# Canonical quantization of noncommutative field theory

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A simple method to canonically quantize noncommutative field theories is proposed. As a result, the elementary excitations of a $(2n+1)$-dimensional scalar field theory are shown to be bilocal objects living in an $(n+1)$-dimensional space-time. Feynman rules for their scattering are derived canonically. They agree, upon suitable redefinitions, with the rules obtained via star-product methods. The IR-UV connection is interpreted within this framework.

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## I. INTRODUCTION AND SUMMARY

Noncommutative field theories [1] are interesting, nonlocal but most probably consistent, extensions of the usual ones. They also arise as a particular low energy limit of string theory [2,3]. The fields are defined over a base space which is noncommutative [1], often obeying relations of the type $[x_\mu, x_\nu] = i\theta_{\mu\nu}$. At the classical level, new physical features appear in these theories. For instance, one encounters solitonic excitations in higher dimensions [4], superluminal propagation [5], or waves propagating on discrete spaces [6]. At the quantum level, one has two superimposed structures: the coordinate space, where $[\hat{x}_\mu, \hat{x}_\nu] \neq 0$, and the dynamical fields' (fiber) space, where canonically conjugate variables do not commute, $[\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{x})] \neq 0$.

This two-level structure hampered the canonical quantization of noncommutative (NC) field theories. Consequently, their perturbative quantum dynamics has been studied via star-product techniques [1], i.e., by replacing operator products with the Groenewold-Moyal one. This leads to deformed theories, living on a commutative space of Weyl symbols. Perturbation theory is then defined in the usual way. Loop calculations performed in this setup pointed to an intriguing mixing between short distance and long distance physics, called the IR-UV connection [7,8,9].

The purpose of this paper is to develop simple operatorial techniques for the direct quantization of noncommutative fields. In addition to naturalness, they present various advantages with respect to Moyal (phase space [10]) methods. First, they allow a simpler derivation of the Feynman rules. Second, canonical methods offer a clear picture of the degrees of freedom of the theory, a picture not yet rigorously established in NC spaces, in spite of many interesting works [11,12]. We are able to prove that the fundamental excitations of a $(2n+1)$-dimensional scalar theory with commuting time are *bilocal* objects living in a *lower*, $(n+1)$-, dimensional space-time. We call them rods, or dipoles, although no charge of any kind enters their description. The information on the remaining $n$ spatial directions is encoded into the length and orientation of the dipoles, through $n$ parameters proportional to the momentum a NC particle would have had in the "lost" (conjugate) directions. This picture puts on firmer ground a general belief [11,12] that NC theories are about dipoles, not particles. Moreover, it shows that the dipoles live in a lower dimensional space. The Feynman rules we obtain for them show, however, that this dimensional reduction is limited to tree level dynamics: loop integrations being taken also over the dipole parameters, the $(2n+1)$-dimensionality of the theory is effectively restored as far as renormalization is concerned. Third, we are able to give a more precise interpretation of the IR-UV mixing than in [8]. Namely, the interaction "vertices" for dipoles have in general a finite area, and a poligonal boundary. As far as this area is kept finite, loop amplitudes are effectively regulated by noncommutativity. However, if the area shrinks to zero (in planar diagrams, or nonplanar ones with zero external momentum), the NC phase is of no effect, and UV infinities are present. They metamorphose into IR divergences if the cause of the vertex shrinking is an external momentum going to zero.

## II. BILOCAL OBJECTS

Let us consider a $(2+1)$-dimensional scalar field $\Phi(t,\hat{x},\hat{y})$ defined over a commutative time $t$ and a pair of NC coordinates satisfying

$$[\hat{x},\hat{y}] = i\theta. \tag{1}$$

The extension to $n$ NC pairs is straightforward. Commutative spatial directions are dropped, for simplicity. The action is

$$S = \frac{1}{2}\int dt\, \mathrm{Tr}_{\mathcal{H}}[\dot{\Phi}^2 - (\partial_x\Phi)^2 - (\partial_y\Phi)^2 - m^2\Phi^2 - 2V(\Phi)]. \tag{2}$$

$\hat{x}$ and $\hat{y}$ act on a harmonic oscillator Hilbert space $\mathcal{H}$ in the usual way. $\mathcal{H}$ may be given a discrete basis $\{|n\rangle\}$ formed by eigenstates of $\hat{x}^2 + \hat{y}^2$, or a continuous one $\{|x\rangle\}$, composed of eigenstates of, say, $\hat{x}$. We will discuss explicitely quartic potentials, $V(\Phi) = g/4!\,\Phi^4$. Cubic potentials are actually simpler, but maybe less relevant physically.

To quantize $\Phi$, start with a usual classical commuting field, expanded into normal modes with coefficients $a$ and $a^*$. Upon usual field quantization, $a$ and $a^*$ become the operators $\hat{a}$ and $\hat{a}^+$ acting on a standard Fock space $\mathcal{F}$. To

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make the underlying space noncommutative, we introduce Eq. (1) and apply the Weyl quantization procedure [13] to the exponentials $e^{i(k_x x+k_y y)}$. The result is

$$\Phi=\int\int\frac{dk_x dk_y}{2\pi\sqrt{2\omega_{\mathbf{k}}}}[\hat{a}_{k_x k_y}e^{i(\omega_{\mathbf{k}}t-k_x\hat{x}-k_y\hat{y})}+\hat{a}^{\dagger}_{k_x k_y}e^{-i(\omega_{\mathbf{k}}t-k_x\hat{x}-k_y\hat{y})}], \tag{3}$$

which means the following: $\Phi$ is a "doubly"-quantum field operator, acting on a direct product of two Hilbert spaces, $\Phi:\mathcal{F}\otimes\mathcal{H}\rightarrow\mathcal{F}\otimes\mathcal{H}$. The NC field theory satisfying Eq. (1) has two superimposed operatorial structures: the NC coordinate space and the quantum field space. Physically, $\Phi$ creates (destroys), via $\hat{a}^{\dagger}_{k_x k_y}$ ($\hat{a}_{k_x k_y}$), an excitation represented by a "plane wave" $e^{i(\omega_{\mathbf{k}}t-k_x\hat{x}-k_y\hat{y})}$. The bilocal nature of such a wave will be demonstrated now.

We could work with $\Phi$ as an operator ready to act on both Hilbert spaces $\mathcal{F}$ and $\mathcal{H}$. It is, however, simpler to "saturate" it on $\mathcal{H}$, working with expectation values $\langle x'|\Phi|x\rangle$, which can still act on $\mathcal{F}$. $|x\rangle$ is an eigenstate of $\hat{x}$, $\hat{x}|x\rangle=x|x\rangle$, $\hat{y}|x\rangle=-i\theta\partial/\partial x|x\rangle$. This means keeping only one coordinate out of a pair of NC spatial directions (for $n$ pairs, commutativity is gained on the reduced space at the expense of strict locality). A key equation is now

$$\langle x'|e^{i(k_x\hat{x}+k_y\hat{y})}|x\rangle=e^{ik_x(x+k_y\theta/2)}\delta(x'-x-k_y\theta)=e^{ik_x[(x+x')/2]}\delta(x'-x-k_y\theta). \tag{4}$$

This is a bilocal expression, and we already see that its span along the $x$ axis, $(x'-x)$, is proportional to the momentum along the conjugate $y$ direction, i.e., $(x'-x)=\theta k_y$. Using Eqs. (3) and (4), one sees that

$$\langle x'|\Phi|x\rangle=\int\frac{dk_x}{2\pi\sqrt{2\omega_{k_x,k_y}}}[\hat{a}_{k_x,k_y}e^{i\{\omega_{\mathbf{k}}t-k_x[(x+x')/2]\}}+\hat{a}^{\dagger}_{k_x,-k_y}e^{-i\{\omega_{\mathbf{k}}t+k_x[(x+x')/2]\}}], \tag{5}$$

where $k_y=(x'-x)/\theta$. Thus, $\Phi$ annihilates a rod of momentum $k_x$ and length $\theta k_y$, and creates a rod of momentum $k_x$ and length $-\theta k_y$. Due to Eq. (1), one degree of freedom apparently disappears from Eq. (5). Its presence shows up in the now $(1+1)$-dimensional space only through the modified dispersion relation

$$\omega_{[k_x,k_y=(x'-x)/\theta]}=\sqrt{k_x^2+\frac{(x'-x)^2}{\theta^2}+m^2}. \tag{6}$$

Of course, this form is seen in the $(1+1)$-dimensional space in which the rods propagate. From the $(2+1)$-dimensional NC point of view, one recovers a standard Klein-Gordon-like dispersion relation through the substitution $k_y=(x'-x)/\theta$. Equation (6) is not a modified dispersion relation in the sense of [14], but a consequence of bilocality *and* dimensional reduction.

## III. CORRELATORS

Let us now calculate two-point correlation functions for such rods. The expectation value of the product of two bilocal fields, taken on the Fock space $\mathcal{F}$ vacuum $|0\rangle$, is

$$\langle 0|\langle x_4|\Phi|x_3\rangle\langle x_2|\Phi|x_1\rangle|0\rangle=\int\frac{dk_x}{8\pi^2\omega_{\mathbf{k}}}e^{ik_x[(x_3+x_4)/2-(x_1+x_2)/2]}\times\delta(x_4-x_3-x_2+x_1), \tag{7}$$

where $k_y=(x'-x)/\theta$, and $\omega_{\mathbf{k}}=\omega_{k_x,k_y}$ obeys Eq. (6) again. Again, there is no integral along $k_y$. More precisely, if one compares Eq. (7) to the $(1+1)$-dimensional commutative correlator of two fields, $\langle 0|\phi(X_2)\phi(X_1)|0\rangle$, with $X_1=(x_1+x_2)/2$ and $X_2=(x_3+x_4)/2$, the only differences are the additional $(x'-x)^2/\theta^2$ term in Eq. (6), and the delta function $\delta([x_4-x_3]-[x_2-x_1])$, which ensures that the length of the rod (the momentum along $y$) is conserved. Thus, our bilocal objects propagate in a $(1+1)$-dimensional space. The extra $y$ direction is accounted for by their length, which contributes to the energy, and orientation. We will also call these rods dipoles, although they have no charges at their ends (at least for real scalar fields), and they are extended objects in the absence of any background. One may speculate on possible relations of these rods with stretched open strings, or with the double index representation for Yang-Mills theories.

## IV. INTERACTIONS

The quartic interaction term in Eq. (2) can be written as

$$\int dt\,\mathrm{Tr}_{\mathcal{H}}V(\Phi)=\frac{g}{4!}\int dt\int_{x,a,b,c}\langle x|\Phi|a\rangle\langle a|\Phi|b\rangle\langle b|\Phi|c\rangle\times\langle c|\Phi|x\rangle. \tag{8}$$

We will have a look at some terms in the Dyson series generated by Eq. (8) to illustrate the canonical derivation of the Feynman rules. Let: $\hat{A}\hat{B}$: denote normal ordering of $\hat{A}\hat{B}$. Once the vacuum correlator (7) is known, the derivation of the diagrammatic rules follows the standard procedure; hence we will not present it in detail. To find the basic "vertex" for four-dipole scattering we evaluate

$$\left\langle -\mathbf{k}_3,-\mathbf{k}_4\right|:\int dt\int_{x,a,b,c}\langle x|\Phi|a\rangle\langle a|\Phi|b\rangle\langle b|\Phi|c\rangle\times\langle c|\Phi|x\rangle:\left|\mathbf{k}_1,\mathbf{k}_2\right\rangle, \tag{9}$$

$|\mathbf{k}_1,\mathbf{k}_2\rangle$ is a Fock space state, meaning two quanta are present, with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$. The momenta $\mathbf{k}_{i,i=1,2,3,4}$ have each two components: $\mathbf{k}_i=(k_i,l_i)$. $k_i$ is the momentum along $x$, whereas $l_i$ represents the dipole extension along $x$ (corresponding to the momentum along $y$). Using Eq. (5) and integrating over $x$, $y$, $z$, and $u$, one obtains the conservation laws $k_1+k_2+k_3+k_4=0$ and $l_1+l_2+l_3+l_4=0$. The final result differs from the four-point scattering vertex of (2+1) commutative particles with momenta $\mathbf{k}_i=(k_i,l_i)$ only through the phase

$$e^{-i\theta/2}\sum_{i<j}(k_il_j-l_ik_j). \tag{10}$$

Interpreting $l_i$ as the $i$th momentum along $y$, this is precisely the star-product modification of the usual Feynman rules. Our approach makes clear that the phase (10) appears due to the bilocal nature of generic $\langle x'|\Phi|x\rangle$'s. Pointlike $\langle x|\Phi|x\rangle$'s would never produce it.

By contracting adjacent (nonadjacent) terms in Eq. (9), one obtains the planar (nonplanar) one-loop correction to the free rod propagator, together with the recipe for calculating loops. Again, the derivation is straightforward. The main result is that one has to integrate over both the momentum and length of the dipole circulating in a loop. This $1/2\pi\int dk_{\mathrm{loop}}\int dl_{\mathrm{loop}}$ integration, together with the dispersion relation (6), brings back into play—as far as divergences are concerned—the $y$ direction. It is easy to extend the above reasoning to $(2n+1)$-dimensions: unconstrained dipoles will propagate in a $(n+1)$-dimensional commutative spacetime; their Feynman rules are obtained as outlined above. Once the dipole lengths are interpreted as momenta in the conjugate directions, our rules are identical to those obtained long ago via star-product calculus. The calculational aspects have been extensively explored [1,7,8,9] in the last years. Our physical interpretation is, however, different, and in this light we will discuss the IR-UV connection.

## V. IR-UV

We have derived directly from the field theory the dipolar character of the NC scalar field excitations. We saw that, in the $\{|x\rangle\}$ basis, the momentum in the conjugate direction becomes the length of the dipole. Thus a connection between ultraviolet (large momentum) and infrared physics (large distances) becomes evident. This puts on a more rigorous basis the argument of [8] concerning the IR-UV connection.

Moreover, we can provide a geometrical view of the differences between planar and nonplanar loop diagrams, and the role of low momenta in nonplanar graphs. Let us go to (4+1) directions, $t$, $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{u}$, and assume $[\hat{x},\hat{y}]=[\hat{z},\hat{w}]=i\theta$. Consider a $\{|x,z\rangle\}$ basis. Then we can speak of a commutative space spanned by the axes $x$ and $z$, on which dipoles with momentum $\mathbf{p}=(p_x,p_z)$ and length $\mathbf{l}=(l_x,l_z)=\theta(p_y,p_w)$ evolve. Consider the scattering of four such dipoles, Their "meeting place" is a poligon with four edges and area $\mathcal{A}$ [Fig. 1(a)]. One has two possibilities for the one-loop correction to the propagator: planar and nonplanar. In the planar case, adjacent dipole fields are contracted. Momentum and length conservation enforce then the poligon to degenerate into a one-dimensional, zero-area object [Fig. 1(b)]. UV divergences persist. In the nonplanar case, due to the nonadjacent contraction the area $\mathcal{A}$ does not go to zero [cf., Fig. 1(c)] unless the external dipole length vanishes [Fig. 1(d)]. $\mathcal{A}\neq 0$ appears thus to be related to the disappearance of UV divergences. Actually, the true regulator is the

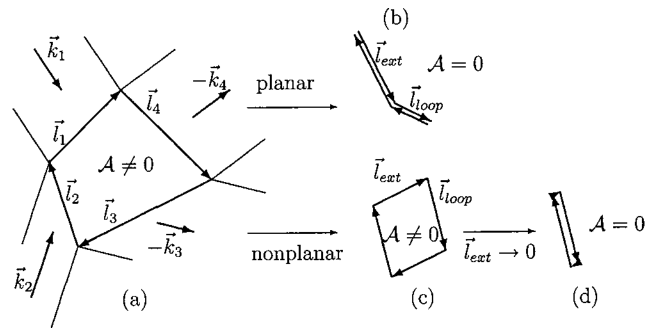


FIG. 1. Area vs finiteness.

phase (10). This is zero, i.e., ineffective, when $\mathcal{A}=0$ in *both* the $|x,z\rangle$ and $|y,u\rangle$ bases. That corresponds to zero external length *and* momentum in the dipole picture, which means that the resulting divergence is half IR ($\mathbf{p}_{\mathrm{ext}}=0$) and half UV ($\mathbf{l}_{\mathrm{ext}}=0$). In Weyl space this is just the usual zero external momentum, say $p_\mu^{\mathrm{ext}}=0$, and one speaks about an IR divergence. For dipoles the divergence comes from having zero vertex area $\mathcal{A}$ in any basis, and is half IR and half UV. NC field theory (NCFT) is somehow between usual field theory and string theory: when the interaction vertex is a point, UV infinities appear; when it opens up, as in string theory, amplitudes are finite.

## VI. REMARKS

We saw that by dropping $n$ coordinates, intuition is gained: the remaining space admits a notion of distance, although bilocal (and in some sense IR-UV dual) objects probe it. Other bases of $\mathcal{H}$ can also be used. For instance, the basis $\{|n\rangle\}$, formed by eigenvectors of $\hat{n}\sim x^2+y^2$, leads to a discrete remnant space [6]. Although the phase operator conjugated to $\hat{n}$ is not easy to define, the multilocal character of the excitations is preserved.

One could put the scalar fields on a torus by imposing periodic boundary conditions. In this case (discrete) high momenta along $y$ would correspond to dipoles which wind around the circle spanned by $x$. This relationship between winding and momentum states is reminescent of T duality, and suggests that the canonical description may be employed in describing Morita equivalence.

An important question is: how do the dimensionality and noncommutativity of space-time depend on the regime in which we probe the theory? To start, we have a NC $(2n+1)$-dimensional theory. Then, at the tree level (i.e., classical plus tree level interference effects), one has $D=n+1$ commuting directions. However, loop effects drive us back to $D=2n+1$. At a scale $r\sim\sqrt{\theta}$, space is surely NC. For $r\gg\sqrt{\theta}$ it is believed to be commutative. However, if $r$ is the radius in the largest available commutative subspace, the IR-UV connection suggests a connection (duality?) between the $r\gg\sqrt{\theta}$ and $r\ll\sqrt{\theta}$ regimes. A clarification of these issues is desirable.

In conclusion, we found a simple way to quantize scalar NCFT through canonical methods. This provides a quantitative description for the kinematics and dynamics of such theories—including limits in the dimensional reduction one may hope for, and a simple reinterpretation of the IR-UV connection. Although the Feynman rules derived in this way were previously known and used, we believe we provided a simple and clear picture for the degrees of freedom of the theory. This alternative point of view may find interesting applications, e.g., along the lines sketched in the above remarks. An extension of the method to gauge theories, as well as a path integral approach, are presently under study.

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