\prod_{f\in F} \mathcal{A}_f\right)\left(\prod_{e\in E} \mathcal{A}_e\right)\prod_{v\in V} \times \left(\prod_{f\in v_0} K_R^{(j_f)}(x^{(+)}_{e_+(f,v)}, x^{(-)}_{e_-(g,v)})\right), \tag{2.2}$$

where the set $v_0 \subseteq F$ includes all faces that contain the vertex $v \in V$ in their boundary, and $e_+(f,v) \in E$ denotes the edge in the boundary of the face $f \in F$ that has the vertex $v = \partial_+(e) \in V$ as its target, similarly $e_-(f,v)$. The function

$$K_R^{(j)}(x,y) := \frac{\sin[(2j+1)d_R(x,y)]}{(2j+1)\sin d_R(x,y)}, \tag{2.3}$$

denotes the (normalized) character of $SU(2)$ in the $(2j+1)$-dimensional irreducible representation. It depends only on the relative polar angle

$$d_R(x,y) := \cos^{-1}(x\cdot y) \geqslant 0 \tag{2.4}$$

on $S^3 \cong SU(2)$ where we write normalized vectors $x,y \in S^3 \subseteq \mathbb{R}^4$ and denote the standard scalar product in $\mathbb{R}^4$ using a center dot.

The model (2.2) contains a sum over all simple[3] (also called balanced) irreducible representations $V_j \otimes V_j^*$ of $SO(4)$ where $V_j \cong \mathbb{C}^{2j+1}$ denotes the irreducible representations of $SU(2)$. There are also continuous variables in the model, namely two integrals over S^3 for each edge which originate from the integral presentation [13] of the Riemannian $10j$ symbols. The last product over the $K_R^{(j)}$ in Eq. (2.2) is the integrand of the $10j$ symbol. For the geometric interpretation of the continuous variables, see Ref. [12].

The functions $\mathcal{A}_f$ and $\mathcal{A}_e$ in the integrand of Eq. (2.2) denote amplitudes for each face and for each edge which are not fixed by the geometric conditions imposed in the construction of the Barrett-Crane model [4]. There exist several proposals for these amplitudes in the literature so that we leave them unspecified in the following calculations.

The Lorentzian Barrett-Crane model whose symmetry group[4] is $SL(2,\mathbb{C})$ in the version in which all tetrahedra are spacelike [5], is defined in complete analogy by the partition function

$$Z_L = \left(\prod_{f\in F} \int_0^\infty p_f^2 dp_f\right)\left(\prod_{e\in E} \int_{H_+^3} dx_e^{(+)} \int_{H_+^3} dx_e^{(-)}\right)\left(\prod_{f\in F} \mathcal{A}_f\right) \times \left(\prod_{e\in E} \mathcal{A}_e\right)\prod_{v\in V}\left(\prod_{f\in v_0} K_L^{(p_f)}(x^{(+)}_{e_+(f,v)}, x^{(-)}_{e_-(f,v)})\right), \tag{2.5}$$

where

$$K_L^{(p)}(x,y) := \frac{\sin[p\, d_L(x,y)]}{p \sinh d_L(x,y)}, \tag{2.6}$$

and

$$d_L(x,y) := \cosh^{-1}(x\cdot y) \geqslant 0 \tag{2.7}$$

denotes the relative rapidity (hyperbolic distance) of $x,y \in H_+^3 \subseteq \mathbb{R}^{1+3}$. Here we denote by

$$H_+^3 := \{x \in \mathbb{R}^{1+3}:\ x\cdot x = 1 \quad \text{and} \quad x^0 > 0\} \tag{2.8}$$

three-dimensional hyperbolic space, written as the set of future pointing timelike unit vectors in Minkowski space $\mathbb{R}^{1+3}$ whose standard scalar product $x\cdot y$ is diagonal with entries

[2]Note that we can equally well start from an $SO(4)$ symmetry without changing the resulting model [12]. Also note that we consider the version of the original proposal [4] that employs the relativistic $10j$ symbols.

[3]We use $V_j \otimes V_j^*$ instead of the isomorphic $V_j \otimes V_j$ in order to remove all unnecessary signs from the expressions, see also Refs. [8,12].

[4]By an analogous argument as before, we could have rather chosen $SO_0(1,3)$, the connected component of the Lorentz group that contains the unit (the proper orthochronous group).

$(1,-1,-1,-1)$. Note that we are following the conventions of Refs. [5,14] here. It should be mentioned that the integrals over H^3_+ in Eq. (2.5) converge only after division by an infinite volume factor [14]. We keep this fact in mind, but leave our formulas unchanged in order to preserve their full symmetry.

III. THE CAUSAL MODELS

A. A construction

Livine and Oriti [9] have proposed a modification of the Lorentzian model (2.5) in which one writes the sine in the numerator of each factor $K_L^{(p)}$ as a sum of two exponentials and keeps only one of the exponentials, discarding the other. What matters is the sign that appears in the exponent. The authors derive consistency conditions under which one can make this choice for the entire triangulation so that one obtains a total amplitude e^{iS} in the integrand in which S is related to the four-dimensional Lorentzian Regge action. In this section, we present a full construction of the relevant signs.

We use the notation introduced in Sec. II A and observe that the condition (85) of Ref. [9] coincides with our Eq. (2.1).

In order to interpret the model (2.5) geometrically as in Ref. [12], we wish to associate two future pointing timelike vectors $x_e^{(\pm)} \in H^3_+$ to each tetrahedron. The only question is what future pointing means for a purely combinatorial triangulation.

There is some additional information required, namely, a partial order "$\succ$" on the set V of four simplices such that the relation is at least defined for each pair of four simplices that share a tetrahedron in their boundary. Without loss of generality, we can then put only tetrahedra e into the set E for which $\partial_+(e) \succ \partial_-(e)$. Otherwise we rather include e^* in the set E and adapt the $\varepsilon(\cdot,\cdot)$ factors of Sec. II A accordingly.

Observe that if we had triangulated an oriented Lorentzian four manifold with only spacelike tetrahedra, we could have derived the partial order "$\succ$" from the causal structure of the metric, i.e., "$\succ$" means "in the causal future of," and the above construction would precisely result in future pointing timelike normal vectors.

The choice of signs in Ref. [9] is based on the Lorentzian Regge action in four dimensions. Therefore one has to calculate the "dihedral rapidities" (generalized defect angles) for all pairs of neighboring tetrahedra $e_1, e_2 \in E$ that cobound a given triangle $f \in F$. Neighboring here means that both tetrahedra are contained in the boundary of the same four simplex $v \in V$. The outward normals are then given by $n_{e_j} := \varepsilon(v,e_j)\varepsilon(v) x_{e_j} \in \mathbb{R}^{1+3}$, $j \in \{1,2\}$, and we also define normals $m_{e_j} = \varepsilon(e_j,f) n_{e_j}$ which go clockwise "around the triangle f." These are used in order to calculate the Lorentzian analogue of the defect angle. Recall that all triangles are spacelike.

Depending on the shape of the four simplex v, we are either in the *thin wedge* situation (see Ref. [15] for the analogous three-dimensional case), in which the rapidity relevant for the action corresponds to an interior angle

$$\xi(x_{e_1}, x_{e_2}) = \cosh^{-1}(m_{e_1} \cdot m_{e_2}), \tag{3.1}$$

or else in the *thick wedge* situation in which it corresponds to an exterior one

$$\xi(x_{e_1}, x_{e_2}) = -\cosh^{-1}(-m_{e_1} \cdot m_{e_2}). \tag{3.2}$$

As all x_e are future pointing, we are in the thin wedge case if and only if

$$\varepsilon(v,e_1)\varepsilon(e_1,f)\varepsilon(v,e_2)\varepsilon(e_2,f) = +1. \tag{3.3}$$

It can thus be shown that the relevant rapidity is always

$$\xi(x_{e_1}, x_{e_2}) := \varepsilon(v,e_1)\varepsilon(e_1,f)\varepsilon(v,e_2) \times \varepsilon(e_2,f)\cosh^{-1}(x_{e_1} \cdot x_{e_2}). \tag{3.4}$$

The contribution to the Lorentzian Regge action from a given triangle $f \in F$ is, therefore,

$$S_f = A_f \sum_{v \in f_0} \xi(x^{(+)}_{e_+(f,v)}, x^{(-)}_{e_-(f,v)}), \tag{3.5}$$

where A_f denotes the area of the triangle $f \in F$ and the set $f_0 \subseteq V$ contains all four simplices in whose boundary the triangle f occurs, so that the sum is over all pairs of tetrahedra $e_+(f,v)$, $e_-(f,v)$ that share the triangle f and that are contained in the boundary of the same four simplex $v \in V$.

While the Regge action S_f for any single triangle is of the form (3.5), each triangle can contribute with a different total sign $\varepsilon(f) \in \{-1,+1\}$ so that the overall action is rather

$$S = \kappa \sum_{f \in F} \varepsilon(f) S_f, \tag{3.6}$$

where we have also introduced a dimensionless "coupling" constant κ.

The $\varepsilon(f)$ are not independent though. There exist relations for each four simplex from Stokes' theorem for the oriented normal vectors to the tetrahedra in its boundary which can be evaluated differentially using the Lorentzian version of the Schläfli identities [16]. For each four simplex $v \in V$ and two tetrahedra $e_1, e_2 \in E$ sharing a common triangle $f \in F$, the condition is

$$\varepsilon(f) = \varepsilon(v,e_1)\varepsilon(e_1 f)\varepsilon(v,e_2)\varepsilon(e_2,f)\mu(v), \tag{3.7}$$

where $\mu(v) \in \{-1,+1\}$ denotes an unspecified sign for each four-simplex. This condition was given in Ref. [16] writing $a_e := \varepsilon(v)\varepsilon(v,e)\varepsilon(e,f)$.

It would now even be possible to absorb the $\mu(v)$ into the orientation of the four simplices of the selected triangulation by putting either v or v^* into the set V so that all $\mu(v) = +1$. It is, however, also possible to keep the orientations of the four simplices as they are and to choose $\mu(v) := \varepsilon(v)$. As we will see below, a four simplex with $\varepsilon(v) = +1$ is then interpreted as a future-pointing contribution to the causal path integral.

According to Ref. [9], the causal Lorentzian Barrett-Crane model is defined by replacing $K_L^{(p)}(x,y)$ in Eq. (2.5) by

$$\tilde{K}_L^{(p)}(x,y) := \frac{\varepsilon(f)e^{i\varepsilon(f)p\chi}}{2ip\sinh\chi}, \quad \chi := \varepsilon(f)\varepsilon(v)d_L(x,y), \tag{3.8}$$

depending on the triangle $f \in F$ and the four simplex $v \in V$. This expression leads precisely to the Lorentzian Regge action (3.6) in the exponent. In order to see this, combine Eq. (3.4) with Eq. (3.7). We remind the reader that we are following the conventions of Refs. [5,14] which differ from those employed in Ref. [9]. By analogy with the Lorentzian model, we also define a causal Riemannian Barrett-Crane model replacing $K_R^{(j)}(x,y)$ in Eq. (2.2) by

$$\tilde{K}_R^{(j)}(x,y) := \frac{\varepsilon(f)e^{i\varepsilon(f)(2j+1)\varphi}}{2i(2j+1)\sin\varphi}, \quad \varphi := \varepsilon(f)\varepsilon(v)d_R(x,y). \tag{3.9}$$

B. The measure

Recall that the integrals over H_+^3 in the original version of the Lorentzian model (2.5) had to be regularized [14], exploiting the $SL(2,\mathbb{C})$ invariance of the $10j$ symbol and dividing by an infinite volume factor, in order to obtain absolutely convergent integrals. It can be seen from Eq. (3.8) that the same procedure will not suffice in order to define the corresponding integrals of the causal model because the overall numerator is of modulus one while the denominator goes to zero linearly as $\chi \to 0$ so that these integrals will diverge. Therefore the four-simplex amplitudes cannot even be defined for a given assignment of representations p_f to the faces.

This situation is different in nature from the expected divergence of the partition function which is already familiar from the Ponzano-Regge model in three dimensions, which originates just from the summation over infinitely many representations and which can be understood as an infrared divergence [17].

In order to proceed, we therefore have to modify Eqs. (3.8) and (3.9) in a suitable way. Let us consider the Riemannian case first. The first observation is that there is some freedom in the splitting performed in Ref. [9] into one factor which is associated with the overall measure, here the denominator $(2j+1)\sin\varphi$, and another factor, here the numerator $\sin[(2j+1)\varphi]$, which is interpreted as the amplitude $e^{iS} \pm e^{-iS}$. The splitting was only motivated by the analogy with the lower dimensional cases, as outlined, for example, in Ref. [18]. The key condition is that the amplitude factor can be written as the sum of two complex conjugate terms of modulus one.

There exists, for example, the following alternative splitting which leads to a bounded measure part and still satisfies the required condition on the amplitude part

$$\frac{\sin[(2j+1)\varphi]}{(2j+1)\sin\varphi} = \underbrace{\frac{\sin\left(\frac{2j+1}{2}\varphi\right)}{(2j+1)\sin\varphi}}_{\text{measure}} \cdot \underbrace{2\cos\left(\frac{2j+1}{2}\varphi\right)}_{\text{amplitude}}. \tag{3.10}$$

The cosine of the amplitude part can then still be written as $e^{iS}+e^{-iS}$. We observe that the numerator of the measure part goes to zero linearly as $\varphi \to 0$ canceling the divergence and at the same time we get the expression $(j+1/2)\varphi$ in the exponent of the amplitude part which uses the area eigenvalue $(j+1/2)$ of Ref. [19]. At this point the Riemannian model is very helpful because this area eigenvalue directly motivates the choice of Eq. (3.10).

The new definitions which replace the causal amplitudes (3.8) and (3.9) are, therefore,

$$K_{R,\text{causal}}^{(j)}(x,y) := \frac{\sin\left(\frac{2j+1}{2}\varphi\right)}{(2j+1)\sin\varphi} e^{i\varepsilon(f)(j+1/2)\varphi}, \tag{3.11}$$

and by analogy the Lorentzian case

$$K_{L,\text{causal}}^{(p)}(x,y) := \frac{\sin\left(\frac{p}{2}\chi\right)}{p\sinh\chi} e^{i\varepsilon(f)(p/2)\chi}, \tag{3.12}$$

where φ and χ are defined as in Eqs. (3.8) and (3.9).

In the Riemannian case, the integrals over the $x_e^{(\pm)} \in S^3$ converge because all $K_{R,\text{causal}}^{(j)}$ are bounded and integrated over a compact manifold $S^3 \times \cdots \times S^3$. In the Lorentzian case, we notice that

$$|K_{L,\text{causal}}^{(p)}(x,y)| \leqslant 2|K_L^{(p/2)}(x,y)|, \tag{3.13}$$

so that the proof of convergence of the integrals of $K_L^{(p)}(x,y)$ over the $x_e^{(\pm)} \in H_+^3$ which was presented in Ref. [14], implies the existence of our integrals of the $K_{L,\text{causal}}^{(p)}$ over the $x_e^{(\pm)} \in H_+^3$. We can use the same regularization procedure.

C. A coupling constant

In our notation, we can simplify Eqs. (3.11) and (3.12) as follows:

$$K_{L,\text{causal}}^{(p)}(x,y) = \varepsilon(v)\varepsilon(f)\frac{\sin\left[\frac{p}{2}d_L(x,y)\right]}{p\sinh d_L(x,y)} e^{i\varepsilon(v)(p/2)d_L(x,y)}, \tag{3.14}$$

$$K_{R,\text{causal}}^{(j)}(x,y) = \varepsilon(v)\varepsilon(f)\frac{\sin\left(\frac{2j+1}{2}d_R(x,y)\right)}{(2j+1)\sin d_R(x,y)} e^{i\varepsilon(v)(j+1/2)d_R(x,y)}, \tag{3.15}$$

so that the exponents depend only on one sign $\varepsilon(v)$ per four simplex. This links the orientation of four simplices with the amplitudes of the causal model. Therefore we can write the partition function of the causal Lorentzian Barrett-Crane model as

$$Z_{L,\mathrm{causal}}=\left(\prod_{f\in F}\int_0^\infty p_f^2 dp_f\right)\left(\prod_{e\in E}\int_{H_+^3} dx_e^{(+)}\int_{H_+^3} dx_e^{(-)}\right)\times\left(\prod_{f\in F}\mathcal{A}_f\right)\left(\prod_{e\in E}\mathcal{A}_e\right)\times\prod_{v\in V}\left(\prod_{f\in v_0}\frac{\sin\left[\frac{p_f}{2}d_L(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})\right]}{p_f\sinh d_L(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})}\right)\times\exp(iS_L),\tag{3.16}$$

where

$$S_L:=\kappa\sum_{v\in V}\varepsilon(v)\sum_{f\in v_0}\frac{p_f}{2}d_L(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})\tag{3.17}$$

denotes the Regge action in the variables of the Lorentzian model in which the areas of the triangles $f\in F$ are given by the $p_f/2\geqslant 0$. Observe that the prefactors $\varepsilon(v)\varepsilon(f)$ of Eq. (3.14) cancel in the product over all vertices and all faces attached to the vertices.

The causal Riemannian model is given by

$$Z_{R,\mathrm{causal}}=\left(\prod_{f\in F}\sum_{j_f\in 1/2\mathbb{N}_0}(2j_f+1)\right)\left(\prod_{e\in E}\int_{S^3} dx_e^{(+)}\int_{S^3} dx_e^{(-)}\right)\times\left(\prod_{f\in F}\mathcal{A}_f\right)\left(\prod_{e\in E}\mathcal{A}_e\right)\times\prod_{v\in V}\left(\prod_{f\in v_0}\frac{\sin\left[\frac{2j_f+1}{2}d_R(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})\right]}{(2j_f+1)\sin d_R(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})}\right)\times\exp(iS_R),\tag{3.18}$$

where

$$S_R:=\kappa\sum_{v\in V}\varepsilon(v)\sum_{f\in v_0}\left(j_f+\frac{1}{2}\right)d_R(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})\tag{3.19}$$

is the Regge action in terms of the variables of the Riemannian model in which the triangle areas are given by $j_f+\frac{1}{2}$.

In both cases, the amplitudes of the causal model lead to the Regge action for the special value $\kappa=1$ which in turn indicates that there could have been be a free parameter κ in the Barrett-Crane model right from the beginning. This observation suggests the following generalization of the original amplitudes (2.3) and (2.6) to

$$K_R^{(j)}(x,y):=\frac{\sin\left[\frac{2j+1}{2}d_R(x,y)\right]}{(2j+1)\sin d_R(x,y)}\cdot 2\cos\left(\frac{2j+1}{2}\kappa d_R(x,y)\right),\tag{3.20}$$

$$K_L^{(p)}(x,y):=\frac{\sin\left[\frac{p}{2}d_L(x,y)\right]}{p\sinh d_L(x,y)}\cdot 2\cos\left(\frac{p}{2}\kappa d_L(x,y)\right),\tag{3.21}$$

and of the causal amplitudes (3.14) and (3.15) to

$$K^{(p)}_{L,\mathrm{causal}}(x,y)=\varepsilon(v)\varepsilon(f)\frac{\sin\left[\frac{p}{2}d_L(x,y)\right]}{p\sinh d_L(x,y)}e^{i\varepsilon(v)(p/2)\kappa d_L(x,y)},\tag{3.22}$$

$$L^{(j)}_{R,\mathrm{causal}}(x,y)=\varepsilon(v)\varepsilon(f)\frac{\sin\left[\frac{2j+1}{2}d_R(x,y)\right]}{(2j+1)\sin d_R(x,y)}e^{i\varepsilon(v)(j+1/2)\kappa d_R(x,y)}.\tag{3.23}$$

Observe that we have inserted the coupling constant κ only into the amplitude part, but not into the measure part, cf. Eq. (3.10).

Once one has accepted the idea for the construction of the causal model, some splitting such as Eq. (3.10) is necessary in order to make the integrals over H_+^3 or S^3 convergent. The constant κ parametrizes in some sense the nonuniqueness of such a splitting. Notice that the original model for $\kappa\neq 1$ no longer satisfies the bivector constraints of Ref. [4] nor does the causal model satisfy them.

At this point, the various models diverge from each other, and the important question is what the physical relevance of the splitting (3.10) and of the constant κ is. In the following sections, we provide some tools in order to study these questions. These are first the definition of an Euclidean version of the causal model in order to apply numerical simulations and second the study of the symmetries and the most general observables of these models.

As far as the significance of the constant κ is concerned, there seem to prevail two opposite philosophies among the experts. If a classical limit can be obtained by studying the large spin (or semiclassical) limit for a single four simplex, the appearance of κ is unlikely to have much impact. If, however, the four simplices are too strongly coupled and the classical limit requires a nonperturbative renormalization by ''block-spin'' or coarse graining transformations in the spirit of statistical mechanics, then the new parameter κ can easily influence the results, even in the original Barrett-Crane model. For numerical results in the original model, see Ref. [20]. Even the fact that the original Barrett-Crane model with $\kappa\neq 1$ does not satisfy the bivector constraints anymore, would not necessarily be fatal. If κ happens to control the

renormalization scale, it is conceivable that the constraints are satisfied only approximately at an effective coarse grained scale.

It was already observed in Ref. [9] that the variables of the path integral, independent areas p_f or j_f and directions $x_e^{(\pm)} \in H_+^3$ or S^3, indicate that the Regge actions (3.17) and (3.19) have to be understood as actions in a first order formalism [21]. This is consistent with the fact that the Regge action appears only in the intermediate stage of the duality transformation [12] in which both types of variables are present. The necessary constraint [21] that the variation of the angles is only over dihedral angles that correspond to actual four-simplex geometries, has been automatically implemented in the construction of the causal model, see Sec. III A or Ref. [9].

IV. EUCLIDEANIZATION

In both partition functions (3.16) and (3.18), the Regge action takes its simplest form, i.e., with the least number of explicit signs, if one sums over all four simplices only in the last step. If one interprets each four simplex with $\varepsilon(v)=+1$ as a future-pointing contribution to the causal path integral, i.e., obtained after integrating only over positive lapse, for details see Ref. [9], there is an obvious candidate for an Euclidean model by a suitable modification of the amplitudes. This can be done four simplex by four simplex. If we substitute $\varepsilon(v)\cdot i$ into all exponents, we turn the oscillations into an exponential damping and arrive at the following proposal for an Euclidean Lorentzian Barrett-Crane model:

$$
\begin{aligned}
Z_{L,\mathrm{Eucl}} = & \left(\prod_{f\in F} \int_0^\infty p_f^2 dp_f \right) \left(\prod_{e\in E} \int_{H_+^3} dx_e^{(+)} \int_{H_+^3} dx_e^{(-)} \right) \\
& \times \left(\prod_{f\in F} \mathcal{A}_f \right) \left(\prod_{e\in E} \mathcal{A}_e \right) \\
& \times \prod_{v\in V} \left(\prod_{f\in v_0} \frac{\sin\left[\frac{p_f}{2} d_L(x_{e_+(f,v)}^{(+)}, x_{e_-(f,v)}^{(-)}) \right]}{p_f \sinh d_L(x_{e_+(f,v)}^{(+)}, x_{e_-(f,v)}^{(-)})} \right) \\
& \times \exp(-S_{L,\mathrm{Eucl}}),
\end{aligned} \tag{4.1}
$$

where

$$
S_{L,\mathrm{Eucl}} := \kappa \sum_{v\in V} \sum_{f\in v_0} \frac{p_f}{2} d_L(x_{e_+(f,v)}^{(+)}, x_{e_-(f,v)}^{(-)}). \tag{4.2}
$$

This means that we have replaced $K_{L,\mathrm{causal}}^{(p)}$ by

$$
K_{L,\mathrm{Eucl}}^{(p)}(x,y) := \frac{\sin\left[\frac{p}{3} d_L(x,y) \right]}{p \sinh d_L(x,y)} e^{-(p/2)\kappa d_L(x,y)}. \tag{4.3}
$$

Similarly for Riemannian signature,

$$
\begin{aligned}
Z_{R,\mathrm{Eucl}} = & \left(\prod_{f\in F} \sum_{j_f \in 1/2\mathbb{N}_0} (2p_f+1) \right) \left(\prod_{e\in E} \int_{S^3} dx_e^{(+)} \int_{S^3} dx_e^{(-)} \right) \\
& \times \left(\prod_{f\in F} \mathcal{A}_f \right) \left(\prod_{e\in E} \mathcal{A}_e \right) \\
& \times \prod_{v\in V} \left(\prod_{f\in v_0} \frac{\sin\left[\frac{2j_f+1}{2} d_R(x_{e_+(f,v)}^{(+)}, x_{e_-(f,v)}^{(-)}) \right]}{(2j_f+1)\sin d_R(x_{e_+(f,v)}^{(+)}, x_{e_-(f,v)}^{(-)})} \right) \\
& \times \exp(-S_{R,\mathrm{Eucl}}),
\end{aligned} \tag{4.4}
$$

where

$$
S_{R,\mathrm{Eucl}} := \kappa \sum_{v\in V} \sum_{f\in v_0} \left(j_f + \frac{1}{2} \right) d_R(x_{e_+(f,v)}^{(+)}, x_{e_-(f,v)}^{(-)}). \tag{4.5}
$$

i.e., we have replaced $K_{R,\mathrm{causal}}^{(j)}$ by

$$
K_{R,\mathrm{Eucl}}^{(j)}(x,y) := \frac{\sin\left[\frac{2j+1}{2} d_R(x,y) \right]}{(2j+1)\sin d_R(x,y)} e^{-(j+1/2)\kappa d_R(x,y)}. \tag{4.6}
$$

We note that $S_{L,\mathrm{Eucl}} \geqslant 0$ and $S_{R,\mathrm{Eucl}} \geqslant 0$ for any $\kappa \geqslant 0$.

Unfortunately, the Euclidean reconstruction [11] does not provide a recipe of how to derive the *Euclidean action*, i.e., how to choose the four-dimensional path integral measure. In a fixed background with a distinguished t coordinate, one is usually guided by the heuristic substitution $t \mapsto -i\tau$ which typically changes some signs in the action wherever there are time derivatives involved, i.e., in the kinetic energy part. With the special ''time'' coordinate used in three-dimensional Lorentzian dynamical triangulations, there is still a similar substitution one can use in order to construct the Euclidean action [10]. The substitution used in those models essentially distinguishes spacelike from timelike simplices and introduces a relative sign.

In the case of spin foam models, we do not have such a ''time'' coordinate at hand. However, since in the Lorentzian case all triangles are spacelike and therefore treated on equal footing, we expect that no relative signs enter the Euclidean action. As long as we capture the relevant local symmetries, one can expect that Euclidean reconstruction will lead to the correct canonical theory by universality arguments.

We observe that if we fix all representations associated to the faces to the same representation, we get regular flat simplices so that we are in a situation very similar to a single configuration in a dynamical triangulation model [10]. The main difference to the dynamical triangulation models is that in those models all simplices have the same geometry, in particular the same size, and that all dynamical properties of the geometry are encoded in the sum over triangulations. In the case of the Barrett-Crane models, we have the additional complication that the geometry of the individual simplices is determined by the assignment of representations j_f or p_f to the triangles. Therefore, the individual simplices can already be arbitrarily large. We also stress that the formulas (4.1) and

(4.4) refer to a two complex with a causal structure imposed on the vertices. This excludes in particular space times with closed timelike curves. Even stronger, if we have Euclidean reconstruction following [11] in mind, we cannot yet deal with topology change and have to require a global space-time topology of $\Sigma\times\mathbb{R}$.

In the following section, we study expectation values of the continuous variables of the Barrett-Crane models. With these definitions, one can easily establish a dictionary in order to compare our constructions with the generalized Euclidean reconstruction of Ref. [11]. Note that the Euclidean measure of Ref. [11] includes the integration measures, the factors which we have called the *measure part* and also what we have called the *amplitude part*. On very regular triangulations, it is possible to check under which conditions the Euclidean path integral measure of the Barrett-Crane model is reflection positive (this refers to what is called *link reflection positive* by lattice gauge theorists). This condition is satisfied provided that the edge and face amplitudes are real, i.e., if they do not change when one dualizes all representations involved.

V. OBSERVABLES

In this section, we consider all Barrett-Crane models mentioned so far as path integrals over the continuous connection variables $x_e^{(\pm)}\in S^3$ or H_+^3. This point of view has already been adopted in Ref. [12] for the Riemannian model where we have shown that one can perform the sums over the representations $j=0,\frac{1}{2},1,...$ as soon as the edge amplitudes are sufficiently simple. This point of view is also more closely related to the Euclidean reconstruction [11] than is the usual picture in which the variables of the path integral are the representations attached to the triangles. In the following, we therefore consider the $x_e^{(\pm)}$ as the variables of the path integral while the rest of the partition function, including the sums over the j_f or the integrals over the p_f, belongs to the amplitudes.

A. Local symmetries

All the versions of the Riemannian signature model, the original one (2.2), the causal (3.18), and the Euclidean one (4.4), are invariant under the following local *Spin*(4) [or $SO(4)$] symmetry [12]

$$x_e^{(+)}\mapsto h_{\partial_+(e)}x_e^{(+)}\tilde{h}_{\partial_+(e)}^{-1},$$

$$x_e^{(-)}\mapsto h_{\partial_-(e)}x_e^{(-)}\tilde{h}_{\partial_-(e)}^{-1}, \tag{5.1}$$

where $(h_v,\tilde{h}_v)\in Spin(4)$, for all $v\in V$, defines a generating function of this local gauge transformation. We have identified $S^3\cong SU(2)$, and the products in Eq. (5.1) are in $SU(2)$. The independence follows from the invariance of the scalar product $x\cdot y$ in $\mathbb{R}^4$ in the definition of $d_R(x,y)$, cf. Eq. (2.4).

The Lorentzian counterpart of this local symmetry is given by

$$x_e^{(+)}\mapsto h_{\partial_+(e)}\cdot x_e^{(+)},$$

$$x_e^{(-)}\mapsto h_{\partial_-(e)}\cdot x_e^{(-)}, \tag{5.2}$$

where $h_v\in SL(2,\mathbb{C})$ for each $v\in V$, and the dot denotes the action of $SL(2,\mathbb{C})$ on Minkowski space $\mathbb{R}^{1+3}$. Again, this symmetry is a consequence of the invariance of the scalar product in Minkowski space under the action of $SL(2,\mathbb{C})$ which appears in the definition of $d_L(x,y)$, cf. Eq. (2.7). All versions of the Lorentzian signature model, Eqs. (2.5), (3.16), and (4.1) are invariant under Eq. (5.2).

B. Most general expectation values

The most general functions of the variables $x_e^{(\pm)}$ that are invariant under these local transformations, can be calculated by standard techniques (see, for example, Ref. [22] for detailed examples). An orthonormal basis for such functions is characterized by *Spin*(4) or $SL(2;\mathbb{C})$ spin networks on the graph (V,E).

For the Riemannian case, let $\ell_e=0,\frac{1}{2},1,...,$ specify a simple irreducible representation $V_\ell\otimes V_\ell^*$ of *Spin*(4) for each edge $e\in E$, and let

$$P^{(v)}:\left(\bigotimes_{\substack{e\in E:\\ v=\partial_-(e)}}(V_{\ell_e}\otimes V_{\ell_e}^*)\right)\otimes\left(\bigotimes_{\substack{e\in E:\\ v=\partial_+(e)}}(V_{\ell_e}\otimes V_{\ell_e}^*)\right)\to\mathbb{C} \tag{5.3}$$

denote (an arbitrary, but suitably normalized) spin(4) intertwiner for each vertex $v\in V$. Then the spin network function

$$\begin{aligned}F_{\ell,P}(\{x_e^{(\pm)}\})=\prod_{v\in V}\Bigg[&\left(\prod_{\substack{e\in E:\\ v=\partial_+(e)}}\sum_{p_e,q_e=1}^{2\ell_e+1}\right)\left(\prod_{\substack{e\in E:\\ v=\partial_-(e)}}\sum_{r_e,s_e=1}^{2\ell_e+1}\right)\\ &\times\left(\prod_{\substack{e\in E:\\ v=\partial_+(e)}}t_{p_eq_e}^{(\ell_e)}(x_e^{(+)})\right)\\ &\times\left(\prod_{\substack{e\in E:\\ v=\partial_-(e)}}\overline{t_{r_es_e}^{(\ell_e)}(x_e^{(-)})}\right)P^{(v)}_{\underbrace{(r_es_e)\cdots}_{e\in E:\ v=\partial_-(e)}\ \underbrace{(p_eq_e)\cdots}_{e\in E:\ v=\partial_+(e)}}\Bigg],\end{aligned} \tag{5.4}$$

is invariant under the local symmetry (5.1). The $t_{pq}^{(\ell)}$ are the representative functions of $SU(2)\cong S^3$ and we follow the conventions of Ref. [12]. Any L^2 function of the $x_e^{(\pm)}$ that is invariant under the local symmetry, is a square summable series over spin network functions of the form (5.4).

For the Lorentzian case, let $V_{(0,q_e)}, q_e\geq 0$, denote a simple irreducible representation of $SL(2,\mathbb{C})$ for each edge $e\in E$. The vectors of these representation spaces can be modeled by functions $H_+^3\to\mathbb{C}$, see Refs. [5,14] for details. Employing the Gel'fand-Graev transform, an orthonormal basis for $V_{(0,q)}$ is given by the functions

$$H^{(q)}_{jm}:H^3_+\to\mathbb{C},\quad x\mapsto H^{(q)}_{jm}(x):=\int_\Gamma Y_{jm}(\xi)(x\cdot\xi)^{-1-ip}d\xi, \tag{5.5}$$

where Γ denotes the two-sphere of future-pointing lightlike vectors whose spatial component are unit vectors, and the integral is performed using the normalized Lebesgue measure of Γ. The indices of the spherical harmonics Y_{jm} are in the range $j=0,1,2,...,$ and $-j\leqslant m\leqslant j$. Let furthermore

$$Q^{(v)}:\left(\bigotimes_{\substack{e\in E:\\ v=\partial_-(e)}} V^*_{(0,q_e)}\right)\otimes\left(\bigotimes_{\substack{e\in E:\\ v=\partial_+(e)}} V_{(0,q_e)}\right)\to\mathbb{C} \tag{5.6}$$

denote (an arbitrary) $SL(2,\mathbb{C})$ intertwiner for each vertex $v\in V$, given in terms of the coefficients

$$Q^{(v)}_{\underbrace{(k_e n_e),\ldots}_{\substack{e\in E:\\ v=\partial_-(e)}},\ \underbrace{(j_e m_e),\ldots}_{\substack{e\in E:\\ v=\partial_+(e)}}}, \tag{5.7}$$

with respect to the above basis. Then the spin network function

$$\begin{aligned}G_{q,Q}(\{x_e^{(\pm)}\})=\prod_{v\in V}\Bigg[&\left(\prod_{\substack{e\in E:\\ v=\partial_+(e)}}\sum_{j_e=0}^{\infty}\sum_{m_e=-j_2}^{j_e}\right)\\ &\times\left(\prod_{\substack{e\in E:\\ v=\partial_-(e)}}\sum_{k_e=0}^{\infty}\sum_{n_e=-k_e}^{k_e}\right)\\ &\times\left(\prod_{\substack{e\in E:\\ v=\partial_+(e)}}H^{(q_e)}_{j_e m_e}(x_e^{(+)})\right)\\ &\times\left(\prod_{\substack{e\in E:\\ v=\partial_-(e)}}\overline{H^{(q_e)}_{k_e n_e}(x_e^{(-)})}\right)\underbrace{Q^{(v)}_{(k_e n_e)\cdots}}_{\substack{e\in E:\\ v=\partial_-(e)}}\ \underbrace{(j_e m_e)\cdots}_{\substack{e\in E:\\ v=\partial_+(e)}}\Bigg],\end{aligned} \tag{5.8}$$

is invariant under the local symmetry (5.2). All L^2 functions of the $x_e^{(\pm)}$ that are invariant under this local symmetry, are Plancherel integrals over spin network functions of the form (5.8). There will, however, arise convergence issues similar to those studied in Ref. [14].

If one views the partition function of the Barrett-Crane model as a path integral over the continuous variables $x_e^{(\pm)}$, the numbers one can extract from the model are precisely the expectation values of spin network functions of the form (5.4) or (5.8), respectively. In the Riemannian case, these expectation values read

$$\begin{aligned}\langle F_{\ell,P}\rangle=\frac{1}{Z_{R,X}}&\left(\prod_{e\in E}\int_{S^3}dx_e^{(+)}\int_{S^3}dx_e^{(-)}\right)F_{\ell,P}(\{x_e^{\pm}\})\\ &\times\left(\prod_{f\in F}\sum_{j_f\in 1/2\mathbb{N}_0}(2j_f+1)\right)\left(\prod_{f\in F}\mathcal{A}_f\right)\\ &\times\left(\prod_{e\in E}\mathcal{A}_e\right)\prod_{e\in E}\left(\prod_{f\in v_0}K^{(j_f)}_{R,X}(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})\right),\end{aligned} \tag{5.9}$$

for spin network functions $F_{\ell,P}$ of the form (5.4). Here the symbol X in $Z_{R,X}$ and $K_{R,X}$ stands for "original," "causal," or "Euclidean," respectively.

In the Lorentzian case, the analogous expectation value reads

$$\begin{aligned}\langle G_{q,Q}\rangle=\frac{1}{Z_{L,X}}&\left(\prod_{e\in E}\int_{H^3_+}dx_e^{(+)}\int_{H^3_+}dx_e^{(-)}\right)G_{q,Q}(\{x_e^{\pm}\})\\ &\times\left(\prod_{f\in F}\int_0^\infty p_f^2 dp_f\right)\left(\prod_{f\in F}\mathcal{A}_f\right)\\ &\times\left(\prod_{e\in E}\mathcal{A}_e\right)\prod_{v\in V}\left(\prod_{f\in v_0}K^{(p_f)}_{L,X}(x^{(+)}_{e_+(f,v)},x^{(-)}_{e_-(f,v)})\right).\end{aligned} \tag{5.10}$$

The Euclidean reconstruction [11] relies on this type of expectation values in the construction of the physical Hilbert space.

The expressions (5.9) and (5.10) can be reformulated in the language of a path integral whose variables are representations assigned to the faces and in which the integrals over the $x_e^{(\pm)}$ are performed, resulting in relativistic $10j$ symbols as the vertex amplitudes. We call this formulation in which the representations j_f or p_f are the variables of the path integral, the *representation picture* as opposed to the *connection picture* in which the continuous variables $x_e^{(\pm)}$ are the variables of the path integral. The transformation from one to the other picture proceeds in complete analogy to the calculation for the partition function presented in Ref. [12].

We can therefore reexpress the expectation values (5.9) and (5.10) in the representation picture. In the case of the original models (2.2) and (2.5) with $\kappa=1$, the result takes the simplest form. If, say in the Riemannian version, the spin network function $F_{\ell,P}$ is supported only on edges in the boundary of the two complex, the expectation value $\langle F_{\ell,P}\rangle$ agrees with a matrix element of spin network states. This means it is calculated by summing over all spin foams of the Barrett-Crane model living on the given two complex, but with additional faces and edges added so that these faces and edges are colored by the same representations and intertwiners $(\{\ell_e\},\{P^{(v)}\})$ as those that characterize the spin network function $F_{\ell,P}$. These expressions are the desired matrix elements between spin network states. A completely analogous result holds for the original Lorentzian model.

Observe that the expectation values (5.9) and (5.10) are more general than just such matrix elements of spin network

states. First, we have shown that the intertwiners $P^{(v)}$ and $Q^{(v)}$ can be generic *Spin*(4) or $SL(2,\mathbb{C})$ intertwiners and are not restricted to the special Barrett-Crane intertwiners. The standard conjecture seems to be that for pure gravity, it is sufficient to employ *relativistic* spin networks, i.e., spin networks whose representations are simple and whose intertwiners are the Barrett-Crane intertwiner. If this happens to be true, then our calculation above parametrizes the most generic way of coupling other fields to pure gravity. This will be of relevance when one studies the coupling of matter to the Barrett-Crane model. Indeed the choice of intertwiners (5.3) and (5.6) is the first occasion where the difference of *Spin*(4) or $SL(2,\mathbb{C})$ versus $SO(4)$ or $SO_0(1,3)$ matters.

The second aspect in which the expectation values (5.9) and (5.10) are more general than matrix elements of spin network states, is the fact that they are not restricted to the boundary of the four manifold. This can be seen as an analogy to the Wilson loop in lattice gauge theory which is used to determine the static potential between a quark-antiquark pair. This loop is not only supported on the spacelike boundary of the four manifold, but it extends in time direction in the interior. This construction serves as a simplified version of a matter coupling which captures only the color properties of the matter field but which neglects its dynamics. Similar constructions may also prove useful in the study of spin foam models of quantum gravity.

In particular, a generic Wilson loop in the connection variables will give access to the curvature of the full *Spin*(4) or $SL(2,\mathbb{C})$ connection and therefore to dynamical properties and not just to its restriction to a spacelike boundary. Finally, we stress that the study of expectation values such as $\langle F_{\ell,P}\rangle$ and $\langle G_{q,Q}\rangle$ is a convenient way of sidestepping the often tedious technicalities when one deals with boundary terms.

VI. TECHNICAL ISSUES

A. Back to the 10*j* symbols

The expectation values $\langle F_{\ell,P}\rangle$ and $\langle G_{q,Q}\rangle$ become more complicated than what we have discussed so far, as soon as we consider the causal or the Euclidean model or $\kappa\neq 1$ in the original model. Recall that in the original partition function (2.5), the integrations over the H_+^3 together with the product of $K_L^{(p)}$ form the relativistic $10j$ symbols (up to the regularization mentioned above). As soon as we replace the $K_L^{(p)}$ by Eqs. (3.21), (3.22), or (4.3), we no longer have a model whose four-simplex amplitudes are the relativistic $10j$ symbols. In the original model for $\kappa=1$, we were able to "solve" the integrations over the H_+^3 and knew that the result of the integration had an abstract definition as a relativistic $10j$ symbol. After the modification of the integrands $K_{L,X}^{(p)}$, there is no obvious analogy available.

In the following we show how the local symmetry of Sec. V A can be exploited in order to expand the modified integrand, a product of factors $K_{L,X}^{(p)}$, into a series of ordinary relativistic $10j$ symbols. The novel feature is that this step requires an additional colouring of the *wedges* (the intersection of a face of the two complex dual to the triangulation with a four simplex of the original triangulation) with simple representations of the symmetry group.

Consider first the Riemannian case. The functions $K_{R,X}^{(j)}(x,y)$ where X stands for "original," "causal," or "Euclidean," are L^2 functions $S^3\times S^3\to\mathbb{C}$. Since they depend only on $\cos d_R(x,y)=\frac{1}{2}\chi^{(1/2)}(g_x\cdot g_y^{-1})$, where $g_x,g_y\in SU(2)$ denote the corresponding elements of $SU(2)\cong S^3$, they are class functions on $SU(2)$ and can be character expanded into a square summable series

$$K_{R,X}^{(j)}(x,y)=\sum_{k=0,1/2,1,\ldots,}\hat{K}_k^{(j)}\chi^{(k)}(g_x\cdot g_y^{-1}),$$

$$\hat{K}_k^{(j)}:=\int_{SU(2)}\overline{\chi^{(k)}(g)}K_{R,X}^{(j)}(g)dg, \tag{6.1}$$

where we write $K_{R,X}^{(j)}(g)$ in order to indicate that $K_{R,X}^{(j)}(x,y)$ is a function of $g=g_x\cdot g_y^{-1}$.

With this expansion performed for each triangle $f\in F$ and all four simplices $v\in f_0\subseteq F$ that contain the triangle f in their boundary, we can apply the techniques of Ref. [12] and obtain

$$\begin{aligned}\langle F_{\ell,P}\rangle=&\frac{1}{Z_{R,X}}\left(\prod_{f\in F}\sum_{j_f=0,\frac{1}{2},1,\ldots,}\right)\left(\prod_{f\in F}\prod_{v\in f_0}\sum_{k_{fv}=0,\frac{1}{2},1,\ldots,}\right)\\&\times\left(\prod_{f\in F}\mathcal{A}_f\right)\left(\prod_{e\in E}\mathcal{A}_e\right)\left(\prod_{f\in F}\prod_{v\in f_0}\hat{K}_{k_{fv}}^{(j_f)}\right)\\&\times\prod_{v\in V}\left[\left(\prod_{\substack{f\in F:\\ v\in f_0}}\sum_{n_f,m_f=1}^{2k_{fv}+1}\right)\left(\prod_{\substack{e\in E:\\ v=\partial_+(e)}}\sum_{p_e,q_e=1}^{2\ell_e+1}\right)\right.\\&\times\left(\prod_{\substack{e\in E:\\ v=\partial_-(e)}}\sum_{r_e,s_e=1}^{2\ell_e+1}\right)P^{(v)}_{\underbrace{(r_es_e)\cdots}_{\substack{e\in E:\\ v=\partial_-(e)}}\underbrace{(p_eq_e)\cdots}_{\substack{e\in E:\\ v=\partial_+(e)}}}\\&\times\left(\prod_{\substack{e\in E:\\ v=\partial_+(e)}}I^{(+,e)}_{\underbrace{(n_fm_f)\cdots}_{f\in e_+}(p_eq_e);\underbrace{(n_fm_f)\cdots}_{f\in e_-}}\right)\\&\left.\times\left(\prod_{\substack{e\in E:\\ v=\partial_-(e)}}I^{(-,e)}_{\underbrace{(n_fm_f)\cdots}_{f\in e_-};\underbrace{(n_fm_f)\cdots}_{f\in e_+}(r_es_e)}\right)\right].\end{aligned} \tag{6.2}$$

This expression looks more complicated than it actually is. We can explain it in words as follows. There are two types of summations over representations. These are first the sum over all colorings of the triangles $f\in F$ with simple representations $V_{(j_f,j_f^*)}=V_{j_f}\otimes V_{j_f}^*$ of *Spin* (4), and second the sum over all colorings of the wedges (f,v) with representations $V_{(k_{fv},k_{fv}^*)}$. Here the wedges are denoted by specifying a dual face $f\in F$ and a four simplex $v\in f_0\subseteq V$ whose intersection forms the wedge.

In addition to the face and edge amplitudes $\mathcal{A}_f$ and $\mathcal{A}_e$ already present in the original model (2.2), there is now an additional amplitude $\hat{K}_{k_{fv}}^{(j_f)}$ for each wedge, namely, a charac-

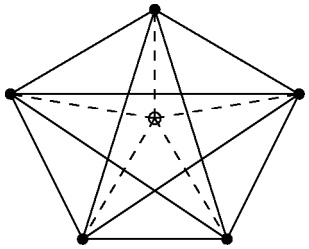

FIG. 1. The four-simplex amplitude of the expectation value $\langle F_{\ell,P}\rangle$ in Eq. (6.2) is the relativistic $10j$ symbol (solid lines) in which a piece of the spin network (ℓ,P) has been inserted (dashed lines). The full dots denote the Barrett-Crane intertwiners and the white dot the intertwiner P of the spin network.

ter expansion coefficient of Eq. (6.1). For the causal model, this amplitude will in general be complex.

The amplitude for each four simplex $v\in V$ is given by the expression inside the square brackets in Eq. (6.2). It is given by the usual relativistic $10j$ symbol with a piece of the spin network (ℓ,P) inserted (Fig. 1). The various summations contract the indices of the Barrett-Crane intertwiners which are denoted by

$$I^{(+,e)}\colon \bigotimes_{f\in e_-} V_{(k_{f\partial_+(e)},k^*_{f\partial_+(e)})}\to\Big(\bigotimes_{f\in e_+} V_{(k_{f\partial_+(e)},k^*_{f\partial_+(e)})}\Big)\otimes V_{(\ell_e,\ell_e^*)},$$

$$I^{(-,e)}\colon \bigotimes_{f\in e_+} V_{(k_{f\partial_-(e)},k^*_{f\partial_-(e)})}\to\Big(\bigotimes_{f\in e_+} V_{(k_{f\partial_-(e)},k^*_{f\partial_-(e)})}\Big)\otimes V_{(\ell_e,\ell_e^*)}, \tag{6.3}$$

using the conventions of Ref. [12]. Here the sets $e_\pm\subseteq F$ contain all triangles f that are contained in the boundary of the tetrahedron $e\in E$ with orientation $\varepsilon(e,f)=\pm 1$. If the intertwiner P of the spin network (ℓ,P) is a Barrett-Crane intertwiner, this amplitude is an evaluated relativistic spin network and therefore non-negative real. For the Riemannian case, this was shown in Ref. [8] whereas for the Lorentzian analogue, this is a plausible conjecture [7].

Observe that the representations j_f attached to the triangles appear only in the expressions for the character expansion coefficients $\hat{K}^{(j_f)}_{k_{fv}}$. The representations for which the $10j$ symbols are evaluated, are no longer the j_f, but rather the representations k_{fv} associated with the wedges (f,v).

Equation (6.2) illustrates the impact that the choice of the causal or Euclidean amplitudes has on the structure and on the symmetries of the model. The central new feature is the additional coloring of the wedges with representations. Only for the original Barrett-Crane model with $\kappa=1$, there exists a significant simplification because in this case $K^{(j)}_R$ is already an $SU(2)$ character. This implies that $\hat{K}^{(j)}_k=\delta_{jk}$ so that all wedges (f,v) of a given dual face $f\in F$ are assigned the same representation $k_{fv}=j_f$. In this special case, Eq. (6.2) reduces to the original Barrett-Crane model [4] with a spin network (ℓ,P) inserted into its $10j$ symbols.

Is there a Lorentzian counterpart of the decomposition (6.1)? In order to derive that formula we have made use of the identification $S^3\cong SU(2)$ which does not have an immediate analogue in the Lorentzian case. Let us reformulate the argument so that we can generalize it.

The functions $K^{(j)}_{R,X}(x,y)$ all have the symmetry

$$K^{(j)}_{R,X}(gx,gy)=K^{(j)}_{R,X}(x,y), \tag{6.4}$$

for all $x,y\in S^3$ and $g\in Spin(4)$ acting on S^3 (Sec. V A). Because of this symmetry, the function is already specified if we know its values $f(x):=K^{(j)}_{R,X}(x,e)$ where $e\in S^3$ denotes the north pole. If we write the function $f:S^3\to\mathbb{C}$ as a function on $Spin(4)$ which is constant on the left cosets gU where $U:=\mathrm{stab}_{Spin(4)}(e)\cong SU(2)$ and $S^3\cong Spin(4)/U$, the invariance condition (6.4) implies that f is also constant on the right cosets Ug and therefore a zonal spherical function. These functions are precisely the characters of $SU(2)$ using the identification $S^3\cong SU(2)$ employed above.

A generalization of Eq. (6.1) to the Lorentzian case is now available since we know that the zonal spherical functions for the quotient $V\backslash SL(2;\mathbb{C})/V$, $V=\mathrm{stab}_{SL(2;\mathbb{C})}(e_t)$, $e_t=(1,0,0,0)$, are precisely the functions $K^{(p)}_L(x,e_t)$ [see Eq. (2.6)]. Therefore we obtain the result that any L^2 function $f:H^3_+\times H^3_+\to\mathbb{C}$ which satisfies

$$f(gx,gy)=f(x,y), \tag{6.5}$$

for all $x,y\in H^3_+$ and $g\in SL(2;\mathbb{C})$, is a Plancherel integral of the form

$$f(x,y)=\int_0^\infty \hat{f}(p)K^{(p)}_L(x,y)p^2dp, \tag{6.6}$$

for a suitable function $\hat{f}:R_+\to\mathbb{C}$.

Therefore the strategy which has lead to Eq. (6.2), can be directly applied to the Lorentzian case. We do not repeat the analogue of Eq. (6.2) here as the required substitutions are now obvious: replace the sums over half integers by integrals $\int_0^\infty p^2dp$ and make use of the integral presentation of the Barrett-Crane intertwiners $I^{(\pm,e)}$. The analogues of the comments listed below Eq. (6.2) also apply to the Lorentzian case.

B. Averaging over the stabilizer

In Ref. [12] we have developed the quantum geometry of the Barrett-Crane model in the connection picture. This includes in particular the interpretation of the integrals over S^3 or H^3_+ that appear in the Barrett-Crane intertwiner, as integrals over possible directions of the vectors normal to the tetrahedra. The fact that there are two such variables for each tetrahedron was interpreted as the consequence of a nontrivial parallel transport which is associated with each tetrahedron and which maps the first normal to the second one. This parallel transport, however, is not a full $Spin(4)$-or $SL(2;\mathbb{C})$-parallel transport, but it is rather specified only up

to elements of the stabilizer which leave the first normal vector fixed. It was then possible to express what is the difference between *BF* theory and the Barrett-Crane model. It is precisely this averaging over the stabilizers. For more details, see Ref. [12].

The geometric interpretation was developed originally for Riemannian signature, see Sec. 4.2 of Ref. [12]. These results can be easily translated to Lorentzian signature by the replacement of $Spin(4)/SU(2) \cong S^3$ by $SL(2,\mathbb{C})/SU(2) \cong H^3_+$. The comparison of the Barrett-Crane model with *BF* theory in Sec. 4.3 of Ref. [12] relies on lemma 4.4 therein whose generalization we sketch in the following.

Let $t^{(n,p)}_{(j_1m_1)(j_2m_2)}$ denote a representative function of $SL(2;\mathbb{C})$ in a representation of type $V_{(n,p)}, n \in \mathbb{Z}, p>0$, of the principal series. We realize the representation $V_{(n,p)}$ as a suitable space of sections of a line bundle $S^3 \to S^2$ and obtain an orthonormal basis from the spherical functions on the total space S^3 so that the range of the indices of the representative functions is given by $j_\ell = |n/2|$, $|n/2|+1$,..., and $m_\ell \in \{-j_\ell, -j_\ell+1,...,j_\ell\}$ for $\ell = 1,2$. It is then possible to show that

$$\int_{SU(2)} t^{(n,p)}_{(j_1m_1)(j_2m_2)}(gu)\,du = \begin{cases} 0 & \text{if } n \neq 0, \\ H^{(p)}_{j_1m_1}\delta_{j_20}\delta_{m_20} & \text{if } n=0. \end{cases} \tag{6.7}$$

Here $SU(2) \subseteq SL(2,\mathbb{C})$ is embedded as the stabilizer of $e_t = (1,0,0,0) \in H^3_+$, and the functions $H^{(p)}_{jm}$ form an orthonormal basis of functions $H^3_+ \to \mathbb{C}$, see (5.5). This shows that there exists an $SU(2)$-invariant subspace of $V_{(n,p)}$ only if the representation is simple, $n=0$, and that this subspace is one dimensional. We therefore obtain basis functions on $H^3_+ \cong SL(2;\mathbb{C})/SU(2)$ from representative functions of $SL(2;\mathbb{C})$ by averaging over the right $SU(2)$ action.

Now we consider two points $x,y \in H^3_+$. For each $z \in H^3_+$, there exists some boost $b_z \in SL(2;\mathbb{C})$ such that $b_z(e_t) = z$. The group elements $g \in SL(2;\mathbb{C})$ that map $gy = x$ are of the form

$$g = b_x b_y^{-1} u_y, \tag{6.8}$$

where $u_y \in \mathrm{stab}_{SL(2;\mathbb{C})}(y) \cong SU(2)$. However, if $u_y \in \mathrm{stab}(y)$, then $b_y^{-1} u_y b_y \in \mathrm{stab}(e_t)$ and conversely, therefore

$$g = b_x u_t^{-1} b_y^{-1}, \tag{6.9}$$

for some $u_t \in \mathrm{stab}(e_t)$. We use this parametrization of g and average over the stabilizer:

$$\int_{SU(2)} t^{(n,p)}_{(j_1m_1)(j_2m_2)}(b_x u b_y^{-1})\,du = \delta_{n0} H^{(p)}_{j_1m_1}(b_x)\overline{H^{(p)}_{j_2m_2}(b_y)}. \tag{6.10}$$

The construction of Ref. [12] then says that for any holonomy $g \in SL(2,\mathbb{C})$ at an edge, it matters only how g acts on H^3_+. Therefore we choose some $x \in H^3_+$, calculate $y = gx$ and average over the stabilizer ambiguity. The integration over the group which is present in the path integral then results in the desired integrations over H^3_+. This is the Lorentzian analogue of lemma 4.4 of Ref. [12].

VII. DISCUSSION

What we have explained in Secs. IV and V, the Euclidean reconstruction and the analysis of the degrees of freedom of the Barrett-Crane model from an understanding of its local symmetries, is only one motivation for studying the observables in the connection picture. Another motivation arises from the observation [12] that for some edge amplitudes, the partition function in the connection picture is particularly simple and resembles a spin model (just ''spin,'' not ''spin foam'') with variables in S^3 or H^3_+ with local interaction terms at the faces. In particular, no evaluations of $10j$ symbols are necessary in this case which makes numerical simulations computationally cheaper. One has just to tabulate the interaction terms.

For the original Riemannian model it was observed in both formulations, in the connection picture [12] and in the representation picture [20], that the dominant configurations of the partition function often correspond to degenerate geometries. With the results presented here, there are two new developments which can modify this conclusion. This is first the introduction of the constant κ (Sec. III C) which provides a natural way of controlling the width of the peaks in the picture of Ref. [12]. This is what a coupling constant [temperature] in a path integral [statistical mechanics] model typically does. The constant κ may play an important role when one tries to locate a critical point at which one can renormalize the model. Second, the causal and the Euclidean model have amplitudes very different from the original model. In the connection picture, the Euclidean version can be studied by exactly the same techniques as the original version so that one can start to investigate and compare the models and their physical interpretation. The transformation of Sec. VI A then allows us to perform the same studies in the representation picture.

Finally, we note that all our formulas for partition functions, matrix elements and expectation values use the language of generic two complexes. These are not restricted to be dual to a given triangulation. The only exceptions were the motivating steps which explicitly involved results from Regge calculus which are available only on triangulations.

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[1] J. C. Baez, in *Geometry and quantum physics*, Lecture Notes in Physics No. 543 (Springer, Berlin, 2000), pp. 25–93.
[2] D. Oriti, Rep. Prog. Phys. **64**, 1703 (2001).
[3] J. C. Baez, Class. Quantum Grav. **15**, 1827 (1998).
[4] J. W. Barrett and L. Crane, J. Math. Phys. **39**, 3296 (1998).
[5] J. W. Barret and L. Crane, Class. Quantum Grav. **17**, 3101 (2000).
[6] J. C. Baez and J. W. Barrett, Adv. Theor. Math. Phys. **3**, 815 (1999).
[7] J. C. Baez and J. D. Christensen, Class. Quantum Grav. **19**, 2291 (2002).
[8] H. Pfeiffer, ''Positivity of relativistic spin network evaluations,'' gr-qc/0211106 [Adv. Theor. Math. Phys. (to be published)].
[9] E. R. Livine and D. Oriti, ''Implementing causality in the spin foam quantum geometry,'' gr-qc/0210064.
[10] J. Ambjørn, J. Jurkiewicz, and R. Loll, Phys. Rev. D **64**, 044011 (2001).
[11] A. Ashtekar, D. Marolf, J. Mourão, and T. Thiemann, Class. Quantum Grav. **17**, 4919 (2000).
[12] H. Pfeiffer, Class. Quantum Grav. **19**, 1109 (2002).
[13] J. W. Barrett, Adv. Theor. Math. Phys. **2**, 593 (1998).
[14] J. C. Baez and J. W. Barrett, Class. Quantum Grav. **18**, 4683 (2001).
[15] J. W. Barrett and T. J. Foxon, Class. Quantum Grav. **11**, 543 (1994).
[16] J. W. Barrett and C. M. Steele, ''Asymptotics of relativistic spin networks,'' gr-qc/0209023.
[17] L. Freidel and D. Louapre, ''Diffeomorphisms and spin foam models,'' gr-qc/0212001.
[18] M. Arnsdorf, Class. Quantum Grav. **19**, 1065 (2002).
[19] A. Alekseev, A. P. Polychronakos, and M. Smedback, ''On area and entropy of a black hole,'' hep-th/0004036.
[20] J. C. Baez, J. D. Christensen, T. R. Halford, and D. C. Tsang, Class. Quantum Grav. **19**, 4627 (2002).
[21] J. W. Barrett, Class. Quantum Grav. **11**, 2723 (1994).
[22] H. Pfeiffer, ''Exact duality transformations for sigma models and gauge theories,'' hep-lat/0205013.
[23] R. De Pietri and L. Freidel, Class. Quantum Grav. **16**, 2187 (1999).