Axially symmetric rotating traversable wormholes

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This paper generalizes the static and spherically symmetric traversable wormhole geometry to a rotating axially symmetric one with a time-dependent angular velocity by means of an exact solution. It was found that the violation of the weak energy condition, although unavoidable, is considerably less severe than in the static spherically symmetric case. The radial tidal constraint is more easily met due to the rotation. Similar improvements are seen in one of the lateral tidal constraints. The magnitude of the angular velocity may have little effect on the weak energy condition violation for an axially symmetric wormhole. For a spherically symmetric one, however, the violation becomes less severe with increasing angular velocity. The time rate of change of the angular velocity, on the other hand, was found to have no effect at all. Finally, the angular velocity must depend only on the radial coordinate, confirming an earlier result.

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I. INTRODUCTION

It was recognized by Flamm [1] in 1916 that our Universe may not be simply connected: there may exist handles or tunnels, now called wormholes, in the spacetime topology linking widely separated regions of our Universe or even connecting us with different universes altogether. That such wormholes may be traversable by humanoid travelers was first conjectured by Morris and Thorne [2], thereby suggesting that interstellar travel and even time travel may some day be possible. For a detailed discussion see the book by Visser [3].

Morris-Thorne (MT) wormholes are static and spherically symmetric and connect asymptotically flat spacetimes, here assumed to be isometric. Adopting units in which $c=G=1$, the metric for this wormhole is given by

$$ds^2=-e^{2\Phi(r)}\,dt^2+\frac{dr^2}{1-b(r)/r}+r^2(d\theta^2+\sin^2\theta\,d\phi^2); \quad (1)$$

$\Phi(r)$ is called the *redshift function* and $b(r)$ the *shape function*. The shape function describes the spatial shape of the wormhole when viewed, for example, in an embedding diagram, described below.

To hold such a wormhole open, violations of certain energy conditions proved to be unavoidable. More precisely, all known forms of matter obey the weak energy condition (WEC) $T_{\alpha\beta}\mu^\alpha\mu^\beta\geqslant 0$ for all timelike vectors and, by continuity, all null vectors (Friedman [4]). Matter that violates this condition is called *exotic* by Morris and Thorne.

Various attempts have been made to generalize the MT wormhole by giving up spherical symmetry [3] or by including time dependence. A particular interesting example of the latter is the inclusion of a de Sitter scale factor multiplying the spatial part of the metric. The goal was to study the possibility of enlarging a wormhole pulled out of the spacetime foam to macroscopic size (Roman [5]). A similar scale factor was used by Kim [6]. Yet another possibility is the use of a conformal factor $\Omega(t)$ [7–10]:

$$ds^2=\Omega(t)[-e^{2\Phi(r)}\,dt^2+e^{2\Lambda(r)}\,dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)]. \quad (2)$$

For a metric with time-dependent functions Φ and Λ, see Ref. [11]. All these studies include a discussion of the WEC violation. Traversability conditions are investigated in [7] and [11].

In this paper we generalize the MT wormhole in another direction by assuming the wormhole to be rotating, not necessarily at a constant rate, and by dropping the assumption of spherical symmetry. Instead, the wormhole is assumed to be axially symmetric, i.e., symmetric with respect to the axis of rotation. Stationary axially symmetric wormholes are discussed by Teo [12]. It is shown that the WEC is indeed violated but that a traveler would not necessarily come into contact with any of the exotic matter. These wormholes may also have an ergoregion, where a particle cannot remain stationary with respect to spatial infinity. This is an extreme example of the well-known dragging effect in general relativity.

The main purpose of this paper is to discuss both the energy violation and the traversability conditions by first finding an exact solution. Possible restrictions on the metric coefficients recently proposed by Perez Bergliaffa and Hibberd [13] are discussed in Sec. VII.

Proposals to search for naturally occurring wormholes, if they exist, go back at least to 1995 [14]. For a summary of these findings see Ref. [15]. While definite conclusions are still lacking, the possible existence of wormholes or the existence of negative mass cannot be ruled out.

II. THE METRIC

The study of spacetimes that are both stationary and axially symmetric has a long history [16–18]. A spacetime is stationary if it possesses a timelike Killing vector field $\eta^a=(\partial/\partial t)^a$ generating invariant time translations. Axially symmetric is formally defined as possessing a spacelike Killing vector field $\xi^a=(\partial/\partial\phi)^a$ generating invariant rotations with respect to ϕ.

Suitable metrics for stationary axially symmetric fields are discussed by Islam [19]. We will adopt the metric suggested by Teo [12], since it appears to be best-suited for obtaining an exact solution:

$$ds^2 = -N^2\,dt^2 + e^{\mu}\,dr^2 + r^2K^2[d\theta^2 + \sin^2\theta(d\phi - \omega\,dt)^2], \tag{3}$$

where N, μ, K, and ω are all functions of r and θ; ω is the angular velocity $d\phi/dt$. More precisely,

$$\omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{u^{\phi}}{u^{t}},$$

referred to in Ref. [20] as the "angular velocity relative to the asymptotic rest frame." The connection between the metric due to a bounded rotating source and the mass and angular momentum of the source is discussed in Ref. [19].

To make our solution as general as possible, we will assume that $\omega=\omega(r,\theta,t)$ is time dependent, so that the wormhole can no longer be called stationary. The reason for proposing this model is a practical one: if an advanced civilization were to succeed in constructing such a wormhole, it is likely to be aspherical and the rate of rotation likely to be varied. Accordingly, we will write our line element as follows:

$$ds^2 = -e^{2\lambda(r,\theta)}\,dt^2 + e^{2\mu(r,\theta)}\,dr^2 + [K(r,\theta)]^2 \times r^2[d\theta^2 - \sin^2\theta(d\phi - \omega(r,\theta,t)dt)^2]. \tag{4}$$

Here $K(r,\theta)$ is a positive dimensionless function of r such that Kr determines the proper radial distance at (r,θ) in the usual manner. In other words, $2\pi(Kr)\sin\theta$ is the proper circumference of the circle through (r,θ).

So far nothing has been said about the shape function. Recall that the wormhole geometry may be conveniently described by means of an embedding diagram in three-dimensional Euclidean space at a fixed moment in time and for a fixed value of θ, the equatorial slice $\theta=\pi/2$ [2,21]. The resulting surface of revolution has the parametric form

$$f(r,\phi) = (r\cos\phi, r\sin\phi, z(r,\theta_1)),$$

where $z=z(r,\theta)$ is some function of r and θ and θ is momentarily held fixed at θ_1. As usual, we think of the surface as connecting two asymptotically flat universes. The radial coordinate decreases from $+\infty$ in the "upper" universe to a minimum value $r=r_0$ at the throat, and then increases again to $+\infty$ in the "lower" universe.

Since the embedding surface must have a vertical tangent at the throat for any value of θ, we require that

$$\lim_{r\to r_0+}\frac{dz}{dr} = +\infty,$$

while $\lim_{r\to\infty} dz/dr = 0$, the meaning of asymptotic flatness. Returning to the line element (4), we further assume that for any fixed θ, $\mu(r,\theta)$ has a vertical asymptote at $r=r_0$: $\lim_{r\to r_0+}\mu(r,\theta)=+\infty$. Also, $\mu(r,\theta)$ is a twice differentiable function of r and θ [as is $\lambda(r,\theta)$] and is a strictly decreasing function of r with $\lim_{r\to\infty}\mu(r,\theta)=0$.

These requirements are met by $z=z(r,\theta)$ for any fixed θ such that

$$\frac{dz}{dr} = \sqrt{e^{2\mu(r,\theta)} - 1}$$

(for the upper universe). Furthermore, $d^2z/dr^2 < 0$ near the throat [since $d\mu(r,\theta_1)/dr < 0$], as required by the "flaring out" condition in Ref. [2]. The shape function is now defined by

$$e^{2\mu(r,\theta)} = \frac{1}{1 - \dfrac{b(r,\theta)}{r}}.$$

It follows that

$$b(r,\theta) = r(1 - e^{-2\mu(r,\theta)}).$$

Finally, at the throat itself, b must be independent of θ. It is readily checked that $\partial b/\partial\theta = 0$ at $r=r_0$.

III. THE SOLUTION

Let us write the line element (4) in slightly more compact form:

$$ds^2 = -e^{2\lambda}\,dt^2 + e^{2\mu}\,dr^2 + K^2r^2[d\theta^2 + \sin^2\theta(d\phi - \omega\,dt)^2]. \tag{5}$$

To make the analysis tractable, we choose an orthonormal basis $\{e_{\hat{\alpha}}\}$ which is dual to the following 1-form basis:

$$\theta^0 = e^{\lambda}\,dt, \quad \theta^1 = e^{\mu}\,dr, \quad \theta^2 = Kr\,d\theta \tag{6}$$

and

$$\theta^3 = Kr\,\sin\theta(d\phi - \omega dt). \tag{7}$$

(See also Ref. [23].) As a result,

$$dt = e^{-\lambda}\theta^0, \quad dr = e^{-\mu}\theta^1, \quad d\theta = \frac{1}{Kr}\theta^2, \tag{8}$$

and

$$d\phi = \frac{1}{Kr\sin\theta}\theta^3 + \omega e^{-\lambda}\theta^0. \tag{9}$$

Furthermore,

$$ds^2 = -(d\theta^0)^2 + (d\theta^1)^2 + (d\theta^2)^2 + (d\theta^3)^2.$$

Since the orthonormal basis is itself rotating, some information regarding ω is going to be lost. We will briefly return to this topic at the end of Sec. VI. (In particular, it will be shown that the magnitude of ω must be restricted.) To do so, we need the components of the fundamental metric tensor $\{g_{\alpha\beta}\}$, as well as $\{g^{\alpha\beta}\}$, in the (t,r,θ,ϕ)-coordinate system:

$$g_{tt}=-e^{2\lambda}+K^2r^2\omega^2\sin^2\theta, \quad g_{t\phi}=-K^2r^2\omega\sin^2\theta,$$
$$g_{rr}=e^{2\mu}, \quad g_{\theta\theta}=K^2r^2, \quad g_{\phi\phi}=K^2r^2\sin^2\theta, \tag{10}$$

and

$$g^{tt}=-e^{-2\lambda}, \quad g^{t\phi}=-\omega e^{-2\lambda}, \quad g^{rr}=e^{-2\mu},$$
$$g^{\theta\theta}=\frac{1}{K^2r^2}, \quad g^{\phi\phi}=\frac{1}{K^2r^2\sin^2\theta}-\omega^2e^{-2\lambda}. \tag{11}$$

The last component, $g^{\phi\phi}$, bears a striking similarity to $d\phi$ in Eq. (9).

To obtain the curvature 2-forms and the components of the Riemann curvature tensor, we use the method of differential forms (Ref. [22]). To that end we calculate the following exterior derivatives in terms of θ^i:

$$d\theta^0=\frac{\partial\lambda}{\partial r}e^{-\mu}\theta^1\wedge\theta^0+\frac{1}{Kr}\frac{\partial\lambda}{\partial\theta}\theta^2\wedge\theta^0,$$

$$d\theta^1=\frac{1}{Kr}\frac{\partial\mu}{\partial\theta}\theta^2\wedge\theta^1,$$

$$d\theta^2=\left(\frac{1}{r}e^{-\mu}+\frac{1}{K}\frac{\partial K}{\partial r}e^{-\mu}\right)\theta^1\wedge\theta^2,$$

and

$$d\theta^3=-Kr\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta\,\theta^1\wedge\theta^0-\frac{\partial\omega}{\partial\theta}e^{-\lambda}\sin\theta\,\theta^2\wedge\theta^0$$
$$+\left(\frac{1}{r}e^{-\mu}+\frac{1}{K}\frac{\partial K}{\partial r}e^{-\mu}\right)\theta^1\wedge\theta^3$$
$$+\left(\frac{1}{Kr}\cot\theta+\frac{1}{K^2r}\frac{\partial K}{\partial\theta}\right)\theta^2\wedge\theta^3.$$

The connection 1-forms $\omega^i{}_k$ have the symmetry

$$\omega^0{}_i=\omega^i{}_0 \quad (i=1,2,3), \quad \text{and} \quad \omega^i{}_j=-\omega^j{}_i \quad (i,j=1,2,3, i\neq j)$$

and are related to the basis θ^i by

$$d\theta^i=-\omega^i{}_k\wedge\theta^k.$$

The solution of this system is found to be

$$\omega^0{}_1=\frac{1}{2}Kr\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta\,\theta^3+\frac{\partial\lambda}{\partial r}e^{-\mu}\theta^0,$$

$$\omega^0{}_2=\frac{1}{Kr}\frac{\partial\lambda}{\partial\theta}\theta^0+\frac{1}{2}\frac{\partial\omega}{\partial\theta}e^{-\lambda}\sin\theta\,\theta^3,$$

$$\omega^0{}_3=\frac{1}{2}Kr\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta\,\theta^1+\frac{1}{2}\frac{\partial\omega}{\partial\theta}e^{-\lambda}\sin\theta\,\theta^2,$$

$$\omega^1{}_2=\frac{1}{Kr}\frac{\partial\mu}{\partial\theta}\theta^1-\left(\frac{1}{r}e^{-\mu}+\frac{1}{K}\frac{\partial K}{\partial r}e^{-\mu}\right)\theta^2,$$

$$\omega^1{}_3=\frac{1}{2}Kr\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta\,\theta^0-\left(\frac{1}{r}e^{-\mu}+\frac{1}{K}\frac{\partial K}{\partial r}e^{-\mu}\right)\theta^3,$$

$$\omega^2{}_3=\frac{1}{2}\frac{\partial\omega}{\partial\theta}e^{-\lambda}\sin\theta\,\theta^0-\left(\frac{1}{Kr}\cot\theta+\frac{1}{K^2r}\frac{\partial K}{\partial\theta}\right)\theta^3.$$

The curvature 2-forms $\Omega^i{}_j$ are calculated directly from the Cartan structural equations

$$\Omega^i{}_j=d\omega^i{}_j+\omega^i{}_k\wedge\omega^k{}_j.$$

The results are given in the Appendix. Since the components of the Riemann curvature tensor can be read off directly using the formula

$$\Omega^i{}_j=-\frac{1}{2}R_{mnj}{}^i\theta^m\wedge\theta^n,$$

there is no need to list them explicitly. As an example, suppose we let $m=0$ and $n=1$ in the equation

$$\Omega^0{}_1=-\frac{1}{2}R_{mn1}{}^0\theta^m\wedge\theta^n.$$

Then

$$\Omega^0{}_1=-\frac{1}{2}R_{011}{}^0\theta^0\wedge\theta^1-\frac{1}{2}R_{101}{}^0\theta^1\wedge\theta^0$$
$$=-R_{011}{}^0\theta^0\wedge\theta^1=R_{011}{}^0\theta^1\wedge\theta^0.$$

Thus $R_{011}{}^0=A(1,0)$ in the Appendix. (As in Ref. [22], we omit the hats whenever numerical indices are used.)

IV. WEC VIOLATION

As with any traversable wormhole, we would expect a violation of the weak energy condition near the throat: $T_{\hat{\alpha}\hat{\beta}}\mu^{\hat{\alpha}}\mu^{\hat{\beta}}\geqslant 0$ for all null vectors. As in Morris and Thorne [2] and Roman [5] we use a radial outgoing null vector $\mu^{\hat{\alpha}}=(\mu^{\hat{t}},\mu^{\hat{r}},0,0)=(1,1,0,0)$. In our orthonormal frame we expect to have the usual stress-energy components $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, $T_{\hat{\theta}\hat{\theta}}$, $T_{\hat{\phi}\hat{\phi}}$, as well as $T_{\hat{t}\hat{\phi}}$, which represents the rotation of the matter distribution [12].

From $G_{\hat{\alpha}\hat{\beta}}=R_{\hat{\alpha}\hat{\beta}}-\frac{1}{2}Rg_{\hat{\alpha}\hat{\beta}}$ and $R_{ab}=R_{acb}{}^c$, we have $G_{00}+G_{11}=R_{00}+R_{11}=R_{011}{}^0+R_{022}{}^0+R_{033}{}^0-R_{011}{}^0-R_{122}{}^1-R_{133}{}^1$. A short calculation yields

$$8\pi(T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}})=R_{\hat{t}\hat{t}}+R_{\hat{r}\hat{r}}=\frac{2}{r}e^{-2\mu}\left(\frac{\partial\lambda}{\partial r}+\frac{\partial\mu}{\partial r}\right)+\frac{2}{K}\frac{\partial K}{\partial r}e^{-2\mu}\left(\frac{\partial\lambda}{\partial r}+\frac{\partial\mu}{\partial r}\right)+\frac{1}{K^2r^2}\left(\frac{\partial\lambda}{\partial\theta}-\frac{\partial\mu}{\partial\theta}\right)\cot\theta+\frac{1}{K^2r^2}\left(\frac{\partial^2\lambda}{\partial\theta^2}-\frac{\partial^2\mu}{\partial\theta^2}\right)+\frac{1}{K^2r^2}\left[\left(\frac{\partial\lambda}{\partial\theta}\right)^2-\left(\frac{\partial\mu}{\partial\theta}\right)^2\right]+\left(-\frac{4}{Kr}\frac{\partial K}{\partial r}e^{-2\mu}\right)+\left(-\frac{2}{K}\frac{\partial^2 K}{\partial r^2}e^{-2\mu}\right)+\left[-\frac{1}{2}\left(\frac{\partial\omega}{\partial\theta}\right)^2e^{-2\lambda}\sin^2\theta\right]. \quad (12)$$

(We have used the Einstein field equations $G_{\hat{\alpha}\hat{\beta}}=8\pi T_{\hat{\alpha}\hat{\beta}}$.)

To put this rather long expression in perspective, consider the static spherical case [11], where only the first term survives:

$$T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}}=\rho^2-\tau=\frac{1}{8\pi}\left(\frac{2}{r}\right)e^{-2\mu}\left(\frac{d\lambda}{dr}+\frac{d\mu}{dr}\right). \quad (13)$$

Referring to Sec. II, recall that the throat corresponds to the value $r=r_0$. So if μ is a smooth function of r and given that $\lim_{r\to r_0+}\mu(r)=+\infty$, it follows that $\lim_{r\to r_0+}d\mu/dr=-\infty$. In Ref. [11], $\lambda(r)=-\kappa/r$, $\kappa>0$, so that $d\lambda/dr=\kappa/r^2$, making $\rho^2-\tau$ negative near the throat. That a violation of the WEC cannot be avoided regardless of the choice of $\lambda(r)$ can be seen geometrically. For if $\lim_{r\to r_0+}d\lambda/dr$ is positive and finite, then the sum on the right side of Eq. (13) is negative near the throat. This might be avoided if $\lim_{r\to r_0+}d\lambda/dr=+\infty$. But then $\lim_{r\to r_0+}\lambda(r)=-\infty$ and $e^{2\lambda(r)}\to 0$, which yields an event horizon.

The main goal in this section is to show that for a rotating axially symmetric wormhole the WEC violation is, in principle, much less severe than for the static spherically symmetric case by a suitable choice of λ and μ. The functions were chosen primarily for convenience, keeping the analysis simple and accommodating the next section at the same time.

Taking k, A, and B to be constants, let

$$\lambda(r,\theta)=-\frac{k}{r}\left[-\frac{1}{3}\left(\frac{\pi}{2}-\theta\right)^3+A\right], \quad 0<\theta\leq\frac{\pi}{2}, \quad k>0, \quad (14)$$

where A is large enough to keep the expression inside the brackets positive, and

$$\mu(r,\theta)=\frac{k\epsilon}{r-r_0}\left[\left(\frac{\pi}{2}-\theta\right)+B\right], \quad 0<\theta\leq\frac{\pi}{2}, \quad k,B>0, \quad (15)$$

where $\epsilon>0$ is a small constant. On the interval $[\pi/2,\pi)$ each function is defined to be the "mirror image," i.e., λ and μ are symmetric about $\theta=\pi/2$. So the discussion may be confined to the interval $(0,\pi/2]$.

The partial derivatives are listed next for easy reference:

$$\frac{\partial\lambda}{\partial r}=\frac{k}{r^2}\left[-\frac{1}{3}\left(\frac{\pi}{2}-\theta\right)^3+A\right],$$

$$\frac{\partial\mu}{\partial r}=-\frac{k\epsilon}{(r-r_0)^2}\left[\left(\frac{\pi}{2}-\theta\right)+B\right],$$

$$\frac{\partial\lambda}{\partial\theta}=-\frac{k}{r}\left(\frac{\pi}{2}-\theta\right)^2,$$

$$\frac{\partial\mu}{\partial\theta}=-\frac{k\epsilon}{r-r_0},$$

$$\frac{\partial^2\lambda}{\partial\theta^2}=\frac{2k}{r}\left(\frac{\pi}{2}-\theta\right),$$

$$\frac{\partial^2\mu}{\partial\theta^2}=0.$$

On the right side of Eq. (12), the first two terms are similar to those in Eq. (13). The first term is therefore negative. The fourth term is strictly positive for $\theta\neq\pi/2$, while the third term is positive near the throat. In the fifth term the expression

$$\left(\frac{\partial\lambda}{\partial\theta}\right)^2-\left(\frac{\partial\mu}{\partial\theta}\right)^2=\frac{k^2}{r^2}\left(\frac{\pi}{2}-\theta\right)^4-\frac{k^2\epsilon^2}{(r-r_0)^2}$$

is similar to the first term but the quadratic factor ϵ^2 reduces the size of the second term, thereby making the fifth term less harmful.

To study the effect of $K(r,\theta)$, we adopt a function similar to one suggested by Teo [12],

$$K(r,\theta)=1+\frac{(4a\sin\theta)^2}{r}.$$

Now the second and sixth terms are positive, as well. The seventh term is close to zero near the throat, thanks to the factor $e^{-2\mu}$, and so is completely overshadowed by the third and fourth terms.

The last term is more of a problem, being strictly negative. However, for Teo's choice, $\omega=2a/r^3$, where a is the total angular momentum of the wormhole, the last term is zero. Requiring ω to be independent of θ may be unavoidable, a conclusion also reached by Khatsymovsky [23], who states that for a macroscopic wormhole to exist, the angular velocity must be independent of θ.

With the last term eliminated, we see that a drastic reduction in the energy condition violation is indeed possible, at least in principle.

Remark. We will see in the next section that the absolute value of the first term exceeds that of the second term. Otherwise the WEC violation would appear to have been eliminated completely.

V. TRAVERSABILITY CONDITIONS

Another area in which improvements over the static spherically symmetric case are possible is in the study of tidal constraints. In particular, for an infalling radial observer the components of the Riemann curvature tensor are found relative to the following orthonormal basis (from the usual Lorentz transformations):

$$e_{\hat{0}'} = \gamma e_{\hat{t}} \mp \gamma\left(\frac{v}{c}\right)e_{\hat{r}}, \quad e_{\hat{1}'} = \mp \gamma e_{\hat{r}} + \gamma\left(\frac{v}{c}\right)e_{\hat{t}},$$
$$e_{\hat{2}'} = e_{\hat{\theta}}, \quad e_{\hat{3}'} = e_{\hat{\phi}}. \tag{16}$$

A traveler should not experience any tidal forces larger than those on Earth. As outlined in Ref. [2], the radial tidal constraint is given by

$$|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| \leq \frac{g_{\oplus}}{c^2 \times 2\text{ m}} \approx \frac{1}{(10^8\text{ m})^2},$$

assuming an observer 2 m tall. We have

$$|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| = |R_{\hat{r}\hat{t}\hat{r}\hat{t}}| = \Bigg| e^{-2\mu}\left[\frac{\partial^2\lambda}{\partial r^2} - \frac{\partial\lambda}{\partial r}\frac{\partial\mu}{\partial r} + \left(\frac{\partial\lambda}{\partial r}\right)^2\right] + \left(-\frac{3}{4}\right)K^2r^2\left(\frac{\partial\omega}{\partial r}\right)^2 e^{-2\lambda}e^{-2\mu}\sin^2\theta + \frac{1}{K^2r^2}\frac{\partial\lambda}{\partial\theta}\frac{\partial\mu}{\partial\theta}\Bigg|. \tag{17}$$

In the static spherically symmetric case only the first term survives. The second term, which involves the angular velocity, reduces the size of $|R_{\hat{r}\hat{t}\hat{r}\hat{t}}|$: the first term,

$$e^{-2\mu}\left[\frac{\partial^2\lambda}{\partial r^2} + \left(-\frac{\partial\lambda}{\partial r}\frac{\partial\mu}{\partial r}\right) + \left(\frac{\partial\lambda}{\partial r}\right)^2\right],$$

is positive near the throat because the positive middle term inside the brackets contains the factor $(r-r_0)^2$ in the denominator. For the same reason the first term is larger than the absolute value of the second (near the throat). Since the second term is negative, the net result, so far, is a reduction in the size of $|R_{\hat{r}\hat{t}\hat{r}\hat{t}}|$.

Concerning the last term, we need to remember that our wormhole is not spherically symmetric. The tidal forces experienced may therefore depend on the direction of approach. So we must ask the traveler to approach the throat in the equatorial plane $\theta = \pi/2$, and here $\partial\lambda/\partial\theta = 0$.

To study the first of the lateral tidal constraints, $|R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| \leq (10^8\text{ m})^{-2}$, we have from Eq. (16),

$$|R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| = \gamma^2|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}| + \gamma^2\left(\frac{v}{c}\right)^2|R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}}|.$$

As usual, this is a constraint on the velocity of the traveler. It is assumed in Ref. [2] that the spaceship decelerates until it comes to rest at the throat. So only the first term needs to be examined:

$$|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}| = |R_{022}{}^0| = \Bigg|\frac{1}{r}\frac{\partial\lambda}{\partial r}e^{-2\mu} + \frac{1}{K^2r^2}\frac{\partial^2\lambda}{\partial\theta^2} + \frac{1}{K^2r^2}\left(\frac{\partial\lambda}{\partial\theta}\right)^2 - \frac{1}{K^3r^2}\frac{\partial K}{\partial\theta}\frac{\partial\lambda}{\partial\theta} + \frac{1}{K}\frac{\partial K}{\partial r}\frac{\partial\lambda}{\partial r}e^{-2\mu} - \frac{3}{4}\left(\frac{\partial\omega}{\partial\theta}\right)^2 e^{-2\lambda}\sin^2\theta\Bigg|. \tag{18}$$

In the static spherically symmetric case only the positive first term survives. Once again requiring that the traveler approach the throat in the equatorial plane $\theta = \pi/2$, the next three terms are zero. The last term is also zero due to the earlier requirement $\partial\omega/\partial\theta = 0$. The fourth term is negative, and for our choice of $K(r,\theta)$

$$\left|\frac{1}{K}\frac{\partial K}{\partial r}\frac{\partial\lambda}{\partial r}e^{-2\mu}\right| < \left|\frac{1}{r}\frac{\partial\lambda}{\partial r}e^{-2\mu}\right|.$$

We therefore have a reduction in the size of $|R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}}|$.

For the remaining lateral tidal constraint we need to examine

$$|R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}}| = |R_{033}{}^0| = \Bigg|\frac{1}{r}\frac{\partial\lambda}{\partial r}e^{-2\mu} + \frac{1}{K}\frac{\partial K}{\partial r}\frac{\partial\lambda}{\partial r}e^{-2\mu} + \frac{1}{4}K^2r^2\left(\frac{\partial\omega}{\partial r}\right)^2 e^{-2\lambda}e^{-2\mu}\sin^2\theta + \frac{1}{K^2r^2}\frac{\partial\lambda}{\partial\theta}\cot\theta + \frac{1}{K^3r^2}\frac{\partial K}{\partial\theta}\frac{\partial\lambda}{\partial\theta} + \frac{1}{4}\left(\frac{\partial\omega}{\partial\theta}\right)^2 e^{-2\lambda}\sin^2\theta\Bigg|. \tag{19}$$

The first two terms appeared in the other lateral constraint; the result is a reduction in the size of $|R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}}|$. Unfortunately, the next term is positive. Although small due to the factor $e^{-2\mu}$, the overall result is hard to quantify and may actually be less favorable than the corresponding static case. (It helps that the next two terms are zero for $\theta = \pi/2$, while the last term vanishes due to the requirement $\partial\omega/\partial\theta = 0$.)

VI. THE EFFECT OF ROTATION

In proposing a model with a time-dependent ω, i.e., an angular acceleration or deceleration, it was hoped that the findings in the last two sections would be strengthened. As it turns out, however, none of the derivatives with respect to time occurring in the curvature 2-forms found their way into the earlier calculations.

This failure suggests that the effect of rotation be studied from a different perspective. Since we are using a rotating basis, some information regarding ω was lost: no ω's appear in any of the curvature 2-forms, although the derivatives do. So it may be useful to examine the weak energy condition violation relative to the (t,r,θ,ϕ)-coordinate system.

The expression for $T_{tt}+T_{rr}$, calculated by the traditional method using the fundamental metric tensor [Eqs. (10) and (11)], contain the old terms in Eq. (12), but not, of course, in the orthonormal frame. The new terms all contain ω:

$$8\pi(T_{tt}+T_{rr})=\text{old terms}+R_{\text{new}},$$

where

$$\begin{aligned}R_{\text{new}}=&-\frac{1}{2}\omega K^3r^4\frac{\partial K}{\partial r}\frac{\partial\omega}{\partial r}e^{-2\lambda}e^{-2\mu}\sin^2\theta\\&+\left(-\frac{1}{2}\right)\omega K^4r^3\frac{\partial\omega}{\partial r}e^{-2\lambda}e^{-2\mu}\sin^2\theta\\&-\omega Kr^2\frac{\partial K}{\partial\theta}\frac{\partial\omega}{\partial\theta}e^{-2\lambda}\sin^2\theta\\&-\frac{1}{2}\omega K^2r^2\frac{\partial^2\omega}{\partial\theta^2}e^{-2\lambda}\sin^2\theta\\&+\frac{1}{2}\omega K^2r^2\frac{\partial\omega}{\partial\theta}\frac{\partial\lambda}{\partial\theta}e^{-2\lambda}\sin^2\theta\\&-\frac{3}{2}\omega K^2r^2\frac{\partial\omega}{\partial\theta}e^{-2\lambda}\sin\theta\cos\theta.\end{aligned}$$

R_{new} obviously vanishes if ω does. The last four terms vanish if we assume, as before, that $\partial\omega/\partial\theta=0$. With the functions used earlier, the first term is negative and the second positive. Unless $\partial K/\partial r$ is very small, this result is, once again, hard to quantify.

The situation is rather different if we return to the assumption of spherical symmetry, while retaining ω, assumed to be positive. Then $K=1$ and λ and μ are independent of θ. In the (t,r,θ,ϕ)-coordinate system, using the outgoing null vector $(1,1,0,0)$, we have

$$\begin{aligned}\rho^2-\tau=&re^{-2\mu}\left(\frac{\partial\lambda}{\partial r}+\frac{\partial\lambda}{\partial r}\sin^2\theta\right)+re^{-2\mu}\left(\frac{\partial\mu}{\partial r}+\frac{\partial\mu}{\partial r}\sin^2\theta\right)\\&+\left(-\frac{1}{2}\right)\omega r^3\frac{\partial\omega}{\partial r}e^{-2\lambda}e^{-2\mu}\sin^2\theta.\end{aligned}\tag{20}$$

Since $\omega>0$, the last term is positive. So if ω is large, the WEC violation is much reduced.

While the resulting reduction in the WEC violation is a welcome surprise, some words of caution are in order. The last term in Eq. (20) suggests that if ω is large enough, the WEC violation can be eliminated altogether. But as explained by Teo [12], for a rapidly rotating wormhole it may not be possible to use a radially outgoing null vector since the g_{tt} component of the fundamental metric tensor may no longer be negative, as can be seen from the line element, Eq. (5).

VII. ADDITIONAL CONSIDERATIONS

It was pointed out in a recent paper by Perez Bergliaffa and Hibberd [13] that the metric (4) used in this paper may require further restrictions. In particular, it is shown that a wormhole of the type studied by Teo [12] cannot be generated by a perfect fluid or by a fluid with anisotropic stresses. In the first case the condition

$$G_{12}=0\tag{21}$$

is violated and in the second case,

$$G_{00}+G_{33}\geqslant 2G_{03}.\tag{22}$$

These conditions seem to be met if the metric (4) includes the functions λ and μ used in this paper, that is, Eqs. (14) and (15), respectively.

For example, to check condition (21), a short calculation yields

$$\begin{aligned}G_{12}=&\frac{1}{2}Kr\frac{\partial\omega}{\partial r}\frac{\partial\omega}{\partial\theta}e^{-2\lambda}e^{-\mu}\sin^2\theta+\frac{1}{Kr^2}\left(\frac{\partial\lambda}{\partial\theta}+\frac{\partial\mu}{\partial\theta}\right)e^{-\mu}\\&+\frac{1}{K^2r}\frac{\partial K}{\partial r}\left(\frac{\partial\lambda}{\partial\theta}+\frac{\partial\mu}{\partial\theta}\right)e^{-\mu}-\frac{1}{Kr}\frac{\partial\lambda}{\partial r}\left(\frac{\partial\lambda}{\partial\theta}-\frac{\partial\mu}{\partial\theta}\right)e^{-\mu}\\&-\frac{1}{Kr}\frac{\partial^2\lambda}{\partial r\,\partial\theta}e^{-\mu}-\frac{1}{K^2r}\frac{\partial^2K}{\partial r\,\partial\theta}e^{-\mu}+\frac{1}{K^3r}\frac{\partial K}{\partial r}\frac{\partial K}{\partial\theta}e^{-\mu}.\end{aligned}$$

Each term contains $e^{-\mu}$, so that $G_{12}\approx 0$ near the throat.

Even more favorable is the outcome of the check on condition (22): five of the terms in G_{03} contain the factor $\partial\omega/\partial\theta$, which is equal to zero. The remaining terms all contain the factor $e^{-2\mu}$, so that $G_{03}\approx 0$ near the throat. The left side, $G_{00}+G_{33}$, contains only two terms that are neither strictly positive, nor zero, nor contain the factor $e^{-2\mu}$. But these two terms are strongly overshadowed by several terms that *are* strictly positive. So condition (22) is easily met in the vicinity of the throat.

Since the vicinity of the throat is the only region that really matters, we can construct (in the usual manner) a solution with a suitable radial cutoff of the stress-energy tensor. Our solution can then be joined, at least in principle, to an external solution also satisfying the desired conditions. (For a discussion of the required junction conditions, see Ref. [20].)

It is indeed surprising that λ and μ, which were chosen for entirely different reasons, are sufficient for satisfying conditions (21) and (22), thereby overcoming the objections raised to the wormhole in Teo [12]. Unfortunately, Ref. [13] discusses other conditions, some of which are not so easily checked. So it may still be necessary to "incorporate more realistic (in the astrophysical sense) features, for example heat flux" (Ref. [13]). Future investigations of these issues could be extended to include stability conditions, especially for rotating wormholes, since such conditions may also require "more realistic features."

VIII. CONCLUSION

In this paper the MT wormhole solution was generalized to wormholes which are both rotating and axially symmetric, i.e., symmetric with respect to the axis of rotation. It was concluded that the unavoidable violation of the weak energy condition is less severe than in the spherically symmetric case. The radial tidal constraint is more easily met due to the rotation. An improvement was also found in one of the lateral tidal constraints. Making the angular velocity ω time dependent does not help since none of the time derivatives in the Appendix appeared in the calculations. Furthermore, ω must be independent of θ, in agreement with Ref. [23]. Finally, the magnitude of the angular velocity may have little effect on the WEC violation for an axially symmetric wormhole. In the spherically symmetric case, however, a rapid rotation will result in a reduction in the WEC violation, as long as ω is not excessively large.

APPENDIX

This appendix lists the curvature 2-forms $\Omega^i{}_j$:

$$\Omega^0{}_1 = A(1,0)\,\theta^1\wedge\theta^0 + A(2,0)\,\theta^2\wedge\theta^0 + A(3,0)\,\theta^3\wedge\theta^0 + A(3,1)\,\theta^3\wedge\theta^1 + A(3,2)\,\theta^3\wedge\theta^2,$$

where

$$A(1,0) = e^{-2\mu}\left[\frac{\partial^2\lambda}{\partial r^2} - \frac{\partial\lambda}{\partial r}\frac{\partial\mu}{\partial r} + \left(\frac{\partial\lambda}{\partial r}\right)^2\right] - \frac{3}{4}K^2r^2\left(\frac{\partial\omega}{\partial r}\right)^2 e^{-2\lambda}e^{-2\mu}\sin^2\theta + \frac{1}{K^2r^2}\frac{\partial\lambda}{\partial\theta}\frac{\partial\mu}{\partial\theta},$$

$$A(2,0) = \frac{1}{Kr}e^{-\mu}\left(\frac{\partial^2\lambda}{\partial r\,\partial\theta} - \frac{\partial\lambda}{\partial r}\frac{\partial\mu}{\partial\theta} + \frac{\partial\lambda}{\partial r}\frac{\partial\lambda}{\partial\theta}\right) - \frac{3}{4}Kr\frac{\partial\omega}{\partial r}\frac{\partial\omega}{\partial\theta}e^{-2\lambda}e^{-\mu}\sin^2\theta - \frac{1}{Kr^2}\frac{\partial\lambda}{\partial\theta}e^{-\mu} - \frac{1}{K^2r}\frac{\partial K}{\partial r}\frac{\partial\lambda}{\partial\theta}e^{-\mu},$$

$$A(3,0) = -\frac{1}{2}Kr\frac{\partial}{\partial t}\left(\frac{\partial\omega}{\partial r}\right)e^{-2\lambda}e^{-\mu}\sin\theta,$$

$$A(3,1) = -\frac{3}{2}K\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-2\mu}\sin\theta - \frac{1}{2}Kr\frac{\partial^2\omega}{\partial r^2}e^{-\lambda}e^{-2\mu}\sin\theta + \frac{1}{2}Kr\frac{\partial\omega}{\partial r}\frac{\partial\lambda}{\partial r}e^{-\lambda}e^{-2\mu}\sin\theta + \frac{1}{2}Kr\frac{\partial\omega}{\partial r}\frac{\partial\mu}{\partial r}e^{-\lambda}e^{-2\mu}\sin\theta - \frac{1}{2Kr}\frac{\partial\omega}{\partial\theta}\frac{\partial\mu}{\partial\theta}e^{-\lambda}\sin\theta - \frac{3}{2}r\frac{\partial K}{\partial r}\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-2\mu}\sin\theta,$$

$$A(3,2) = \frac{1}{2}\frac{\partial\omega}{\partial r}\left(\frac{\partial\lambda}{\partial\theta} + \frac{\partial\mu}{\partial\theta}\right)e^{-\lambda}e^{-\mu}\sin\theta - \frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\cos\theta - \frac{1}{K}\frac{\partial K}{\partial\theta}\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta - \frac{1}{2}\frac{\partial^2\omega}{\partial r\,\partial\theta}e^{-\lambda}e^{-\mu}\sin\theta.$$

$$\Omega^0{}_2 = B(1,0)\,\theta^1\wedge\theta^0 + B(2,0)\,\theta^2\wedge\theta^0 + B(3,0)\,\theta^3\wedge\theta^0 + B(3,1)\,\theta^3\wedge\theta^1 + B(3,2)\,\theta^3\wedge\theta^2,$$

where $B(1,0) = A(2,0)$,

$$B(2,0) = \frac{1}{r}\frac{\partial\lambda}{\partial r}e^{-2\mu} + \frac{1}{K^2r^2}\frac{\partial^2\lambda}{\partial\theta^2} + \frac{1}{K^2r^2}\left(\frac{\partial\lambda}{\partial\theta}\right)^2 - \frac{1}{K^3r^2}\frac{\partial K}{\partial\theta}\frac{\partial\lambda}{\partial\theta} + \frac{1}{K}\frac{\partial K}{\partial r}\frac{\partial\lambda}{\partial r}e^{-2\mu} - \frac{3}{4}\left(\frac{\partial\omega}{\partial\theta}\right)^2 e^{-2\lambda}\sin^2\theta,$$

$$B(3,0) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial\theta}\right)e^{-2\lambda}\sin\theta,$$

$$B(3,1) = -\frac{1}{2r}\frac{\partial\omega}{\partial\theta}e^{-\lambda}e^{-\mu}\sin\theta - \frac{1}{2}\frac{\partial^2\omega}{\partial r\partial\theta}e^{-\lambda}e^{-\mu}\sin\theta + \frac{1}{2}\frac{\partial\omega}{\partial\theta}\frac{\partial\lambda}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta + \frac{1}{2}\frac{\partial\omega}{\partial r}\frac{\partial\mu}{\partial\theta}e^{-\lambda}e^{-\mu}\sin\theta - \frac{1}{2}\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\cos\theta - \frac{1}{2K}\frac{\partial K}{\partial r}\frac{\partial\omega}{\partial\theta}e^{-\lambda}e^{-\mu}\sin\theta - \frac{1}{2K}\frac{\partial K}{\partial\theta}\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta,$$

$$B(3,2) = -\frac{1}{2Kr}\frac{\partial^2\omega}{\partial\theta^2}e^{-\lambda}\sin\theta + \frac{1}{2Kr}\frac{\partial\omega}{\partial\theta}\frac{\partial\lambda}{\partial\theta}e^{-\lambda}\sin\theta - \frac{3}{2Kr}\frac{\partial\omega}{\partial\theta}e^{-\lambda}\cos\theta - \frac{1}{2}K\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-2\mu}\sin\theta - \frac{1}{2}r\frac{\partial K}{\partial r}\frac{\partial\omega}{\partial r}e^{-\lambda}e^{-2\mu}\sin\theta$$

$$-\frac{1}{K^2 r}\frac{\partial K}{\partial \theta}\frac{\partial \omega}{\partial \theta}e^{-\lambda}\sin\theta.$$

$$\Omega^0{}_3 = C(1,0)\,\theta^1\wedge\theta^0 + C(2,0)\,\theta^2\wedge\theta^0 + C(3,0)\,\theta^3\wedge\theta^0 + C(2,1)\,\theta^2\wedge\theta^1,$$

where

$$C(1,0) = -\frac{1}{2}Kr\frac{\partial}{\partial t}\left(\frac{\partial \omega}{\partial r}\right)e^{-2\lambda}e^{-\mu}\sin\theta,$$

$$C(2,0) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \omega}{\partial \theta}\right)e^{-2\lambda}\sin\theta,$$

$$C(3,0) = \frac{1}{r}\frac{\partial \lambda}{\partial r}e^{-2\mu} + \frac{1}{K}\frac{\partial K}{\partial r}\frac{\partial \lambda}{\partial r}e^{-2\mu} + \frac{1}{4}K^2r^2\left(\frac{\partial \omega}{\partial r}\right)^2 e^{-2\lambda}e^{-2\mu}\sin^2\theta + \frac{1}{K^2r^2}\frac{\partial \lambda}{\partial \theta}\cot\theta + \frac{1}{K^3r^2}\frac{\partial K}{\partial \theta}\frac{\partial \lambda}{\partial \theta} + \frac{1}{4}\left(\frac{\partial \omega}{\partial \theta}\right)^2 e^{-2\lambda}\sin^2\theta,$$

$$C(2,1) = -\frac{1}{2}\frac{\partial \omega}{\partial r}\frac{\partial \lambda}{\partial \theta}e^{-\lambda}e^{-\mu}\sin\theta + \frac{1}{2}\frac{\partial \omega}{\partial \theta}\frac{\partial \lambda}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta - \frac{1}{2r}\frac{\partial \omega}{\partial \theta}e^{-\lambda}e^{-\mu}\sin\theta + \frac{1}{2}\frac{\partial \omega}{\partial r}e^{-\lambda}e^{-\mu}\cos\theta - \frac{1}{2K}\frac{\partial K}{\partial r}\frac{\partial \omega}{\partial \theta}e^{-\lambda}e^{-\mu}\sin\theta + \frac{1}{2K}\frac{\partial K}{\partial \theta}\frac{\partial \omega}{\partial r}e^{-\lambda}e^{-\mu}\sin\theta.$$

$$\Omega^1{}_2 = D(3,0)\,\theta^3\wedge\theta^0 + D(2,1)\,\theta^2\wedge\theta^1,$$

where $D(3,0) = -C(2,1)$,

$$D(2,1) = \frac{2}{Kr}\frac{\partial K}{\partial r}e^{-2\mu} - \frac{1}{r}\frac{\partial \mu}{\partial r}e^{-2\mu} + \frac{1}{K^2r^2}\frac{\partial^2 \mu}{\partial \theta^2} + \frac{1}{K^2r^2}\left(\frac{\partial \mu}{\partial \theta}\right)^2 + \frac{1}{K}\frac{\partial^2 K}{\partial r^2}e^{-2\mu} - \frac{1}{K}\frac{\partial K}{\partial r}\frac{\partial \mu}{\partial r}e^{-2\mu} - \frac{1}{K^3r^2}\frac{\partial K}{\partial \theta}\frac{\partial \mu}{\partial \theta}.$$

$$\Omega^1{}_3 = E(1,0)\,\theta^1\wedge\theta^0 + E(2,0)\,\theta^2\wedge\theta^0 + E(3,1)\,\theta^3\wedge\theta^1 + E(3,2)\,\theta^3\wedge\theta^2,$$

where $E(1,0) = -A(3,1)$, $E(2,0) = -B(3,1)$,

$$E(3,1) = \frac{2}{Kr}\frac{\partial K}{\partial r}e^{-2\mu} - \frac{1}{r}\frac{\partial \mu}{\partial r}e^{-2\mu} + \frac{1}{K}\frac{\partial^2 K}{\partial r^2}e^{-2\mu} - \frac{1}{K}\frac{\partial K}{\partial r}\frac{\partial \mu}{\partial r}e^{-2\mu} + \frac{1}{4}K^2r^2\left(\frac{\partial \omega}{\partial r}\right)^2 e^{-2\lambda}e^{-2\mu}\sin^2\theta + \frac{1}{K^2r^2}\frac{\partial \mu}{\partial \theta}\cot\theta + \frac{1}{K^3r^2}\frac{\partial K}{\partial \theta}\frac{\partial \mu}{\partial \theta},$$

$$E(3,2) = -\frac{1}{Kr^2}\frac{\partial \mu}{\partial \theta}e^{-\mu} - \frac{1}{K^2r}\frac{\partial K}{\partial r}\frac{\partial \mu}{\partial \theta}e^{-\mu} + \frac{1}{K^2r}\frac{\partial^2 K}{\partial r\,\partial \theta}e^{-\mu} + \frac{1}{4}Kr\frac{\partial \omega}{\partial r}\frac{\partial \omega}{\partial \theta}e^{-2\lambda}e^{-\mu}\sin^2\theta - \frac{1}{K^3r}\frac{\partial K}{\partial r}\frac{\partial K}{\partial \theta}e^{-\mu}.$$

$$\Omega^2{}_3 = F(1,0)\,\theta^1\wedge\theta^0 + F(2,0)\,\theta^2\wedge\theta^0 + F(3,1)\,\theta^3\wedge\theta^1 + F(3,2)\,\theta^3\wedge\theta^2,$$

where $F(1,0) = -A(3,2)$, $F(2,0) = -B(3,2)$, $F(3,1) = E(3,2)$,

$$F(3,2) = -\frac{1}{K^2r^2} + \frac{1}{r^2}e^{-2\mu} + \frac{2}{Kr}\frac{\partial K}{\partial r}e^{-2\mu} + \frac{1}{K^2}\left(\frac{\partial K}{\partial r}\right)^2 e^{-2\mu} - \frac{1}{K^4r^2}\left(\frac{\partial K}{\partial \theta}\right)^2 + \frac{1}{K^3r^2}\frac{\partial^2 K}{\partial \theta^2} + \frac{1}{K^3r^2}\frac{\partial K}{\partial \theta}\cot\theta + \frac{1}{4}\left(\frac{\partial \omega}{\partial \theta}\right)^2 e^{-2\lambda}\sin^2\theta.$$

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