Geodesics in spacetimes with expanding impulsive gravitational waves

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We study geodesic motion in expanding spherical impulsive gravitational waves propagating in a Minkowski background. Employing the continuous form of the metric we find and examine a large family of geometrically preferred geodesics. For the special class of axially symmetric spacetimes with the spherical impulse generated by a snapping cosmic string we give a detailed physical interpretation of the motion of test particles.

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I. INTRODUCTION

In his classical work [1] Penrose constructed impulsive spherical gravitational waves in a Minkowski background using his vivid ''cut and paste'' method. It is based on cutting the spacetime along a null cone and then reattaching the two pieces with a suitable warp. An explicit solution using coordinates in which the metric is continuous was later on given by Nutku and Penrose [2] and Hogan [3,4], but was only recently related explicitly to the impulsive limit of Robinson-Trautman type N solutions [5,6]. However, the latter has to be considered as only formal since the metric tensor contains terms proportional to the square of the Dirac δ, and the transformation relating this coordinate system to the continuous one mentioned above is necessarily discontinuous. Nevertheless, this transformation is analogous to the one relating the distributional and the continuous form of the metric tensor for impulsive pp-waves (plane-fronted gravitational waves with parallel rays [7]) which was also introduced in [1] and has recently been analyzed *rigorously* [8,9] using nonlinear theories of generalized functions (Colombeau algebras) [10,11]. It is thus a natural open question whether a similar mathematically sound treatment can also be found for expanding spherical impulses. This indeed is one main motivation for the present work in which we study the motion of test particles in spacetimes with spherical impulsive waves.

On the other hand, this work is motivated by the quest of a physical interpretation of radiative Robinson-Trautman spacetimes, one of the most interesting nonstatic exact solutions of Einstein's equations which admit a geodesic, shear-free and twist-free null congruence of diverging rays [7]. This large family involves not only spacetimes of Petrov type N (investigated in the impulsive limit in the present paper) but also type II solutions describing bodies which radiate away their asymmetries and approach a Schwarzschild black hole, or the C-metric of type D which represents gravitational radiation generated by uniformly accelerated black holes. By studying these explicit exact solutions one may acquire an intuition necessary for investigation of more general and realistic situations.

This work is organized as follows. In Sec. II we review the class of spacetimes under consideration and describe the geometry of the expanding impulses. By employing the continuous form of the metric in Sec. III we find a large class of privileged and simple geodesics which can be related to explicit geodesics in the distributional form of the metric ''in front'' and ''behind'' the spherical impulse. This may allow one to lay the foundations for a rigorous (distributional) treatment of impulsive Robinson-Trautman solutions of type N as well as the transformation relating the latter to the continuous form of the metric. Moreover, assuming the geodesics to be C^1 across the impulse (in the continuous system) we completely solve the problem of geodesic motion in spacetimes with expanding impulsive gravitational waves. In Sec. IV we focus on impulsive waves generated by a snapping cosmic string. This interesting solution of Einstein's equations was previously constructed by Gleiser and Pullin [12] and Nutku and Penrose [2] using the ''cut and paste'' method. An independent approach was used by Bičák [13,14] (with recent generalizations in [15]) who obtained the same spacetime by considering a null limit of particular solutions with boost-rotational symmetry representing a pair of particles uniformly accelerating due to semi-infinite strings attached to them. We discuss in detail the physical interpretation of the motion of test particles influenced by such an impulse.

II. EXPANDING IMPULSIVE WAVES IN A MINKOWSKI BACKGROUND

As mentioned above, Penrose [1] has described a ''cut and paste'' method for constructing expanding spherical gravitational waves in a Minkowski background. The procedure can be performed explicitly as follows. One starts with the Minkowski line element

$$ds_0^2 = 2\,d\eta\,d\bar{\eta} - 2\,d\mathcal{U}\,d\mathcal{V} = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.1}$$

where the relation between the coordinates is given by

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$$\mathcal{U}=\frac{1}{\sqrt{2}}(t+z),\quad \mathcal{V}=\frac{1}{\sqrt{2}}(t-z),\quad \eta=\frac{1}{\sqrt{2}}(x+\mathrm{i}y). \tag{2.2}$$

We may now perform the transformation

$$\mathcal{V}=\frac{V}{p}-\epsilon U,\qquad \mathcal{U}=\frac{Z\bar{Z}}{p}V-U,\qquad \eta=\frac{Z}{p}V, \tag{2.3}$$

where

$$p=1+\epsilon Z\bar{Z},\quad \epsilon=-1,0,+1. \tag{2.4}$$

(The parameter ϵ is related to the Gaussian curvature of the 2-surfaces given by $U=\text{const}$, $V=\text{const}$, cf. [5].) Using Eq. (2.3), the metric (2.1) takes the form

$$ds_0^2=2\frac{V^2}{p^2}\,dZ\,d\bar{Z}+2\,dU dV-2\epsilon\,dU^2. \tag{2.5}$$

On the other hand, we consider the alternative, more involved transformation given by

$$\mathcal{V}=AV-DU,\qquad \mathcal{U}=BV-EU,\qquad \eta=CV-FU, \tag{2.6}$$

where

$$A=\frac{1}{p|h'|},\quad B=\frac{|h|^2}{p|h'|},\quad C=\frac{h}{p|h'|},$$

$$D=\frac{1}{|h'|}\left\{\frac{p}{4}\left|\frac{h''}{h'}\right|^2+\epsilon\left[1+\frac{Z}{2}\frac{h''}{h'}+\frac{\bar{Z}}{2}\frac{\bar{h}''}{\bar{h}'}\right]\right\},$$

$$E=\frac{|h|^2}{|h'|}\left\{\frac{p}{4}\left|\frac{h''}{h'}-2\frac{h'}{h}\right|^2+\epsilon\left[1+\frac{Z}{2}\left(\frac{h''}{h'}-2\frac{h'}{h}\right)+\frac{\bar{Z}}{2}\left(\frac{\bar{h}''}{\bar{h}'}-2\frac{\bar{h}'}{\bar{h}}\right)\right]\right\},$$

$$F=\frac{h}{|h'|}\left\{\frac{p}{4}\left(\frac{h''}{h'}-2\frac{h'}{h}\right)\frac{\bar{h}''}{\bar{h}'}+\epsilon\left[1+\frac{Z}{2}\left(\frac{h''}{h'}-2\frac{h'}{h}\right)+\frac{\bar{Z}}{2}\frac{\bar{h}''}{\bar{h}'}\right]\right\}. \tag{2.7}$$

Here $h\equiv h(Z)$ is an arbitrary function, and the derivative with respect to its argument Z is denoted by a prime. With this, the Minkowski metric (2.1) becomes

$$ds_0^2=2\left|\frac{V}{p}\,dZ+Up\bar{H}\,d\bar{Z}\right|^2+2\,dU\,dV-2\epsilon\,dU^2, \tag{2.8}$$

where H is the Schwarzian derivative of h, i.e.,

$$H(Z)=\frac{1}{2}\left[\frac{h'''}{h'}-\frac{3}{2}\left(\frac{h''}{h'}\right)^2\right]. \tag{2.9}$$

In the coordinates used in Eq. (2.5), as well as in the ones used in Eq. (2.8), the null hypersurface $U=0$ represents a null cone $\eta\bar{\eta}-\mathcal{U}\mathcal{V}=0$, i.e., an expanding sphere $x^2+y^2+z^2=t^2$ in the Minkowski background. Moreover, the reduced 2-metrics on this cone are identical. Following the Penrose "cut and paste" method, we attach the line element (2.5) for $U<0$ to the line element Eq. (2.8) for $U>0$. The resulting metric takes the form

$$ds^2=2\left|\frac{V}{p}\,dZ+U\Theta(U)p\bar{H}\,d\bar{Z}\right|^2+2\,dU dV-2\epsilon\,dU^2, \tag{2.10}$$

where $\Theta(U)$ is the Heaviside step function. This combined metric, which was first presented in [2,4], is explicitly continuous everywhere, including the null hypersurface $U=0$. However, the discontinuity in the derivatives of the metric across $U=0$ yields an impulsive gravitational wave term proportional to the Dirac δ function. More precisely, the only non-vanishing component of the Weyl tensor in the Newman-Penrose formalism [7,16] is $\Psi_4=(p^2H/V)\delta(U)$, which is the component $\Psi_4\equiv C_{abcd}l^a\bar{m}^bl^c\bar{m}^d$ with respect to the null tetrad $\mathbf{k}=\partial_V$, $\mathbf{l}=-\epsilon\partial_V-\partial_U$, $\mathbf{m}=p^2(V^2-U^2\Theta p^4H\bar{H})^{-1}[(V/p)\partial_{\bar{Z}}-U\Theta p\bar{H}\partial_Z]$. The spacetime is thus flat everywhere except on the wave surface $U=0$. Also, as shown in [17], the only nonvanishing tetrad component of the Ricci tensor is $\Phi_{22}\equiv\frac{1}{2}R_{ab}l^al^b=(p^4H\bar{H}/V^2)U\delta(U)$. This demonstrates that the spacetime is vacuum everywhere (except on the impulse at $V=0$ and at possible singularities of the function p^2H).

For later use we also remark that the inverse relation to Eq. (2.3) is given by

$$U=\frac{\eta\bar{\eta}}{\mathcal{V}}-\mathcal{U},\qquad V=\mathcal{V},\qquad Z=\frac{\eta}{\mathcal{V}}, \tag{2.11}$$

when $\epsilon=0$, and by

$$U=-\epsilon\mathcal{V}-\frac{2\eta\bar{\eta}}{(\mathcal{V}-\epsilon\mathcal{U})\mp\sqrt{(\mathcal{V}-\epsilon\mathcal{U})^2+4\epsilon\eta\bar{\eta}}},$$

$$V=-(\mathcal{V}-\epsilon\mathcal{U})-\frac{4\epsilon\eta\bar{\eta}}{(\mathcal{V}-\epsilon\mathcal{U})\mp\sqrt{(\mathcal{V}-\epsilon\mathcal{U})^2+4\epsilon\eta\bar{\eta}}},$$

$$Z=-\frac{\epsilon}{2\bar{\eta}}[(\mathcal{V}-\epsilon\mathcal{U})\mp\sqrt{(\mathcal{V}-\epsilon\mathcal{U})^2+4\epsilon\eta\bar{\eta}}], \qquad (2.12)$$

for $\epsilon\neq 0$.

We may define the functions U_{inv}, V_{inv}, and Z_{inv} of $(U,V,Z,\bar{Z})$ as the composition of Eq. (2.11) [or Eqs. (2.12) for $\epsilon\neq 0$] with Eqs. (2.6), (2.7), which transforms the metric (2.8) to (2.5). Consequently, a discontinuous transformation

$$u=U+\Theta(U)[U_{inv}(U,V,Z,\bar{Z})-U],$$

$$w=V+\Theta(U)[V_{inv}(U,V,Z,\bar{Z})-V], \qquad (2.13)$$

$$\xi=Z+\Theta(U)[Z_{inv}(U,V,Z,\bar{Z})-Z],$$

relates Eq. (2.10) for all $U\neq 0$ to Minkowski spacetime in the form

$$ds_0^2=2\frac{w^2}{\psi^2}\,d\xi\,d\bar{\xi}+2\,du\,dw-2\epsilon\,du^2, \qquad (2.14)$$

where $\psi=1+\epsilon\xi\bar{\xi}$. Interestingly, by considering the transformation (2.13) of the metric (2.10) for arbitrary U, i.e., keeping also the distributional terms, one (formally) obtains the metric

$$ds^2=2\frac{w^2}{\psi^2}|d\xi-f\delta(u)du|^2+2\,du\,dw-2\epsilon\,du^2$$
$$+w\left[(f_{\xi}+\bar{f}_{\bar{\xi}})-\frac{2\epsilon}{\psi}(f\bar{\xi}+\bar{f}\xi)\right]\delta(u)du^2, \qquad (2.15)$$

in which $f(\xi)$ is related to $h(Z)$ through the identification $f\equiv Z_{inv}-Z$ evaluated on $U=0$. Although this form of the metric is only formal since it contains the square of the delta function, it explicitly shows that expanding impulsive gravitational waves (2.10) arise as the impulsive limits of the Robinson-Trautman type *N* spacetimes, expressed in the coordinates introduced in [18] (see [5,19] for the details).

Let us now conclude this review section by a brief description of the geometry of the above expanding impulsive waves localized along the wave surfaces $U=0$, i.e., $u=0$. This will also elucidate the meaning of the parameter ϵ.

From the Robinson-Trautman form of the metric (2.15) it is obvious that the impulse splits the spacetime into two flat regions, $u>0$ and $u<0$. In the following we shall call the Minkowski half-space $u>0$ as being "in front of the wave," and the other Minkowski half-space $u<0$ (note that $u\equiv U$ for $U<0$) as being "behind the wave." The "background" metric on both sides of the impulse is given by Eq. (2.14). For *arbitrary* $u\neq 0$, this metric can be put into explicit Minkowski form (2.1) by the transformations (2.2) and (2.11) [or (2.12)] and the (trivial) identification $u=U$, $w=V$, $\xi=Z$. Using these relations, we can easily analyze the geometry of the null hypersurfaces $u=u_0=$const in Minkowski coordinates, which are geometrically privileged and thus allow for a clear physical interpretation.

We start with the subclass of solutions for which $\epsilon=0$. Substituting Eq. (2.2) into Eq. (2.11) and setting $U=u_0$, we get the relation

$$x^2+y^2+\left(z+\frac{1}{\sqrt{2}}u_0\right)^2=\left(t+\frac{1}{\sqrt{2}}u_0\right)^2 \qquad (2.16)$$

(if $t\neq z$, i.e., for $x\neq 0$, $y\neq 0$). For various values of u_0 this represents a family of *null cones* with vertices at $(-(1/\sqrt{2})u_0,0,0,-(1/\sqrt{2})u_0)$ localized along a singular null line $t=z$, $x=0$, $y=0$. Also, $V=w_0=$const is a set of parallel hyperplanes $t=z+\sqrt{2}w_0$ in the Minkowski background. This reveals the geometrical meaning of the coordinates u,w used in the metric (2.14) with $\epsilon=0$. Note that $w=0$ represents a physical singularity in the Robinson-Trautman spacetimes [20] which can be interpreted as the source of the wave surfaces $u=u_0$. At any time *t*, these surfaces are spheres of the radius $R=|t+(1/\sqrt{2})u_0|$. In particular, the impulse localized on $u=0$ is a null cone with the vertex in the origin which, at any time, is a sphere of radius $R=\sqrt{x^2+y^2+z^2}=|t|$.

Analogous results can similarly be obtained for the remaining two subclasses $\epsilon=\pm 1$. In this case, using Eqs. (2.2) and (2.12), the hypersurfaces $u_0=$const are given by

$$x^2+y^2+z^2=(t+\sqrt{2}u_0)^2, \qquad (2.17)$$

for $\epsilon=+1$ and

$$x^2+y^2+(z+\sqrt{2}u_0)^2=t^2, \qquad (2.18)$$

for $\epsilon=-1$, respectively. Again, these are families of null cones in Minkowski space with vertices shifted in the *t* direction for $\epsilon=+1$, and in the *z* direction for $\epsilon=-1$. Note that these vertices form a singular timelike line $x=0=y$, $z=0$ if $\epsilon=+1$, and a spacelike line $x=0=y$, $t=0$ if $\epsilon=-1$. These lines are given by $\eta=0$, $\mathcal{V}-\epsilon\mathcal{U}=0$, and correspond to the physical singularity of the Robinson-Trautman spacetime at $w=0$.

It is obvious that for the above spacetimes with a *single* expanding impulsive wave localized at $u=u_0=0$, i.e., at $U=0$, the null cones of the three classes of wave surfaces given by Eqs. (2.16), (2.17), and (2.18) *coincide*. In fact, in the impulsive limit the three (generically different) subclasses $\epsilon=0,+1,-1$ of the Robinson-Trautman class of solutions are *locally* equivalent [5].

It can also be observed that for $\epsilon=0$ the physical singularity at $V=0$ is a *singular null line* on the wavefront surface $U=0$. For a physical interpretation, it would be better to remove this singularity from the spacetime. This can be achieved by considering solutions with $\epsilon \neq 0$: In these cases, the singularity at $V=0$ appears *only at the vertex of* the null cone, $x=y=z=t=0$, which may be considered as the "origin" of the spherical wave.

III. GEODESIC MOTION IN SPACETIMES WITH EXPANDING IMPULSIVE WAVES

The purpose of this paper is to investigate the effect of expanding impulsive waves of Robinson-Trautman type on the motion of freely moving test particles. It is natural to start with geodesics in (local, see below) Minkowski space $U>0$ in front of the wave, i.e., "outside" the null cone $U=0$ corresponding to $x^2+y^2+z^2=t^2$. (Note that at $t=0$ the spherical impulse is just "created" at the origin.) Obviously, general geodesics are given by

$$\begin{aligned} t^+ &= \gamma\tau, \\ x^+ &= \dot{x}_0(\tau-\tau_i)+x_0, \\ y^+ &= \dot{y}_0(\tau-\tau_i)+y_0, \\ z^+ &= \dot{z}_0(\tau-\tau_i)+z_0, \end{aligned} \tag{3.1}$$

with $\gamma=\sqrt{\dot{x}_0^2+\dot{y}_0^2+\dot{z}_0^2-e}$, i.e., τ is a normalized affine parameter of timelike ($e=-1$) or spacelike ($e=+1$) geodesics. For null geodesics ($e=0$) it is always possible to scale the factor γ to unity. The constants x_0,y_0,z_0 and $\dot{x}_0,\dot{y}_0,\dot{z}_0$ characterize the position and velocity, respectively, of each test particle at the instant

$$\tau_i=\sqrt{x_0^2+y_0^2+z_0^2}/\gamma, \tag{3.2}$$

when the geodesic intersects the null cone. At τ_i each particle is hit by the impulse and its trajectory is refracted and (possibly) shifted. The geodesics (3.1) in front of the impulse can also be written as

$$\begin{aligned} \mathcal{V}^+ &= \dot{\mathcal{V}}_0^+(\tau-\tau_i)+\mathcal{V}_0^+, \\ \mathcal{U}^+ &= \dot{\mathcal{U}}_0^+(\tau-\tau_i)+\mathcal{U}_0^+, \\ \eta^+ &= \dot{\eta}_0^+(\tau-\tau_i)+\eta_0^+, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \dot{\mathcal{V}}_0^+ &= \frac{1}{\sqrt{2}}(\gamma-\dot{z}_0), \quad \mathcal{V}_0^+=\frac{1}{\sqrt{2}}(\gamma\tau_i-z_0), \\ \dot{\mathcal{U}}_0^+ &= \frac{1}{\sqrt{2}}(\gamma+\dot{z}_0), \quad \mathcal{U}_0^+=\frac{1}{\sqrt{2}}(\gamma\tau_i+z_0), \\ \dot{\eta}_0^+ &= \frac{1}{\sqrt{2}}(\dot{x}_0+\mathrm{i}\dot{y}_0), \quad \eta_0^+=\frac{1}{\sqrt{2}}(x_0+\mathrm{i}y_0). \end{aligned} \tag{3.4}$$

Now, we wish to investigate the influence of the impulse on the geodesics, and to determine explicitly formulas for corresponding "refraction" and "shift." In the region $U<0$ behind the impulse the particles again move in Minkowski space (2.1) so that the trajectories also have to be straight lines of the form

$$\begin{aligned} \mathcal{V}^- &= \dot{\mathcal{V}}_0^-(\tau-\tau_i)+\mathcal{V}_0^-, \\ \mathcal{U}^- &= \dot{\mathcal{U}}_0^-(\tau-\tau_i)+\mathcal{U}_0^-, \\ \eta^- &= \dot{\eta}_0^-(\tau-\tau_i)+\eta_0^-. \end{aligned} \tag{3.5}$$

It remains to express the constants appearing in Eqs. (3.5) in terms of the initial data introduced in Eqs. (3.1), and the structural function $h(Z)$, which characterizes specific expanding impulsive waves.

The key idea is to *employ the continuous form* of the solution (2.10). It can easily be observed that in this coordinate system the Christoffel symbols Γ^μ_{UU}, Γ^μ_{UV}, and Γ^μ_{VV} vanish identically. Therefore, the geodesic equations *always admit privileged global solutions* of the form

$$\begin{aligned} Z &= Z_0=\text{const}, \\ U &= \dot{U}_0(\tau-\tau_i), \\ V &= \dot{V}_0(\tau-\tau_i)+V_0. \end{aligned} \tag{3.6}$$

Here we have set $U_0=0$, so that each geodesic reaches the impulse localized at $U=0$ at parameter time $\tau=\tau_i$.

Using Eqs. (2.6), it is possible to express the geodesics (3.6) *in front* of the impulse in the Minkowski form (3.3) where the coefficients (3.4) are

$$\begin{aligned} \dot{\mathcal{V}}_0^+ &= A\dot{V}_0-D\dot{U}_0, \quad \mathcal{V}_0^+=AV_0, \\ \dot{\mathcal{U}}_0^+ &= B\dot{V}_0-E\dot{U}_0, \quad \mathcal{U}_0^+=BV_0, \\ \dot{\eta}_0^+ &= C\dot{V}_0-F\dot{U}_0, \quad \eta_0^+=CV_0. \end{aligned} \tag{3.7}$$

The constants *A*, *B*, *C*, *D*, *E*, and *F* are given by the values of the functions (2.7) at $Z=Z_0$. The relations (3.4) and (3.7) enable us to relate the parameters Z_0, $\dot{U}_0$, $\dot{V}_0$, and V_0 to the natural initial data introduced in Eqs. (3.1) by

$$\begin{aligned} x_0 &= \sqrt{2}V_0 \text{ Re } C, \\ y_0 &= \sqrt{2}V_0 \text{ Im } C, \\ z_0 &= \frac{1}{\sqrt{2}}V_0(B-A), \\ \dot{x}_0 &= \sqrt{2}[\dot{V}_0 \text{ Re } C-\dot{U}_0 \text{ Re } F], \\ \dot{y}_0 &= \sqrt{2}[\dot{V}_0 \text{ Im } C-\dot{U}_0 \text{ Im } F], \\ \dot{z}_0 &= \frac{1}{\sqrt{2}}[\dot{V}_0(B-A)-\dot{U}_0(E-D)]. \end{aligned} \tag{3.8}$$

The three position parameters x_0, y_0, z_0 are thus related to the three independent constants $Z_0, \bar{Z}_0, V_0$. The normalization condition implies the constraint $(\dot{V}_0 - \epsilon \dot{U}_0)\dot{U}_0 = \frac{1}{2}e$ (which can be obtained using the identities $AB - C\bar{C} = 0$, $DE - F\bar{F} = \epsilon$, and $AE + BD - C\bar{F} - \bar{C}F = 1$) on the parameters $\dot{V}_0$ and $\dot{U}_0$. However, in general it is possible to set at least one of the velocities $\dot{x}_0$, $\dot{y}_0$, or $\dot{z}_0$ to zero by a suitable choice of $\dot{U}_0$.

Finally, we transform the geodesics (3.6) *behind* the impulse using Eqs. (2.3), which gives the uniform Minkowskian motion (3.5) with the coefficients

$$\dot{\mathcal{V}}_0^- = \frac{\dot{V}_0}{p} - \epsilon \dot{U}_0, \quad \mathcal{V}_0^- = \frac{V_0}{p},$$
$$\dot{\mathcal{U}}_0^- = \frac{Z_0\bar{Z}_0}{p}\dot{V}_0 - \dot{U}_0, \quad \mathcal{U}_0^- = \frac{Z_0\bar{Z}_0}{p} V_0, \tag{3.9}$$
$$\dot{\eta}_0^- = \frac{Z_0}{p}\dot{V}_0, \quad \eta_0^- = \frac{Z_0}{p} V_0.$$

Substituting for Z_0, V_0, $\dot{V}_0$, and $\dot{U}_0$ from the inverse of the relations (3.8), we obtain an explicit result which can be used for discussion of the effect of expanding impulsive waves on the privileged family of geodesics (3.6).

However, it should be emphasized that in the above construction we started with the initial data (3.1) *outside* the impulse in the region $U>0$, which is a Minkowski space *only locally*. Because of the complicated form of the generating complex function h, there are topological defects such as cosmic strings (corresponding to the presence of deficit angles) outside the null cone, see, e.g., [17] for more details. Therefore, for better physical interpretation it may be useful to set the initial data for the geodesics in the region $U<0$ *inside* the cone, which is considered to be a "complete" Minkowski space without topological defects. Evolving these data,

$$t^- = \gamma\tau,$$
$$x^- = \dot{x}_0(\tau - \tau_i) + x_0,$$
$$y^- = \dot{y}_0(\tau - \tau_i) + y_0, \tag{3.10}$$
$$z^- = \dot{z}_0(\tau - \tau_i) + z_0,$$

"backward" in time (i.e., for decreasing affine parameter τ), it is possible to prolong the geodesics across the spherical impulse to the "incomplete" Minkowski region $U>0$ with cosmic strings outside the impulse.

In this case, the *explicit geodesics outside* the impulse have the form (3.3), (3.7), in which the constants Z_0, V_0, $\dot{V}_0$, and $\dot{U}_0$ have to be expressed in terms of the data (3.10). Using (2.11) for $\epsilon = 0$, and Eqs. (2.12) for $\epsilon \neq 0$ we obtain

$$Z_0 = \frac{x_0 + \mathrm{i}y_0}{\gamma\tau_i - z_0},$$
$$V_0 = \frac{1}{\sqrt{2}}[(1+\epsilon)\gamma\tau_i - (1-\epsilon)z_0],$$
$$\dot{V}_0 = \frac{1}{\sqrt{2}}[(1-\epsilon)\gamma - (1+\epsilon)\dot{z}_0]\frac{(1+\epsilon)\gamma\tau_i - (1-\epsilon)z_0}{(1-\epsilon)\gamma\tau_i - (1+\epsilon)z_0}, \tag{3.11}$$
$$\dot{U}_0 = \frac{\sqrt{2}\gamma(z_0 - \tau_i\dot{z}_0)}{(1-\epsilon)\gamma\tau_i - (1+\epsilon)z_0},$$

with the constraint

$$[(1-\epsilon)\gamma\tau_i - (1+\epsilon)z_0](\dot{x}_0 + \mathrm{i}\dot{y}_0) = [(1-\epsilon)\gamma - (1+\epsilon)\dot{z}_0](x_0 + \mathrm{i}y_0). \tag{3.12}$$

In Eq. (3.11) we assumed that $(1-\epsilon)\gamma\tau_i \neq (1+\epsilon)z_0$. The case $(1-\epsilon)\gamma\tau_i = (1+\epsilon)z_0$ requires $x_0 = y_0 = z_0 = 0$. Such geodesics reach the singular vertex $x=y=z=0$ of the impulsive null cone at $t=\tau=0$, and it is thus unphysical to investigate their continuation across $U=0$. On the other hand, if $(1-\epsilon)\gamma = (1+\epsilon)\dot{z}_0$ then $\dot{x}_0 = \dot{y}_0 = 0$, and $\dot{V}_0 = 0$. These geodesics are *null* for $\epsilon = 0$ (with $\dot{U}_0 = -\sqrt{2}\dot{z}_0$), *timelike* for $\epsilon = 1$ (with $\gamma = 1$, $\dot{z}_0 = 0$, $\dot{U}_0 = -1/\sqrt{2}$), and *spacelike* for $\epsilon = -1$.

Note that the relations (3.11) simplify for each particular choice of the parameter ϵ. In particular, in the case $\epsilon = 0$ we obtain

$$Z_0 = \frac{x_0 + \mathrm{i}y_0}{\gamma\tau_i - z_0}, \quad V_0 = \frac{1}{\sqrt{2}}(\gamma\tau_i - z_0),$$
$$\dot{V}_0 = \frac{1}{\sqrt{2}}(\gamma - \dot{z}_0), \quad \dot{U}_0 = \frac{1}{\sqrt{2}}\frac{e}{\gamma - \dot{z}_0}. \tag{3.13}$$

Let us finally recall that the above class of geodesics is privileged and very special since $Z = Z_0 = \text{const}$ (cf. (3.6)). However, it is possible to find *general* geodesics *assuming them to be* C^1 *across the impulse* in the continuous coordinate system (2.10). With this assumption, the constants

$$Z_i \equiv Z(\tau_i), \quad V_i \equiv V(\tau_i), \quad U_i \equiv U(\tau_i) = 0,$$
$$\dot{Z}_i \equiv \dot{Z}(\tau_i), \quad \dot{V}_i \equiv \dot{V}(\tau_i), \quad \dot{U}_i \equiv \dot{U}(\tau_i), \tag{3.14}$$

describing positions and velocities at τ_i, the instant of interaction with the impulsive wave, have the *same* values when evaluated in the limits $U \to 0$ both from the region in front $(U>0)$ and behind the impulse $(U<0)$. Starting now with the general initial data (3.10) in the region $U<0$, in which Eqs. (2.11), (2.12) apply, it is straightforward to derive

$$Z_i=\frac{x_0+iy_0}{\gamma\tau_i-z_0},$$

$$V_i=\frac{1}{\sqrt{2}}[(1+\epsilon)\gamma\tau_i-(1-\epsilon)z_0],$$

$$\dot{Z}_i=\frac{\dot{x}_0+i\dot{y}_0}{\gamma\tau_i-z_0}-\frac{x_0+iy_0}{(\gamma\tau_i-z_0)^2}\times\frac{[(1-\epsilon)\gamma-(1+\epsilon)\dot{z}_0](\gamma\tau_i-z_0)+2\epsilon(x_0\dot{x}_0+y_0\dot{y}_0)}{(1+\epsilon)\gamma\tau_i-(1-\epsilon)z_0}, \tag{3.15}$$

$$\dot{V}_i=\frac{1}{\sqrt{2}}\Big\{[(1-\epsilon)\gamma-(1+\epsilon)\dot{z}_0][(1-\epsilon)\gamma\tau_i-(1+\epsilon)z_0]+4\epsilon(x_0\dot{x}_0+y_0\dot{y}_0)\Big\}[(1+\epsilon)\gamma\tau_i-(1-\epsilon)z_0]^{-1},$$

$$\dot{U}_i=\sqrt{2}\,\frac{x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0-\gamma^2\tau_i}{(1+\epsilon)\gamma\tau_i-(1-\epsilon)z_0}.$$

Then, using Eqs. (2.6) the geodesics outside the impulse in the (local) Minkowski space $U>0$ are given by the expressions (3.3) with

$$\begin{aligned}\mathcal{V}_0^+&=AV_i,\\ \mathcal{U}_0^+&=BV_i,\\ \eta_0^+&=CV_i,\\ \dot{\mathcal{V}}_0^+&=A\dot{V}_i-D\dot{U}_i+(A_{,Z}\dot{Z}_i+A_{,\bar{Z}}\dot{\bar{Z}}_i)V_i,\\ \dot{\mathcal{U}}_0^+&=B\dot{V}_i-E\dot{U}_i+(B_{,Z}\dot{Z}_i+B_{,\bar{Z}}\dot{\bar{Z}}_i)V_i,\\ \dot{\eta}_0^+&=C\dot{V}_i-F\dot{U}_i+(C_{,Z}\dot{Z}_i+C_{,\bar{Z}}\dot{\bar{Z}}_i)V_i,\end{aligned} \tag{3.16}$$

in which the coefficients and their derivatives are given by the functions (2.7), evaluated at Z_i.

Of course, with the constraint (3.12) we obtain $\dot{Z}_i=0$, $Z_i=Z_0$, $V_i=V_0$, $\dot{V}_i=\dot{V}_0$, $\dot{U}_i=\dot{U}_0$, and the above geodesics reduce to the privileged family presented in Eq. (3.11).

IV. GEODESICS IN SPACETIMES WITH A SNAPPING COSMIC STRING

In this section we apply the above general results to an interesting particular class of spacetimes in which the expanding spherical impulsive wave is generated by a snapping cosmic string (identified outside the impulse by a deficit angle). This solution was previously introduced and discussed in a number of works [2,12,13,15]. It can be written as the metric (2.10) with

$$H(Z)=\frac{\frac{1}{2}\delta\left(1-\frac{1}{2}\delta\right)}{Z^2}, \tag{4.1}$$

which is generated by Eq. (2.9) with $h(Z)=Z^{1-\delta}$. Here δ is a real positive constant, $\delta<1$, which characterizes the deficit angle $2\pi\delta$ of the snapped string localized outside the impulse along the axis $\eta=0$ (see, e.g., [2,17]). It is straightforward to calculate the coefficients (2.7) and their derivatives for such a function h, i.e.,

$$\begin{aligned}&A=\frac{|Z|^{\delta}}{(1-\delta)p},\quad B=\frac{|Z|^{2-\delta}}{(1-\delta)p},\quad C=\frac{Z^{1-\delta}|Z|^{\delta}}{(1-\delta)p},\\ &D=\frac{|Z|^{\delta-2}}{1-\delta}[(\tfrac{1}{2}\delta)^2+(1-\tfrac{1}{2}\delta)^2\epsilon|Z|^2],\\ &E=\frac{|Z|^{-\delta}}{1-\delta}[(1-\tfrac{1}{2}\delta)^2+(\tfrac{1}{2}\delta)^2\epsilon|Z|^2],\\ &F=\frac{Z^{1-\delta}|Z|^{\delta-2}\frac{1}{2}\delta(1-\frac{1}{2}\delta)p}{1-\delta},\\ &A_{,Z}=\frac{|Z|^{\delta}[\frac{1}{2}\delta-(1-\frac{1}{2}\delta)\epsilon|Z|^2]}{(1-\delta)p^2Z},\\ &B_{,Z}=\frac{|Z|^{2-\delta}[(1-\frac{1}{2}\delta)-\frac{1}{2}\delta\epsilon|Z|^2]}{(1-\delta)p^2Z},\\ &C_{,Z}=\left(\frac{\bar{Z}}{Z}\right)^{\delta/2}\frac{(1-\frac{1}{2}\delta)-\frac{1}{2}\delta\epsilon|Z|^2}{(1-\delta)p^2},\\ &C_{,\bar{Z}}=\left(\frac{Z}{\bar{Z}}\right)^{\delta/2-1}\frac{\frac{1}{2}\delta-(1-\frac{1}{2}\delta)\epsilon|Z|^2}{(1-\delta)p^2}.\end{aligned} \tag{4.2}$$

A. Privileged geodesics with $Z=Z_0$

We first consider the family of geometrically preferred geodesics (3.6) for which $Z=Z_0=\text{const}$. It is convenient to use the global axial symmetry of the solution (2.10), (4.1) corresponding to a coordinate freedom $Z\to Z\exp(i\phi)$, where ϕ is a constant. Therefore, without loss of generality we can assume Z_0 to be a *real* positive constant, in which case the coefficients (4.2) reduce to

$$\begin{aligned}&A=\frac{Z_0^{\delta}}{(1-\delta)p},\quad B=\frac{Z_0^{2-\delta}}{(1-\delta)p},\quad C=\frac{Z_0}{(1-\delta)p},\\ &D=\frac{Z_0^{\delta-2}}{1-\delta}\left[\left(\frac{1}{2}\delta\right)^2+\left(1-\frac{1}{2}\delta\right)^2\epsilon Z_0^2\right],\\ &E=\frac{Z_0^{-\delta}}{1-\delta}\left[\left(1-\frac{1}{2}\delta\right)^2+\left(\frac{1}{2}\delta\right)^2\epsilon Z_0^2\right],\\ &F=\frac{\frac{1}{2}\delta(1-\frac{1}{2}\delta)p}{(1-\delta)Z_0},\end{aligned} \tag{4.3}$$

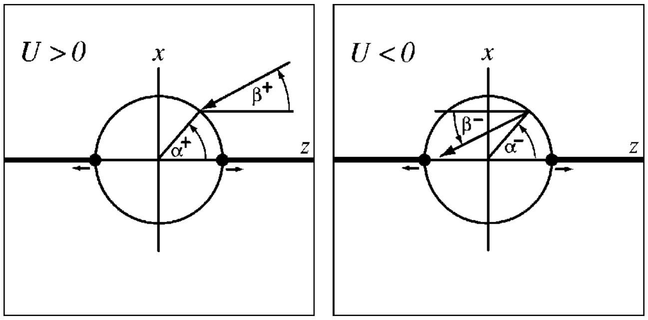

FIG. 1. The angle α identifies the point where the particle interacts with the impulse (indicated by a circle) generated by a snapped string localized on the z axis. The angle β characterizes the inclination of its trajectory. The superscripts "+" and "−" correspond to the respective values in the (local) Minkowskian coordinate system outside $(U>0)$ and the (different) Minkowskian system inside $(U<0)$ the impulse.

where $p=1+\epsilon Z_0^2$. Substituting Eqs. (4.3) into Eqs. (3.8), we obtain the relations between the parameters Z_0, $\dot{U}_0$, $\dot{V}_0$, and V_0, characterizing the family of geodesics (3.6) and the initial data *outside* the spherical impulse (cf. Eqs. (3.1)). Obviously, these geodesics all have $y^+\equiv 0$, since $\dot{y}_0=0=y_0$ due to vanishing imaginary parts of C and F. The remaining relations (3.8) yield

$$
\begin{aligned}
Z_0^{1-\delta}&=\frac{z_0}{x_0}+\sqrt{1+\frac{z_0^2}{x_0^2}},\\
\dot{U}_0&=\sqrt{2}\,\frac{z_0\dot{x}_0-x_0\dot{z}_0}{(E-D)x_0-2Fz_0},\\
\frac{Z_0}{p}V_0&=\frac{1}{\sqrt{2}}(1-\delta)x_0,\\
\frac{Z_0}{p}\dot{V}_0&=\frac{1}{\sqrt{2}}(1-\delta)x_0\frac{(E-D)\dot{x}_0-2F\dot{z}_0}{(E-D)x_0-2Fz_0}.
\end{aligned}
\tag{4.4}
$$

(Note that "initially static" observers $\dot{x}_0=0=\dot{z}_0$ are excluded from the family (3.6) as this would give $\dot{U}_0=0=\dot{V}_0$, i.e., the geodesic is constant.) Finally, substituting Eqs. (4.4) into Eqs. (3.9) we obtain

$$
\begin{aligned}
x_0^-&=(1-\delta)x_0,\\
y_0^-&=0,\\
z_0^-&=\tfrac{1}{2}(1-\delta)x_0(Z_0-Z_0^{-1}),\\
t_0^-&=\tfrac{1}{2}(1-\delta)x_0(Z_0+Z_0^{-1}),\\
\dot{x}_0^-&=(1-\delta)x_0\frac{(E-D)\dot{x}_0-2F\dot{z}_0}{(E-D)x_0-2Fz_0},\\
\dot{y}_0^-&=0,\\
\dot{z}_0^-&=\frac{z_0^-[(E-D)\dot{x}_0-2F\dot{z}_0]-(1-\epsilon)(z_0\dot{x}_0-x_0\dot{z}_0)}{(E-D)x_0-2Fz_0},\\
\dot{t}_0^-&=\frac{t_0^-[(E-D)\dot{x}_0-2F\dot{z}_0]-(1+\epsilon)(z_0\dot{x}_0-x_0\dot{z}_0)}{(E-D)x_0-2Fz_0},
\end{aligned}
\tag{4.5}
$$

where the coefficients $E-D$ and F are

$$
\begin{aligned}
(1-\delta)(E-D)&=(1-\tfrac{1}{2}\delta)^2(Z_0^{-\delta}-\epsilon Z_0^{\delta})\\
&\quad-(\tfrac{1}{2}\delta)^2(Z_0^{\delta-2}-\epsilon Z_0^{2-\delta}),\\
(1-\delta)2F&=\delta(1-\tfrac{1}{2}\delta)(Z_0^{-1}+\epsilon Z_0),
\end{aligned}
\tag{4.6}
$$

and Z_0 is given by Eqs. (4.4).

The above relations explicitly express the effect of an expanding impulsive wave generated by a snapping cosmic string on geodesics (3.1) of the privileged family (3.6). These start outside the impulse $(U>0)$ with the initial data entering the right-hand side of Eqs. (4.5), and continue inside the spherical impulse in the Minkowski space without the string $(U<0)$, where the positions and velocities at $\tau=\tau_i$ are now given by the left-hand side of Eqs. (4.5). For $\delta=0$ we obtain $E-D=1-\epsilon$, $F=0$, $Z_0-Z_0^{-1}=2z_0/x_0$, $Z_0+Z_0^{-1}=2\gamma\tau_i/x_0$, so that $x^-=x^+$, $y^-=0=y^+$, $z^-=z^+$, $t_0^-=\gamma\tau_i$, and $\dot{t}_0^-=\gamma$ (to derive the last relation we have used the constraint (3.12)). Obviously, these are *global* geodesics in *complete* Minkowski space without the strings and the impulse.

Let us now investigate in some more detail the effect of the spherical gravitational impulse on free test particles. To describe the "refraction" and the "shift" of geodesic trajectories it is convenient to introduce angles α and β, whose geometrical meaning is indicated in Fig. 1. Recall that for the special family of geodesics (3.6) we have $y^-=0=y^+$, so that the motion is confinded to th x,z plane which contains the string (located along the z axis). Hence α and β represent the *position* of the particle at the instant τ_i of interaction with the impulse and the direction of its *velocity* (inclination of the trajectory) in the (x,z)-plane, respectively.

In the region $U>0$ outside the impulsive wave these parameters are defined as

$$\cot\alpha^+=z_0/x_0,\quad \cot\beta^+=\dot{z}_0/\dot{x}_0. \tag{4.7}$$

Similarly, behind the impulse in the region $U<0$ we have

$$\cot\alpha^-=z_0^-/x_0^-,\quad \cot\beta^-=\dot{z}_0^-/\dot{x}_0^-. \tag{4.8}$$

Straightforward calculations using Eqs. (4.4) give

$$Z_0^{1-\delta}=\cot(\tfrac{1}{2}\alpha^+), \tag{4.9}$$

which implies the relation $\frac{1}{2}(Z_0^{1-\delta}-Z_0^{\delta-1})=\cot\alpha^+$. From Eqs. (4.5) and (4.9) we immediately obtain

$$\cot\alpha^- = \tfrac{1}{2}\,(Z_0 - Z_0^{-1}) = \tfrac{1}{2}\,[\cot^q(\tfrac{1}{2}\,\alpha^+) - \cot^{-q}(\tfrac{1}{2}\,\alpha^+)], \tag{4.10}$$

where $q = 1/(1-\delta)$. This expression gives the relation $\alpha^-(\alpha^+)$ which *identifies the points* on both sides of the impulse in the natural Minkowskian coordinate systems.

Analogously we derive the following relation for the velocities:

$$\cot\beta^- - \cot\alpha^- = \mathcal{N}(\cot\beta^+ - \cot\alpha^+), \tag{4.11}$$

where

$$\mathcal{N} = \mathcal{N}(\alpha^+,\beta^+) = \frac{1-\epsilon}{(1-\delta)(E-D) - (1-\delta)2F\cot\beta^+}. \tag{4.12}$$

This is the *refraction formula* for trajectories of free test particles which cross the spherical impulse.

Notice that the above considerations also apply to geodesics propagating in the privileged directions $\beta^+ = 0$ and $\beta^+ = \pi/2$. For geodesics *parallel to the string*, i.e., in the case $\beta^+ = 0$ (implying $\dot{x}_0 = 0$) the right-hand side of Eq. (4.11) has to be replaced by the simple expression $(\epsilon - 1)/[(1-\delta)2F]$. For trajectories with $\beta^+ = \pi/2$, which are *perpendicular* to the string ($\dot{z}_0 = 0$), the right-hand side simplifies to $[(\epsilon-1)\cot\alpha^+]/[(1-\delta)(E-D)]$.

Several interesting observations can immediately be done. For $\delta = 0$ representing a complete Minkowski space without the impulse and topological defects, one obtains $\alpha^- = \alpha^+$, $\mathcal{N} = 1$, and consequently $\beta^- = \beta^+$. There is thus no "shift" and "refraction," as expected.

For a general δ, it follows from Eq. (4.11) that if $\alpha^+ = \beta^+$ then $\alpha^- = \beta^-$. This means physically that the *radial geodesics* ("perpendicular" to the spherical impulse) *remain radial* also behind the impulse.

Moreover, it can be observed from Eq. (4.12) that the coefficient $\mathcal{N}$ identically vanishes for spacetimes *with the parameter* $\epsilon = +1$. Consequently, $\alpha^- = \beta^-$, which means that the geodesics (3.6) are refracted by the impulse in such a way that their *trajectories become radial*. These are thus either exactly focused towards the origin $x = 0 = z$, or defocused directly from it.

The typical behavior of geodesics affected by the impulsive gravitational wave [Eqs. (2.10), (4.1)] as described by the refraction formula (4.11), is shown in Fig. 2 for various choices of α^+, β^+, and ϵ. Each test particle follows in the region $U>0$ a trajectory with the inclination angle β^+ until it reaches the spherical impulse at the point represented by α^+. The impulse influences the particle in such a way that it emerges in the region $U<0$ at the point given by α^- and continues to move uniformly along the straight trajectory with inclination β^-. Note that the lines in Fig. 2 represent just the inclination of the geodesic trajectories, not the speed and orientation of the motion—these will be investgated later on.

From Fig. 2 it becomes obvious that the dependence of β^- on the data α^+, β^+, and the parameter ϵ is rather delicate. To shed some light on the details of this dependence we introduce two special "incoming" inclination angles, denoted by $\beta^+_\parallel(\alpha^+)$ and $\beta^+_\perp(\alpha^+)$, that are defined by the property that the corresponding geodesic *behind* the impulse is *parallel* ($\beta^- = 0$) and *perpendicular* ($\beta^- = \pm\pi/2$), respectively, to the strings localized along the z axis. It follows immediately from Eq. (4.11) that

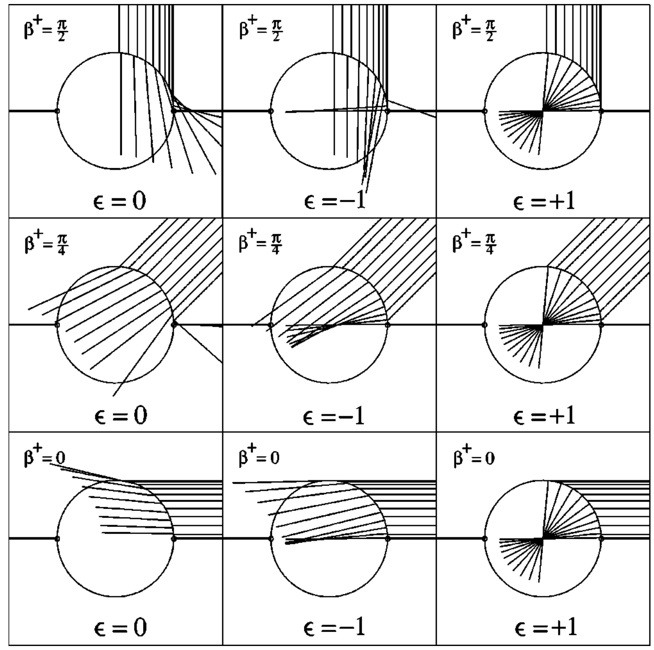

FIG. 2. Typical behavior of geodesic trajectories, which are refracted and shifted by the expanding spherical impulse, for various values of α^+, β^+ and the parameter ϵ. Here $\delta = 0.3$.

$$\cot\beta^+_\parallel = \frac{E-D}{2F},$$
$$\cot\beta^+_\perp = \frac{(1-\epsilon)\cot\alpha^+ - (1-\delta)(E-D)\cot\alpha^-}{(1-\epsilon) - (1-\delta)2F\cot\alpha^-}, \tag{4.13}$$

where the functions D,E,F are given by Eq. (4.6), Z_0 by Eq. (4.9), and α^- by Eq. (4.10). The functions (4.13) are drawn in Fig. 3 for the three types of spacetimes given by $\epsilon = 0, -1, +1$, and for a "typical" value of the deficit-angle parameter $\delta = 0.3$. Both $\beta^+_\parallel$ and $\beta^+_\perp$ vanish at $\alpha^+ = 0$. For $\epsilon = 0$ the functions $\beta^+_\parallel(\alpha^+) < \alpha^+ < \beta^+_\perp(\alpha^+)$ monotonically increase to the values $\cot\beta^+_\parallel = (1-\delta)/[\delta(1-\frac{1}{2}\delta)]$, $\beta^+_\perp = \pi/2$ at $\alpha^+ = \pi/2$. For $\epsilon = -1$ the relation is $\beta^+_\parallel > \beta^+_\perp > \alpha^+$, and the corresponding values at $\alpha^+ = \pi/2$ are $\beta^+_\parallel = \pi$, $\beta^+_\perp = \pi/2$. In the case $\epsilon = 1$ these two functions *coincide* for all values of α^+, which directly follows from Eqs. (4.13). The functions grow to a maximum value and then decrease to $\beta^+_\parallel = \beta^+_\perp = \pi/2$ at $\alpha^+ = \pi/2$. The graphs presented in Fig. 3 provide a qualitative picture of the character of the trajectories depending on the choice of the initial angles α^+, β^+. Trajectories close to $\beta^+_\parallel(\alpha^+)$ become "nearly horizontal" behind the impulse, whereas those close to $\beta^+_\perp(\alpha^+)$ become "nearly vertical." Since for $\beta^+_\parallel$ we have $\dot{x}_0^- = 0$ whereas $\beta^+_\perp$ implies

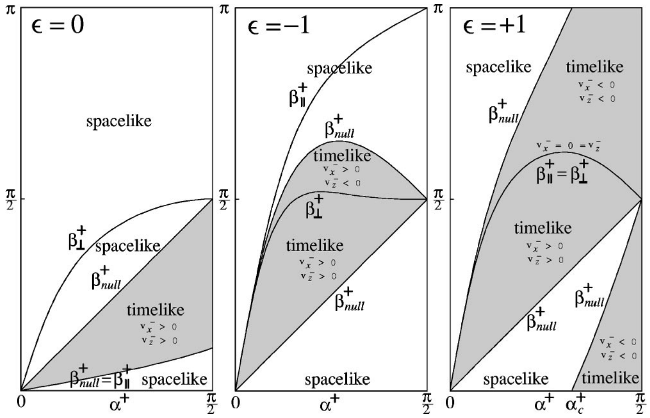

FIG. 3. Plots of $\beta_{\parallel}^{+}$, $\beta_{\perp}^{+}$, and β_{null}^{+}, introduced in the text, as functions of the angle α^{+} (again $\delta=0.3$). The causal character of the geodesics with specific initial data is indicated by the shading of the respective regions: gray and white correspond to timelike and spacelike geodesics, respectively.

$\dot{z}_0^{-}=0$, it follows that the particles actually *stop* behind the impulse if we choose $\beta^{+}=\beta_{\parallel}^{+}(=\beta_{\perp}^{+})$ for a given α^{+} in the impulsive spacetime with $\epsilon=+1$.

Note, however, that the refraction formula (4.11), while relating the *inclination* of the trajectories behind the wave to its initial values, does not provide any information on the specific *speed* and *orientation* of the motion. Of course, the velocity is proportional to the parameters $\dot{x}_0, \dot{y}_0, \dot{z}_0$, which are the derivatives of space coordinates with respect to the affine parameter τ, see Eqs. (3.1). However, for physical interpretation of the motion we need the *velocity with respect to the Minkowski frame behind the impulse*, which is given by

$$(v_x^{-}, v_y^{-}, v_z^{-}) = \left(\frac{\dot{x}_0^{-}}{\dot{t}_0^{-}}, \frac{\dot{y}_0^{-}}{\dot{t}_0^{-}}, \frac{\dot{z}_0^{-}}{\dot{t}_0^{-}} \right). \tag{4.14}$$

From Eqs. (4.5) using Eqs. (4.7) we obtain

$$\begin{aligned} v_x^{-} &= \frac{G}{\frac{1}{2}(Z_0+Z_0^{-1})G-(1+\epsilon)(\cot\alpha^{+}-\cot\beta^{+})}, \\ v_z^{-} &= \frac{\frac{1}{2}(Z_0-Z_0^{-1})\,G-(1-\epsilon)(\cot\alpha^{+}-\cot\beta^{+})}{\frac{1}{2}(Z_0+Z_0^{-1})G-(1+\epsilon)(\cot\alpha^{+}-\cot\beta^{+})}, \end{aligned} \tag{4.15}$$

and $v_y^{-}=0$, where $G=(1-\delta)[(E-D)-2F\cot\beta^{+}]$. The above expressions determine the velocity of the particle (including its orientation) in the region behind the impulse as a function of the initial parameters α^{+},β^{+}: The particle moves from the point $\alpha^{-}(\alpha^{+})$ in the refracted direction $\beta^{-}(\alpha^{+},\beta^{+})$ given by Eqs. (4.10),(4.11) in terms of the parameters α^{+}, β^{+} and its velocity is given by Eqs. (4.15).

Each geodesic belonging to the family (3.6) following the trajectory determined by α^{+},β^{+} has a specific causal character. If the magnitude of the velocity is such that $v^{-} \equiv \sqrt{(v_x^{-})^2+(v_z^{-})^2}<1$, the geodesic is *timelike* ($e=-1$). When $v^{-}=1$ it is *null* ($e=0$), and for $v^{-}>1$ it is *spacelike* ($e=+1$). Moreover, we can express the condition for the null geodesics explicitly. Substituting from (4.15) we obtain a quadratic equation for $\cot\beta^{+}$ which can be solved. In the range $\alpha^{+}\in(0,\pi/2]$ there are always two real roots, namely

$$\begin{aligned} &(1) \qquad \beta_{null}^{+}=\alpha^{+}, \\ &(2) \qquad \cot\beta_{null}^{+}=\frac{\epsilon\cot\alpha^{+}-\frac{1}{2}(Z_0^{-1}+\epsilon Z_0)(1-\delta)(E-D)}{\epsilon-\frac{1}{2}(Z_0^{-1}+\epsilon Z_0)(1-\delta)2F}. \end{aligned} \tag{4.16}$$

The first equation in (4.16) implies that *all geodesics of the family* (3.6) *which move radially "outside" the impulse are null*. In fact, it follows from Eqs. (4.4) and (4.7) that $\dot{U}_0=0$, i.e., $U\equiv 0$. These are exactly those null geodesics which *generate the spherical impulse itself*. Nontrivial null geodesics which cross the impulsive wave are thus given by the second root in Eqs. (4.16). Figure 3 shows the functions $\beta_{null}^{+}(\alpha^{+})$ for the three spacetimes characterized by $\epsilon=0$, $\epsilon=-1$, and $\epsilon=+1$, respectively.

For $\epsilon=0$ it follows immediately from Eqs. (4.16) and (4.13) that $\beta_{null}^{+}=\beta_{\parallel}^{+}$. Therefore, these *null* geodesics are refracted to rays which are *parallel* to the strings behind the impulse ($v_x^{-}=0, v_z^{-}=1$). All geodesics with trajectories given by α^{+}, β^{+} such that $\beta_{\parallel}^{+}<\beta^{+}<\alpha^{+}$ are *timelike* with $v_x^{-}>0, v_z^{-}>0$. All other geodesics are spacelike.

In the case $\epsilon=-1$ it can be shown that the nontrivial root β_{null}^{+} in Eqs. (4.16) satisfies the relation $\beta_{\parallel}^{+}>\beta_{null}^{+}>\beta_{\perp}^{+}>\alpha^{+}$, as shown in Fig. 3. Again, in the region $\beta^{+}\in(\alpha^{+},\beta_{null}^{+}(\alpha^{+}))$ all the geodesics are timelike with $v_x^{-}>0$. At $\beta_{\perp}^{+}$ the velocity v_z^{-} changes sign: for $\beta^{+}<\beta_{\perp}^{+}$ we have $v_z^{-}>0$, and for $\beta^{+}>\beta_{\perp}^{+}$ we obtain $v_z^{-}<0$.

Finally, the most interesting case is $\epsilon=+1$. In this case the graph shown in Fig. 3 is even more delicate. For a particular value α_c^{+} given by the condition $\cot(\frac{1}{2}\alpha_c^{+})=(\Delta+\sqrt{\Delta^2-1})^{1-\delta}$, where $\Delta^{-2}=2\delta(1-\frac{1}{2}\delta)$, the denominator on the right-hand side of Eqs. (4.16) vanishes. Therefore, the value of β_{null}^{+} jumps at α_c^{+} from $\beta_{null}^{+}(\alpha_{c-}^{+})=\pi$ to $\beta_{null}^{+}(\alpha_{c+}^{+})=0$. (Note that α_c^{+} monotonically increases from 0 to $\pi/2$ as the string parameter δ grows from the value 0 to 1.) Consequently, there are *two disconnected regions of timelike geodesics*. In the "upper" region ($\alpha^{+}<\beta^{+}<\beta_{null}^{+}$) lies the line $\beta_{\parallel}^{+}=\beta_{\perp}^{+}$: particles moving along timelike geodesics with the "incoming position" α^{+} and a suitable "inclination" $\beta^{+}=\beta_{\parallel}^{+}(\alpha^{+})=\beta_{\perp}^{+}(\alpha^{+})$ will *exactly stop* at the point x_0^{-}, z_0^{-} in the region behind the impulse ($v_x^{-}=0=v_z^{-}$), see Eqs. (4.15). Particles below this boundary ($\beta^{+}<\beta_{\parallel}^{+}=\beta_{\perp}^{+}$) *move radially outward* since $v_x^{-}>0, v_z^{-}>0$. On the other hand, timelike particles with $\beta^{+}>\beta_{\parallel}^{+}=\beta_{\perp}^{+}$ for a given α^{+} have $v_x^{-}<0, v_z^{-}<0$ so that these *radially approach the origin*. Thus there is an *exact focusing effect of the impulse* on these timelike geodesics. The same is true for all timelike geodesics in the "lower region" corresponding to initial values α^{+} close to $\pi/2$ and small β^{+} (see Fig. 3). The time $t_f^{-}=2x_0(\cot\alpha^{+}-\cot\beta^{+})/(E-D$

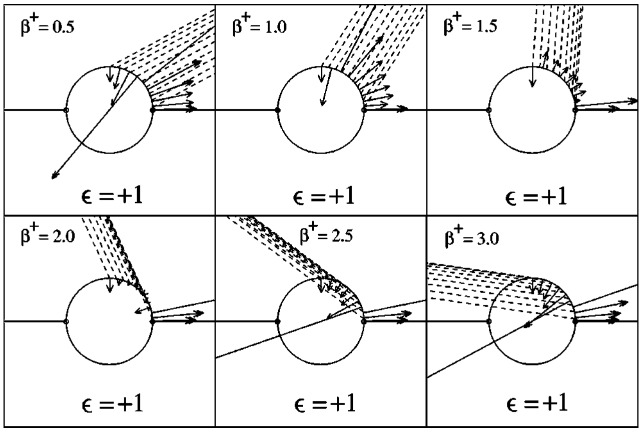

FIG. 4. In the region $U<0$ behind the expanding spherical impulse with $\epsilon=+1$, the motion of test particles is always radial, i.e., exactly (de)focusing. Incoming trajectories for various β^+ and α^+ in the region $U>0$, with the deficit angle parameter $\delta=0.3$, are indicated by dashed lines. The velocity vectors behind the wave are indicated by arrows of the corresponding length and orientation (tachyons are denoted by double arrows).

$-2F\cot\beta^+$), when each individual particle in the spacetime $\epsilon=+1$ reaches the origin, depends on the particular initial data. These geodesics are explicitly drawn in Fig. 4 with arrows indicating the precise value of the particle velocity behind the impulsive wave. Double arrows correspond to tachyons moving along spacelike geodesics. Notice that for large values of β^+ and small α^+ some of the incoming trajectories (dashed lines) are drawn inside the circle which indicates the impulse. However, this is not a contradiction as the figure represents just a snapshot at a given time. In fact, the corresponding timelike particles move in the outer region $U>0$ until they are hit by the expanding impulse (which at previous instants of time is a smaller circle). The tachyons move ''acausally'' and thus their motion is neither intuitive nor represents the motion of a test particle; this case is included for the sake of completeness.

All the above results can equivalently be obtained also by the ''inverse approach,'' i.e., starting with the initial data (3.10) behind the impulse (in the region $U<0$) and evolving these ''backward'' in time into $U>0$. The solution is given by Eqs. (3.11) which has to be substituted into Eqs. (3.7). Let us demonstrate this method by considering a simple yet important particular example. Consider a geodesic motion of *timelike* particles which are *at rest behind* the impulse generated by a snapping cosmic string, $\dot{x}_0=\dot{y}_0=\dot{z}_0=0$ (so that $\gamma=1$). In other words, we investigate the motion of those particles which are exactly stopped by the impulse. Again, we can without loss of generality assume that $y_0=0$. Then the constraint (3.12) implies $\epsilon=+1$ so that such a situation may occur only in the spacetime with this value of the parameter ϵ. Relations (3.11) then immediately yield

$$Z_0=\frac{x_0}{\tau_i-z_0},\quad V_0=\sqrt{2}\,\tau_i\,,\quad \dot{V}_0=0,\quad \dot{U}_0=-\frac{1}{\sqrt{2}}. \tag{4.17}$$

The motion of the particles in front of the wave is thus given by (3.7) with the parameters substituted from Eqs. (4.17) and (4.3). We obtain

$$x_0^+ + i y_0^+ = 2\tau_i C,\qquad \dot{x}_0^+ + i\dot{y}_0^+ = F,$$

$$z_0^+ = \tau_i(B-A),\qquad \dot{z}_0^+ = \tfrac{1}{2}(E-D), \tag{4.18}$$

$$t_0^+ = \tau_i(B+A),\qquad \dot{t}_0^+ = \tfrac{1}{2}(E+D).$$

As expected, $y_0^+=0=\dot{y}_0^+$ since the coefficients C and F are real. From the remaining relations we easily derive (using the definitions (4.13) and the fact that $\epsilon=+1$)

$$\cot\alpha^+ \equiv \frac{z_0^+}{x_0^+} = \frac{B-A}{2C} = \frac{1}{2}(Z_0^{1-\delta}-Z_0^{\delta-1}),$$

$$\cot\beta^+ \equiv \frac{\dot{z}_0^+}{\dot{x}_0^+} = \frac{E-D}{2F} \equiv \cot\beta_\parallel \equiv \cot\beta_\perp\,. \tag{4.19}$$

Of course, these results are identical to those obtained previously using the ''direct'' approach. However, now we know *explicitly* how to choose the initial data α^+, β^+ to put the particle at rest behind the impulse at time τ_i in the specific point x_0, z_0. For this, one simply substitutes $Z_0=x_0/(\tau_i-z_0)$ into Eqs. (4.19).

B. General C^1-geodesics

Let us recall again that all the geodesics in the spacetime (2.10) with the impulsive gravitational wave generated by a snapping cosmic string (4.1) which we have investigated so far, are very special, i.e., $Z=Z_0=\text{const}$ (cf. Eqs. (3.6)). They are geometrically preferred since they are *restricted to a single plane* (taken above as $y=0$) which also contains the snapping string localized along the z-axis. This fact immediately follows from the constraint (3.12). Therefore, the corresponding particles move—although not necessarily parallel—''along'' the strings. To investigate more general geodesics which ''bypass'' the strings, we have to relax the condition $Z=Z_0$. However, these general geodesics with $\dot{Z}_0=0$ cannot be found easily in the continuous form of the metric (2.10). Nevertheless, in Eqs. (3.14)–(3.16) we presented an *explicit form of general geodesics* which was derived under the assumption that these are C^1 in the continuous coordinate system (2.10).

As an interesting particular example, which can be investigated using these expressions, let us now consider geodesics in the $z=0$ plane only. This is the plane of symmetry *perpendicular to the strings*. We assume that $z_0=0=\dot{z}_0$ in the region $U<0$ behind the wave. With this, the relations (3.15) simplify to

$$Z_i=\frac{x_0+\mathrm{i}y_0}{\gamma\tau_i},\quad V_i=\frac{1}{\sqrt{2}}(1+\epsilon)\gamma\tau_i\,,$$

$$\dot{Z}_i=\frac{\dot{x}_0+\mathrm{i}\dot{y}_0}{\gamma\tau_i}-(x_0+\mathrm{i}\ y_0)\frac{(1-\epsilon)\gamma^2\tau_i+2\epsilon(x_0\dot{x}_0+y_0\dot{y}_0)}{(1+\epsilon)(\gamma\tau_i)^3},$$
$$\dot{V}_i=\frac{(1-\epsilon)^2\gamma^2\tau_i+4\epsilon(x_0\dot{x}_0+y_0\dot{y}_0)}{\sqrt{2}(1+\epsilon)\gamma\tau_i},\tag{4.20}$$
$$\dot{U}_i=\sqrt{2}\,\frac{x_0\dot{x}_0+y_0\dot{y}_0-\gamma^2\tau_i}{(1+\epsilon)\gamma\tau_i},$$

from which follows that $|Z_i|=1$. Therefore the coefficients (4.2) entering (3.16) take the following form:

$$A=B=\frac{1}{(1-\delta)(1+\epsilon)},\quad C=\frac{Z_i^{1-\delta}}{(1-\delta)(1+\epsilon)},$$
$$D=\frac{(\frac{1}{2}\ \delta)^2+\epsilon(1-\frac{1}{2}\ \delta)^2}{1-\delta},\quad E=\frac{(1-\frac{1}{2}\ \delta)^2+\epsilon(\frac{1}{2}\ \delta)^2}{1-\delta},$$
$$F=\frac{Z_i^{1-\delta}\frac{1}{2}\ \delta(1-\frac{1}{2}\ \delta)(1+\epsilon)}{1-\delta},\tag{4.21}$$
$$A_{,Z}=\frac{\frac{1}{2}\ \delta-\epsilon(1-\frac{1}{2}\ \delta)}{(1-\delta)(1+\epsilon)^2Z_i},\quad B_{,Z}=\frac{(1-\frac{1}{2}\ \delta)-\epsilon\frac{1}{2}\ \delta}{(1-\delta)(1+\epsilon)^2Z_i},$$
$$C_{,Z}=\left(\frac{\bar{Z}_i}{Z_i}\right)^{\delta/2}\frac{(1-\frac{1}{2}\ \delta)-\epsilon\frac{1}{2}\ \delta}{(1-\delta)(1+\epsilon)^2},$$
$$C_{,\bar{Z}}=\left(\frac{\bar{Z}_i}{Z_i}\right)^{\delta/2-1}\frac{\frac{1}{2}\ \delta-\epsilon(1-\frac{1}{2}\ \delta)}{(1-\delta)(1+\epsilon)^2}.$$

Substituting Eqs. (4.20), (4.21) into Eqs. (3.16) we obtain an explicit solution which describes the behavior in the region $U>0$ outside the impulse. In particular, we easily derive that

$$z_0^+\equiv\frac{1}{\sqrt{2}}(\mathcal{U}_0^+-\mathcal{V}_0^+)=0,$$
$$\dot{z}_0^+\equiv\frac{1}{\sqrt{2}}(\dot{\mathcal{U}}_0^+-\dot{\mathcal{V}}_0^+)=0.\tag{4.22}$$

Therefore, the geodesics *remain in the plane* $z=0$ also in the outside region, as is expected from the symmetry of the spacetime. Straightforward but somewhat lengthy calculations for $\eta_0^+\equiv(1/\sqrt{2})\,(x_0^++\mathrm{i}y_0^+)$, $\dot{\eta}_0^+\equiv(1/\sqrt{2})\,(\dot{x}_0^++\mathrm{i}\dot{y}_0^+)$ yield

$$x_0^++\mathrm{i}y_0^+=\frac{(\gamma\tau_i)^\delta}{1-\delta}(x_0+\mathrm{i}y_0)^{1-\delta},\tag{4.23}$$

$$\dot{x}_0^++\mathrm{i}\dot{y}_0^+=\left(\frac{x_0-\mathrm{i}y_0}{x_0+\mathrm{i}y_0}\right)^{\delta/2}[(\dot{x}_0+\mathrm{i}\dot{y}_0)-\mathcal{P}(x_0+\mathrm{i}y_0)],\tag{4.24}$$

where

$$\mathcal{P}=-\frac{\delta}{1-\delta}\left(\frac{\frac{1}{2}\ \delta(x_0\dot{x}_0+y_0\dot{y}_0)}{x_0^2+y_0^2}+\frac{1-\frac{1}{2}\ \delta}{\tau_i}\right),\tag{4.25}$$

and $\tau_i=\sqrt{(x_0^2+y_0^2)/(\dot{x}_0^2+\dot{y}_0^2-e)}$. Equations (4.23) and (4.24) describe the identification of points on the impulse, and the refraction formula in the transverse plane $z=0$, respectively. These admit a natural geometrical interpretation. If we introduce a ‘‘polar’’ representation of positions and velocities by $x_0+\mathrm{i}y_0\equiv\rho_0\exp(\mathrm{i}\phi_0)$, $\dot{x}_0+\mathrm{i}\dot{y}_0\equiv\dot{\rho}_0\exp(\mathrm{i}\dot{\phi}_0)$, we can conclude from Eq. (4.23) that $\phi_0^+=(1-\delta)\phi_0$. As the range of ϕ_0 inside (behind) the spherical impulse spans the whole Minkowski space, $\phi_0\in[0,2\pi)$, the range of the angular parameter ϕ_0^+ outside is $[0,2\pi(1-\delta))$. Therefore, there is a *deficit angle* $2\pi\delta$ in front of the impulse corresponding to the presence of the (snapped) cosmic string. This is in full agreement with the geometrical construction of the spacetime presented, e.g., in [17]. The relation (4.24) is the *refraction formula* for geodesics in the symmetry plane $z=0$ perpendicular to the strings. Interestingly, here the effect is totally *independent* of the parameter ϵ, i.e., the differences between the spacetimes characterized by $\epsilon=0,-1,+1$ disappear in this plane of symmetry. Of course, for $\delta=0$ we obtain a trivial solution $\dot{y}_0^+=\dot{y}_0$, $\dot{x}_0^+=\dot{x}_0$ in the complete Minkowski space without string and impulse. Note also that the factor

$$\left(\frac{x_0-\mathrm{i}y_0}{x_0+\mathrm{i}y_0}\right)^{\delta/2}\equiv\exp(-\mathrm{i}\delta\phi_0)\tag{4.26}$$

in Eq. (4.24) is just an appropriate ‘‘rectifying’’complex unit factor which ensures the one-to-one correspondence between the identified points on both sides of the impulse (analogously to the function $\alpha^-(\alpha^+)$ given by Eq. (4.10) for longitudinal motion). This can be seen easily if we consider two infinitesimally close parallel null geodesics $y_0=0=\dot{y}_0$ in the Minkowski region $U<0$ without topological defects. The first geodesic is given by $\phi_0=0$, the second one by an angle ϕ_0 near 2π. However, from the formula (4.24), which reads $\dot{\rho}_0^+\exp(\mathrm{i}\dot{\phi}_0^+)=\mathcal{F}\exp(-\mathrm{i}\delta\phi_0)$, where $\mathcal{F}$ is a real factor, it follows that $\dot{\phi}_0^+=-\delta\phi_0$. Therefore, outside the impulse the two geodesics which *remain parallel* are described by $\dot{\phi}_0^+=0$ and $\dot{\phi}_0^+$ near to $-2\pi\delta$, respectively. The difference $2\pi\delta$ exactly corresponds to the deficit angle in the (locally) flat space with the string outside the spherical impulse. Therefore, the ‘‘pure’’ physical refraction effect of the impulse on geodesics is described just by the expression in the square bracket on the right-hand side of the Eq. (4.24).

The above relations can easily be applied to investigate the effect of the impulsive wave on a *ring of free test particles*. Let us consider a ring in the $z=0$ plane, centered around $x=0=y$, consisting of particles which are *at rest in front* of the wave, $\dot{x}_0^+=0=\dot{y}_0^+$, in the (locally) flat Minkowski region $U>0$. All the particles are simultaneously hit by the impulse at the instant τ_i and the ring starts to deform according to Eqs. (4.23), (4.24). Obviously, it follows from Eq. (4.24) that the velocities of the particles $\dot{x}_0$, $\dot{y}_0$ behind the impulse $(U<0)$ are given by

$$\dot{x}_0=\mathcal{P}x_0, \quad \dot{y}_0=\mathcal{P}y_0, \tag{4.27}$$

with $\mathcal{P}$ given by Eq. (4.25) and $\tau_i^{-1}=\sqrt{\mathcal{P}^2+(x_0^2+y_0^2)^{-1}}$. This yields a self-consistent solution only if

$$\mathcal{P}=-\frac{\delta(1-\frac{1}{2}\delta)}{(1-\delta)\sqrt{x_0^2+y_0^2}}. \tag{4.28}$$

Thus, all the particles of the ring move *radially* towards the origin in the $z=0$ plane, with the *same velocity* $v\equiv\sqrt{\dot{x}_0^2+\dot{y}_0^2}=\delta(1-\frac{1}{2}\delta)/(1-\delta)$. The ring is deformed by the impulse into *contracting and concentric circles*. Of course, this is in accordance with the axial symmetry of the spacetime.

A more general situation in which the impulse deforms a *sphere* of test particles (around the origin) initially at rest is, however, more difficult to investigate explicitly. We can again employ the coordinate freedom $Z\rightarrow Z\exp(\mathrm{i}\phi)$ related to the axial symmetry of the spacetime which corresponds to a simple rotation of the (x,y)-plane around the z-axis. Using Eqs. (4.2) and (3.16) we conclude $\eta_0^+\rightarrow\eta_0^+\exp[\mathrm{i}(1-\delta)\phi]$, $\dot{\eta}_0^+\rightarrow\dot{\eta}_0^+\exp[\mathrm{i}(1-\delta)\phi]$. Therefore, without loss of generality we can always set for each *individual* test particle η_0^+ to be real, i.e., $y_0^+=0$. Moreover, we are considering the motion of test particles which are *at rest outside* the expanding impulsive wave, $\dot{\eta}_0^+=0$, $\dot{z}_0^+=0$. From Eqs. (3.16), (4.2), and (3.15) it then follows that Z_i and $\dot{Z}_i$ are real so that $y_0=0=\dot{y}_0$. The sphere of test particles is thus deformed into an *axially symmetric* surface which is fully described by the section $y=0$.

Setting $y_0=0=\dot{y}_0$ in Eqs. (3.15) we can now simplify $\dot{Z}_i$ to

$$\dot{Z}_i=\frac{[(1-\epsilon)\gamma\tau_i-(1+\epsilon)z_0]\dot{x}_0-[(1-\epsilon)\gamma-(1+\epsilon)\dot{z}_0]x_0}{(\gamma\tau_i-z_0)[(1+\epsilon)\gamma\tau_i-(1-\epsilon)z_0]}. \tag{4.29}$$

Consequently, for these geodesics $\dot{Z}_i=0$ if and only if the constraint (3.12) is satisfied. In such a case, the geodesics reduce to the privileged family (3.6) for which $Z_i=Z_0=\text{const}$, which we investigated in detail above. However, these special geodesics *exclude* observers which are static in the Minkowski region outside the impulse. Indeed, from the conditions $\dot{x}_0^+=0=\dot{z}_0^+$ we obtain using (3.16) the relation $(A-B)F=(D-E)C$. Substituting from (4.3) this reduces to $Z_0^{\delta-1}(\frac{1}{2}\delta p-\epsilon Z_0^2)=Z_0^{1-\delta}(\frac{1}{2}\delta p-1)$, which has no solution except for observers in the plane $z=0$ in spacetime with $\epsilon=+1$, which we investigated in Eqs. (4.27), (4.28).

Therefore, to obtain a nontrivial family of geodesics corresponding to initially static test particles, one has to consider the more complicated situation in which $\dot{Z}_i\neq0$. It is difficult to obtain the description of these geodesics in an explicit form. Nevertheless one can immediately argue that the motion cannot be spherically symmetric. For example, for the case $\epsilon=+1$ we observe from Eq. (4.29) that $z_0\dot{x}_0\neq x_0\dot{z}_0$ which, in terms of Eqs. (4.8), can be expressed as $\alpha^-\neq\beta^-$. Obviously, the trajectories of such geodesics behind the impulse *are not radial*, i.e., these do not "point" towards the origin. A sphere of free test particles which are at rest in the Minkowski region outside the expanding impulsive wave is thus not deformed into spherical shapes, but to a more complicated (axially symmetric) surface.

V. CONCLUDING REMARKS

We presented a complete solution of geodesic motion—although not always in closed explicit form—which describes the effect on free particles of expanding spherical impulsive gravitational waves propagating in a flat background. In particular, we discussed in detail the geodesics in the axially symmetric spacetimes with the impulse generated by a snapping cosmic string. The above results can be used not only for physical interpretation of the behavior of free test particles but also as a starting point for a mathematically rigorous distributional treatment of impulsive Robinson-Trautman spacetimes. To be more specific, the geodesics of the special family (3.6) provide the key to understanding the discontinuous transformation relating the distributional and the continuous form or the metric (analogous to the case of impulsive *pp*-waves; cf. [9]). These interesting questions will, however, be investigated elsewhere.

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