Explicit solutions to the time-independent perturbation equations of the Reissner-Nordström geometry

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It is proven that the equations for the time-independent, negative-parity, gravitational and electromagnetic perturbations of the Reissner-Nordström geometry have (for each angular momentum) one solution in the form of a finite power series in the (Schwarzschild-like) radial coordinate r. The explicit form of these solutions is given. Therefrom the second fundamental solutions of these equations, and the positive-parity perturbations, can also be constructed explicitly.

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I. INTRODUCTION

In 1974 four different groups derived, obviously independently, the differential equations obeyed by the first order perturbations of the Reissner-Nordström (RN) geometry: Zerilli [1] worked in the Regge-Wheeler gauge [2] and derived coupled second-order equations between the gravitational and the electromagnetic multipoles for both the odd parity (or magnetic) and the even parity (or electric) case, but he did not decouple these equations. Sibgatullin and Alekseev [3] derived in a different gauge decoupled, Schrödinger-like equations for combined gravitational and electromagnetic perturbations, again for both parities. Moncrief applied his Hamiltonian formalism [4], in which the gauge invariant perturbations are explicitly singled out, separately to the odd-parity case [5] and to the even-parity case [6]. He indicated that the resulting equations can be decoupled without working this out explicitly. Lun [7] worked in the Newman-Penrose spin coefficient formalism [8] and derived coupled equations, but only for the case of odd parity. (The structurally similar differential equation for scalar perturbations of the RN geometry was already derived in 1972 by Bičák [9].)

The RN geometry is, in Schwarzschild-like coordinates, given by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

$$= -F(r)dt^2 + F(r)^{-1}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$

with $F(r) = 1 - 2M/r + q^2/r^2$. The perturbed metric is usually denoted by $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, similarly the perturbed electromagnetic field by $\tilde{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu}$, where $F_{\mu\nu}$ has only one nontrivial component $F_{tr} = -q/r^2$. It is advantageous to expand the gravitational perturbations $h_{\mu\nu}$ in tensor harmonics (with respect to ϑ and φ), and the electromagnetic perturbations in vector harmonics, in addition to expanding both perturbations in their Fourier components $\sim e^{i\sigma t}$. Since the RN metric is static and spherically symmetric, all these perturbation modes decouple. Furthermore only axisymmetric modes with angular momentum "quantum number" l and magnetic "quantum number" $m = 0$ have to be considered. However, for each l (and σ) there exists an axial mode (with negative parity) and a polar mode (with positive parity). According to the analysis of Moncrief [4–6], for given $l > 1$, σ, and given parity ($\pm$), there exists only one gauge invariant metric perturbation, and one gauge invariant electromagnetic perturbation. In the notation of Chandrasekhar [10] these are called $H_2^{(\pm)}(r)$ and $H_1^{(\pm)}(r)$. By introducing the following linear combinations between gravitational and electromagnetic perturbations

$$Z_1^{(\pm)} = q_1 H_1^{(\pm)} + \sqrt{-q_1 q_2} H_2^{(\pm)}, \tag{2a}$$

$$Z_2^{(\pm)} = -\sqrt{-q_1 q_2} H_1^{(\pm)} + q_1 H_2^{(\pm)}, \tag{2b}$$

with

$$q_{1,2} = 3M \pm \sqrt{9M^2 + 4q^2(l-1)(l+2)},$$

the Einstein-Maxwell equations for the perturbations of the RN metric reduce (for each l and σ) to the following four Schrödinger-like differential equations

$$\frac{d}{dr}\left[F(r)\frac{d}{dr}\right]Z_a^{(\pm)} + \frac{\sigma^2}{F(r)} Z_a^{(\pm)} = V_a^{(\pm)}(r) Z_a^{(\pm)}, \tag{3}$$

with

$$V_a^{(-)}(r) = \frac{L}{r^2} - \frac{q_b}{r^3} + \frac{4q^2}{r^4},$$

and

$$V_a^{(+)}(r) = \frac{1}{[(L-2)r + q_b]^2}\left\{(L-2)\left[L(L-2) + \frac{8q^2}{r^3}\right.\right.$$
$$\left.\times\left(M - \frac{q^2}{r}\right)\right] + q_b\left[\frac{(L-2)^2}{r} + \frac{6M}{r^2}\right.$$
$$\left.\left.\times\left(L - 2 + \frac{2M}{r} - \frac{2(L-2)q^2}{3Mr} - \frac{2q^2}{r^2}\right)\right]\right\}$$

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where $a \neq b = 1,2$, and $L = l(l+1)$. Obviously, the equations for the odd-parity perturbations $Z_a^{(-)}$ are much simpler, and in Secs. II and III we will first look for explicit solutions to these equations (in the time-independent case $\sigma = 0$). It turns out that the more complicated equations for $Z_a^{(+)}$ do not have to be solved separately, because Chandrasekhar ([11]; see also [10]) has succeeded in expressing $Z_a^{(+)}$ by $Z_a^{(-)}$ and its r-derivative. [Our expression for $V_a^{(+)}(r)$ is considerably simpler than the expressions in [3] and [10]. Obviously nobody has seen hitherto that the denominators in their potentials $V_a^{(+)}(r)$ can be factorized into $[(L-2)r+q_a]^2[(L-2)r+q_b]^2$, and that the factor $[(L-2)r+q_a]^2$ cancels with the same factor in the numerator. This cancellation is also necessary in order that Chandrasekhar's solution for $Z_a^{(+)}$—see our Eq. (19) in Sec. III—can satisfy Eq. (3).]

Concerning explicit solutions to Eq. (3), it seems that no general analysis has been performed hitherto, not even in the time-independent case. However, in special cases explicit solutions have been found: Bičák [12] has considered time-independent perturbations of the extreme RN geometry ($q^2 = M^2$), and has given all odd-parity solutions as finite power series in $(r-M)$ and $(r-M)^{-1}$, and the even-parity solutions as similar power series multiplied by a factor $[(L-2)r^2+6Mr-4M^2]^{-1}$. (In these expressions again some cancellations lead to the much simpler formulas $R_+^{(I)} = \xi^{l+1}(\xi+1)/[(l+2)\xi+l]$, and $R_-^{(II)} = \xi^{-l}(\xi+1)/[(l-1)\xi+l+1]$.) For general q and M, and $l=1$ (and $\sigma=0$) Bičák and co-workers [13,14] have found, for both parities, one explicit solution of Eq. (3). Briggs et al. [15] (see also [16]) have considered stationary and odd-parity ($l=1$)-perturbations, i.e., rotational perturbations of a general RN geometry. They have found one fundamental solution $Z_a^{(-)}$ as a finite power series in r and r^{-1} and the other fundamental solution as a similar power series, combined with the function

$$S(r;M,q) = \begin{cases} \dfrac{1}{2\sqrt{M^2-q^2}} \log\left(\dfrac{r-M+\sqrt{M^2-q^2}}{r-M-\sqrt{M^2-q^2}}\right) & \text{for } q^2 < M^2, \\ (r-M)^{-1} & \text{for } q^2 = M^2, \\ \dfrac{1}{\sqrt{q^2-M^2}} \operatorname{arccot}\left(\dfrac{r-M}{\sqrt{q^2-M^2}}\right) & \text{for } q^2 > M^2. \end{cases} \tag{4}$$

A similar result, but for even parity, $q^2=0$, and general l has already been indicated by Fackerell [17], but no explicit solutions as power series have been provided.

In Sec. II it is proven that in the case of negative parity Eq. (3) for $\sigma=0$ allows one solution of the form $\bar{Z}_a^{(-)}(r) = \Sigma_{k=-1}^{l+1} \alpha_k^{(l,a)}(-r/2M)^k$, with r-independent coefficients $\alpha_k^{(l,a)}$, and with $\alpha_1^{(l,a)}=0$ (as indicated already in [16]). Furthermore it is shown that the second fundamental solution has the form $\bar{\bar{Z}}_a^{(-)}(r) = M\bar{Z}_a^{(-)}(r)S(r;M,q) + \hat{Z}_a^{(-)}(r)$, with $S(r;M,q)$ from Eq. (4), and with $\hat{Z}_a^{(-)}(r) = \Sigma_{k=-1}^{l}\beta_k^{(l,a)}(-r/2M)^k$. [In [9] it has been shown that the differential equation for the scalar perturbations of the RN geometry can, by the substitution $r \to \xi = (r-M)/\sqrt{M^2-q^2}$, be converted to the Legendre equation whose fundamental solutions are polynomials, respectively polynomials combined with the function S from Eq. (4). In the Schwarzschild case $q^2=0$ a similar observation was already made by Israel [18]. Our results for $\bar{Z}_a^{(-)}$ and $\bar{\bar{Z}}_a^{(-)}$ can be considered as generalizations of the Legendre functions.]

In Sec. III the coefficients $\alpha_k^{(l,a)}$ and $\beta_k^{(l,a)}$ are explicitly calculated, and also the simpler coefficients for the time-independent perturbations of the Schwarzschild geometry are given. These results are applicable to the late time behavior of a slightly nonspherical collapse to a charged or uncharged black hole (BH), to the motion of such a BH in weak electromagnetic and/or gravitational fields, to the motion of small charged objects in the vicinity of a charged BH, and as a test bed of numerical codes for BH physics. In special cases such applications have already been worked out in [3,9,13,14,19,20].

II. PROOF FOR THE EXISTENCE OF TIME-INDEPENDENT SOLUTIONS AS FINITE POWER SERIES IN r AND r^{-1}

We confine ourselves here to the odd-parity solutions $Z_a^{(-)}$, and to the time-independent case $\sigma=0$. Obviously, the differential equation (3) for $Z_a^{(-)}$ has four singular points: $r=0$, $r=\infty$, and the zeros of $F(r)$, resulting in the horizons $r_{1,2} = M \pm \sqrt{M^2-q^2}$. In principle, such a differential equation can be reduced to the Heun equation [21]. Its solutions will in general be more complicated than the solutions of the hypergeometric differential equation (with three singular points). It is easily seen that near $r=0$ and $r=\infty$ the solutions can be represented by power series in r and r^{-1} (at each point one of the fundamental solutions being regular, the other being singular). In contrast, at the horizons $r_{1,2} = M \pm \sqrt{M^2-q^2}$ there is, besides one regular solution, a fundamental solution with logarithmic behavior which, combining the two horizons, can be represented by

$$\tilde{Z}_a^{(-)}(r) = f_1(r) S(r;M,q) + f_2(r), \tag{5}$$

with $S(r;M,q)$ from Eq. (4), and with functions $f_1(r)$, $f_2(r)$ being regular at $r = r_{1,2}$. Now, concerning exact solutions of Eq. (3) for $Z_a^{(-)}$, it is expected that normally infinite power series in r and/or r^{-1} are necessary for representing $Z_a^{(-)}$, and that equally the functions $f_1(r), f_2(r)$ in Eq. (5) will be infinite power series. (It is known from quantum mechanics that only for a few special potential functions and then only for discrete energy values there exist solutions of the time-independent radial Schrödinger equation which are essentially—besides factors like $e^{-\lambda r}$ or $e^{-\kappa r^2}$—given by finite power series.)

However, we will now show that for Eq. (3) for $Z_a^{(-)}$, and for $\sigma = 0$, the potential $V_a^{(-)}(r)$ has such a special form, and such a special relation to the function $F(r)$ that there exists one fundamental solution $\overline{Z}_a^{(-)}$ as a finite power series in r and r^{-1}, and that the second fundamental solution has the form

$$\overline{\overline{Z}}_a^{(-)}(r) = M \overline{Z}_a^{(-)}(r) S(r;M,q) + \hat{Z}_a^{(-)}(r), \tag{6}$$

with another finite power series $\hat{Z}_a^{(-)}$ in r and r^{-1}. The structure of the first term on the right-hand side (RHS) of Eq. (6) is also suggested by d'Alembert's reduction procedure, giving $\overline{\overline{Z}}_a^{(-)}$ from $\overline{Z}_a^{(-)}$:

$$\overline{\overline{Z}}_a^{(-)}(r) = \overline{Z}_a^{(-)}(r) \int^r \frac{dr'}{F(r') [\overline{Z}_a^{(-)}(r')]^2}.$$

The *Ansätze* for the solutions $\overline{Z}_a^{(-)}$ and $\overline{\overline{Z}}_a^{(-)}$ are of course also suggested by the results [12,15–17] for special cases. (It is quite obvious that the same "miracle" cannot happen for general values of σ, and we have not checked whether there exist discrete values $\sigma_n \neq 0$ with similar power series solutions.)

So we start with the *Ansatz*

$$\overline{Z}_a^{(-)}(r) = \sum_k \alpha_k^{(l,a)} \left(\frac{-r}{2M} \right)^k, \tag{7}$$

for one of the fundamental solutions of Eq. (3). The range of the k values will be determined in the following. Inserting this *Ansatz* into Eq. (3) with $\sigma = 0$ leads to the 3-term recursion relation

$$(l+k)(l+1-k)\alpha_k^{(l,a)} - c_k^{(l,a)} \alpha_{k+1}^{(l,a)} - \frac{q^2}{4M^2}(k+3) \times (k-2)\alpha_{k+2}^{(l,a)} = 0 \tag{8}$$

with

$$c_k^{(l,a)} = k^2 - \frac{5}{2} \pm \sqrt{\frac{9}{4} + \frac{q^2}{M^2}(l-1)(l+2)}.$$

[For the Schwarzschild case $q^2 = 0$, Eq. (8) reduces to a simpler 2-term recursion relation.] In the following we will omit the upper indices (l,a) in the coefficients α_k and in the expressions c_k. They should be re-introduced in the final results. In order that a solution of Eq. (8) is also valid for $q^2 = 0$, it is advisable to start with the highest coefficient α_{k+2}, and to calculate from Eq. (8) the lower coefficients. Due to the factor $(l+1-k)$ in the first term, there obviously exists a solution with $\alpha_{l+1} \neq 0$ but with $\alpha_{l+2} = \alpha_{l+3} = \cdots = 0$. Then the recursion relation is trivially satisfied for $k = l+1, l+2, \ldots$. For $k = l$ we get $\alpha_l = (c_l/2l)\alpha_{l+1}$, and similarly we can successively calculate all lower coefficients as multiples of α_{l+1}. If we arrive at $k = 2$, the last term of Eq. (8) vanishes (for all q values), with the result (for $l > 1$) $\alpha_2 = [c_2/(l-1)(l+2)]\alpha_3$. Because of this relation, and due to $c_1 c_2 = (q^2/M^2)(l-1)(l+2)$, in the recursion relation for $k = 1$ the last two terms cancel, with the result $\alpha_1 = 0$. From $k = 0$ we get $\alpha_0 = -[3q^2/2M^2 l(l+1)]\alpha_2$, and from $k = -1$: $\alpha_{-1} = [c_{-1}/(l-1)(l+2)]\alpha_0$ (for $l > 1$). Because of this relation, and due to $c_{-1}c_{-2} = (q^2/M^2)(l-1)(l+2)$, in the recursion relation for $k = -2$ again the last two terms cancel, with the result $\alpha_{-2} = 0$ (compulsory for $l \neq 2$, but valid also for $l = 2$). For $k = -3$ the last term in Eq. (8) vanishes, with the result $\alpha_{-3} = 0$, and consequently also $\alpha_{-4} = \alpha_{-5} = \ldots = 0$. In summary, we have proven that for $l > 1$ one fundamental solution of Eq. (3) with $\sigma = 0$ has the form

$$\overline{Z}_a^{(-)}(r) = \sum_{k=-1}^{l+1} \alpha_k^{(l,a)} \left(\frac{-r}{2M} \right)^k, \tag{9}$$

with $\alpha_1^{(l,a)} = 0$. The case $l = 1$ obviously has to be considered separately. It turns out that for $a = 1$ the resulting Eq. (9) remains valid but the arguments are somewhat different: The recursion relation for $k = -1$ leaves the coefficient α_{-1} undetermined, but α_{-1} is "later" determined by the demand that the relations for $k = -2, k = -3, \ldots$ result in $\alpha_{-2} = \alpha_{-3} = \ldots = 0$. For $a = 2$ the recursion relations for $k = 1$, $k = 0$, and $k = -1$ are in contradiction to each other (for $\alpha_2 \neq 0$ and $q^2 \neq 2M^2$), i.e., no solution as a finite power series in r and r^{-1} exists. There does not even exist any gauge-invariant solution $\overline{Z}_{a=2}^{(-)}$ for $l = 1$, because, as was already stressed by Bičák [22], there exist only electromagnetic but no gravitational gauge-invariant dipole degrees of freedom. [For $l = 1$, we have $q_2 = 0$ in Eqs. (2a), (2b), i.e., no coupling between gravitational and electromagnetic perturbations.]

Coming now to the second fundamental solution of Eq. (3) for $\overline{Z}_a^{(-)}$ and $\sigma = 0$, we have the task to show that the form of Eq. (6) really solves Eq. (3) for $\sigma = 0$, and that $\hat{Z}_a^{(-)}$ is again a finite power series in r and r^{-1}. If we insert the *Ansatz* of Eq. (6) with $\hat{Z}_a^{(-)}(r) = \Sigma_k \beta_k^{(l,a)}(-r/2M)^k$ into Eq. (3) with $\sigma = 0$, we get, again omitting the upper indices (l,a), the recursion relation

$$(l+k)(l+1-k)\beta_k - c_k\beta_{k+1} - \frac{q^2}{4M^2}(k+3)(k-2)\beta_{k+2} = k\alpha_{k+1}. \tag{10}$$

The LHS of Eq. (10) is of course equivalent to the LHS of Eq. (8) [because $\overline{Z}_a^{(-)}$ and $\overline{\overline{Z}}_a^{(-)}$ have to satisfy the same differential equation (3)], however, Eq. (10) has a nontrivial RHS (an "inhomogeneity"), resulting from the derivatives of the function $S(r;M,q)$ [see Eq. (6)]. We will now analyze the recursion relation (10) in a similar way as we did with Eq. (8), and we try to prove that there exists again a solution with a finite number of terms β_k. [Since the solution $\overline{Z}_a^{(-)}$ behaves asymptotically like r^{l+1}, and we know from the analysis of the asymptotic behavior of Eq. (3) that the second fundamental solution $\overline{\overline{Z}}_a^{(-)}$ behaves like r^{-l}, we could also determine the highest $2l$ coefficients β_k through the condition that they cancel the terms coming from a power series expansion of $M\overline{Z}_a^{(-)}(r)S(r;M,q)$.] Due to $\alpha_{l+2}=0$, the recursion relation for $k=l+1$ can be satisfied with $\beta_{l+1}=\beta_{l+2}=\cdots=0$. For $k=l$ we get $\beta_l=\frac{1}{2}\alpha_{l+1}\neq 0$, and similarly we can successively calculate all lower coefficients β_k as multiples of α_{l+1}. If we arrive at $k=2$, the last term of the LHS of Eq. (10) vanishes, with the result (for $l>1$) $\beta_2=(c_2\beta_3+2\alpha_3)/(l-1)(l+2)$. Due to this relation, and due to $c_1c_2=(q^2/M^2)(l-1)(l+2)$, in the recursion relation for $k=1$ the contributions proportional to β_3 cancel, with the result $\beta_1=(c_2+2c_1)\alpha_3/(l-1)l(l+1)(l+2)$, generally being nonzero, in contrast to $\alpha_1=0$. From $k=0$ we get $\beta_0=[c_0\beta_1-(3q^2/2M^2)\beta_2]/l(l+1)$, and from $k=-1$: $\beta_{-1}=[c_{-1}\beta_0-(3q^2/2M^2)\beta_1-\alpha_0]/(l-1)(l+2)$ (for $l>1$). From $k=-2$ we get in the first place $(l-2)(l+3)\beta_{-2}=c_{-2}\beta_{-1}-(q^2/M^2)\beta_0-2\alpha_{-1}$. Inserting here the results for β_{-1} and β_1, all the β_k-terms cancel, and also the remaining α_k-terms cancel, due to the relations between $\alpha_{-1},\alpha_0,\alpha_2$, and α_3. So we get $\beta_{-2}=0$. For $k=-3$ the last term of the LHS of Eq. (10) cancels, and, together with $\alpha_{-2}=0$, we get $\alpha_{-3}=0$, and consequently also $\alpha_{-4}=\alpha_{-5}=\ldots=0$. In summary, we have proven that for $l>1$

$$\hat{Z}_a^{(-)}(r)=\sum_{k=-1}^{l}\beta_k^{(l,a)}\left(\frac{-r}{2M}\right)^k. \tag{11}$$

For $l=1$ and $a=1$, this formula is still valid, although due to somewhat different arguments. For $l=1$ and $a=2$, again the argument counts that no gauge-invariant gravitational dipole degrees of freedom exist.

III. EXPLICIT CALCULATION OF THE FINITE POWER SERIES SOLUTIONS

According to Sec. II, in the series *Ansatz* of Eq. (7) for $\overline{Z}_a^{(-)}$ the coefficient $\alpha_{l+1}\neq 0$ is arbitrary [because Eq. (3) is homogeneous]. The lower coefficients $\alpha_l,\alpha_{l-1},\ldots$ can be calculated recursively from Eq. (8). Performing this for a few steps, it is seen that a useful abbreviation is

$$\eta_j^{(l,a)}=\frac{j(2l+1-j)(l+3-j)(l-2-j)}{c_{l+1-j}^{(l,a)}c_{l-j}^{(l,a)}}. \tag{12}$$

The values $\eta_{l-1}=\eta_{l+2}=-4M^2/q^2$ are especially simple; all other η_j contain the square root appearing in c_k. With the abbreviation (12) it can then be seen and proven that the general formula for the coefficients α_k reads

$$\alpha_{l-k}=\frac{(2l-k-1)!\prod_{i=0}^{k}c_{l-i}}{(2l)!(k+1)!}\sum_{n=0}^{[(k+1)/2]}\left\{\left[\frac{q^2}{4M^2}\right]^n\sum_{j_1=1}^{k-2(n-1)}\eta_{j_1}\left[\sum_{j_2=j_1+2}^{k-2(n-2)}\eta_{j_2}\left(\ldots\left[\sum_{j_{n-1}=j_{n-2}+2}^{k-2}\eta_{j_{n-1}}\times\left(\sum_{j_n=j_{n-1}+2}^{k}\eta_{j_n}\right)\right]\ldots\right)\right]\right\}\alpha_{l+1}. \tag{13}$$

Here, as usual, $[(k+1)/2]$ denotes the largest integer being smaller or equal to $(k+1)/2$, and for $n=0$ the expression $\{\cdots\}$ should have the value 1. In formula (13) one does not see explicitly that $\alpha_1=0$, and $\alpha_{-2}=\alpha_{-3}=\cdots=0$, and the formula may be formally inapplicable for some of the highest (relevant) k-values, especially for low l values. But these coefficients have already been considered explicitly in Sec. II.

For the Schwarzschild case $q^2=0$ formula (13) of course simplifies considerably: We get, for $a=1$ (and $k\leqslant l-2$), i.e., for electromagnetic test fields on the Schwarzschild geometry,

$$\alpha_{l-k}^{(l,1)}=\frac{(2l-k-1)!(l+1)!(l-1)!}{(k+1)!(l-k)!(l-k-2)!(2l)!}\alpha_{l+1}^{(l,1)}, \tag{14}$$

and, for $a=2$ (and $k\leqslant l-3$, and $l>1$),

$$\alpha_{l-k}^{(l,2)}=\frac{(2l-k-1)!(l+2)!(l-2)!}{(k+1)!(l-k+1)!(l-k-3)!(2l)!}\alpha_{l+1}^{(l,2)}. \tag{15}$$

Furthermore one gets in this case $\alpha_0^{(l,1)}=\alpha_{-1}^{(l,1)}=0$, i.e., the series (9) begins with a term $\alpha_2^{(l,1)}\neq 0$. Similarly $\alpha_2^{(l,2)}=\alpha_0^{(l,2)}=\alpha_{-1}^{(l,2)}=0$, i.e., this series begins with a term $\alpha_3^{(l,2)}\neq 0$.

The coefficients β_k, appearing in Eq. (11) for $\hat{Z}_a^{(-)}$ have, with $\beta_l=\frac{1}{2}\alpha_{l+1}\neq 0$, similarly been calculated recursively from Eq. (10). Here it is advantageous to introduce, besides $\eta_j^{(l,a)}$ from Eq. (12), the abbreviation

$$f_j^{(l,a)}=\frac{l-j}{c_{l-j}^{(l,a)}}. \tag{16}$$

Then one conjectures from the first few recursion steps, and can prove generally that the coefficients β_k have the form

$$\beta_{l-k}=\frac{(2l-k-1)!\prod_{i=0}^{k} c_{l-i}}{(2l)!(k+1)!}\sum_{n=0}^{[k/2]}\left\{\left[\frac{q^2}{4M^2}\right]^n\sum_{j_1=1}^{k-2(n-1)}\eta_{j_1}\left[\sum_{j_2=j_1+2}^{k-2(n-2)}\eta_{j_2}\right.\right.$$
$$\left.\left.\times\left(\cdots\left[\sum_{j_{n-1}=j_{n-2}+2}^{k-2}\eta_{j_{n-1}}\left(\sum_{j_n=j_{n-1}+2}^{k}\eta_{j_n}\sum_{\substack{j=0\\ j\neq j_1,j_{1-1}\\ \vdots\\ j\neq j_n,j_{n-1}}}^{k}f_j\right)\right]\cdots\right)\right]\right\}\alpha_{l+1}, \tag{17}$$

where for $n=0$ the expression $\{\cdots\}$ should have the value $\Sigma_{j=0}^{k}f_j$. For the Schwarzschild case $q^2=0$ formula (17) simplifies to

$$\beta_{l-k}=\left(\sum_{j=0}^{k}f_j\right)\alpha_{l-k}, \tag{18}$$

with α_{l-k} from Eqs. (14) and (15). In contrast to the vanishing of the lowest coefficients α_k, all coefficients β_k until β_{-1} stay nonzero.

As indicated in the Introduction, the positive-parity solutions $Z_a^{(+)}$ can be explicitly constructed from the above negative-parity solutions $Z_a^{(-)}$. In detail the formula reads ([10,11]) for $\sigma=0$, and $l>1$, and with $K=(l-1)l(l+1)(l+2)$

$$Z_a^{(+)}=\left\{1+\frac{2F(r)q_b^2}{Kr[(L-2)r+q_b]}\right\}Z_a^{(-)}+\frac{2F(r)q_b}{K}\frac{d}{dr}Z_a^{(-)},\quad a\neq b=1,2. \tag{19}$$

The case $l=1$ has again to be considered separately, but only the subcase $a=1$ is relevant. In this case our expression for $V_1^{(+)}(r)$ formally diverges. A consistent differential equation for $Z_1^{(+)}$ for $l=1$ can, however, be found in [3] and [22]:

$$\frac{d}{dr}\left[F(r)\frac{d}{dr}\right]Z_1^{(+)}+\frac{\sigma^2}{F(r)}Z_1^{(+)}=\frac{2}{(3Mr-2q^2)^2}\left(9M^2-\frac{18M^2q^2}{r^2}+\frac{16Mq^4}{r^3}-\frac{4q^6}{r^4}\right)Z_1^{(+)}. \tag{20}$$

According to [13] one fundamental solution of Eq. (20) for $\sigma=0$ reads

$$\overline{Z_1^{(+)}}(r)=\frac{3M^2r^3-6Mq^2r^2+(4q^2-M^2)q^2r}{3Mr-2q^2}, \tag{21}$$

simplifying to $\overline{Z_1^{(+)}}(r)=Mr^2$ for $q^2=0$. The other fundamental solution results in analogy to Eq. (6), or by integrating the corresponding d'Alembert formula, as

$$\overline{\overline{Z_1^{(+)}}}(r)=\overline{Z_1^{(+)}}(r)S(r;M,q)-\frac{3Mr(Mr+M^2-2q^2)}{3Mr-2q^2}, \tag{22}$$

and it simplifies to $\overline{\overline{Z_1^{(+)}}}(r)=-Mr[1+(r/2M)\log(1-2M/r)]$ for $q^2=0$.

If one is interested in the explicit form of all (gauge-dependent) components $h_{\mu\nu}$ of the metric perturbations, and the components $f_{\mu\nu}$ of the electromagnetic perturbations, these can be calculated from the gauge invariant perturbations $Z_a^{(\pm)}$ by formulas which can be found in all detail, e.g., in [22].

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