# Casimir effect in de Sitter and anti–de Sitter braneworlds

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We discuss the bulk Casimir effect (effective potential) for a conformal or massive scalar when the bulk represents five-dimensional anti–de Sitter (AdS) or de Sitter (dS) space with one or two four-dimensional dS branes, which may correspond to our Universe. Using zeta regularization, the interesting conclusion is reached that for both bulks in the one-brane limit the effective potential corresponding to the massive or to the conformal scalar is zero. The radion potential in the presence of quantum corrections is found. It is demonstrated that both the dS and the AdS braneworlds may be stabilized by using the Casimir force only. A brief study indicates that bulk quantum effects are relevant for brane cosmology, because they do deform the de Sitter brane. They may also provide a natural mechanism yielding a decrease of the four-dimensional cosmological constant on the physical brane of the two-brane configuration.

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## I. INTRODUCTION

If our world is really multidimensional, as *M* (string) theory predicts, then one of the most economical possibilities for its realization is the braneworld paradigm. Indeed, in the case when string theory is taken in its exact vacuum state, with the five-dimensional (asymptotically) anti–de Sitter (AdS) sector, in a full ten-dimensional space, the corresponding effective five-dimensional theory represents some (gauged) supergravity. Adding the four-dimensional surface terms predicted by the AdS conformal field theory (CFT) correspondence to such five-dimensional AdS (super)gravity, one arrives at the dynamical four-dimensional boundary (brane) of this five-dimensional manifold. Depending on the structure of the surface terms, the choice of (bulk and brane) matter, the assumptions about the general structure of the brane and bulk manifold, fields content, etc., our four-dimensional universe can be realized in a particular way as such a brane. The brane universe can be consistent with observational data even when the radius of the extra dimension is quite significant. Moreover, the braneworld point of view of our Universe may bring about a number of interesting mechanisms to resolve such well-known problems as the cosmological constant and the hierarchy problems.

As the braneworld corresponds to a five-dimensional (usually AdS) manifold with a four-dimensional dynamical boundary, it is clear that, when five-dimensional quantum field theory (QFT) is considered, the nontrivial vacuum energy (Casimir effect, see, e.g., Ref. [1] for a recent review) should appear. Moreover, when brane QFT is considered, the nontrivial brane vacuum energy also appears. The bulk Casimir effect should conceivably play a quite remarkable role in the construction of the consistent braneworlds. Indeed, it gives a contribution to both the brane and the bulk cosmological constants. Hence it is expected that it may help in the resolution of the cosmological constant problem.

For consistency, the five-dimensional braneworld should be stabilized (radion stabilization) [2], and the challenging idea is that a very fundamental quantity, the bulk vacuum energy (Casimir contribution), may be used explicitly for realizing the radion stabilization. This has been checked in a number of models [4–16], although mainly with flat branes only. An interesting connection between the bulk Casimir effect and supersymmetry breaking in braneworld [17] or moving branes [18] also exists. On the other hand, the brane Casimir effect may be used for a braneworld realization [19] of the anomaly driven (also called Starobinsky) inflation [20].

The works mentioned above discuss mainly the Casimir effect in the situation when the brane is flat space. But also the situation in which the brane is more realistic, say a de Sitter (dS) universe, has been discussed in Refs. [5,14]. It has been shown there that, in an AdS bulk, the Casimir energy for the bulk conformal scalar field in a one-brane configuration is zero. However, in situations where the bulk is different, a nonzero contribution of the Casimir energy is not excluded and even a possibility may exist of gravity trapping on the brane itself.

In the present work we study the bulk Casimir effect for a conformal or massive scalar when the bulk is a five-dimensional AdS or a dS space and the brane is a four-dimensional dS space. We show that zeta-regularization techniques at its full power [21] can be used in order to calculate the bulk effective potential in such braneworlds, in a quite general setting. One interesting result we got is that, for both bulks (AdS and dS) under discussion with one brane, the bulk effective potential is zero for a conformal as well as for a massive scalar. Applications of our results to the stabilization of the radion and to the brane dynamics are presented as well.

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The paper is organized as follows. The next section is devoted to the discussion of a general effective potential (Casimir effect) for bulk conformal scalar on AdS when the brane is a de Sitter space. The small distance behavior is investigated and the one-brane limit of the potential, which turns out to be zero, is worked out. As an application, we discuss the role of the leading term of the effective potential to the brane dynamics. It is shown here that the Casimir force only slightly deforms the shape of the four-dimensional sphere $S_4$. The radion potential (in two limits), with account of the Casimir term, is found and the stabilization of the braneworld is discussed. Using an explicit short distance expansion for the effective potential, it is demonstrated that the brane may indeed be stabilized using the Casimir force only.

In Sec. III similar questions are investigated for a conformal scalar when the brane is $S_4$, and the bulk is a five-dimensional dS space. It is interesting that the effective potential turns out to be the same as in the case of the previous section (AdS). Also, the one-brane limit of the effective potential is again zero. From the study of brane dynamics it turns out that the role of the Casimir force is again that of inducing some deformation of the $S_4$ brane (especially close to the poles).

In Sec. IV the effective potential for a massive scalar (also with scalar-gravitational coupling) is presented, for both a dS and an AdS bulk, when the brane is $S_4$. The small and large mass limits are found. The one-brane limit of the potential is again zero, even in the massive case, but the main nonzero correction to this limit is obtained explicitly. Brane stabilization due to the Casimir force for a massive scalar is discussed when the bulk is five-dimensional dS.

In Sec. V the potential for a massive scalar without a scalar-gravitational coupling is briefly studied for dS and AdS braneworlds. It is shown that it is again zero in the one-brane limit. Finally, a short summary and an outlook are presented in Sec. VI.

## II. CASIMIR EFFECT FOR A de SITTER BRANE IN A FIVE-DIMENSIONAL ANTI–de SITTER BACKGROUND

### A. Effective potential for the brane

In this section, we review the calculation of the effective potential for a de Sitter brane in a five-dimensional anti–de Sitter background, following Refs. [4,5,14]. First, we start with the action for a conformally invariant massless scalar with scalar-gravitational coupling,

$$\mathcal{S}=\frac{1}{2}\int d^5x\sqrt{g}[-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\xi_5R^{(5)}\phi^2], \tag{2.1}$$

where $\xi_5=-\frac{3}{16}$, $R^{(5)}$ being the five-dimensional scalar curvature. This action is conformally invariant under the conformal transformations:[1]

$$g_{\mu\nu}=e^{\alpha\sigma(x^\mu)}\hat{g}_{\mu\nu},\quad \phi=e^{\beta\sigma(x^\mu)}\hat{\phi}, \tag{2.2}$$

where $-\frac{3}{4}\alpha=\beta$.

Let us recall the expression for the Euclidean metric of the five-dimensional AdS bulk:

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=\frac{l^2}{\sinh^2 z}(dz^2+d\Omega_4^2), \tag{2.3}$$

$$d\Omega_4^2=d\xi^2+\sin^2\xi d\Omega_3^2, \tag{2.4}$$

where $l$ is the AdS radius which is related to the cosmological constant of the AdS bulk, and $d\Omega_3$ is the metric on the three-sphere. Two dS branes, which are four-dimensional spheres, are placed in the AdS background. If we put one brane at $z_1$, which is fixed, and the other brane at $z_2$, the distance between the branes is given by $L=|z_1-z_2|$. When $z_2$ tends to $\infty$, namely $L=\infty$, the two-brane configuration becomes a one-brane configuration.

We can see that the action Eq. (2.1) is conformally invariant under the conformal transformations for the metric Eq. (2.3) and the scalar field, which are given by

$$g_{\mu\nu}=\sinh^{-2}zl^2\hat{g}_{\mu\nu},\quad \phi=\sinh^{3/2}zl^{-3/2}\hat{\phi}. \tag{2.5}$$

The action (2.1) is not changed by the conformal transformation, Eqs. (2.5). The corresponding transformed Lagrangian looks like

$$\mathcal{L}=\phi(\partial_z^2+\Delta^{(4)}+\xi_5R^{(4)})\phi, \tag{2.6}$$

where $R^{(4)}=12$. Since we are interested in the Casimir effect for the bulk scalar in the AdS background, we shall use this Lagrangian hereafter.

The one-loop effective potential can be written as [5,14]

$$V=\frac{1}{2L\,\mathrm{Vol}(M_4)}\ln\det(L_5/\mu^2), \tag{2.7}$$

where $L_5=-\partial_z^2-\Delta^{(4)}-\xi_5R^{(4)}=L_1+L_4$. To calculate the effective potential in Eq. (2.7), we use $\zeta$ function regularization [21,25], as was done in Refs. [4,5,14]. Being precise, the very first step in this procedure consists in the introduction of a mass parameter in order to work with dimensionless eigenvalues, thus we should write at every instance $L_5/\mu^2$, etc. However, as is often done for the sake of the simplicity of the notation, we will just keep in mind the presence of this $\mu$ factor, to recover it explicitly only in the final formulas.

First, we assume that the eigenvalues of $L_1$ and $L_4$ are of the form $\lambda_n^2$, $\lambda_\alpha^2\geqslant 0$ (with $n$, $\alpha=1,2,\ldots$) respectively. In terms of these eigenvalues, $\ln\det L_5$ can be rewritten as follows:

$$\ln\det L_5=\mathrm{Tr}\ln L_5=\mathrm{Tr}\ln(L_1+L_4)=\sum_{n,\alpha}\ln(\lambda_n^2+\lambda_\alpha^2). \tag{2.8}$$

[1]Note that there is a relation between $\alpha$ and $\beta$, namely $-[(D-2)/4]\alpha=\beta$, and $\xi_D$ depends on the dimensions as $-(D-2)/4(D-1)$, for the general $D$-dimensional bulk.

Since the $\zeta$ function for an arbitrary operator $A$ is defined by

$$\zeta(s|A) \equiv \sum_m (\lambda_m^2)^{-s} = \sum_m e^{-s\ln\lambda_m^2}, \tag{2.9}$$

it turns out that $\mathrm{Tr}\ln L_5$ can be rewritten as

$$\mathrm{Tr}\ln L_5 = -\partial_s \zeta(s|L_5)|_{s=0}. \tag{2.10}$$

Furthermore, the $\zeta$ function is related to the $\Gamma$ function and heat kernel $K_t(A)$:

$$\zeta(s|A) = \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1} K_t(A), \quad K_t(A) = \sum_m e^{-\lambda_m^2 t}. \tag{2.11}$$

$L_1$ is a one-dimensional Laplace operator, and the boundary conditions result in that the brane separation $L$ can be taken as the width of a one-dimensional potential well. As a consequence, the eigenvalues of $L_1$ are given by

$$\lambda_n^2 = \left(\frac{\pi n}{L}\right)^2 \tag{2.12}$$

for finite $L$.

### B. One-brane limit ($L \to \infty$)

The above formula leads to the heat kernel $K_t(L_1)$:

$$K_t(L_1) \sim \sum_n e^{-t(\pi n/L)^2} \sim \int_0^\infty dy e^{-t(\pi y/L)^2} = \frac{L}{2\sqrt{\pi t}}, \tag{2.13}$$

where the large-$L$ limit has been taken, namely, the continuous limit of $n$. The heat kernel for $L_5$ is written in terms of $K_t(L_1)$ and $K_t(L_4)$ [25], as

$$K_t(L_5) = K_t(L_1) K_t(L_4). \tag{2.14}$$

By using Eqs. (2.11), (2.13), and (2.14), we obtain $\zeta(s|L_5)$:

$$\begin{aligned}
\zeta(s|L_5) &= \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1} K_t(L_1) K_t(L_4),\\
&\sim \frac{L}{2\sqrt{\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{1}{\Gamma\left(s-\frac{1}{2}\right)}\\
&\quad \times \int_0^\infty dt t^{(s-1/2)-1} K_t(L_4) + \mathcal{O}\left(\frac{1}{L}\right)\\
&= \frac{L}{2\sqrt{\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \zeta\left(s-\frac{1}{2}\middle| L_4\right) + \mathcal{O}\left(\frac{1}{L}\right).
\end{aligned} \tag{2.15}$$

Combined with Eq. (2.10), we obtain the effective potential in the large $L$ limit:

$$\begin{aligned}
V &= -\frac{1}{2L\,\mathrm{Vol}(M_4)}\{\zeta'(0|L_5/\mu^2) + \ln\mu^2 \zeta(0|L_5/\mu^2)\}\\
&= \frac{1}{2L\,\mathrm{Vol}(M_4)} \zeta\left(-\frac{1}{2}|L_4/\mu^2\right) + \mathcal{O}\left(\frac{\mu^2}{L}\right).
\end{aligned} \tag{2.16}$$

Note that the $\mu^2$ factor has to be taken into account for obtaining the derivative and, as discussed before, it is in fact everywhere present in each Lagrangian and its eigenvalues (although it is usually not written down in order to simplify the notation). For the spherical brane $S_4$ whose radius is $\mathcal{R}$, the four-dimensional zeta function $\zeta(s|L_4)$ is given by

$$\zeta(s|L_4) = \frac{\mathcal{R}^{2s}}{3}\left[\zeta_H\left(2s-3,\frac{3}{2}\right) - \frac{1}{4}\zeta_H\left(2s-1,\frac{3}{2}\right)\right]. \tag{2.17}$$

Here we used a Hurwitz zeta function and a Bernoulli polynomial as in Ref. [5]. This equation leads to

$$\zeta\left(-\frac{1}{2}\middle| L_4\right) = \frac{1}{3\mathcal{R}}\left[\zeta_H\left(-4,\frac{3}{2}\right) - \frac{1}{4}\zeta_H\left(-2,\frac{3}{2}\right)\right] = 0. \tag{2.18}$$

As a result, the effective potential Eq. (2.16) becomes zero (as first has been observed in Ref. [5] and has been confirmed in Ref. [14]) as $L \to \infty$. This situation corresponds to the case of a one-brane configuration.

### C. Small distance expansion

Using the power of the zeta regularization formulas [21,22], a much more precise (albeit involved) calculation can be carried out which respects at every step the complete structure of the five-dimensional zeta function. That is, the full zeta function is preserved until the end, and the final expression is given in terms of an expansion on the brane distance $L$ over the brane compactification radius $\mathcal{R}$, valid for $L/\mathcal{R} \leqslant 1$, which complements the one for large brane distance obtained above. A detailed calculation follows.

As to the specific zeta formulas employed, adhering to the classification that has been given in Ref. [22], the case at hand is indeed to be found there (even if at first sight it would not seem so). It corresponds to a two-dimensional quadratic plus linear form with truncated spectrum. In fact, this is clear from the structure of the spectrum yielding the zeta function

$$\zeta(s|L_5) = \mu^{-2s} \sum_{n,l=0}^{\infty} (\lambda_n^2 + \lambda_l^2)^{-s}, \tag{2.19}$$

where $\mu$ is a dimensional regularization scale that renders the argument of the zeta function dimensionless. In the case of the four-dimensional spherical brane of radius $\mathcal{R}$ considered above, this reduces to

$$\zeta(s|L_5)=\frac{\mu^{-2s}}{6}\sum_{n,l=0}^{\infty}(l+1)(l+2)(2l+3)\times\left[\left(\frac{\pi n}{L}\right)^2+\mathcal{R}^{-2}\left(l^2+3l+\frac{9}{4}\right)\right]^{-s}. \quad (2.20)$$

This zeta function looks awkward, at first sight. But after some reshuffling it can be brought to exhibit the standard structure mentioned. Specifically,

$$\begin{aligned}\zeta(s|L_5)=&\frac{\mathcal{R}^{2s}}{6\mu^{2s}}\sum_{n,l=0}^{\infty}2\left(l+\frac{3}{2}\right)\left\{\left[\left(l+\frac{3}{2}\right)^2+\frac{\pi^2n^2\mathcal{R}^2}{L^2}\right]^{1-s}\right.\\&\left.-\left(\frac{\pi^2n^2}{L^2}+\frac{1}{4}\right)\left[\left(l+\frac{3}{2}\right)^2+\frac{\pi^2n^2\mathcal{R}^2}{L^2}\right]^{-s}\right\}\\\equiv&\frac{\mathcal{R}^{2s}}{6\mu^{2s}}[Z_1(s)+Z_2(s)],\end{aligned} \quad (2.21)$$

where both $Z_1(s)$ and $Z_2(s)$ are obtained by taking derivatives (see Ref. [23] for a discussion of this issue, nontrivial when asymptotic expansions are involved), with respect to $x$ at $x=\frac{3}{2}$, of a zeta function of the class just mentioned, e.g.,

$$\sum_{l=0}^{\infty}[(l+x)^2+q]^{-s},\quad q\equiv\frac{\pi^2n^2\mathcal{R}^2}{L^2}. \quad (2.22)$$

In Refs. [22], explicit formulas for the analytical continuation of this class of zeta functions are given. To be brief (and forgetting for the moment about the $n$ sum, for simplicity), we just have to recall the useful asymptotic expansion

$$\begin{aligned}\sum_{n=0}^{\infty}[(n+c)^2+q]^{-s}\sim&\left(\frac{1}{2}-c\right)q^{-s}+\frac{q^{-s}}{\Gamma(s)}\\&\times\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+s)}{n!}q^{-n}\zeta_H\\&\times(-2n,c)+\frac{\sqrt{\pi}\Gamma(s-1/2)}{2\Gamma(s)}q^{1/2-s}\\&+\frac{2\pi^s}{\Gamma(s)}q^{1/4-s/2}\sum_{n=1}^{\infty}n^{s-1/2}\\&\times\cos(2\pi nc)\,K_{s-1/2}(2\pi n\sqrt{q}).\end{aligned} \quad (2.23)$$

After some calculations, we get, for $Z_1(s)$ and $Z_2(s)$,

$$\begin{aligned}Z_1(s)=&-\frac{1}{2-s}\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{2-s}\zeta(2s-4)\\&-\frac{1}{\Gamma(s-1)}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+s-2)}{n!}\\&\times\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{2-n-s}\zeta(2s+2n-4)\zeta_H'(-2n,3/2),\end{aligned} \quad (2.24)$$

$$\begin{aligned}Z_2(s)=&\frac{1}{1-s}\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{1-s}\left(\frac{\pi^2\mathcal{R}^2}{L^2}\zeta(2s-4)\right.\\&\left.+\frac{1}{4}\zeta(2s-2)\right)+\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+s-1)}{n!}\\&\times\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{1-n-s}\left(\frac{\pi^2\mathcal{R}^2}{L^2}\zeta(2s+2n-4)\right.\\&\left.+\frac{1}{4}\zeta(2s+2n-2)\right)\zeta_H'(-2n,3/2).\end{aligned} \quad (2.25)$$

Finally, for the derivative of the five-dimensional zeta function at $s=0$, we obtain

$$\begin{aligned}\zeta'(0|L_5)=&\frac{\zeta'(-4)}{6}\frac{\pi^4\mathcal{R}^4}{L^4}+\frac{\zeta'(-2)}{12}\frac{\pi^2\mathcal{R}^2}{L^2}\\&+\frac{1}{24}\left(\zeta_H'(-4,3/2)-\frac{1}{2}\zeta_H'(-2,3/2)\right)\ln\frac{\pi^2\mathcal{R}^2}{L^2}\\&+\frac{\zeta'(0)}{6}\left(\zeta_H'(-4,3/2)-\frac{1}{2}\zeta_H'(-2,3/2)\right)\\&+\frac{1}{24}\zeta_H'(-4,3/2)+\frac{1}{36}\left(\frac{1}{8}\zeta_H'(-4,3/2)\right.\\&\left.-\frac{1}{3}\zeta_H'(-6,3/2)\right)\frac{L^2}{\mathcal{R}^2}+\mathcal{O}\left(\frac{L^4}{\pi^4\mathcal{R}^4}\right)\\\simeq&0.129\,652\frac{\mathcal{R}^4}{L^4}-0.025\,039\frac{\mathcal{R}^2}{L^2}-0.002\,951\ln\frac{\mathcal{R}^2}{L^2}\\&-0.017\,956-0.000\,315\frac{L^2}{\mathcal{R}^2}+\cdots.\end{aligned} \quad (2.26)$$

### D. Dynamics of the brane

We now consider the dynamics of the dS brane, which is taken to be the four-dimensional sphere $S_4$, as in Ref. [5]. The bulk part is given by five-dimensional Euclidean Anti–de Sitter space, Eq. (2.3), which can be rewritten as

$$ds^2_{\mathrm{AdS}_5}=dy^2+l^2\sinh^2\frac{y}{l}\,d\Omega_4^2. \quad (2.27)$$

One also assumes that the boundary (brane) lies at $y=y_0$ and the bulk space is obtained by gluing two regions, given by $0\leqslant y<y_0$ (see Ref. [19] for more details).

We start with the action $S$ which is the sum of the Einstein-Hilbert action $S_{\mathrm{EH}}$, the Gibbons-Hawking surface term $S_{\mathrm{GH}}$ [24], and the surface counterterm $S_1$, e.g.,

$$S=S_{\mathrm{EH}}+S_{\mathrm{GH}}+2S_1, \quad (2.28)$$

$$S_{\mathrm{EH}}=\frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left(R_{(5)}+\frac{12}{l^2}\right), \quad (2.29)$$

$$S_{\rm GH}=\frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu, \tag{2.30}$$

$$S_1=-\frac{3}{8\pi Gl}\int d^4x\sqrt{g_{(4)}}. \tag{2.31}$$

Hereafter the quantities in the five-dimensional bulk spacetime are specified by the subindices (5) and those in the boundary four-dimensional spacetime are (4). The factor 2 in front of $S_1$ in Eq. (2.28) is coming from the fact that we have two bulk regions, which are connected with each other by the brane. In Eq. (2.30), $n^\mu$ is the unit vector normal to the boundary.

If we change the coordinate $\xi$ in the metric of $S_4$, Eq. (2.4), to $\sigma$ by

$$\sin\zeta=\pm\frac{1}{\cosh\sigma}, \tag{2.32}$$

we obtain

$$d\Omega_4^2=\frac{1}{\cosh^2\sigma}(d\sigma^2+d\Omega_3^2). \tag{2.33}$$

For later convenience, one can rewrite the metric of the five-dimensional space, Eqs. (2.27) and (2.33), as follows:

$$ds^2=dy^2+e^{2A(y,\sigma)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu,$$
$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu\equiv l^2(d\sigma^2+d\Omega_3^2). \tag{2.34}$$

From Eq. (2.34), the actions (2.29)–(2.31), have the following forms:

$$S_{\rm EH}=\frac{l^4V_3}{16\pi G}\int dy d\sigma\Bigg\{[-8\partial_y^2A-20(\partial_yA)^2]e^{4A}$$
$$+[-6\partial_\sigma^2A-6(\partial_\sigma A)^2+6]\frac{e^{2A}}{l^2}+\frac{12}{l^2}e^{4A}\Bigg\}, \tag{2.35}$$

$$S_{\rm GH}=\frac{4l^4V_3}{8\pi G}\int d\sigma e^{4A}\partial_yA, \tag{2.36}$$

$$S_1=-\frac{3l^3V_3}{8\pi G}\int d\sigma e^{4A}. \tag{2.37}$$

Here $V_3=\int d\Omega_3$ is the volume (or area) of the unit three-dimensional sphere.

As it follows from the discussion in the previous subsections, there is a gravitational Casimir contribution coming from bulk quantum fields. As one sees in the simple example of a bulk scalar, $S_{\rm Csmr}$ (leading term) has typically the following form:

$$S_{\rm Csmr}=\frac{cV_3}{\mathcal{R}^5}\int dyd\sigma e^{-A}. \tag{2.38}$$

Here $c$ is some coefficient, whose value and sign depend on the type of bulk field (scalar, spinor, vector, graviton, etc.) and on parameters of the bulk theory (mass, scalar-gravitational coupling constant, etc.). In a previous subsection we have found this coefficient for a conformal scalar. For the following discussion it is more convenient to consider this coefficient to be some parameter of the theory. Doing so, the results are quite common and may be applied to an arbitrary quantum bulk theory. We also assume that there are no background bulk fields in the theory (except for the bulk gravitational field).

Adding the quantum bulk contribution to the action $S$ in Eq. (2.28), one can regard

$$S_{\rm total}=S+S_{\rm Csmr} \tag{2.39}$$

as the total action. In Eq. (2.38), $\mathcal{R}$ is the radius of $S_4$.

In the bulk, one obtains the following equation of motion from $S_{\rm EH}+S_{\rm Csmr}$ by variation over $A$:

$$0=\left(-24\partial_y^2A-48(\partial_yA)^2+\frac{48}{l^2}\right)e^{4A}$$
$$+\frac{1}{l^2}[-12\partial_\sigma^2A-12(\partial_\sigma A)^2+12]\sigma^{2A}+\frac{16\pi Gc}{\mathcal{R}^5l^4}e^{-A}. \tag{2.40}$$

Let us discuss the solution in the situation when the scale factor depends on both coordinates: $y$ and $\sigma$. In Ref. [5], the solution of Eq. (2.40) given by an expansion with respect to $e^{-y/l}$ was found by assuming that $y/l$ is large:

$$e^A=\frac{\sinh\dfrac{y}{l}}{\cosh\sigma}-\frac{32\pi Gcl^3}{15\mathcal{R}^5}\cosh^4\sigma e^{-4y/l}+\mathcal{O}(e^{-5y/l}) \tag{2.41}$$

for the perturbation from the solution where the brane is $S_4$. On the brane at the boundary, one gets the following equation:

$$0=\left(\partial_yA-\frac{1}{l}\right)e^{4A}. \tag{2.42}$$

Substituting the solution (2.41) into Eq. (2.42), we find that

$$0\sim\left(\frac{1}{\mathcal{R}}\sqrt{1+\frac{\mathcal{R}^2}{l^2}}+\frac{2\pi Gl^2c}{3\mathcal{R}^{10}}\cosh^5\sigma-\frac{1}{l}\right). \tag{2.43}$$

Equation (2.43) tells us that the Casimir force deforms the shape of $S_4$, since $\mathcal{R}$ depends on $\sigma$. The effect becomes larger for large $\sigma$. In the case of a $S_4$ brane, the effect becomes large if the distance from the equator becomes large, since $\sigma$ is related to the angle coordinate $\xi$ by Eq. (2.32). In particular, at the north and south poles ($\xi=0,\pi$), $\cosh\sigma$ diverges and then $\mathcal{R}$ should vanish. This is not coordinate singularity. In fact, when $\sigma\to\pm\infty$, the 5D scalar curvature behaves as

$$R_{(5)} \sim -\frac{20}{l^2} + \frac{12\pi G c l}{\mathcal{R}^5} e^{7|\sigma|} e^{-7y/l} + \mathcal{O}(e^{-9y/l}). \quad (2.44)$$

This only tells, however, that the perturbation with respect to $c$ or $e^{-y/l}$ breaks down. In fact, when $\sigma$ is large, the corrections appear in the combination of the power of $e^{|\sigma|}e^{-y/l}$. Then the singularity at the poles is not a real one but if we can sum up the correction terms in all orders with respect to $e^{-y/l}$, the singularity would vanish. Then we have demonstrated that bulk quantum effects do have the tendency to support the creation of a de Sitter braneworld Universe.

The original Euclidean 5D AdS space has a isometry of SO(5, 1), which is identical with the Euclidean 4D conformal symmetry. The existence of the $S_4$ brane breaks the isometry into SO(5) rotational symmetry, which makes $S_4$ invariant. If there is no the Casimir effect, Eq. (2.43) has the SO(5) symmetry. The result in Eq. (2.43) seems to indicate that the Casimir force breaks the SO(5) symmetry. We should note that the effective action (2.38) does not seem to be invariant under the rotational symmetry since the action seems to depend on the choice of the axis connecting the north and south poles although the calculation of the Casimir effect should be invariant under the SO(5) symmetry. Since the Casimir effect is the nonlocal effect, the exact form of the effective action should be nonlocal. Then a more exact form of the effective action might be obtained, for example, by averaging the action (2.38) with respect to the choice of the axis. Such a symmetry can be, in general, broken spontaneously as in Eq. (2.43). The breakdown would occur by choosing the time direction to be parallel with the axis. Then the SO(5) symmetry is broken to SO(4), which preserves the rotations making the axis, that is, also north and south poles, invariant.

We now consider the case when the bulk quantum effects are the leading ones. From Eq. (2.43), one obtains

$$\mathcal{R}^8 \sim -\frac{4\pi G l c}{3}\cosh^5\sigma. \quad (2.45)$$

Here we only consider the leading term with respect to $c$, which corresponds to the large $\mathcal{R}$ approximation. Thus we have demonstrated that bulk quantum effects do not violate (in some cases they even support) the creation of a de Sitter brane living in a five-dimensional AdS background.

### E. Dynamics of two branes at small distance

In this subsection, we consider the dynamics of two dS branes when the distance between them is small. Before including the Casimir effect, we consider the following actions:

$$S = S_{\mathrm{EH}} + \sum_{a=\pm} (S_{\mathrm{GH}} + 2S_1), \quad (2.46)$$

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int d^5x\sqrt{g_{(5)}}\left(R_{(5)} + \frac{12}{l^2}\right), \quad (2.47)$$

$$S_{\mathrm{GH}}^{\pm} = \pm\frac{1}{8\pi G}\int d^4x\sqrt{g_{(4)}}\nabla_\mu n^\mu, \quad (2.48)$$

$$S_1^{\pm} = \mp\frac{3}{8\pi G l^{\pm}}\int d^4x\sqrt{g_{(4)}}. \quad (2.49)$$

Here the index $a=\pm$ distinguishes the two branes and we assume that the radius $\mathcal{R}^+$ ($\mathcal{R}^-$) corresponds to the larger (smaller) brane. The bulk space is AdS again and, on the branes, we obtain the following equations:

$$\frac{1}{\mathcal{R}^{\pm}}\sqrt{1-\frac{\mathcal{R}^{\pm 2}}{l^2}} = \frac{1}{l^{\pm}}. \quad (2.50)$$

The left-hand side in Eq. (2.50) is a monotonically decreasing function with respect to $\mathcal{R}$. Since the left-hand side becomes $+\infty$ when $\mathcal{R}\to 0$ and $1/l$ when $\mathcal{R}\to+\infty$, there is a solution when

$$l > l^+ > l^-. \quad (2.51)$$

We now include the Casimir effect. First, we consider the backreaction to the bulk geometry. As we assume the distance between the branes is small, the radius of the branes are almost constant. The distance $L$ in Eq. (2.26) is given by $|z^+ - z^-|$, the energy density by the Casimir effect would be proportional to $e^{-5A}/L^5$. Then the effective action would be

$$S_{\mathrm{Csmr}} = \frac{\tilde{c}V_3}{L^5}\int dy\, d\sigma e^{-A}. \quad (2.52)$$

Therefore, as in the previous section, the bulk geometry would be deformed as

$$e^A = \frac{\sinh\frac{y}{l}}{\cosh\sigma} - \frac{32\pi G\tilde{c}l^3}{15L^5}\cosh^4\sigma e^{-4y/l} + \mathcal{O}(e^{-5y/l}). \quad (2.53)$$

In this case, the equation of the brane corresponding to Eq. (2.43), has the following form:

$$0 \sim \left(\frac{1}{\mathcal{R}^{\pm}}\sqrt{1+\frac{\mathcal{R}^{\pm 2}}{l^2}} \pm \frac{2\pi G l^2\tilde{c}}{3L^{10}}\cosh^5\sigma - \frac{1}{l^{\pm}}\right). \quad (2.54)$$

Equation (2.54) tells us that the Casimir force deforms the shape of $S_4$ and the effect becomes larger for large $\sigma$, again, as in the previous section. We should note, however, that the signs of the contribution from the Casimir effect are different for the larger and smaller branes. Then if the radius of the large brane becomes large (small), that in the smaller one becomes small (large). It is interesting that if larger brane is the physical Universe, this may serve as a dynamical mechanism of decreasing the cosmological constant.

### F. Stabilization of the radion potential

In this subsection, we consider the stabilization of the radion potential following Ref. [2]. As first setup, we prepare the suitable metric and action for the discussion of the stabilization of the radion potential,

$$ds^2=e^{-2kr_c|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu-r_c^2d\phi^2. \tag{2.55}$$

Here $\phi$ is the coordinate on five dimensions, $x^\mu$ are the coordinates on the four-dimensional surfaces of constant $\phi$, and $-\pi\leqslant\phi\leqslant\pi$ with $(x,\phi)$ and $(x,-\phi)$ identified. The coordinate $z$ in the metric (2.3) corresponds to $e^{kr_c\phi}/k$ in Eq. (2.55), and the distance between two branes $L$ corresponds to $(e^{\pi kr_c}-e^{-\pi kr_c})/k$.

We assume that a potential can arise classically from the presence of a bulk scalar with interaction terms that are localized at the two three-branes. The action of the model with scalar field $\Phi$ is given by

$$S_b=\frac{1}{2}\int dx^4\int_{-\pi}^{\pi}d\phi\sqrt{G}(G^{AB}\partial_A\Phi\partial_B\Phi-m^2\Phi^2), \tag{2.56}$$

where $G_{AB}$ with $A$, $B=\mu$, $\phi$ as in Eq. (2.55). The interaction terms on the hidden and visible branes (at $\phi=0$ and $\phi=\pi$, respectively) are also given by

$$S_h=-\int d^4x\sqrt{-g_h}\lambda_h(\Phi^2-v_h^2)^2, \tag{2.57}$$

$$S_v=-\int d^4x\sqrt{-g_v}\lambda_v(\Phi^2-v_v^2)^2, \tag{2.58}$$

where $g_h$ and $g_v$ are the determinants of the induced metric on the hidden and visible branes, respectively.

The general solution for $\Phi$ which only depends on the coordinate $\phi$ is taken from the equation of motion of the action with respect to $\Phi$ to have the following form:

$$\Phi(\phi)=e^{2\sigma}[Ae^{\nu\sigma}+Be^{-\nu\sigma}], \tag{2.59}$$

where $\sigma=kr_c|\phi|$ and $\nu=\sqrt{4+m^2/k^2}$. Substituting this solution (2.59) into the action and integrating over $\phi$ yields an effective four-dimensional potential for $r_c$ which has the form [2]

$$\begin{aligned}V_\Phi(r_c)=&k(\nu+2)A^2(e^{-2\nu kr_c\pi}-1)+k(\nu-2)\\&\times B^2(1-e^{-2\nu kr_c\pi})+\lambda_v e^{-4kr_c\pi}[\Phi(\pi)^2-v_v^2]^2\\&+\lambda_h[\Phi(0)^2-v_h^2]^2.\end{aligned} \tag{2.60}$$

The unknown coefficients $A$ and $B$ are determined by imposing appropriate boundary conditions on the three-branes. Recalling Ref. [2], the coefficients $A$ and $B$ are given by

$$A=v_v e^{-(2+\nu)kr_c\pi}-v_h e^{-2\nu kr_c\pi}, \tag{2.61}$$

$$B=v_h(1+e^{-2\nu kr_c\pi})-v_v e^{-(2+\nu)kr_c\pi}, \tag{2.62}$$

for a large $kr_c$ limit. Here we take $\Phi(0)=v_h$ and $\Phi(\pi)=v_v$.

We now consider the case that $kr_c$ is large and $m/k\ll 1$ for simplicity as in Ref. [2], so that $\nu=2+\epsilon$ with $\epsilon\sim m^2/4k^2$ being a small quantity. Small $m/k$ should generate correct hierarchy [3]. We also assume $\epsilon kr_c$ is the $\mathcal{O}(1)$ quantity. Then the potential (2.60) becomes

$$\begin{aligned}V_\Phi(r_c)=&k\epsilon v_h^2+4ke^{-4kr_c\pi}(v_v-v_he^{-\epsilon kr_c\pi})^2\left(1+\frac{\epsilon}{4}\right)\\&-k\epsilon v_he^{-(4+\epsilon)kr_c\pi}(2v_v-v_he^{\epsilon kr_c\pi}),\end{aligned} \tag{2.63}$$

and its minimum is given by

$$r_0=\left(\frac{4}{\pi}\right)\frac{k}{m^2}\ln\left[\frac{v_h}{v_v}\right]. \tag{2.64}$$

When $kr_c$ is large, $L=(e^{\pi kr_c}-e^{-\pi kr_c})/k\sim e^{\pi kr_c}/k$ is also large. Then one may assume that the effective potential includes the term induced by the Casimir effect as $\alpha/L$ discussed in Sec. II B, where $\alpha$ is some constant.[2] Thus we shall add this term to the potential (2.63) and consider the first-order correction to $r_c$ with respect to $\alpha$. Then by assuming $r_c=r_0+\delta r_c$, we find the minimum of the potential is shifted by

$$\delta r_c=-\frac{\alpha e^{(3+\epsilon)\pi kr_0}}{16k\pi v_hv_v\epsilon}+\frac{1}{4k\pi}\sim-\frac{\alpha e^{3\pi kr_0}}{16k\pi v_hv_v\epsilon}, \tag{2.65}$$

where terms of order $\epsilon^2$ and the higher-order terms with respect to $e^{-\pi kr_0}$ are neglected. The role of Casimir effect is in only to shift slightly the minimum.

In the small $kr_c$ limit, which corresponds to the small $L$ limit as well, the coefficients $A$ and $B$ in the radion potential (2.60) are changed as follows:

$$A=\frac{1}{2kr_c\pi\nu}\{v_v[1+kr_c\pi(\nu-2)]-v_h\}, \tag{2.66}$$

$$\begin{aligned}B=&\frac{1}{2kr_c\pi\nu}\{-v_v[1+kr_c\pi(\nu-2)]\\&+v_h(1+2\nu kr_c\pi)\}.\end{aligned} \tag{2.67}$$

In this limit, we suppose that $m/k\gg 1$, so that $\nu\sim m/k$, which makes the situation simple. The effective potential might include the term induced by the Casimir effect as $\beta/L^5$ discussed in Sec. II C, where $\beta$ is some constant. Then, the radion potential in the small $kr_c$ limit is

$$\begin{aligned}V_\Phi(r_c)=&2mr_c\pi k\left(\frac{m}{k}+2\right)A^2+2mr_c\pi k\left(\frac{m}{k}-2\right)B^2+\frac{\beta}{L^5}\\&\sim\frac{1}{r_c\pi}(v_v-v_h)^2+\frac{\beta}{(2\pi r_c)^5},\end{aligned} \tag{2.68}$$

being here $L\sim 2\pi r_c$. To obtain the minimum of the potential, we differentiate Eq. (2.68) with respect to $r_c$:

$$\frac{d}{dr_c}V_\Phi(r_c)=-\frac{1}{r_c^2\pi}(v_v-v_h)^2-\frac{5\beta}{(2\pi)^5r_c^6}. \tag{2.69}$$

Then, if $\beta\leqslant 0$, the extremum of the potential is reached at

[2]Note that a Casimir term may be induced by other bulk fields.

$$r_c = \pm \frac{1}{2\pi(v_v - v_h)^{1/2}} \left( -\frac{5\beta}{2} \right)^{1/4}. \tag{2.70}$$

The extremum is, however, maximum then the stabilization should be local. Let us give some numbers. If $v_v$, $v_h \sim (10^{19}\ \mathrm{GeV})^{3/2}$ and $\beta \sim (10^{19}\ \mathrm{GeV})^{-1}$, we have that $r_c \sim (10^{19}\ \mathrm{GeV})^{-1}$ and $kr_c$ could be of $\mathcal{O}(1)$. Thus it is not so unnatural for the hierarchy problem.

For the short $r_c$ case, we may not include the scalar field $\Phi$ in Eq. (2.56) but instead we may include the next-to-leading order of the effective potential (2.26), induced by the Casimir effect, although the next-to-leading term should be neglected for the flat brane corresponding to $\mathcal{R} \to +\infty$:

$$V_C(r_c) = \frac{\beta_1}{(2\pi r_c)^5} + \frac{\beta_2}{(2\pi r_c)^3}. \tag{2.71}$$

From Eq. (2.26), we see that $\beta_1 > 0$ and $\beta_2 < 0$. As a consequence, in the above potential, there is a minimum at

$$r_c = \frac{1}{2\pi} \sqrt{-\frac{5\beta_1}{3\beta_2}} \simeq 0.4675 l. \tag{2.72}$$

The result in Eq. (2.26) is not for flat brane but for de Sitter brane and only including the contribution from massless scalar. We also put a length parameter $l$ in Eq. (2.72). Then the numerical value in Eq. (2.72) would be changed but hopefully the main structure would not be changed. We conclude therefore that with the only consideration of the Casimir effect, the brane might get stabilized, which is a nice result.[3]

As we will see later in Eq. (4.10), when one considers the massive scalar with small mass, there appears the correction to the effective potential. Motivated with such result, one considers the following correction to the effective potential, which corresponds to the leading term in Eq. (4.10) when $L$ is small:

$$\Delta V_C(r_c) = \frac{\beta_3 m^2}{2\pi r_c}. \tag{2.73}$$

Here $m$ expresses the mass of the scalar field. The result in Eq. (4.10) suggests that $\beta_3$ is negative. By assuming that the correction term (2.73) is dominant compared with the third (logarithmic) term in Eq. (2.26), the minimum in Eq. (2.72) is shifted as

$$r_c = \frac{1}{2\pi} \sqrt{-\frac{5\beta_1}{3\beta_2}} \left( 1 + \frac{5\beta_1\beta_3 m^2}{18\beta_2^2} + \mathcal{O}(m^4) \right). \tag{2.74}$$

Then the contribution from small mass has a tendency to make the distance between the two branes smaller.

[3]Note, however, that thermal effects [6] may significantly change the above discussion.

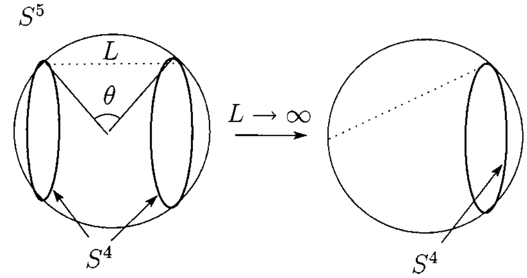


FIG. 1. The two dS branes are placed in the $dS_5$ background. The two-brane configuration becomes a one-brane configuration as $L \to \infty$.

## III. CASIMIR EFFECT FOR THE DE SITTER BRANE IN A FIVE-DIMENSIONAL DE SITTER BACKGROUND

### A. Effective potential for the brane

Next, we use the Euclideanized form of the five-dimensional de Sitter (dS) metric for a four-dimensional dS brane as follows:

$$ds^2 = l^2(d\theta^2 + \sin^2\theta d\Omega_4^2) = \frac{l^2}{\cosh^2 z}(dz^2 + d\Omega_4^2), \tag{3.1}$$

where $l$ is the dS radius, which is related to the cosmological constant of the dS bulk.

We place two dS branes—which are four-dimensional spheres, as in the AdS bulk case—in a dS background as the one depicted in Fig. 1. Since the parameter $\theta$ in Eq. (3.1) takes values between 0 and $\pi$, the parameter $z$ takes values between $-\infty$ and $\infty$. As in the AdS bulk case, the distance between the branes can be defined as $L = |z_1 - z_2|$. When $z_2$ is placed at $\infty$, namely $L = \infty$, the two-brane configuration becomes a one-brane configuration, as seen in Fig. 1.

The Casimir effect for the bulk scalar in dS background can be calculated by using the same method as in AdS bulk.

Namely, the Lagrangian for a conformally invariant massless scalar with scalar-gravitational coupling is obtained by conformal transformation of the action Eq. (2.1) for the metric and the scalar field given by

$$g_{\mu\nu} = \cosh^{-2} z l^2 \hat{g}_{\mu\nu}, \quad \phi = \cosh^{3/2} z l^{-3/2} \hat{\phi}. \tag{3.2}$$

Then the Lagrangian is of the same form of Eq. (2.6).

The one-loop effective potential is calculated by means of $\zeta$-function regularization techniques. Then, the calculated result for the effective potential in the large $L$ limit is of the same form of Eq. (2.16). Since the effective potential in Eq. (2.16) becomes zero at $L \to \infty$, the effective potential of the one-brane configuration becomes zero. Note that this means that the effective potential for $B_5$, which is the right part in Fig. 1, is zero. Concerning the small distance expansion, for a potential corresponding to a conformally invariant scalar we have an expression as Eq. (2.26). No essential difference is encountered in this case.

### B. Dynamics of the brane

The dynamics of dS brane in a five-dimensional Euclidean de Sitter bulk can be considered in a similar way as for the AdS bulk. The brane is de Sitter, and is taken to be a four-dimensional sphere $S_4$, as in the previous section. The five-dimensional Euclidean de Sitter space Eq. (3.1) can be rewritten as

$$ds^2_{\mathrm{dS}_5} = dy^2 + \sin^2\frac{y}{l}\, d\Omega_4^2. \tag{3.3}$$

Here, we adopt Eq. (2.33) for the metric of $S_4$. We assume that the brane lies at $y = y_0$ and that the bulk is obtained by gluing two regions given by $0 \leqslant y < y_0$.

The total action $S$ is the sum of the Einstein-Hilbert action $S_{\mathrm{EH}}$, the Gibbons-Hawking surface term $S_{\mathrm{GH}}$, and the surface counter term $S_1$, like in the AdS bulk case:

$$S = S_{\mathrm{EH}} + S_{\mathrm{GH}} + 2S_1. \tag{3.4}$$

The Einstein-Hilbert action $S_{\mathrm{EH}}$ is

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int d^5x\,\sqrt{g_{(5)}}\left(R_{(5)} - \frac{12}{l^2}\right). \tag{3.5}$$

The Gibbons-Hawking surface term $S_{\mathrm{GH}}$ and the surface counter term $S_1$ are of the same forms as in Eqs. (2.30) and (2.31).

For later convenience, we rewrite the metric of the five-dimensional dS space, Eqs. (3.3) and (2.33), as follows:

$$ds^2 = dy^2 + e^{2A(y,\sigma)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu,$$

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu \equiv l^2(d\sigma^2 + d\Omega_3^2). \tag{3.6}$$

By using Eq. (3.6), the action Eq. (3.5) becomes

$$S_{\mathrm{EH}} = \frac{l^4V_3}{16\pi G}\int dy\,d\sigma\Bigg([-8\partial_y^2A - 20(\partial_yA)^2]e^{4A} + [-6\partial_\sigma^2A - 6(\partial_\sigma A)^2 + 6]\frac{e^{2A}}{l^2} - \frac{12}{l^2}e^{4A}\Bigg), \tag{3.7}$$

which is similar to the AdS bulk case, Eq. (2.35), except for the last term, i.e., the cosmological constant. The Gibbons-Hawking surface term $S_{\mathrm{GH}}$ and the surface counter term $S_1$, Eqs. (2.30) and (2.31), have also the same form of Eqs. (2.36) and (2.37). We also consider the gravitational Casimir contribution due to bulk quantum fields. So we add the action of the Casimir effect $S_{\mathrm{Csmr}}$, Eq. (2.38), to the total action $S$, Eq. (3.4).

In the bulk, we obtain the following equation of motion from $S_{\mathrm{EH}} + S_{\mathrm{Csmr}}$ by variation over $A$:

$$0 = \left(-24\partial_y^2A - 48(\partial_yA)^2 - \frac{48}{l^2}\right)e^{4A} + \frac{1}{l^2}[-12\partial_\sigma^2A - 12(\partial_\sigma A)^2 + 12]e^{2A} + \frac{16\pi Gc}{\mathcal{R}^5l^4}e^{-A}. \tag{3.8}$$

For the AdS bulk case, the solution of Eq. (3.8) can be found as an expansion with respect to $e^{-y/l}$, assuming that $y/l$ is large. But for the dS bulk case, we cannot adopt the same method, since the function $\sin y/l$ cannot be regarded as an expansion with respect to $e^{-y/l}$. Thus we assume the solution to have the following form:

$$e^A = \frac{\sin\frac{y}{l}}{\cosh\sigma} + \delta A. \tag{3.9}$$

Substituting Eq. (3.9) into Eq. (3.8), we obtain

$$0 = \frac{1}{l^2}\left(-\frac{6\sin\frac{y}{l}}{\cosh\sigma} + \frac{\cosh\sigma}{\sin\frac{y}{l}} - \frac{2}{\cosh\sigma\sin\frac{y}{l}}\right)\delta A - \frac{4}{l}\frac{\cos\frac{y}{l}}{\cosh\sigma}\partial_y(\delta A) - \frac{2\sin\frac{y}{l}}{\cosh\sigma}\partial_y^2(\delta A) - \frac{\cosh\sigma}{l^2\sin\frac{y}{l}}\partial_\sigma^2(\delta A) - \frac{4\pi Gc}{3\mathcal{R}^5l^4}\left(\frac{\cosh\sigma}{\sin\frac{y}{l}}\right)^3. \tag{3.10}$$

We now investigate the behavior of Eq. (3.10) at the north and south poles ($\xi = 0,\ \pi$), that is, as $\cosh\sigma$ diverges. In this case, Eq. (3.10) is approximated as

$$0 \sim \frac{e^\sigma}{2l^2\sin\frac{y}{l}}\delta A - \frac{e^\sigma}{2l^2\sin\frac{y}{l}}\partial_\sigma^2(\delta A) - \frac{4\pi Gc}{3\mathcal{R}^5l^4}\left(\frac{e^\sigma}{2\sin\frac{y}{l}}\right)^3, \tag{3.11}$$

and then

$$\delta A - \partial_\sigma^2(\delta A) \propto \frac{\pi Gc}{3\mathcal{R}^5l^2}\frac{e^{2\sigma}}{\sin^2\frac{y}{l}}. \tag{3.12}$$

Here, we have used the approximation $\cosh\sigma \sim e^\sigma/2$. From Eq. (3.12), we assume

$$\delta A = \alpha\frac{e^{2\sigma}}{\sin^2\frac{y}{l}}, \tag{3.13}$$

where $\alpha$ is the constant which is obtained by substituting Eq. (3.13) into Eq. (3.12), thus

$$\alpha=-\frac{\pi Gc}{9\mathcal{R}^5 l^2}. \tag{3.14}$$

The region of the equator $\xi=\pi/2$, namely, $\cosh\sigma\sim 1+1/2\sigma^2$, Eq. (3.10), is approximated as

$$0\sim-\left[\frac{1}{l^2}\left(6\sin\frac{y}{l}+\frac{2}{\sin\frac{y}{l}}\right)\delta A+\frac{4}{l}\cos\frac{y}{l}\partial_y(\delta A)+2\sin\frac{y}{l}\partial_y^2(\delta A)\right]\left(1-\frac{1}{2}\sigma^2\right). \tag{3.15}$$

On the brane at the boundary, we get the same Eq. (2.42):

$$0=\left(\partial_y A-\frac{1}{l}\right)e^{4A}. \tag{3.16}$$

Finally, by substituting the solutions (3.9) into Eq. (3.16), we find

$$0=\frac{1}{l\cosh\sigma}\left(\cos\frac{y}{l}-\sin\frac{y}{l}\right)+\partial_y(\delta A). \tag{3.17}$$

In the region at the north and south poles, $\cosh\sigma\sim e^{|\sigma|}/2$, if we assume $y=(\pi/4)l+\delta y$, from Eq. (3.17), $\delta y$ is obtained by

$$\delta y=\frac{\sqrt{2}\pi Gc}{9\mathcal{R}^5 l}e^{3|\sigma|}. \tag{3.18}$$

Thus the deformation of the brane seems to become large at the north and south pole.

We should note the expression in Eq. (3.18) diverges at north and south poles where $\sigma\to\pm\infty$. As in case of AdS bulk in the previous section, this indicates that the perturbation with respect to $c$ breaks down. The original Euclidean 5D dS space has a isometry of SO(6), which is broken by the existence of the $S_4$ brane into SO(5). Due to the Casimir effect, the SO(5) symmetry seems to be broken to SO(4), again.

## IV. EFFECTIVE POTENTIAL FOR A MASSIVE SCALAR FIELD IN THE AdS AND dS BULKS

Until now we have dealt with a massless scalar. In this section we will consider a massive scalar field in AdS and dS backgrounds. Let us start with the action for a massive scalar with scalar-gravitational coupling,

$$\mathcal{S}=\frac{1}{2}\int d^5x\sqrt{g}[-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2\phi^2+\xi_5 R^{(5)}\phi^2]. \tag{4.1}$$

For the AdS background with the metric Eq. (2.3), under the conformal transformations (2.5), the action changes as

$$\mathcal{S}=\frac{1}{2}\int d^5x\sqrt{g}[-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2l^2\sinh^{-2}z\,\phi^2+\xi_5R^{(5)}\phi^2], \tag{4.2}$$

which yields the Lagrangian for the massive scalar field with scalar-gravitational coupling in an AdS background as

$$\mathcal{L}=\phi(\partial_z^2+\Delta^{(4)}-m^2l^2\sinh^{-2}z+\xi_5R^{(4)})\phi. \tag{4.3}$$

In the above Lagrangian, there appears a singularity at $z=0$. The point $z=0$ corresponds to $\infty$, where the warp factor blows up to infinity. Then by putting a brane as the boundary of the bulk, say putting a brane at $z=z_0<0$ (or $z_0>0$) and considering the region $z<z_0$ (or $z>z_0$) as bulk space, the singularity does not appear. And as we can see in the Appendix, if we include the singular point $z=0$, half of the solutions are excluded but there remain the other half of the solutions. From this Lagrangian, we can calculate the one-loop effective potential like in the case of a massless scalar field. The form of the effective potential from the massive scalar field is given by

$$\begin{aligned}V&=\frac{1}{2L\,\mathrm{Vol}(M_4)}\ln\det(L_5/\mu^2),\\ L_5&\equiv-\partial_z^2+m^2l^2\sinh^{-2}z-\Delta^{(4)}-\xi_5R^{(4)}\\ &=L_1+L_4,\end{aligned} \tag{4.4}$$

where the mass term is included in $L_1$. The eigenvalue of $L_1$ is different from that in Eq. (2.12), for finite $L$, since $L_1$ in Eq. (4.4) is the one-dimensional Schrödinger operator with the potential term $m^2l^2\sinh^{-2}z$. But this potential term, which is positive valued and has no bound state, becomes zero in the limit $z_2\to\infty$, that is, when the distance between branes $L$ becomes $\infty$. In this case, the eigenvalue of $L_1$ reduces to the same form of Eq. (2.12) and thus the effective potential becomes zero at the limit of a one-brane configuration.

For the case of a dS background, Eq. (3.1), the conformal transformations, Eqs. (3.2) change the action (4.1) as follows:

$$\mathcal{S}=\frac{1}{2}\int d^5x\sqrt{g}[-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2\cosh^{-2}z\,\phi^2+\xi_5R^{(5)}\phi^2]. \tag{4.5}$$

Then, the Lagrangian for a massive scalar field in the dS background is given by

$$\mathcal{L}=\phi(\partial_z^2+\Delta^{(4)}-m^2\cosh^{-2}z+\xi_5R^{(4)})\phi. \tag{4.6}$$

Similarly, the effective potential for the massive scalar field in the dS bulk can be calculated as in Eqs. (2.7) and (4.4), by using the operators:

$$L_5\equiv-\partial_z^2+m^2\cosh^{-2}z-\Delta^{(4)}-\xi_5R^{(4)}=L_1+L_4, \tag{4.7}$$

where the mass term is included in $L_1$. The potential term of $L_1$, $m^2\cosh^{-2}z$, has always a positive value and no bound state like in the AdS case. It becomes zero in the limit $z_2\to\infty$ as well. Therefore the effective potential for the massive scalar field in a dS background also becomes zero in the limit of a one-brane configuration.

### A. Small mass limit (with *L* not large)

Continuing with the massive scalar field, and for a de Sitter brane in an AdS bulk, in the case of the two brane configuration we just need to supplement the calculation carried out in the Appendix, which can be done exactly, with the boundary conditions imposed on the two branes. We thus obtain a modification of a perfectly solvable model which appears in several textbooks (namely, an hyperbolic variant of the celebrated Pöschl-Teller potential), albeit with reverse sign and supplemented with the infinite well created by the branes (as in the massless case). Since we shall deal with the low and high mass approximations, the WKB method turns out to be well suited to carry out the analysis.

Setting the branes at $z=\pm L/2$ (for the sake of symmetry) we get the following results. In the small mass limit, we obtain a modification of the eigenvalues of the $L_1$ Lagrangian, in the form

$$\lambda_n^2\simeq\frac{\pi^2n^2}{\mu^2L^2}+m^2l^2\frac{\tanh(\mu L/2)}{\mu L/2}. \tag{4.8}$$

Carrying this into the zeta function, after a further approximation one gets that the elementary zeta functions in the formulas are modified in the way, e.g.,

$$\zeta(2s)\to\zeta(2s)-s\zeta(2s+2)\rho+\frac{s(1+s)}{2}\zeta(2s+4)\rho^2+\mathcal{O}(m^6),$$

$$\rho\equiv\frac{m^2l^2\mu^2L^2}{\pi^2}\,\frac{\tanh(\mu L/2)}{\mu L/2}. \tag{4.9}$$

Thus in the case here considered, when $m$ is small and $L$ is not very large, for the derivative of the zeta function at $z=0$ we obtain the following additional terms ($l^2\mu^2=1$):

$$\Delta\zeta'(0|L_5)\simeq\frac{a\rho+a^2\rho^2}{48}-\frac{\pi^2}{144}\left(\frac{a\rho^2}{2}+[2\zeta'(-4,3/2)-\zeta'(-2,3/2)]\rho\right)-\frac{\pi^4}{4370}[2\zeta'(-4,3/2)-\zeta'(-2,3/2)]\rho^2+\mathcal{O}(m^6),$$

$$a\equiv\frac{\pi^2\mathcal{R}^2}{L^2},\quad \rho\equiv\frac{m^2l^2}{\pi^2}\,\frac{\tanh(L/2l)}{L/2l}. \tag{4.10}$$

These terms have just to be added to the derivative of the zeta function at $z=0$, Eq. (2.26), corresponding to the de Sitter brane in AdS bulk, in order to obtain the corresponding effective potential. In a full-fledged analysis of the different contributions to the effective potential, one has to take into account the relative importance of the different dimensionless ratios involved here. The working hypothesis has been that $m^2$ was "small." In fact, we see from the final result that $m^2$ most naturally goes with $l^2$, which also serves as a unit for $L$ and, indirectly, for $\mathcal{R}$. The ordering in Eq. (4.10) assumes that $a\rho\sim1$, $\rho<1$, but a lot more information can be extracted from this small-mass expansion.

The calculation in the same case of a massive scalar field but for a de Sitter brane in a dS bulk (two- and one-brane configurations) proceeds in a quite similar fashion. Only, an additional coordinate change is required at the beginning, to deal with the problem of the singularity of the potential of the Schrödinger equation at $z=0$ in the initial coordinates, as carefully explained in the Appendix.

### B. Large mass limit (with *L* not small)

In this case the calculation turns out to be more involved. The eigenvalues get modified as follows:

$$\lambda_n^2\simeq\frac{\pi^2n^2l^2}{L^2}+\frac{2\,\arctan(\sinh L/2l)}{\sinh(L/2l)}m^2l^2+\frac{\pi nml^2}{L\sinh(L/2l)}+\cdots. \tag{4.11}$$

However, we will be interested in the dominant contribution only. Thus in the approximation which is opposite to the previous one, namely when $m^2$ is large and $L$ is not very small, we get a simple modification of the relevant zeta function of the form

$$\zeta(s|L_5)=\frac{L}{2l\sqrt{\pi}}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\times\zeta\left(s-\frac{1}{2}\middle|L_4+2m^2\frac{\arctan(\sinh(L/2l))}{\sinh(L/2l)}\right)+\cdots. \tag{4.12}$$

And this leads to the following result, for the derivative of the zeta function at $z=0$, which is valid for sufficiently large scalar mass and $L$:

$$\zeta'(0|L_5)=-\frac{4m^2l^3}{3\mathcal{R}}\,\frac{\arctan(\sinh L/2l)}{\sinh(L/2l)}+\cdots. \tag{4.13}$$

Again, this is the additional contribution to the derivative of the full zeta function at $z=0$, the same as Eq. (2.18) but corresponding to the de Sitter case. However, as this derivative was equal to zero in the massless case, the above expression yields now the *whole* value of the derivative and, correspondingly, of the effective potential. Note in fact that this reduces to zero, exponentially fast, in the one-brane limit ($L\to\infty$), in perfect accordance with Eq. (2.18). Also in this case we are allowed to play with the relative values of the different dimensionless fractions appearing in our expression.

### C. Braneworld stabilization by the Casimir force

In Ref. [13], the brane stabilization via study of radion potential in the Lorentzian de Sitter bulk space was discussed in direct analogy with the AdS case. The branes are spacelike and the distance between two branes is timelike and we denote the distance by $T$. As in Eqs. (2.71)–(2.74), we now consider the contribution from the Casimir effect to the stabilization. For simplicity, we do not include the massive scalar field $\Phi$ as in Eq. (2.56) but we take the next-to-leading order of the effective potential (2.26), induced by the Casimir effect, and we assume

$$V_C(T)=\frac{\beta_1^{\mathrm{dS}}}{T^5}+\frac{\beta_2^{\mathrm{dS}}}{T^3}. \tag{4.14}$$

If $\beta_1^{\mathrm{dS}}>0$ and $\beta_2^{\mathrm{dS}}<0$ as in Eq. (2.26), there is a minimum at

$$T=\sqrt{-\frac{5\beta_1^{\mathrm{dS}}}{3\beta_2^{\mathrm{dS}}}}. \tag{4.15}$$

Then even for the branes in the de Sitter bulk, only by the Casimir effect, the brane might get stabilized.

As in Eq. (4.10), when we consider the Casimir effect from the massive scalar with small mass, we may consider the following correction to the effective potential:

$$\Delta V_C(T)=\frac{\beta_3 m^2}{T}. \tag{4.16}$$

Here $m$ expresses the mass of the scalar field. Then the minimum in Eq. (4.15) is shifted as

$$r_c=\sqrt{-\frac{5\beta_1^{\mathrm{dS}}}{3\beta_2^{\mathrm{dS}}}}\left(1+\frac{5\beta_1^{\mathrm{dS}}\beta_3^{\mathrm{dS}}m^2}{18\beta_2^{\mathrm{dS}^2}}+\mathcal{O}(m^4)\right). \tag{4.17}$$

Then again the contribution from small mass has a tendency to make the distance between the two branes smaller. Thus the possibility of dS braneworld stabilization occurs in the same way as with AdS bulk.

## V. EFFECTIVE POTENTIAL FOR A MASSIVE SCALAR WITHOUT SCALAR-GRAVITATIONAL COUPLING

In this section we will consider a more simple case, which does not include a scalar-gravitational coupling term, $\xi_5 R^{(5)}\phi^2$. The action is simply

$$\mathcal{S}=\frac{1}{2}\int d^5x\sqrt{g}[-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2\phi^2]. \tag{5.1}$$

This action is not conformally invariant under the conformal transformations (2.2), which change it as

$$\begin{aligned}\mathcal{S}=&\frac{1}{2}\int d^5x\sqrt{\hat{g}}[-e^{3\sigma}\hat{g}^{\mu\nu}\partial_\mu(e^{-(3/2)\sigma}\hat{\phi})\partial_\nu(e^{-(3/2)\sigma}\hat{\phi})\\&-m^2e^{2\sigma}\hat{\phi}^2]\\=&\frac{1}{2}\int d^5x\sqrt{\hat{g}}\left(-\hat{g}^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}-\frac{9}{4}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\hat{\phi}^2\right.\\&\left.+3\hat{\phi}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\hat{\phi}-m^2e^{2\sigma}\hat{\phi}^2\right),\end{aligned} \tag{5.2}$$

where we take $\alpha=2$ and $\beta=-3/2$ for simplicity. The third term in Eq. (5.2) can be rewritten as

$$\hat{\phi}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\hat{\phi}=\frac{1}{2}D^\mu(\hat{\phi}^2\partial_\mu\sigma)-\frac{1}{2}\hat{\phi}^2\Delta^{(5)}\sigma \tag{5.3}$$

and using partial integration, we obtain

$$\begin{aligned}\mathcal{S}=&\frac{1}{2}\int d^5x\sqrt{\hat{g}}\Big[-\hat{g}^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}\\&-\left(\frac{9}{4}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma+\frac{3}{2}\Delta^{(5)}\sigma\right)\hat{\phi}^2-m^2e^{2\sigma}\hat{\phi}^2\Big].\end{aligned} \tag{5.4}$$

If we now introduce the AdS background, which has the metric Eq. (2.3), under the conformal transformations (2.5), namely $e^{2\sigma}=l^2\sinh^{-2}z$, the action changes as

$$\begin{aligned}\mathcal{S}=&\frac{1}{2}\int d^5x\sqrt{g}\Big[-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\left(\frac{9}{4}+\frac{15}{4}\sinh^{-2}z\right)\phi^2\\&-m^2l^2\sinh^{-2}z\,\phi^2\Big].\end{aligned} \tag{5.5}$$

This action leads the Lagrangian for the massive scalar field without scalar-gravitational coupling in an AdS background as

$$\mathcal{L}=\phi\left[\partial_z^2+\Delta^{(4)}-\left(\frac{9}{4}+\frac{15}{4}\sinh^{-2}z\right)-m^2l^2\sinh^{-2}z\right]\phi. \tag{5.6}$$

Note that the third term in Eq. (5.6),

$$-\left(\frac{9}{4}+\frac{15}{4}\sinh^{-2}z\right), \tag{5.7}$$

corresponds to

$$\zeta_5(R^{(4)}-R^{(5)}e^{2\sigma}), \tag{5.8}$$

coming from Eqs. (2.1) and (2.6), where $e^{2\sigma}=l^2\sinh^{-2}z$, because if we put $\xi_5=-\frac{3}{16}$, $R^{(4)}=12$, $R^{(5)}=-20/l^2$, which are the scalar curvatures of $S_4$ and $\mathrm{AdS}_5$, respectively, into Eq. (5.8), then Eq. (5.8) coincides with Eq. (5.7) exactly.

The one-loop effective potential can be written as

$$V=\frac{1}{2L\,\mathrm{Vol}(M_4)}\ln\det(L_5/\mu^2),$$

$$L_5 = -\partial_z^2 - \Delta^{(4)} + \left(\frac{9}{4} + \frac{15}{4}\sinh^{-2} z\right) + m^2 l^2 \sinh^{-2} z = L_1 + L_4,$$

$$L_1 = -\partial_z^2 + \frac{15}{4}\sinh^{-2} z + m^2 l^2 \sinh^{-2} z,$$

$$L_4 = \frac{9}{4} - \Delta^{(4)}. \tag{5.9}$$

Then, the eigenvalue of $L_1$ agrees with Eq. (2.12) in the limit when the distance between the two-brane becomes infinity, $L \to \infty$, because the potential terms of Eq. (5.9), $\frac{15}{4}\sinh^{-2} z + m^2 l^2 \sinh^{-2} z$, become zero in this limit. Therefore the effective potential for the massive scalar field without scalar-gravitational coupling in an AdS background becomes zero in the limit of the one-brane configuration.

Similarly, the Lagrangian for the massive scalar field without scalar-gravitational coupling in a dS background can be seen to be

$$\mathcal{L} = \phi\left[\partial_z^2 + \Delta^{(4)} - \left(\frac{9}{4} - \frac{3}{4}\cosh^{-2} z\right) - m^2 l^2 \cosh^{-2} z\right]\phi. \tag{5.10}$$

In the limit $L \to \infty$, the eigenvalue of $L_1$ and the heat kernel $K_t(L_1)$ have the same form of Eqs. (2.12) and (2.13) as in the AdS case. Thus the effective potential becomes zero too, in the limit when the distance between the two branes becomes infinite.

## VI. DISCUSSION AND CONCLUSIONS

To summarize, in this paper we have shown how one can bring the calculation of the effective potential for a massive or conformal bulk scalar, in an AdS or dS braneworld with a dS brane, down to well-known cases corresponding to zeta-function expansions [21]. In this way, a complete and detailed analysis of the different situations can be given, and corrections to the limiting cases are obtainable at any order. As our four-dimensional universe is (or will be) in a dS phase, our results have, potentially, very interesting applications to primordial cosmology. What is also important, our method and results here open the door to corresponding calculations for other quantum fields as spinors, vectors, graviton, gravitino, etc. As we see it, this will only need some more involved calculations, but no new conceptual problems are expected, at least at the level of the one-loop efective potential. In the case of several spin fields, the bulk Casimir effect may also be found in this way, at least in principle, for supersymmetric theories, including supergravity too. It is quite possible then that a five-dimensional AdS gauged supergravity can be constructed, with AdS being the vacuum state but still having a dynamically realized de Sitter brane, which represents our observable universe.

Another issue where bulk quantum effects may play a dominant role involves moving, curved branes. The corresponding bulk effective potential might sometimes be a measure of supersymmetry breaking, and thus be of primordial cosmological importance in the study of the very early brane universe.

Finally, the bulk effective potential in realistic supersymmetric theories gives a nontrivial contribution to the effective cosmological constant, in five as well as in four dimensions. Hence it is conceivable to use it in a relaxation of the cosmological constant problem.

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## APPENDIX

We consider the following Schrödinger equation:

$$\left(-\frac{d^2}{dz^2} + \frac{m^2 l^2}{\sinh^2 z}\right)\phi = \lambda\phi. \tag{A1}$$

This equation is the $z$-dependent part of the Klein-Gordon equation in $\mathrm{AdS}_5$ and $\hat{\phi} = \sinh^{-3/2} z\,\phi$ corresponds to the original scalar field in the action. The limit $z=\infty$ corresponds to the infinity in $\mathrm{AdS}_5$ at which the warp factor vanishes, and $z=0$ corresponds to the infinity where the warp factor grows up to infinity. In Eq. (A1) there appears a singularity at $z=0$. At the point $z=0$ corresponding to $\infty$, by putting a brane as the boundary of the bulk, say putting a brane at $z=z_0<0$ (or $z_0>0$), and considering the region $z<z_0$ (or $z>z_0$) as bulk space, the singularity does not appear.

With the redefinitions

$$\phi = \sinh^{1/2} z\,\psi, \quad x = \cosh z, \tag{A2}$$

Eq. (A1) can be rewritten as

$$0 = (x^2-1)\frac{d^2\psi}{dx^2} + 2x\frac{d\psi}{dx} - \left(-\lambda - \frac{1}{4} + \frac{m^2 l^2 + \frac{1}{4}}{x^2-1}\right)\psi, \tag{A3}$$

whose solutions are given by the associated Legendre functions $P_\nu^{\pm\mu}(x)$, which are defined in terms of the Gauss hypergeometric function:

$$P_\nu^\mu(z)=\frac{1}{\Gamma(1-\mu)}\left(\frac{x+1}{x-1}\right)^{\mu/2}F\left(-\nu,\nu+1,1-\mu;\frac{1-x}{2}\right). \tag{A4}$$

The parameters $\mu$ and $\nu$ are here given by

$$\mu^2=l^2m^2+\frac{1}{4},\quad \nu(\nu+1)=-\lambda-\frac{1}{4}\text{ or }\ \nu=\frac{-1\pm\sqrt{-4\lambda}}{2}. \tag{A5}$$

When $x$ is large, $P_\nu^\mu(x)$ behaves as

$$P_\nu^\mu(x)\sim\frac{1}{\sqrt{\pi}}\left[\frac{\Gamma\left(\nu+\frac{1}{2}\right)(2x)^\nu}{\Gamma(\nu-\mu+1)}+\frac{\Gamma\left(-\nu+\frac{1}{2}\right)}{\Gamma(-\nu-\mu)(2x)^{\nu+1}}\right]. \tag{A6}$$

Since $\phi\sim x^{1/2}\psi$, then in order that $\phi$ is regular there, we have the constraint that

$$-4\lambda\leqslant 0\quad\text{or}\quad\lambda\geqslant 0, \tag{A7}$$

which is identical with what we have in the massless case. When we include the point $z=0$, which corresponds to $x=1$, when $\sqrt{x-1}\sim z\to 0$, Eq. (A4) becomes singular for positive $\mu$ as $(x-1)^{-\mu/2}\sim z^{-\mu}$. As $\phi\sim z^{1/2}\psi\sim z^{(1/2)-\mu}=z^{(1/2)(1-\sqrt{1+4l^2m^2})}$, the positive branch of $\mu$ should be excluded and we must have $\mu=-\sqrt{l^2m^2+1/4}$.

If we do not include the brane, the spectrum for the massive case is not changed. In order to investigate the effect of the mass, we put a brane at $x=x_0\gg 1$ (or $z=z_0$). On the brane, we impose the Neumann boundary condition for $\phi$:

$$\frac{d\phi}{dz}=0,\quad\left(\Leftrightarrow\frac{d\phi}{dx}=0\right). \tag{A8}$$

For simplicity, we consider the model where the bulk space includes the point $x=1$ $(z=0)$; hence $\mu=-\sqrt{l^2m^2+1/4}$. We write $\mu$ and $\nu$ in Eq. (A5) as

$$\mu=-\omega-\frac{1}{2},\quad \nu=-\frac{1}{2}+i\omega. \tag{A9}$$

Then we have $\lambda=\omega^2$. By using Eq. (A6), we find, for large $x$,

$$\phi(x)\sim\frac{\Gamma(i\omega)}{\Gamma(i\omega+k)}(2x)^{i\omega}+\frac{\Gamma(-i\omega)}{\Gamma(-i\omega+k)}(2x)^{-i\omega}. \tag{A10}$$

Then the boundary condition (A8) yields

$$\frac{\Gamma(i\omega)}{\Gamma(i\omega+k)}(2x_0)^{i\omega}-\frac{\Gamma(-i\omega)}{\Gamma(-i\omega+k)}(2x_0)^{-i\omega}. \tag{A11}$$

If we assume $\omega$ and $k$ to be small, the gamma function can be approximated by $\Gamma(\pm i\omega)\sim\pm 1/i\omega$ and $\Gamma(\pm i\omega+k)\sim 1/\pm i\omega+k$. Then, Eq. (A11) can be rewritten as

$$\ln\left(\frac{1+i\dfrac{k}{\omega}}{1-i\dfrac{k}{\omega}}\right)=i\omega\ln(2x_0)+2\pi in\quad(n=0,\pm1,\pm2,\ldots). \tag{A12}$$

For large $x_0$, the solution for $n=0$ is given by

$$\omega\sim\frac{\pi}{\ln(2x_0)} \tag{A13}$$

for nonvanishing $k$ $(m\neq 0)$, which gives the following lower bound for $\lambda$:

$$\lambda=\omega^2\geqslant\left(\frac{\pi}{\ln(2x_0)}\right)^2\sim\frac{\pi^2}{z_0^2}. \tag{A14}$$

We now consider the equation for the dS case:

$$\left(-\frac{d^2}{dz^2}+\frac{m^2l^2}{\cosh^2 z}\right)\phi=\lambda\phi. \tag{A15}$$

This equation is the $z$-dependent part of the Klein-Gordon equation in $S_5$ or Euclidean de Sitter space, and $\hat{\phi}=\cosh^{-3/2}z\,\phi$ corresponds to the original scalar field in the action. The limit of $z=\pm\infty$ corresponds to the south and north poles in $S_5$. With the following redefinitions:

$$\phi=\cosh^{1/2}z\,\psi,\quad x=\cosh z, \tag{A16}$$

Eq. (A15) can be rewritten as

$$0=(x^2+1)\frac{d^2\psi}{dx^2}+2x\frac{d\psi}{dx}-\left(-\lambda-\frac{1}{4}+\frac{m^2l^2+\frac{1}{4}}{x^2+1}\right)\psi. \tag{A17}$$

If we replace $x$ by $x=iy$, the above equation (A17) turns into

$$0=(y^2-1)\frac{d^2\psi}{dy^2}+2x\frac{d\psi}{dx}-\left(-\lambda-\frac{1}{4}-\frac{m^2l^2+\frac{1}{4}}{y^2-1}\right)\psi. \tag{A18}$$

Finally, if we choose, as in Eq. (A5),

$$\mu^2=-\left(l^2m^2+\frac{1}{4}\right),\quad \nu(\nu+1)=-\lambda-\frac{1}{4}\ \text{ or}$$

$$\nu=\frac{-1\pm\sqrt{-4\lambda}}{2}, \tag{A19}$$

the solution of Eq. (A18) or Eq. (A17) is given by the associated Legendre functions $P_\nu^{\pm\mu}(ix)$, again. Note that $\mu$ in Eq. (A19) is imaginary, in general. Anyhow, in order that $\hat{\phi}$ be regular there, we must impose again the same constraint (A17).

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