

Dissipation in equations of motion of scalar fields

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The methods of nonequilibrium quantum field theory are used to investigate the possibility of representing dissipation in the equation of motion for the expectation value of a scalar field by a friction term, such as is commonly included in phenomenological inflaton equations of motion. A sequence of approximations is exhibited which reduces the nonequilibrium theory to a set of local evolution equations. However, the adiabatic solution to these evolution equations which is needed to obtain a local equation of motion for the expectation value is not well defined; nor, therefore, is the friction coefficient. Thus, a nonequilibrium treatment is essential, even for a system that remains close to thermal equilibrium, and the formalism developed here provides one means of achieving this numerically.

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I. INTRODUCTION

Inflationary universe scenarios frequently refer to an inflaton equation of motion of the form

$$\ddot{\phi} + 3H\dot{\phi} + \eta(\phi)\dot{\phi} + V_{\text{eff}}(\phi) = 0. \quad (1.1)$$

The friction term $\eta(\phi)\dot{\phi}$ represents one mechanism through which the inflationary universe might be reheated, converting the inflaton energy into particles [1–3], although the process of parametric amplification [4–6] is in some cases a more important effect. Moreover, it was pointed out long ago by Moss [7] that a frictional effect large enough to maintain a thermal bath of particles during the inflationary era would have a significant effect both on the inflaton evolution and on the primordial perturbation spectrum, alleviating the fine-tuning problems that afflict many inflationary models. More recently, and independently, a similar idea has been extensively investigated by Berera and co-workers [8–10].

While some frictional effect is to be expected in any interacting field theory, it is by no means clear at a fundamental level that this effect can properly be represented by a local equation of motion such as (1.1), and the purpose of the present paper is to study this question in some detail. (A brief account of our findings has been given in Ref. [11].) To be fairly generic, we may consider a scalar field Φ , which couples to other scalar fields and to fermions, denoted collectively by ξ and ψ respectively. In Minkowski spacetime, a schematic Lagrangian density for this collection of fields is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m_\Phi^2 \Phi^2 - \frac{1}{4!} \lambda \Phi^4 - \frac{1}{2} g_1 \Phi^2 \xi^2 - g_2 \Phi \bar{\psi} \psi + \Delta \mathcal{L}(\xi, \psi) \quad (1.2)$$

where $\Delta \mathcal{L}$ is that part of the Lagrangian that does not depend on Φ . (The considerations developed here can probably be extended with little difficulty to a theory in which Φ also couples to gauge fields. They are essentially unchanged in a

Robertson-Walker spacetime, as we discuss in Sec. IX.) On splitting $\Phi(t, \mathbf{x})$ into its expectation value $\phi(t)$, which we take to be spatially homogeneous, and a fluctuation field $\varphi(t, \mathbf{x})$ with $\langle \varphi(t, \mathbf{x}) \rangle = 0$, we find that the equation of motion for $\phi(t)$ is

$$\begin{aligned} \ddot{\phi} + m_\Phi^2 \phi + \frac{1}{6} \lambda \phi^3 + \frac{1}{2} \lambda \phi \langle \varphi^2 \rangle + g_1 \phi \langle \xi^2 \rangle + g_2 \langle \bar{\psi} \psi \rangle \\ + \frac{1}{6} \lambda \langle \varphi^3 \rangle = 0, \end{aligned} \quad (1.3)$$

while at the tree level, the quantum fields φ , ξ , and ψ acquire effective masses

$$\begin{aligned} m_\varphi^2(t) = m_\Phi^2 + \frac{1}{2} \lambda \phi^2(t), \quad m_\xi^2(t) = m_\xi^2 + g_1 \phi^2(t), \\ m_\psi(t) = m_\psi + g_2 \phi(t) \end{aligned} \quad (1.4)$$

which depend on time through $\phi(t)$. Dissipative effects in the equation of motion (1.3) can arise from the final term $\frac{1}{6} \lambda \langle \varphi^3 \rangle$, but we concentrate here on those arising from the other expectation values, which are quadratic in the quantum fields. Generically, the equation of motion is

$$\ddot{\phi}(t) + m_\Phi^2 \phi(t) + g \langle \chi(t, \mathbf{x})^2 \rangle \phi(t) + \dots = 0 \quad (1.5)$$

where χ may stand for φ , the fluctuating part of Φ , for another scalar field ξ or for a fermionic field ψ , and has a $\phi(t)$ -dependent mass of the form (1.4). In general, there will be a sum of such contributions from the fields appearing in a specific model, but our considerations are substantially model independent.

Superficially, it is quite plausible that a system which remains fairly close to equilibrium can be treated by using equilibrium statistical mechanics. In Sec. II, we review the relevant dissipative mechanisms and the results for the friction coefficient $\eta(\phi)$ which have been obtained on this basis; we also indicate why this approach might be questioned, when applied to a system that is not maintained in exact thermal equilibrium. Sections III–V develop a strategy for

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tackling the problem in nonequilibrium scalar field theory, exhibiting the sequence of approximations necessary to obtain a local equation of motion of the form (1.1). We find that this local equation of motion contains coefficients whose values must be found from the solution of auxiliary kinetic equations. The resulting set of local evolution equations is suitable for a numerical solution. However, we investigate in Sec. VI whether an adiabatic treatment of these evolution equations leads to a well defined friction coefficient $\eta(\phi)$. We find that it does not, and Sec. VII exhibits numerical evidence that nonequilibrium effects may be quantitatively quite significant. Section VIII briefly discusses frictional effects due to fermions, showing that the formal situation is quite similar to that developed in detail for scalar fields. Finally, our principal conclusions are summarized in Sec. IX, where we also discuss the extent to which they depend on the methods of approximation we have utilized.

II. MECHANISMS OF ENERGY TRANSFER AND DISSIPATION

Roughly speaking, we can identify two mechanisms through which energy may be transferred between the classical field ϕ and the system of χ particles. One is the creation of new particles, which we will refer to as type I. The other, to which we refer as type II, involves changing the energies of particles that are already present. In the following subsections, we review the arguments which purport to derive from these two mechanisms friction terms in the equation of motion for ϕ .

A. Particle creation

A crude argument given by Morikawa and Sasaki [12] takes χ in the first instance to be a free field, except that the interaction with ϕ leads to a time-dependent effective mass. Thus, the energy of a single χ particle is $\omega_k(t) = \sqrt{k^2 + m^2 + g\phi^2(t)}$. According to standard arguments, this field can be represented as

$$\chi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} [e^{i\mathbf{k}\cdot\mathbf{x}} f_k(t; \hat{t}) a_k(\hat{t}) + e^{-i\mathbf{k}\cdot\mathbf{x}} f_k^*(t; \hat{t}) a_k^\dagger(\hat{t})] \quad (2.1)$$

where the mode functions $f_k(t; \hat{t})$ are solutions of the equation

$$[\partial_{\hat{t}}^2 + \omega_k^2(t)] f_k(t; \hat{t}) = 0 \quad (2.2)$$

obeying the boundary conditions

$$f_k(\hat{t}; \hat{t}) = [2\omega_k(\hat{t})]^{-1/2}, \quad \partial_t f_k(t; \hat{t})|_{t=\hat{t}} = -i[\omega_k(\hat{t})/2]^{1/2} \quad (2.3)$$

imposed at the reference time \hat{t} . In principle, \hat{t} is quite arbitrary, but we shall eventually choose it to be the instant at which we wish to evaluate the equation of motion (1.5). These mode functions satisfy the Wronskian condition

$$f_k(t; \hat{t}) \dot{f}_k^*(t; \hat{t}) - \dot{f}_k(t; \hat{t}) f_k^*(t; \hat{t}) = i, \quad (2.4)$$

where the overdot indicates differentiation with respect to the first argument, t . They can be expressed as

$$f_k(t; \hat{t}) = [2\Omega_k(t; \hat{t})]^{-1/2} \exp\left[-i \int_{\hat{t}}^t \Omega_k(t'; \hat{t}) dt'\right] \quad (2.5)$$

where the frequency $\Omega_k(t; \hat{t})$ is the solution of

$$\frac{1}{2} \ddot{\Omega}_k - \frac{3}{4} \frac{\dot{\Omega}_k^2}{\Omega_k} + \Omega_k^2 = \omega_k^2 \quad (2.6)$$

subject to the boundary conditions $\Omega_k(\hat{t}; \hat{t}) = \omega_k(\hat{t})$ and $\dot{\Omega}_k(\hat{t}; \hat{t}) = \ddot{\Omega}_k(\hat{t}; \hat{t}) = 0$. At times close to \hat{t} , therefore, $f_k(t; \hat{t})$ is the positive-energy solution,

$$f_k(t; \hat{t}) \approx [2\omega_k(\hat{t})]^{-1/2} \exp[-i\omega_k(\hat{t})(t - \hat{t})], \quad (2.7)$$

but at other times the frequency $\Omega_k(t; \hat{t})$ does not necessarily correspond to a single-particle energy. Mode functions referred to different reference times are related by a Bogoliubov transformation. To be concrete, let $f_k^0(t) = f_k(t; 0)$. Then $f_k(t; \hat{t})$ is a linear combination of $f_k^0(t)$ and $f_k^{0*}(t)$, say $f_k(t; \hat{t}) = A(\hat{t}) f_k^0(t) + B(\hat{t}) f_k^{0*}(t)$. It is a simple exercise using Eqs. (2.4) and (2.3) to find the Bogoliubov coefficients $A(\hat{t})$ and $B(\hat{t})$ and thus the dependence on \hat{t} of $f_k(t; \hat{t})$, which is

$$\partial_{\hat{t}} f_k(t; \hat{t}) = i\omega_k(\hat{t}) f_k(t; \hat{t}) - \frac{1}{2} \frac{\dot{\omega}_k(\hat{t})}{\omega_k(\hat{t})} f_k^*(t; \hat{t}). \quad (2.8)$$

The creation and annihilation operators in Eq. (2.1) have the commutation relation $[a_k(\hat{t}), a_{k'}^\dagger(\hat{t})] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$. Their dependence on the reference time \hat{t} follows from the fact that $\chi(t, \mathbf{x})$ itself is independent of \hat{t} ; using the fact that $f_k(t; \hat{t})$ depends only on $k = |\mathbf{k}|$, we find

$$\partial_{\hat{t}} a_k(\hat{t}) = -i\omega_k(\hat{t}) a_k(\hat{t}) + \frac{1}{2} \frac{\dot{\omega}_k(\hat{t})}{\omega_k(\hat{t})} a_{-k}^\dagger(\hat{t}). \quad (2.9)$$

With this formalism in hand, we evaluate the expectation value in Eq. (1.5) by choosing the reference time \hat{t} to be t . For a spatially homogeneous state, the result is

$$\langle \chi(t, \mathbf{x})^2 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k(t)} [1 + 2n_k(t) + 2 \operatorname{Re} \nu_k(t)], \quad (2.10)$$

where the functions $n_k(t)$ and $\nu_k(t)$ are defined by

$$\langle a_k^\dagger(t) a_{k'}(t) \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') n_k(t), \quad (2.11)$$

$$\langle a_k(t) a_{k'}(t) \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \nu_k(t). \quad (2.12)$$

The functions $n_k(t)$ can be interpreted as occupation numbers for the single-particle modes $f_k(t; \hat{t})$ when t is close to \hat{t} , while $\nu_k(t)$ measures the off-diagonality of the density matrix in the representation defined by these modes. The time dependence of these functions is easily established from (2.9), and we find

$$\partial_t n_k(t) = \frac{\dot{\omega}_k(t)}{\omega_k(t)} \text{Re } \nu_k(t), \quad (2.13)$$

$$\partial_t \nu_k(t) = -2i\omega_k(t)\nu_k(t) + \frac{\dot{\omega}_k(t)}{2\omega_k(t)}[1 + 2n_k(t)]. \quad (2.14)$$

These evolution equations are valid only in the approximation that $\chi(t, \mathbf{x})$ is a free field. It is argued in [12] that interactions will give rise to an imaginary part of the χ self-energy, which can be taken into account by replacing $\omega_k(t)$ with $\omega_k(t) - i\Gamma_k(t)$. With the further assumption that $\omega_k(t)$, $\Gamma_k(t)$, and $n_k(t)$ change negligibly on a time scale of order $\Gamma_k(t)^{-1}$, integration of Eq. (2.14) yields

$$\text{Re } \nu_k(t) \approx \frac{\dot{\omega}_k(t)\Gamma_k(t)\omega_k(t)}{2[\omega_k^2(t) + \Gamma_k^2(t)]^2} [1 + 2n_k(t)] \quad (2.15)$$

if Γ_k is negligible by comparison with $\dot{\omega}_k$. Since $\dot{\omega}_k = g\phi\dot{\phi}/\omega_k$, we identify a contribution of the form $\eta_I(\phi)\dot{\phi}$ in the equation of motion (1.5), with

$$\eta_I(\phi) = \frac{g^2\phi^2(t)}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\Gamma_k(t)}{\omega_k(t)[\omega_k^2(t) + \Gamma_k^2(t)]^2} \times [1 + 2n_k(t)]. \quad (2.16)$$

B. Changing particle energies

To the extent that the χ particles can be considered as free, the energy of this system of particles can be estimated as

$$E(t) \approx \int \frac{d^3k}{(2\pi)^3} \omega_k(t)n_k(t). \quad (2.17)$$

Clearly, this energy is altered by a change in the classical field $\phi(t)$, which causes a change in the single-particle energies $\omega_k(t)$, and this provides a mechanism whereby energy may be exchanged. However, a frictional term in the equation of motion for ϕ arises only from an irreversible transfer of energy. This is a secondary effect, brought about by the fact that a change in $\omega_k(t)$ alters the scattering and decay rates of χ particles, and hence affects the evolution of the occupation numbers $n_k(t)$. A crude argument for estimating the friction coefficient that arises from this mechanism was given by Hosoya and Sakagami [13], who take the time evolution of $n_k(t)$ to be governed by a phenomenological kinetic equation of the form

$$\partial_t n_k(t) = -2\Gamma_k[n_k(t) - n_k^{\text{eq}}(t)], \quad (2.18)$$

which is sometimes referred to as the relaxation-time approximation to the Boltzmann equation. It is implied that the system of particles is always close to an equilibrium state characterized by a fixed temperature β^{-1} , with the distribution function

$$n_k^{\text{eq}}(t) = 1/[\exp(\beta\omega_k(t)) - 1]. \quad (2.19)$$

If we once more assume that $n_k^{\text{eq}}(t)$ changes slowly on a time scale Γ_k^{-1} , then Eq. (2.18) can be approximately integrated to yield $n_k(t) = n_k^{\text{eq}}(t) + \delta n_k(t)$, with

$$\begin{aligned} \delta n_k(t) &\approx -(2\Gamma_k)^{-1} \partial_t n_k^{\text{eq}}(t) \\ &\approx \beta(2\Gamma_k)^{-1} \dot{\omega}_k(t) n_k^{\text{eq}}(t) [1 + n_k^{\text{eq}}(t)], \end{aligned} \quad (2.20)$$

and we obtain a contribution to the equation of motion (1.5) of the form $\eta_{II}(\phi)\dot{\phi}$, with

$$\eta_{II}(\phi) = \frac{\beta g^2 \phi^2(t)}{2} \int \frac{d^3k}{(2\pi)^3} \frac{n_k^{\text{eq}}(t)[1 + n_k^{\text{eq}}(t)]}{\Gamma_k \omega_k^2(t)}. \quad (2.21)$$

C. Linear response theory

The crude arguments given above can in some respects be improved by restricting attention to a situation in which the system of χ particles is in thermal equilibrium, apart from a small time-dependent perturbation which is treated to linear order. Different versions of this treatment have been given by Hosoya and Sakagami [13], by Morikawa and Sasaki [12,14] and more recently by Berera, Gleiser and Ramos [15–17]. It is necessary to suppose that over a sufficiently long period of time, the classical field $\phi(t)$ can be decomposed as $\phi(t) = \phi_0 + \delta\phi(t)$, where ϕ_0 is constant and $\delta\phi(t)$ is small. Correspondingly, the Hamiltonian for χ will be decomposed as

$$H(\chi, t) = H_0(\chi) + g\phi_0\delta\phi(t) \int d^3x \chi^2(t, \mathbf{x}) + \text{O}(\delta\phi^2), \quad (2.22)$$

the second term being treated as a perturbation. To linear order in this perturbation, the standard Kubo formalism then yields

$$\begin{aligned} \langle \chi^2(t, \mathbf{x}) \rangle &\approx \langle \chi^2(t, \mathbf{x}) \rangle^{\text{eq}} + ig\phi_0 \int_{-\infty}^t dt' \delta\phi(t') \\ &\times \int d^3x' \langle [\chi^2(t', \mathbf{x}'), \chi^2(t, \mathbf{x})] \rangle^{\text{eq}} \end{aligned} \quad (2.23)$$

where “eq” denotes the thermal average in the equilibrium ensemble determined by $H_0(\chi)$. Although $H_0(\chi)$ has no explicit time dependence, it still contains interactions, and the remaining thermal averages can be computed systematically in perturbation theory. To obtain a well-defined answer for the linear response term in Eq. (2.23), it proves necessary to effect a partial resummation of the χ propagator, in particular identifying the thermal width Γ_k as the imaginary part of a suitable self-energy (as discussed in more detail below). As

first pointed out in Ref. [14], the net result is a contribution to the equation of motion (1.5) of the form $\eta(\phi)\dot{\phi}$, with $\eta(\phi) \approx \eta_I(\phi_0) + \eta_{II}(\phi_0)$, provided that $\Gamma_k \ll \omega_k$ and that $n_k(t)$ in Eq. (2.16) and $n^{\text{eq}}(t)$ in Eq. (2.21) are identified as the constant equilibrium distribution associated with $H_0(\chi)$. As in the previous calculations, it is also necessary to assume that $\delta\phi(t')$ varies slowly on a time scale of order Γ_k^{-1} and can be approximated as $\delta\phi(t') \approx \dot{\phi}(t)(t' - t)$.

D. Open questions

Each of the calculations outlined above has its own deficiencies. The type-I analysis of Sec. II A is valid arbitrarily far from equilibrium, but treats interactions in an incomplete and *ad hoc* manner. The type-II analysis of Sec. II B is explicitly restricted to states very close to local equilibrium and rests on a kinetic equation which is little more than a guess. The linear response treatment of Sec. II C is much more systematic, insofar as the expectation value in Eq. (2.23) can in principle be evaluated at any desired order of equilibrium perturbation theory. However, this calculation depends in an essential way on analytic properties of thermal Green's functions that are meaningful only in a state of exact thermal equilibrium. Specifically, the Wightman functions $G^>(t - t'; \mathbf{x} - \mathbf{x}') = \langle \chi(t, \mathbf{x}) \chi(t', \mathbf{x}') \rangle^{\text{eq}}$ and $G^<(t - t'; \mathbf{x} - \mathbf{x}') = \langle \chi(t', \mathbf{x}') \chi(t, \mathbf{x}) \rangle^{\text{eq}}$ have Fourier transforms which satisfy the Kubo-Martin-Schwinger (KMS) condition $G^>(k_0, \mathbf{k}) = \exp(\beta k_0) G^<(k_0, \mathbf{k})$ (see, for example, Ref. [18]). In a state which departs even slightly from thermal equilibrium, neither the temporal Fourier transform nor the KMS condition has any meaning.

The primary question addressed in the remainder of this paper is whether the apparently plausible result of linear response theory is recovered, for a nonequilibrium system, in the limit of slow time evolution. This will be possible, at best, only if we have a local approximation to expectation values such as $\langle \chi^2(t, \mathbf{x}) \rangle$, which are inherently nonlocal in time [as evidenced by Eq. (2.23) even in the linear response approximation], and devising such an approximation is the key feature of the analysis that follows.

An important issue that is not addressed in this paper arises from the expression (2.21). This contribution to the friction coefficient depends inversely on the thermal width Γ_k , which is typically of the order of the square of a coupling constant, and appears to call into question the reliability of perturbation theory as applied to this problem. In fact, this is typical of expressions obtained in the application of linear response theory to the estimation of transport coefficients, and it is known that infinite classes of diagrams contribute at each order of perturbation theory [19–21]. How the requisite resummation might be effected for a nonequilibrium system is beyond the scope of this paper.

III. LOCAL APPROXIMATION FOR SELF-ENERGIES

The exact two-point functions for the quantum field $\chi(t, \mathbf{x})$ will be denoted in a standard notation by

$$G_{ab}(t, t'; \mathbf{k}) = \int d^3x e^{ik \cdot x} G_{ab}(t, \mathbf{x}; t', \mathbf{0}) \quad (3.1)$$

(for $a, b = 1, 2$) where

$$G(t, \mathbf{x}; t', \mathbf{x}') = \begin{pmatrix} \langle T[\chi(t, \mathbf{x}) \chi(t', \mathbf{x}')] \rangle & \langle \chi(t', \mathbf{x}') \chi(t, \mathbf{x}) \rangle \\ \langle \chi(t, \mathbf{x}) \chi(t', \mathbf{x}') \rangle & \langle \bar{T}[\chi(t, \mathbf{x}) \chi(t', \mathbf{x}')] \rangle \end{pmatrix}. \quad (3.2)$$

The self-energies $\Sigma_{ab}(t, t'; \mathbf{k})$ can be defined by the Dyson-Schwinger equations

$$G_{ab}(t, t'; \mathbf{k}) = g_{ab}^{(F)}(t, t'; \mathbf{k}) - i \int dt'' dt''' g_{ac}^{(F)}(t, t''; \mathbf{k}) \times \Sigma_{cd}(t'', t'''; \mathbf{k}) G_{db}(t''', t'; \mathbf{k}), \quad (3.3)$$

in which the free-field propagators $g_{ab}^{(F)}(t, t'; \mathbf{k})$ are solutions of the equations

$$\mathcal{D}_{ac}^{(F)}(t, \partial_t; \mathbf{k}) g_{cb}^{(F)}(t, t'; \mathbf{k}) = g_{ac}^{(F)}(t, t'; \mathbf{k}) \tilde{\mathcal{D}}_{cb}^{(F)}(t', \partial_{t'}; \mathbf{k}) = -i \delta_{ab} \delta(t - t'), \quad (3.4)$$

where the differential operator $\mathcal{D}^{(F)}(t, \partial_t; \mathbf{k})$ is given by

$$\mathcal{D}^{(F)} = \begin{pmatrix} \partial_t^2 + k^2 + m^2(t) & 0 \\ 0 & -\partial_t^2 - k^2 - m^2(t) \end{pmatrix} \quad (3.5)$$

with $m^2(t) = m^2 + g\phi^2(t)$. The form of the self-energy matrix Σ_{ab} is constrained by some general considerations. First, the full propagators $G_{ab}(t, t'; \mathbf{k})$ defined by Eqs. (3.1) and (3.2) have the properties

$$G_{ab}(t, t'; \mathbf{k}) = G_{ba}(t', t; \mathbf{k}), \quad (3.6)$$

$$G_{11}^*(t, t'; \mathbf{k}) = G_{22}(t, t'; \mathbf{k}), \quad G_{12}^*(t, t'; \mathbf{k}) = G_{21}(t, t'; \mathbf{k}). \quad (3.7)$$

Second, causality requires the integrand in Eq. (3.3) to vanish (after summing over the closed-time-path indices c and d) if either t'' or t''' is larger than both t and t' . From these observations, it is not hard to show that the self-energy matrix has the general form

$$\Sigma(t, t'; \mathbf{k}) = \Sigma^L(t; \mathbf{k}) \delta(t - t') + \Sigma^>(t, t'; \mathbf{k}) \theta(t - t') + \Sigma^<(t, t'; \mathbf{k}) \theta(t' - t) \quad (3.8)$$

with

$$\Sigma^L(t; \mathbf{k}) = \begin{pmatrix} \beta_k(t) - i\alpha_k(t) & i\alpha_k(t) \\ i\alpha_k(t) & -\beta_k(t) - i\alpha_k(t) \end{pmatrix}, \quad (3.9)$$

$$\Sigma^>(t, t'; \mathbf{k}) = \begin{pmatrix} A_k(t, t') & A_k^*(t, t') \\ -A_k(t, t') & -A_k^*(t, t') \end{pmatrix}, \quad (3.10)$$

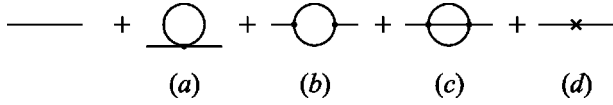


FIG. 1. Diagrams contributing to the 2-point function of $\lambda\Phi^4$ theory. Diagram (a) gives a local contribution to the self-energy; diagram (c) gives rise to the principal dissipative effects; diagram (d) represents the contribution of \mathcal{M}_{ab} .

$$\Sigma^<(t, t'; \mathbf{k}) = \begin{pmatrix} A_k(t', t) & -A_k(t', t) \\ A_k^*(t', t) & -A_k^*(t', t) \end{pmatrix}. \quad (3.11)$$

In the local part Σ^L , the functions $\alpha_k(t)$ and $\beta_k(t)$ are real. If χ has, for example, a quartic self-coupling χ^4 , or a biquadratic coupling $\chi^2\xi^2$ to another scalar field ξ , then these couplings generate Feynman diagrams [for example, diagram (a) in Fig. 1] that are manifestly local; such diagrams contribute to $\beta_k(t)$, but not to $\alpha_k(t)$. More generally, it will be advantageous to extract local contributions from non-local diagrams, and it is easily seen that a contribution to $A_k(t, t')$ of the form $[\beta_k(t) - i\alpha_k(t)]\delta(t - t')$ is equivalent to Eq. (3.9), provided that we identify $\theta(0) = \frac{1}{2}$. We take $\alpha_k(t) = 0$ for now, but will later use $\alpha_k(t)$ to denote a local contribution to the imaginary part of $A_k(t, t')$.

In fact, the goal of this section is to develop a plausible *Ansatz* for approximating the whole self-energy matrix as a local quantity, containing terms proportional only to $\delta(t - t')$ and $\partial_t \delta(t - t')$. To this end, it is convenient to deal with the commutator and anticommutator (or correlation) functions

$$\begin{aligned} \rho(t, t'; \mathbf{k}) &= i \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \langle [\chi(t, \mathbf{x}), \chi(t', \mathbf{0})] \rangle \\ &= i[G_{21}(t, t'; \mathbf{k}) - G_{12}(t, t'; \mathbf{k})], \end{aligned} \quad (3.12)$$

$$\begin{aligned} C(t, t'; \mathbf{k}) &= \frac{1}{2} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \langle \{\chi(t, \mathbf{x}), \chi(t', \mathbf{0})\} \rangle \\ &= \frac{1}{2} [G_{21}(t, t'; \mathbf{k}) + G_{12}(t, t'; \mathbf{k})] \end{aligned} \quad (3.13)$$

from which all the G_{ab} can be constructed. For these functions, the Dyson-Schwinger equations (3.3) together with Eq. (3.4) imply the equations of motion (see, for example, Ref. [22])

$$[\partial_t^2 + \omega_k^2(t)]\rho(t, t'; \mathbf{k}) = - \int_{t'}^t dt'' \Sigma_\rho(t, t''; \mathbf{k})\rho(t'', t'; \mathbf{k}), \quad (3.14)$$

$$\begin{aligned} [\partial_t^2 + \omega_k^2(t)]C(t, t'; \mathbf{k}) &= - \int_0^t dt'' \Sigma_\rho(t, t''; \mathbf{k})C(t'', t'; \mathbf{k}) \\ &\quad + \int_0^{t'} dt'' \Sigma_C(t, t''; \mathbf{k})\rho(t'', t; \mathbf{k}), \end{aligned} \quad (3.15)$$

with

$$\omega_k^2(t) = k^2 + m^2(t) + \beta_k(t), \quad (3.16)$$

$$\begin{aligned} \Sigma_\rho(t, t'; \mathbf{k}) &= [A_k(t, t') + A_k^*(t, t')] \theta(t - t') - [A_k(t', t) \\ &\quad + A_k^*(t', t)] \theta(t' - t), \end{aligned} \quad (3.17)$$

$$\begin{aligned} \Sigma_C(t, t'; \mathbf{k}) &= -\frac{1}{2} i [A_k(t, t') - A_k^*(t, t')] \theta(t - t') \\ &\quad - \frac{1}{2} i [A_k(t', t) - A_k^*(t', t)] \theta(t' - t). \end{aligned} \quad (3.18)$$

In a state of thermal equilibrium, for which $\rho(t, t'; \mathbf{k}) = \rho^{\text{eq}}(t - t'; \mathbf{k})$, one can define the spectral density

$$\rho^{\text{eq}}(\omega, \mathbf{k}) = -i \int dt e^{i\omega t} \rho^{\text{eq}}(t; \mathbf{k}). \quad (3.19)$$

A strategy frequently adopted in the context of linear response theory is to assume that this spectral density can be approximated by the Breit-Wigner form

$$\rho^{\text{eq}}(\omega, \mathbf{k}) \approx \frac{4\Gamma_k \omega}{(\omega^2 - \Omega_k^2 - \Gamma_k^2)^2 + 4\Gamma_k^2 \omega^2} \quad (3.20)$$

where Ω_k and Γ_k are a quasiparticle energy and width to be abstracted from the real and imaginary parts of the self-energies. The numerical calculations of Ref. [22] for a one-dimensional scalar field theory suggest that such an approximation is reasonable (though certainly not exact) for a nonequilibrium state also. To realize this approximation in the present context, observe that the real-time commutator function is given by the Breit-Wigner approximation as

$$\rho^{\text{eq}}(t - t'; \mathbf{k}) \approx \frac{i}{2\Omega_k} [e^{-i\Omega_k(t-t')} - e^{i\Omega_k(t-t')}] e^{-\Gamma_k|t-t'|}. \quad (3.21)$$

For the non-equilibrium state, we adopt an *Ansatz* which is a natural generalization of this expression, namely $\rho(t, t'; \mathbf{k}) \approx \rho^{(0)}(t, t'; \mathbf{k})$, where

$$\begin{aligned} \rho^{(0)}(t, t'; \mathbf{k}) &= i[f_k(t; \hat{t})f_k^*(t'; \hat{t}) - f_k^*(t; \hat{t})f_k(t'; \hat{t})][\Theta_k(t, t') \\ &\quad + \Theta_k(t', t)], \end{aligned} \quad (3.22)$$

$$\Theta_k(t, t') = \exp\left[-\int_{t'}^t dt'' \Gamma_k(t'')\right] \theta(t - t'), \quad (3.23)$$

and $f_k(t; \hat{t})$ are the mode functions introduced in Sec. II A, except that the quasiparticle energies $\omega_k(t)$ may include loop corrections. It is easily checked that $\rho^{(0)}(t, t'; \mathbf{k})$ is real and antisymmetric in its time arguments, as it should be. It also has the property $\partial_t \rho^{(0)}(t, t'; \mathbf{k})|_{t'=t} = 1$ as required by the canonical commutator. Finally, as can be verified from Eq. (2.8), it is independent of the reference time \hat{t} , provided that

$\Gamma_k(t)$ is. The approximate version of the equation of motion (3.14) obeyed by $\rho^{(0)}(t, t'; \mathbf{k})$ is conveniently expressed by writing

$$\rho^{(0)}(t, t'; \mathbf{k}) = \rho^{(0)>}(t, t'; \mathbf{k}) \theta(t - t') + \rho^{(0)<}(t, t'; \mathbf{k}) \theta(t' - t), \quad (3.24)$$

with $\rho^{(0)<}(t, t'; \mathbf{k}) = -\rho^{(0)>}(t', t; \mathbf{k})$. By differentiating Eq. (3.22), we find

$$[\partial_t^2 + 2\Gamma_k(t)\partial_t + \omega_k^2(t) + \Gamma_k^2(t) + \dot{\Gamma}_k(t)]\rho^{(0)>}(t, t'; \mathbf{k}) = 0, \quad (3.25a)$$

$$[\partial_t^2 - 2\Gamma_k(t)\partial_t + \omega_k^2(t) + \Gamma_k^2(t) - \dot{\Gamma}_k(t)]\rho^{(0)<}(t, t'; \mathbf{k}) = 0, \quad (3.25b)$$

together with the equal-time conditions

$$\partial_t \rho^{(0)>}(t, t'; \mathbf{k})|_{t'=t} = \partial_t \rho^{(0)<}(t, t'; \mathbf{k})|_{t'=t} = 1. \quad (3.26)$$

Comparing these equations with Eq. (3.14), we see that they imply a corresponding *Ansatz* for the self-energy (3.17), or for the real part of $A_k(t, t')$. It is

$$\text{Re} A_k^{(0)}(t, t') = [\Gamma_k(t)\Gamma_k(t')]^{1/2} \partial_t \delta(t - t' - \epsilon), \quad (3.27)$$

where the positive infinitesimal ϵ is included to ensure that the delta function is satisfied inside the range of integration in Eq. (3.14).

Given a nonzero quasiparticle width $\Gamma_k(t)$, the non-local self-energies may be expected to decay roughly exponentially with $|t - t'|$. For $\Sigma_C(t, t'; \mathbf{k})$, which is symmetric in its time arguments, or, equivalently, for the imaginary part of $A_k(t, t')$, a suitable local *Ansatz* is

$$\text{Im} A_k^{(0)}(t, t') = -\alpha_k(t) \delta(t - t'). \quad (3.28)$$

With this approximation, the equation of motion (3.15) for the correlation function becomes

$$\begin{aligned} & [\partial_t^2 + 2\Gamma_k(t)\partial_t + \omega_k^2(t) + \Gamma_k^2(t) + \dot{\Gamma}_k(t)]C^{(0)}(t, t'; \mathbf{k}) \\ & = -\alpha_k(t)\rho^{(0)}(t, t'; \mathbf{k})\theta(t' - t). \end{aligned} \quad (3.29)$$

Alternatively, defining

$$\begin{aligned} C^{(0)}(t, t'; \mathbf{k}) & = C^{(0)>}(t, t'; \mathbf{k})\theta(t - t') \\ & + C^{(0)<}(t, t'; \mathbf{k})\theta(t' - t), \end{aligned} \quad (3.30)$$

with $C^{(0)<}(t, t'; \mathbf{k}) = C^{(0)>}(t', t; \mathbf{k})$, this may be written as

$$\begin{aligned} & [\partial_t^2 + 2\Gamma_k(t)\partial_t + \omega_k^2(t) + \Gamma_k^2(t) + \dot{\Gamma}_k(t)]C^{(0)>}(t, t'; \mathbf{k}) \\ & = 0, \end{aligned} \quad (3.31a)$$

$$\begin{aligned} & [\partial_t^2 + 2\Gamma_k(t)\partial_t + \omega_k^2(t) + \Gamma_k^2(t) + \dot{\Gamma}_k(t)]C^{(0)<}(t, t'; \mathbf{k}) \\ & = -\alpha_k(t)\rho^{(0)<}(t, t'; \mathbf{k}), \end{aligned} \quad (3.31b)$$

together with the equal-time condition

$$\partial_t C^{(0)>}(t, t'; \mathbf{k})|_{t'=t} = \partial_t C^{(0)<}(t, t'; \mathbf{k})|_{t'=t}. \quad (3.32)$$

Finally, we assemble the approximate correlation and commutator functions into a single complex function,

$$C^{(0)>}(t, t', \mathbf{k}) - \frac{1}{2}i\rho^{(0)>}(t, t'; \mathbf{k}) = h_k(t, t'), \quad (3.33a)$$

$$C^{(0)<}(t, t', \mathbf{k}) + \frac{1}{2}i\rho^{(0)<}(t, t'; \mathbf{k}) = h_k(t', t). \quad (3.33b)$$

Then, denoting by $g_{ab}(t, t'; \mathbf{k})$ the approximation to the propagator matrix (3.1) that embodies the *Ansätze* (3.27) and (3.28), this matrix can be summarized by

$$g_{ab}(t, t'; \mathbf{k}) = h_b(t, t'; \mathbf{k})\theta(t - t') + h_a(t', t; \mathbf{k})\theta(t' - t), \quad (3.34)$$

with $h_1(t, t'; \mathbf{k}) = h_k(t, t')$ and $h_2(t, t'; \mathbf{k}) = h_k^*(t, t')$. The approximate equations of motion (3.25) and (3.31) are

$$[\partial_t^2 + 2\Gamma_k(t)\partial_t + \omega_k^2(t) + \Gamma_k^2(t) + \dot{\Gamma}_k(t)]h_k(t, t') = 0, \quad (3.35)$$

$$\begin{aligned} & [\partial_t^2 + \omega_k^2(t) + \Gamma_k^2(t) - i\alpha_k(t)]h_k(t', t) + [2\Gamma_k(t)\partial_t + \dot{\Gamma}_k(t) \\ & + i\alpha_k(t)]h_k^*(t', t) = 0, \end{aligned} \quad (3.36)$$

while the equal-time conditions (3.26) and (3.32) become

$$\partial_t [h_k(t, t') - h_k(t', t)]|_{t'=t} = -i,$$

$$\partial_t [h_k^*(t, t') - h_k(t', t)]|_{t'=t} = 0. \quad (3.37)$$

At this point, the *Ansätze* (3.27) and (3.28) have yielded local equations of motion for the approximate two-point functions, but the functions $\alpha_k(t)$ and $\Gamma_k(t)$ that appear in these equations and the function $\beta_k(t)$ that appears in the single-particle energy (3.16) are unknown. Given a specific Lagrangian, approximations to the self-energies Σ_{ab} can be obtained—for example, from some version of perturbation theory. The function $\beta_k(t)$ can be identified from the local part of Σ_{ab} , but a prescription is needed for extracting from the calculated Σ_ρ and Σ_C local contributions of the kind indicated in Eqs. (3.27) and (3.28) so as to identify $\Gamma_k(t)$ and $\alpha_k(t)$. This issue will be addressed in Sec. V.

IV. LOCAL KINETIC EQUATIONS

The approximation summarized by Eqs. (3.34)–(3.37) is equivalent to that obtained from somewhat different considerations in Ref. [23]. There it was shown that the general solution to the local equations of motion can be expressed in the form

$$h_k(t, t') = \frac{1}{2} \exp \left[- \int_{t'}^t dt'' \Gamma_k(t'') \right] \tilde{h}_k(t, t'), \quad (4.1)$$

$$\begin{aligned} \tilde{h}_k(t, t') = & [1 + Q_k(t'; \hat{t})] f_k(t; \hat{t}) f_k^*(t'; \hat{t}) \\ & + [-1 + Q_k^*(t'; \hat{t})] f_k^*(t; \hat{t}) f_k(t'; \hat{t}). \end{aligned} \quad (4.2)$$

The function $Q_k(t; \hat{t})$ is a solution of the equation

$$\left[\partial_t + 2\Gamma_k(t) + 2 \frac{f_k^*(t; \hat{t})}{f_k^*(t; \hat{t})} \right] [\partial_t + 2\Gamma_k(t)] Q_k(t; \hat{t}) = 2i\alpha_k(t) \quad (4.3)$$

subject to the constraint

$$(\partial_t + 2\Gamma_k)(Q_k + Q_k^*) + i(f_k f_k^*)^{-1}(Q_k - Q_k^*) = 0, \quad (4.4)$$

which is preserved by Eq. (4.3). For our present purpose, it is useful to observe that $Q_k(t; \hat{t})$ can be decomposed as

$$Q_k(t; \hat{t}) = Q_k^{(1)}(t; \hat{t}) + \frac{f_k(t; \hat{t})}{f_k^*(t; \hat{t})} Q_k^{(2)}(t; \hat{t}) \quad (4.5)$$

where $Q_k^{(1)}(t; \hat{t})$ is real. If $Q_k^{(1)}$ and $Q_k^{(2)}$ are taken to obey the first-order equations

$$[\partial_t + 2\Gamma_k(t)] Q_k^{(1)}(t; \hat{t}) = 2f_k(t; \hat{t}) f_k^*(t; \hat{t}) \alpha_k(t), \quad (4.6)$$

$$[\partial_t + 2\Gamma_k(t)] Q_k^{(2)}(t; \hat{t}) = -2f_k^{*2}(t; \hat{t}) \alpha_k(t), \quad (4.7)$$

then both Eqs. (4.3) and (4.4) are satisfied and the decomposition (4.5) is unique.

The propagator function $\tilde{h}_k(t, t')$ is now given by

$$\begin{aligned} \tilde{h}_k(t, t') = & [1 + Q_k^{(1)}(t'; \hat{t})] f_k(t; \hat{t}) f_k^*(t'; \hat{t}) \\ & + [-1 + Q_k^{(1)}(t'; \hat{t})] f_k^*(t; \hat{t}) f_k(t'; \hat{t}) \\ & + Q_k^{(2)}(t'; \hat{t}) f_k(t; \hat{t}) f_k(t'; \hat{t}) \\ & + Q_k^{(2)*}(t'; \hat{t}) f_k^*(t; \hat{t}) f_k^*(t'; \hat{t}). \end{aligned} \quad (4.8)$$

It must be independent of the reference time \hat{t} , and this determines the dependence on \hat{t} of the functions $Q_k^{(i)}(t; \hat{t})$. We find

$$\partial_t Q_k^{(1)}(t; \hat{t}) = \frac{\dot{\omega}_k(\hat{t})}{\omega_k(\hat{t})} \text{Re } Q_k^{(2)}(t; \hat{t}), \quad (4.9)$$

$$\begin{aligned} \partial_t Q_k^{(2)}(t; \hat{t}) = & -2i\omega_k(\hat{t}) Q_k^{(2)}(t; \hat{t}) \\ & + \frac{\dot{\omega}_k(\hat{t})}{\omega_k(\hat{t})} Q_k^{(1)}(t; \hat{t}). \end{aligned} \quad (4.10)$$

In the equation of motion (1.5), our approximation to $\langle \chi^2(t, \mathbf{x}) \rangle$ is

$$\langle \chi^2(t, \mathbf{x}) \rangle \approx \int \frac{d^3k}{(2\pi)^3} h_k(t, t) \quad (4.11)$$

and this is conveniently evaluated by choosing the reference time \hat{t} to be the time t of interest:

$$h_k(t, t) = Q_k^{(1)}(t; t) + \text{Re } Q_k^{(2)}(t; t). \quad (4.12)$$

In fact, let us define

$$n_k(t) = \frac{1}{2} [Q_k^{(1)}(t; t) - 1], \quad (4.13)$$

$$v_k(t) = \frac{1}{2} Q_k^{(2)}(t; t). \quad (4.14)$$

We have

$$h_k(t, t) = [2\omega_k(t)]^{-1} [1 + 2n_k(t) + 2\text{Re } v_k(t)] \quad (4.15)$$

and will loosely identify the functions $n_k(t)$ and $v_k(t)$ with the quantities denoted by the same symbols in Sec. II. By combining Eqs. (4.6) and (4.7) with Eq. (4.9) and (4.10), we obtain the evolution equations

$$\partial_t n_k(t) = \frac{\alpha_k(t)}{2\omega_k(t)} - \Gamma_k(t) [1 + 2n_k(t)] + \frac{\dot{\omega}_k(t)}{\omega_k(t)} \text{Re } v_k(t), \quad (4.16)$$

$$\begin{aligned} \partial_t v_k(t) = & -2i[\omega_k(t) - i\Gamma_k(t)] v_k(t) - \frac{\alpha_k(t)}{2\omega_k(t)} \\ & + \frac{\dot{\omega}_k(t)}{2\omega_k(t)} [1 + 2n_k(t)]. \end{aligned} \quad (4.17)$$

These are clearly generalizations of the free-field equations (2.13) and (2.14), the extra terms involving the functions $\alpha_k(t)$ and $\Gamma_k(t)$ whose exact meanings are explored further below. On the other hand, the tentative identification

$$\alpha_k(t) \approx 2\omega_k(t)\Gamma_k(t)[1 + 2n_k^{\text{eq}}(t)] \quad (4.18)$$

(which we will later find to be too naive) brings Eq. (4.16) into the form

$$\partial_t n_k(t) \approx -2\Gamma_k(t) [n_k(t) - n_k^{\text{eq}}(t)] + \frac{\dot{\omega}_k(t)}{\omega_k(t)} \text{Re } v_k(t), \quad (4.19)$$

which is a generalization of the kinetic equation (2.18) including a source term to account for particle creation.

V. DETERMINATION OF LOCAL SELF-ENERGIES

To give substance to the kinetic equations (4.16) and (4.17), we need a concrete method of determining the functions $\beta_k(t)$ [which appears in the quasiparticle energy (3.16)], $\alpha_k(t)$ and $\Gamma_k(t)$. A prescription for doing this was given in [23]; here we describe a refinement of that prescription which is convenient for the problem at hand. The approximate two-point functions (3.34) which solve (3.35) and (3.36) are the exact propagators of an approximate theory defined by the closed-time-path action

$$S_{\text{CTP}}^{(0)}(\chi_1, \chi_2) = -\frac{1}{2} \int dt \int d^3x \chi_a(t, \mathbf{x}) \mathcal{D}_{ab}(t, \partial_t, \nabla) \chi_b(t, \mathbf{x}), \quad (5.1)$$

where the differential operator \mathcal{D} is given, after a spatial Fourier transformation, by

$$\mathcal{D} = \begin{pmatrix} \partial_t^2 + \omega_k^2(t) + \Gamma_k^2(t) - i\alpha_k(t) & 2\Gamma_k(t)\partial_t + \dot{\Gamma}_k(t) + i\alpha_k(t) \\ -2\Gamma_k(t)\partial_t - \dot{\Gamma}_k(t) + i\alpha_k(t) & -\partial_t^2 - \omega_k^2(t) - \Gamma_k^2(t) - i\alpha_k(t) \end{pmatrix}. \quad (5.2)$$

(An effective action having essentially this structure also describes an open system, coupled to environmental degrees of freedom, which can be integrated out by the Feynman-Vernon influence functional method [24–26]. Here, one may think of a single field mode having an environment that consists of all the other modes, but this environment is treated in a self-consistent manner, rather than being integrated out.) If the complete theory has the action $S(\chi)$, and the corresponding closed-time-path action $S_{\text{CTP}}(\chi_1, \chi_2) = S(\chi_1) - S(\chi_2)$, then a partly resummed perturbation expansion can be defined by writing

$$S_{\text{CTP}}(\chi_1, \chi_2) = S_{\text{CTP}}^{(0)}(\chi_1, \chi_2) + \Delta S_{\text{CTP}}(\chi_1, \chi_2) \quad (5.3)$$

and treating ΔS_{CTP} as the perturbation. Included in ΔS_{CTP} is the counterterm $\frac{1}{2} \int dt d^3x \chi_a \mathcal{M}_{ab} \chi_b$, with

$$\mathcal{M}(t, \partial_t; \mathbf{k}) = \mathcal{D}(t, \partial_t; \mathbf{k}) - \mathcal{D}^{(\text{F})}(t, \partial_t; \mathbf{k}) = \begin{pmatrix} \beta_k(t) + \Gamma_k^2(t) - i\alpha_k(t) & 2\Gamma_k(t)\partial_t + \dot{\Gamma}_k(t) + i\alpha_k(t) \\ -2\Gamma_k(t)\partial_t - \dot{\Gamma}_k(t) + i\alpha_k(t) & -\beta_k(t) - \Gamma_k^2(t) - i\alpha_k(t) \end{pmatrix}, \quad (5.4)$$

which accounts for the difference between g_{ab} and $g_{ab}^{(\text{F})}$. In the context of this expansion, self-energies are defined by replacing $g_{ab}^{(\text{F})}$ in the Dyson-Schwinger equation (3.3) with g_{ab} . They have the form

$$\Sigma_{ab}(t, t'; \mathbf{k}) = \Sigma_{ab}^{\text{loop}}(t, t'; \mathbf{k}) - \mathcal{M}_{ab}(t, \partial_t; \mathbf{k}) \delta(t - t'), \quad (5.5)$$

where Σ^{loop} consists of loop diagrams in which the propagators are g_{ab} .

We would like to choose $\alpha_k(t)$, $\beta_k(t)$, and $\Gamma_k(t)$ in such a way that the propagators $g_{ab}(t, t'; \mathbf{k})$ approximate the exact two-point functions as closely as possible. Loosely speaking, this means making the self-energies (5.5) as small as possible. More precisely, it is necessary to obtain a local approximation to $\Sigma_{ab}^{\text{loop}}$, which can be cancelled by an appropriate choice of \mathcal{M}_{ab} . To this end, suppose that t and t' are both close to the reference time \hat{t} . We define time-translation-invariant propagators which approximate $g_{ab}(t, t'; \mathbf{k})$ in this region by introducing the function

$$\begin{aligned} \bar{h}_k(\hat{t}, \tau) &= \frac{e^{-\hat{\Gamma}_k \tau}}{2\hat{\omega}_k} \{ [1 + n_k(\hat{t}) + \nu_k(\hat{t})] e^{-i\hat{\omega}_k \tau} \\ &\quad + [n_k(\hat{t}) + \nu_k^*(\hat{t})] e^{i\hat{\omega}_k \tau} \}, \end{aligned} \quad (5.6)$$

where $\tau = t - t'$, $\hat{\omega}_k = \omega_k(\hat{t})$, and $\hat{\Gamma}_k = \Gamma_k(\hat{t})$. This function is obtained from the one defined in (4.1) by taking $\bar{h}_k(\hat{t}, \tau) = h_k(\hat{t} + \tau, \hat{t})$, using the approximation (2.7) for $f_k(t; \hat{t})$, and replacing $\Gamma_k(t')$ with $\hat{\Gamma}_k$ in the prefactor. The approximate propagators are then given by

$$\bar{g}_{ab}(\hat{t}, \tau; \mathbf{k}) = \bar{h}_b(\hat{t}, \tau; \mathbf{k}) \theta(\tau) + \bar{h}_a(\hat{t}, -\tau; \mathbf{k}) \theta(-\tau), \quad (5.7)$$

where, as in Eq. (3.34), we use the notation $\bar{h}_1(\hat{t}, \tau; \mathbf{k}) = \bar{h}_k(\hat{t}, \tau)$ and $\bar{h}_2(\hat{t}, \tau; \mathbf{k}) = \bar{h}_k^*(\hat{t}, \tau)$.

By replacing g_{ab} with \bar{g}_{ab} in the diagrams that constitute $\Sigma_{ab}^{\text{loop}}$, we obtain a time-translation invariant approximation to these self-energies, $\bar{\Sigma}_{ab}^{\text{loop}}(\hat{t}, \tau; \mathbf{k})$, valid when t and t' are both close to \hat{t} , which can be used to determine the functions $\alpha_k(\hat{t})$, $\beta_k(\hat{t})$, and $\Gamma_k(\hat{t})$. After a Fourier transform on τ , our approximation to the right-hand side of Eq. (5.5) is

$$\int \frac{d\omega}{2\pi} [\bar{\Sigma}_{ab}^{\text{loop}}(\hat{t}; \omega, \mathbf{k}) - \mathcal{M}_{ab}(\hat{t}, -i\omega; \mathbf{k})] e^{-i\omega\tau}. \quad (5.8)$$

Although $\bar{\Sigma}^{\text{loop}}(\hat{t}, \tau; \mathbf{k})$ is time-translation invariant, it is not, in general, a distribution concentrated at $\tau=0$. In Fourier transformed language, $\bar{\Sigma}^{\text{loop}}(\hat{t}; \omega, \mathbf{k})$ is a non-linear function of ω , while $\mathcal{M}_{ab}(\hat{t}, -i\omega; \mathbf{k})$ [which results from replacing ∂_t in Eq. (5.4) with $-i\omega$] is linear in ω . Therefore, the integrand in Eq. (5.8) cannot be made to vanish for all values of ω . A reasonable prescription for determining $\alpha_k(\hat{t})$, $\beta_k(\hat{t})$, and $\Gamma_k(\hat{t})$ is to demand that

$$\mathcal{M}_{ab}(\hat{t}, \pm i\hat{\omega}_k; \mathbf{k}) = \bar{\Sigma}_{ab}^{\text{loop}}(\hat{t}; \mp \hat{\omega}_k, \mathbf{k}). \quad (5.9)$$

To the extent that the considerations of Sec. III are valid, this ensures that $g_{ab}(t, t'; \mathbf{k})$ are the propagators for free quasi-particles whose energies and widths are approximately those determined by the peaks of the true nonequilibrium spectral density. Explicitly, the prescription implied by Eq. (5.9) is

$$\alpha_k(\hat{t}) = -\frac{i}{2} [\bar{\Sigma}_{21}^{\text{loop}}(\hat{t}; \hat{\omega}_k, \mathbf{k}) + \bar{\Sigma}_{21}^{\text{loop}}(\hat{t}; -\hat{\omega}_k, \mathbf{k})], \quad (5.10)$$

$$\Gamma_k(\hat{t}) = -\frac{i}{4\hat{\omega}_k} [\bar{\Sigma}_{21}^{\text{loop}}(\hat{t}; \hat{\omega}_k, \mathbf{k}) - \bar{\Sigma}_{21}^{\text{loop}}(\hat{t}; -\hat{\omega}_k, \mathbf{k})], \quad (5.11)$$

$$\beta_k(\hat{t}) = \text{Re} \bar{\Sigma}_{11}^{\text{loop}}(\hat{t}; \hat{\omega}_k, \mathbf{k}) - \Gamma_k^2(\hat{t}). \quad (5.12)$$

VI. FRICTION TERMS IN THE EQUATION OF MOTION

The nature and purpose of the approximations we have introduced so far can usefully be summarized as follows. By partitioning the closed-time-path action as in Eq. (5.3), we obtain a reorganized perturbation theory in which the unperturbed propagators are the g_{ab} defined by Eqs. (3.34)–(3.37). Summed to all orders, this perturbation theory would (presumably) be equivalent to the usual expansion based on the free-particle propagators $g_{ab}^{(F)}$, and this assertion is essentially independent of how we choose the functions $\alpha_k(t)$, $\beta_k(t)$, and $\Gamma_k(t)$ that enter the definition of g_{ab} . However, the reorganized perturbation theory cannot in practice be summed to all orders. Its utility rests on the possibility of making g_{ab} a better approximation than $g_{ab}^{(F)}$ to the full two-point functions; in particular, we wish to estimate the expectation value $\langle \chi^2(t, \mathbf{x}) \rangle$ by retaining only the lowest-order term as indicated in Eq. (4.11). To do this, we choose $\alpha_k(t)$, $\beta_k(t)$, and $\Gamma_k(t)$ to be local contributions to the true self-energies, which are thereby resummed in the reorganized perturbation expansion. For the purpose of extracting these local contributions (and *only* for this purpose) we introduced the time-translation-invariant propagators \bar{g}_{ab} in Eq. (5.7), which enabled us to formulate the prescription recorded in Eqs. (5.10)–(5.12). Of course, this prescription can be implemented only approximately, by evaluating $\bar{\Sigma}^{\text{loop}}$ to some finite order of perturbation theory.

In principle, we are now in a position to evaluate the right-hand sides of the evolution equations (4.16) and (4.17), to solve these equations for $n_k(t)$ and $\nu_k(t)$ and hence to estimate the expectation value (4.11) in which we are principally interested. There is, however, a practical difficulty. It is that the self-energy on the right-hand side of Eq. (5.11) is itself a function of Γ_k , and this equation cannot be solved analytically to obtain a concrete expression for Γ_k . A numerical solution is feasible, and this is no doubt the best way of estimating the time evolution, given a specific model. Here, though, we wish to investigate the circumstances under which dissipation might be represented by the frictional terms in the equation of motion exhibited in Sec. II. To that end, we now introduce two further approximations. First, we take the limit $\Gamma_k \rightarrow 0$ in the propagators \bar{g}_{ab} used to calculate $\bar{\Sigma}_{ab}$ in Eqs. (5.10)–(5.12). This is reasonably well justified in a weakly coupled theory, where Γ_k is of order g^2 , say, and $\bar{\Sigma}_{ab}$ itself contains an overall factor of g^2 . Second, we will

also set $\nu_k = 0$ in \bar{g}_{ab} . A justification for this step will appear below [see Eq. (6.7)].

With these approximations, the propagators \bar{g}_{ab} assume the form familiar from the equilibrium theory (see, for example, Ref. [18]). In particular, the temporal Fourier transform of \bar{g}_{12} is

$$\bar{g}_{21}(\hat{t}; \omega, \mathbf{k}) = \frac{\pi}{\hat{\omega}_k} \{ [1 + n_k(\hat{t})] \delta(\omega - \hat{\omega}_k) + n_k(\hat{t}) \delta(\omega + \hat{\omega}_k) \}. \quad (6.1)$$

As a standard example, we consider in what follows the $\lambda\Phi^4$ theory, whose Lagrangian density consists of the first three terms of Eq. (1.2), identifying χ as the fluctuation field φ and the coupling g as $g = \lambda/2$. The 2-point functions for χ contain, among others, the diagrams shown in Fig. 1. At 2-loop order, with \bar{g}_{21} given by Eq. (6.1), the only contribution to $\bar{\Sigma}_{21}$ comes from diagram (c). [The 1-loop diagram (b) contains products of δ functions which cannot be simultaneously satisfied. It would give an off-shell contribution if we were to retain a non-zero width Γ_k in \bar{g}_{21} .] Evaluating this diagram, we find from Eqs. (5.10) and (5.11)

$$\begin{aligned} \alpha_k(\hat{t}) &= \frac{\lambda^2}{32(2\pi)^5} \int d^3k_1 d^3k_2 d^3k_3 \\ &\quad \times \frac{\delta(\hat{\omega}_{k_1} + \hat{\omega}_{k_2} - \hat{\omega}_{k_3} - \hat{\omega}_k) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k})}{\hat{\omega}_{k_1} \hat{\omega}_{k_2} \hat{\omega}_{k_3}} \\ &\quad \times [(1 + \hat{n}_{k_1})(1 + \hat{n}_{k_2})\hat{n}_{k_3} + \hat{n}_{k_1}\hat{n}_{k_2}(1 + \hat{n}_{k_3})], \end{aligned} \quad (6.2)$$

$$\begin{aligned} \Gamma_k(\hat{t}) &= \frac{\lambda^2}{64(2\pi)^5} \int d^3k_1 d^3k_2 d^3k_3 \\ &\quad \times \frac{\delta(\hat{\omega}_{k_1} + \hat{\omega}_{k_2} - \hat{\omega}_{k_3} - \hat{\omega}_k) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k})}{\hat{\omega}_k \hat{\omega}_{k_1} \hat{\omega}_{k_2} \hat{\omega}_{k_3}} \\ &\quad \times [(1 + \hat{n}_{k_1})(1 + \hat{n}_{k_2})\hat{n}_{k_3} - \hat{n}_{k_1}\hat{n}_{k_2}(1 + \hat{n}_{k_3})] \end{aligned} \quad (6.3)$$

where $\hat{n}_{k_i} = n_{k_i}(\hat{t})$. In particular, the quantity

$$\begin{aligned} S_k(\hat{t}) &= \frac{\alpha_k(\hat{t})}{2\hat{\omega}_k} - \Gamma_k(\hat{t})[1 + 2n_k(\hat{t})] \\ &= \frac{\lambda^2}{32(2\pi)^5} \int d^3k_1 d^3k_2 d^3k_3 \\ &\quad \times \frac{\delta(\hat{\omega}_{k_1} + \hat{\omega}_{k_2} - \hat{\omega}_{k_3} - \hat{\omega}_k) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k})}{\hat{\omega}_k \hat{\omega}_{k_1} \hat{\omega}_{k_2} \hat{\omega}_{k_3}} \\ &\quad \times [\hat{n}_{k_1}\hat{n}_{k_2}(1 + \hat{n}_{k_3})(1 + \hat{n}_k) - (1 + \hat{n}_{k_1}) \\ &\quad \times (1 + \hat{n}_{k_2})\hat{n}_{k_3}\hat{n}_k] \end{aligned} \quad (6.4)$$

which appears in the kinetic equation (4.16) is precisely the 2-particle elastic scattering integral that ought to appear in a genuine Boltzmann equation. (More generally, since the $\lambda\Phi^4$ theory has no conserved particle number, inelastic contributions should also be expected, and these will indeed appear if we extend the evaluation of the self-energies beyond 2-loop order.)

With these explicit expressions for $\alpha_k(t)$ and $\Gamma_k(t)$, we now have concrete forms for the evolution equations (4.16) and (4.17). These evolution equations are local in time, but their solutions will be non-local. To extract contributions to $\langle\chi^2(t,\mathbf{x})\rangle$ which are proportional to $\dot{\phi}(t)$, we must resort to some adiabatic approximation of the kind considered in Sec. II. To do this systematically, we rewrite the evolution equations as

$$\epsilon\partial_t n_k(t) = \bar{\alpha}_k(t) - \Gamma_k(t)[1 + 2n_k(t)] + \epsilon \frac{\dot{\omega}_k(t)}{\omega_k(t)} \text{Re } \nu_k(t), \quad (6.5)$$

$$\begin{aligned} \epsilon\partial_t \nu_k(t) = & -2i[\omega_k(t) - i\Gamma_k(t)]\nu_k(t) - \bar{\alpha}_k(t) \\ & + \epsilon \frac{\dot{\omega}_k(t)}{2\omega_k(t)} [1 + 2n_k(t)], \end{aligned} \quad (6.6)$$

where $\bar{\alpha}_k(t) = \alpha_k(t)/2\omega_k(t)$ and we have introduced a formal expansion parameter ϵ multiplying terms with time derivatives. By expanding in powers of ϵ and finally setting $\epsilon = 1$, we generate expansions of $n_k(t)$ and $\nu_k(t)$ in powers of the time derivatives of $\phi(t)$. On substituting these expansions in the expression (4.11) for $\langle\chi^2(t,\mathbf{x})\rangle$, the next-to-leading terms, proportional to $\dot{\phi}$, yield an estimate of the friction coefficient $\eta(\phi)$.

At leading order, Eq. (6.5) reduces to the equation $S_k(t) = 0$ whose solution is well-known to be the Bose-Einstein distribution $n_k(t) = n_k^{\text{eq}}(t)$ for some temperature β^{-1} . [We do not allow for a non-zero chemical potential, because the inelastic contributions to $S_k(t)$ expected at higher orders of perturbation theory would constrain the equilibrium chemical potential to vanish.] The corresponding solution of Eq. (6.6) is

$$\nu_k^{\text{eq}}(t) = i\bar{\alpha}_k^{\text{eq}}/2[\omega_k(t) - i\Gamma_k^{\text{eq}}(t)] \quad (6.7)$$

where $\bar{\alpha}_k^{\text{eq}}$ and Γ_k^{eq} are obtained from Eqs. (6.2) and (6.3) by setting $n_k = n_k^{\text{eq}}$. We see that, near equilibrium, ν_k is smaller than n_k by a factor of order Γ_k/ω_k , so setting $\nu_k = 0$ in the calculation of the self-energies should be a fair approximation.

At next-to-leading order, we set $n_k \approx n_k^{\text{eq}} + \epsilon\delta n_k$ and $\nu_k \approx \nu_k^{\text{eq}} + \epsilon\delta\nu_k$. The linear-response approximation to $\eta(\phi)$ [the sum of Eqs. (2.16) and (2.21)] can be recovered only at the expense of further approximations. The first is to replace $\Gamma_k(n)$ with $\Gamma_k^{\text{eq}} = \Gamma_k(n^{\text{eq}})$ and similarly for $\bar{\alpha}_k(n)$: that is, to ignore the fluctuations in these quantities brought about by fluctuations in $n_k(t)$. The second (when calculating $\partial_t \nu^{\text{eq}}$) is to take the time dependence of $\Gamma_k^{\text{eq}}(t)$ to be Γ_k^{eq}

$= \text{const}/\omega_k(t)$. These approximations amount to replacing the self-energies on the right-hand sides of Eqs. (5.10) and (5.11) with constant equilibrium values. With the additional approximation that $\Gamma_k \ll \omega_k$, we find

$$2(\delta n_k + \delta\nu_k) \approx -\frac{\partial_t n_k^{\text{eq}}}{\Gamma_k^{\text{eq}}} + \frac{\dot{\omega}_k \Gamma_k^{\text{eq}}(1 + 2n_k^{\text{eq}})}{\omega_k^3}, \quad (6.8)$$

which reproduces the sum of Eqs. (2.15) and (2.20).

It is not surprising that the above strategy agrees with linear response theory only when $\Gamma_k \ll \omega_k$. This indicates only that the methods available for resumming self-energies in the equilibrium and nonequilibrium theories differ beyond the leading order in Γ_k/ω_k . The fact that we can recover the linear response result only by ignoring the fluctuations in self-energies induced by those in the n_k is, however, rather more significant, as we discover by attempting to solve the next-to-leading order equations without this extra approximation. To simplify matters, we continue to retain only the leading terms in Γ_k/ω_k , in which case the order ϵ equations are

$$\text{Re } \delta\nu_k \approx \frac{1}{2\omega_k^2} \left[-\omega_k \partial_t (\text{Im } \nu_k^{\text{eq}}) + \frac{\dot{\omega}_k \Gamma_k^{\text{eq}}(1 + 2n_k^{\text{eq}})}{2\omega_k} \right], \quad (6.9)$$

$$\int_0^\infty dk' K(k, k') \delta n_{k'} = \partial_t n_k^{\text{eq}} - \frac{\dot{\omega}_k}{\omega_k} \text{Re } \nu_k^{\text{eq}}, \quad (6.10)$$

where

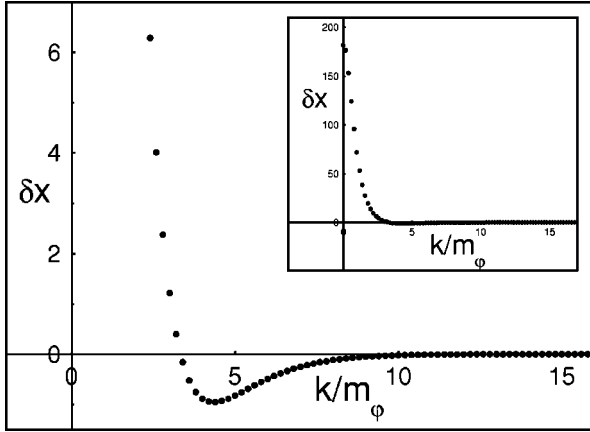
$$K(k, k') = \left. \frac{\delta S_k}{\delta n_{k'}} \right|_{n=n^{\text{eq}}}. \quad (6.11)$$

While the first of these gives $\delta\nu_k$ explicitly, the second is an integral equation to be solved for δn_k . It turns out that *this equation has no solution*. The reason is that the scattering processes described by S_k conserve both particle number and energy. One easily finds that this implies the two sum rules

$$\int_0^\infty dk k^2 K(k, k') = \int_0^\infty dk k^2 \omega_k K(k, k') = 0 \quad (6.12)$$

valid for all k' . The source terms on the right-hand side of Eq. (6.10) do not respect these sum rules, so the equation is not self-consistent and has, in principle, no solution. At higher orders of perturbation theory, particle number is not conserved; only the energy sum rule remains, but that is sufficient to invalidate Eq. (6.10).

It is important to emphasize that the evolution equation (6.5) with $\epsilon = 1$ is perfectly sound: it is a Boltzmann equation with a source term, which presumably has a satisfactory solution for $n_k(t)$. What we have found is that this solution does not have a time-derivative expansion. That is, it cannot be expressed as $n_k(t) = n_k^{(0)}(\phi) + n_k^{(1)}(\phi)\dot{\phi} + \dots$. Nor, therefore, can the equation of motion (1.5). Our principal conclusion, then, is that *the friction coefficient $\eta(\phi)$ does*

FIG. 2. The solution of Eq. (7.1) for $\beta m_\phi = 1$ and $c = 1$.

not exist. As it stands, this conclusion rests on an approximate treatment of a particular model, the $\lambda\Phi^4$ theory. It is likely, however, to be quite generic, as we discuss in Sec. IX.

VII. NUMERICAL INVESTIGATION

Although we have just reached the conclusion that the friction coefficient $\eta(\phi)$ does not exist, we have also seen that the linear-response result for $\eta(\phi)$ can be recovered by ignoring fluctuations in the self-energies—an approximation that, at first sight, would seem not to be severe for a system reasonably close to equilibrium. Thus, although $\eta(\phi)$ formally does not exist, the local equation of motion (1.1) with $\eta(\phi)$ as given by linear response theory might be a reasonable approximation to the true equation of motion. We have obtained numerical results that may bear on this question by taking advantage of the following circumstance. A discretized and truncated version of Eq. (6.10) that one might attempt to solve numerically is

$$\sum_{k'=0}^{k_{\max}} K_{k,k'} \delta n_{k'} = b_k, \quad (7.1)$$

where b_k stands for the source terms on the right-hand side. Because the kernel $K_{k,k'}$ now involves only values of k and k' up to the cutoff value k_{\max} , it does not exactly obey the sum-rule constraints (6.12) and the truncated equation may have a solution.

In fact, we find that it has a very well defined solution, as illustrated in Fig. 2, where the quantity plotted is $\delta x = (4m^3c/\lambda\phi\phi)\delta n$ and we define the naturally occurring coupling constant $c = \lambda^2\pi^2/64(2\pi)^5$. The example shown in Fig. 2 is for $\beta m_\phi = c = 1$. The kernel $K_{k,k'}$ decays rapidly for $k \gg k'$, but is not small when $k \lesssim k'$. Thus, when both k and k' are bounded by the cutoff k_{\max} , the sum rules may be verified for $k' \ll k_{\max}$, but they fail for larger values of k' . Figure 3 illustrates this for the energy sum rule, with $\beta m_\phi = 1$ and a cutoff $k_{\max} = 30m_\phi$. The circles in this figure represent the integral $\int_0^{k_{\max}} dk k^2 \omega_k K(k,k')$, while the squares show, for comparison, the quantity $0.1 \int_0^{k_{\max}} dk k^2 \omega_k |K(k,k')|$. The source b_k in Eq. (7.1) be-

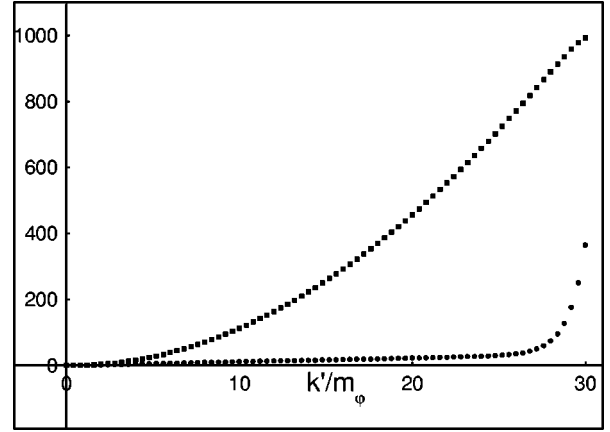


FIG. 3. Illustration of the energy sum rule (6.12); the quantities plotted are explained in the text.

comes very small when k is greater than a few times m_ϕ , and Fig. 2 shows that the same is true of the solution δn_k . In effect, we see that, regardless of the cutoff, only a “self-truncated” kernel, with k and k' restricted to values smaller than a few times m_ϕ , contributes significantly to the solution of Eq. (7.1). The fact that this “self-truncated” kernel does not in itself respect the sum rules accounts for the existence of a well defined solution and, because of the self-truncation, we are able to verify that this solution converges to a cutoff-independent form as k_{\max} is increased.

Formally, we can use this numerical solution in Eq. (4.11) to obtain an estimate for the friction coefficient η . The result of doing this for a range of coupling strengths is shown in Fig. 4, where the quantity plotted is $\sigma = \eta m_\phi / (128\pi\phi^2)$. Figure 5 shows the ratio η/η_{LR} , where η_{LR} is the friction coefficient calculated in linear response theory from Eqs. (2.16) and (2.21) in the limit $\Gamma_k \ll \omega_k$. (The haphazard appearance of this figure results from the different dependencies of η_{I} and η_{II} on temperature and coupling strength.) The formal quantity shown in Fig. 4 has negative values at weak coupling (where our perturbative methods are most likely to make sense), and clearly cannot be interpreted as a genuine friction coefficient. This, of course, is consistent with our earlier argument that the friction coefficient is not well de-

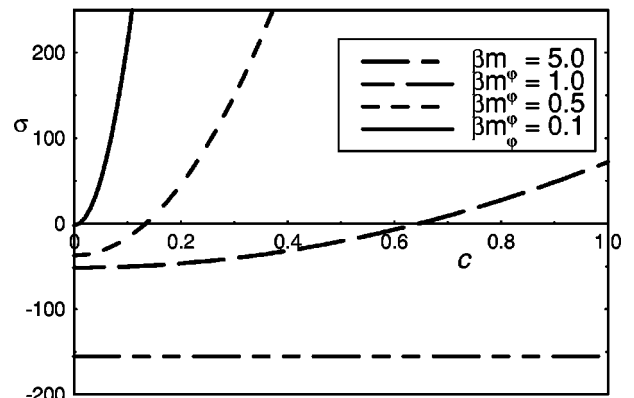


FIG. 4. Friction coefficient calculated from the solution of Eq. (7.1).

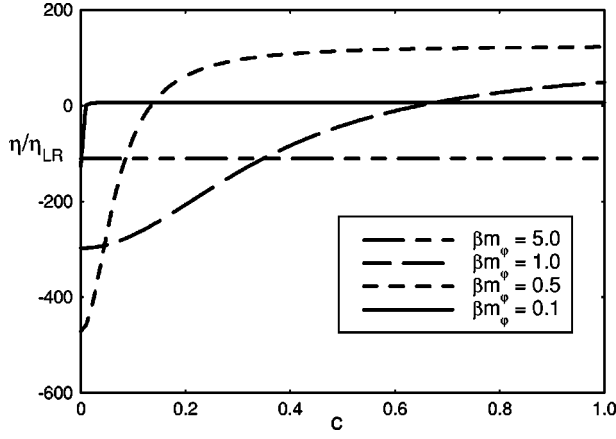


FIG. 5. Ratio of the formal friction coefficient of Fig. 4 to that calculated from linear response theory.

finer. The conclusion is that nonequilibrium methods are needed to investigate the real effect of dissipation in the equation of motion (1.3), even for a system that may be quite close to equilibrium. One approximate method of doing this is to integrate this equation of motion simultaneously with the evolution equations (4.16) and (4.17) and we plan to report on such calculations in future work. The large discrepancies apparent in Fig. 5 suggest that quantitatively significant deviations from the predictions of linear response theory may be expected.

VIII. FRICTION ARISING FROM FERMIONS

We comment briefly on the frictional effect of a Yukawa coupling to fermions. Non-equilibrium perturbation theory for fermions is discussed in Ref. [27], to which we refer the reader for the somewhat cumbersome details. With approximations analogous to those described above for scalar fields, we find for the relevant term in Eq. (1.3)

$$\langle \bar{\psi}(t, \mathbf{x}) \psi(t, \mathbf{x}) \rangle = 4 \int \frac{d^3 k}{(2\pi)^3 2\omega_k^{(\psi)}} [m_\psi (2n_k^{(\psi)} - 1) + 2|\mathbf{k}| \text{Re } \nu_k^{(\psi)}] \quad (8.1)$$

where $n_k^{(\psi)}(t)$ and $\nu_k^{(\psi)}(t)$ are the fermionic analogues of the functions $n_k(t)$ and $\nu_k(t)$. If the fermions have a mass $m_\psi(t) < \frac{1}{2}m_\phi(t)$, then the on-shell processes $\phi \rightleftharpoons \bar{\psi}\psi$ are kinematically allowed. We then obtain kinetic equations of the form

$$\partial_t n_k^{(\psi)} = -\Gamma_k^{(\psi)} (2n_k^{(\psi)} - 1) + \frac{\omega_k^{(\psi)}}{|\mathbf{k}|} \alpha_k^{(\psi)} + \frac{\dot{\omega}_k^{(\psi)}}{\omega_k^{(\psi)}} \frac{|\mathbf{k}|}{m_\psi} \text{Re } \nu_k^{(\psi)}, \quad (8.2)$$

$$\begin{aligned} \partial_t \nu_k^{(\psi)} = & -2i(\omega_k^{(\psi)} - i\Gamma_k^{(\psi)}) \nu_k^{(\psi)} - \left(\frac{\omega_k^{(\psi)} - m_\psi}{\omega_k^{(\psi)}} \right) \alpha_k^{(\psi)} \\ & - \frac{\dot{\omega}_k^{(\psi)}}{2\omega_k^{(\psi)}} \frac{|\mathbf{k}|}{m_\psi} (2n_k^{(\psi)} - 1) \end{aligned} \quad (8.3)$$

where $\alpha_k^{(\psi)}$ and $\Gamma_k^{(\psi)}$ are extracted as in Eqs. (5.10) and (5.11) from the fermion self-energy. Naturally, the scalar self-energy now acquires a contribution from a fermion loop. Taking this into account, the kinetic equations (4.16) and (8.2) are a consistent pair of Boltzmann equations, in which the scattering integrals preserve the particle number $2N_\phi + N_\psi$ [we have derived Eqs. (8.2) and (8.3) only for zero chemical potential, in which case fermions and antifermions are equally abundant and N_ψ means the total number of these particles] and the total energy of scalar and fermionic particles. These conservation laws (or, at higher orders, just energy conservation) again imply that neither the solution of the Boltzmann equations nor the equation of motion for ϕ has a time-derivative expansion.

IX. DISCUSSION

The equation of motion (1.3) is inherently non-local in time; it represents a non-Markovian process in which evolution depends on the history $\phi(t')$ at all times prior to the time t of interest. When the state of the system is not too far from thermal equilibrium, it is tempting to suppose that a local equation of motion (1.1) might be approximately valid, the friction coefficient being estimated from equilibrium statistical mechanics. In this paper, we have examined the approximations needed to extract a local equation of motion from the non-local one, and concluded that this cannot in fact be done. Under suitable conditions [the principal requirement is the existence of a relaxation time Γ_k^{-1} short enough to ensure that correlations decay rapidly compared with the characteristic time scale on which $\phi(t)$ changes] the expectation values in Eq. (1.3) can be approximated by local expressions of the form (4.11) or (8.1), in which the auxiliary functions $n_k(t)$ and $\nu_k(t)$ themselves obey local kinetic equations, such as Eqs. (4.16), (4.17), (8.2), and (8.3). However, this set of local evolution equations can be reduced further to a single local equation for $\phi(t)$ only if the kinetic equations admit a solution in the form of a time-derivative expansion—and we find that they do not. The obstruction arises from fluctuations in self-energies, of which the equilibrium theory takes no account.

Although our explicit computations focussed on the simplest example of a single, self-coupled scalar field, we have indicated in Secs. I and VIII that the situation is quite generic. The above conclusion emerges from an approximate treatment of nonequilibrium dynamics, which can hardly be regarded as a rigorous proof. It would seem that some approximation more or less equivalent to the local *Ansätze* (3.27) and (3.28) for self-energies is an inevitable step toward the derivation of a local equation of motion; without some such approximation, the expectation values in Eq. (1.3) remain non-Markovian and, *a fortiori*, cannot be represented by a local friction term. A subsidiary approximation made in Sec. VI was to set the quasiparticle width Γ_k to zero for the purposes of estimating self-energies. Although this approximation greatly simplified our analytical analysis, it can and should be avoided in a comprehensive numerical study. The effect of this approximation is to restrict the scattering processes in Boltzmann equations to on-shell processes. Now,

the inclusion of off-shell processes might well invalidate the sum rules (6.12) from which we concluded that the friction coefficient η does not exist. Formally, then, by including off-shell processes, we might after all be able to obtain a time-derivative expansion of the equation of motion. However, the friction coefficient implied by this expansion would be quantitatively similar to that obtained in Sec. VII. At weak coupling, it is negative (and thus physically unacceptable) and quite different from the one yielded by linear response theory. At strong coupling, the perturbative methods employed both here and in linear response theory are quantitatively, and perhaps also qualitatively, unreliable. Our practical conclusion, then, is that even for a system quite close to thermal equilibrium the local equation of motion (1.1) does not furnish a reliable account of dissipation, whether or not off-shell processes serve to recover a formal time-derivative expansion. A thorough numerical investigation of the non-equilibrium evolution is therefore essential. At weak coupling, a numerical implementation of the evolution equations developed here is quite widely applicable, and may well be quantitatively adequate, though recently developed methods based on the 2PI-1/N formalism [22,28] are probably more powerful in situations where they can be applied.

Finally, we observe that the analysis given here applies to

quantum field theory in an expanding spacetime with only minor modifications. In a spatially homogeneous Robertson-Walker spacetime, the field redefinitions $\Phi \rightarrow a^{-1}\Phi$ and $\psi \rightarrow a^{-3/2}\psi$, where $a(t)$ is the scale factor, serve to cast the theories we deal with in the form of a Minkowski-space theory with time-dependent masses, provided that t is taken as the conformal time coordinate. In the case of a spinor field or a conformally coupled scalar field, these masses are given simply by $m(t) = a(t)m$. Consequently, the formalism we have constructed changes only insofar as the masses in Eq. (1.4) depend on t through both $\phi(t)$ and $a(t)$. This additional time dependence modifies evolution equations in a way that may be cosmologically important. However, its net effect on, say, Eqs. (6.9) and (6.10) is just that the right-hand sides of these equation contain terms proportional to \dot{a} in addition to those proportional to $\dot{\phi}$. These induce additional contributions to δn_k and $\delta \nu_k$, but do not affect our conclusions concerning the terms proportional to $\dot{\phi}$.

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