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Domain wall dynamics of phase interfaces

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The statics and dynamics of a surface separating two phases of a relativistic quantum field theory at or near the critical temperature typically make use of a free energy as a functional of an order parameter. This free energy functional also affords an economical description of states away from equilibrium. The similarities and differences between using a scalar field as the order parameter versus the energy density are examined, and a peculiarity is noted. We also point out several conceptual errors in the literature dealing with the dynamical prefactor in the nucleation rate.

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I. INTRODUCTION

Different phases of matter are separated in space and time by dividing layers called domain walls. The dynamics of the domain walls is the determining mechanism governing phase transitions in an intermediate range between very slow, quasistatic and very rapid, dynamical processes. In this intermediate range the phase transition speed and the speed of external constraints are comparable to each other.

The dynamics of phase transitions is an involved subject even in macroscopic systems. First of all, phase transitions can be different. They may include slow burning or deflagration, detonation, condensation, evaporation, and many other forms of transition. Nevertheless, the basic conditions of all these transitions have some similarities. These arise from the basic conservation laws and from the requirement of local, or at least approximately local, equilibrium.

In a dynamical situation the approach using the equation of state with a first order phase transition is identical both in compression and in expansion. If the compression is supersonic, shock or detonation waves are formed where the final new phase is immediately created. The phase transition speed influences only the width of the shock front. On the other hand, for slow dynamics and rapid phase transition the shock front width is primarily determined by the transport coefficients, viscosity and heat conductivity, and not by the phase transition speed.

At high energies a relativistic treatment is frequently necessary. It is important to mention that a system is also relativistic if the matter is radiation dominated, meaning that the rest mass of the constituent particles is zero or negligible compared to the fourth root of the energy density. These types of systems must be treated as relativistic even if the collective velocities are small. This was one of the important new features recognized in Ref. [1]. If conserved charges do not exist the application of the conventional theory of phase transition dynamics [2,3] is not possible, and this leads to essential differences in the phase transition dynamics. Most importantly the flow is tied to the energy flow; thus energy or heat current and the fluid flow are identical and heat conduction (with respect to the flow) may not occur. Therefore the coefficient of heat conductivity does not exist, and cannot govern the phase transition speed.

The spatial configuration of phase transition instabilities varies in large homogeneous systems. When the conditions for the occurrence of a new phase are established the most common form of the appearance of the new phase happens via the formation of small critical size bubbles or droplets. (Subcritical size bubbles or droplets will shrink and vanish but supracritical size ones will grow.) This configuration is called homogeneous nucleation.

As the level of supercooling increases and the amount of the new phase increases other geometries become energetically more favorable, such as elongated cylindrical objects (spagetti) or layers (lasagna). The phase transition dynamics is then referred to as spinodal decomposition, indicating that systems which supercool (or superheat) and reach the adiabatic or isothermal spinodals on the phase diagram start the formation of the new phase in these configurations immediately.

Finally, when the two phases are about equally abundant and/or the transition is extremely rapid the two phases form a somewhat random occupation of the configuration space called percolation.

Most frequently explicit dynamical calculations are performed for the homogeneous nucleation geometry as this is usually the initial and the slowest of all. Nevertheless, the domain walls and their dynamical properties play an important role in all of the above mentioned configurations. One important feature of the nucleation studies that is not always recognized is that within the dynamical domain wall or droplet wall we do not have thermal equilibrium. Here, quantities like energy, entropy or particle density make sense, while other quantities that rely on thermal equilibrium, such as the equation of state, may not.

If the fluid is perfect and consists of a single phase of matter in full phase equilibrium, the flow is adiabatic [4]. Nevertheless entropy production is still possible if the system is out of phase equilibrium [4], or out of thermal equilibrium. Thus, in domain walls entropy may be produced even in perfect fluids as here we do not have an equation of state which would satisfy the requirements of equilibrium thermodynamics.

In this paper we would like to discuss connections between the two most frequently encountered approaches to phase transitions using effective field theoretical models and phenomenological thermal and fluid dynamical methods. In Sec. II we first demonstrate with a simple effective field theoretical model how one can obtain a Landau-model type of phenomenological description of critical dynamics. In Sec. III we elaborate on the assumptions and error of different approaches used recently in the literature. We present a summary and conclusion in Sec. IV.

II. THE FREE ENERGY FUNCTIONAL

A common approach to dealing with a phase transition in relativistic quantum field theory is to assume a uniform condensate ϕ for some scalar field and then to compute quantum and thermal fluctuations about that condensate [5]. This is especially so for systems exhibiting spontaneous symmetry breaking where the condensate field serves as an order parameter to distinquish the two phases. Examples include the Higgs field in electroweak theory and the sigma field or scalar quark condensate in strong interaction physics. The resulting free energy density f is obtained from the partition function Z in the usual way:

$$f(\phi,T)=-T\ln Z(\phi,T,\mathcal{V})/\mathcal{V}. \tag{1}$$

Here $\mathcal{V}$ is the volume. Since the high temperature symmetric phase usually corresponds to $\phi=0$, it is customary to define the effective potential as the deviation of the free energy from its value in the symmetric phase:

$$V(\phi,T)=f(\phi,T)-f(0,T). \tag{2}$$

The point $\phi=0$ is either a global minimum or only a local minimum depending on whether T is greater than or less than the critical temperature T_c, respectively. Therefore the results of computations are often found to be, or parametrized as, a fourth order polynomial in ϕ:

$$V(\phi,T)=\sum_{n=2}^{4} a_n(T)\phi^n. \tag{3}$$

Occasionally one will find additional terms of order $\phi^4\ln(\phi/\Lambda)$ or higher powers of ϕ, but these are not common and will not change our analysis qualitatively.

To give a specific example we will use the parametrization of Ref. [6]:

$$V(\phi,T)=\frac{1}{2}\gamma(T^2-T_0^2)\phi^2-\frac{1}{3}\alpha T\phi^3+\frac{1}{4}\lambda\phi^4. \tag{4}$$

The γ, T_0^2, α and λ are temperature independent constants, to be specified shortly. In an equilibrium state the free energy density is the negative of the pressure, $f(T)=-p(T)$. Normalizing to the high temperature symmetric phase, and denoting the equilibrium pressure of that phase by $p_h(T)$, we have

$$f(\phi,T)=-p_h(T)+V(\phi,T). \tag{5}$$

It is not enough to specify the effective potential; the equilibrium pressure as a function of temperature must be specified too.

If one knows the free energy as a function of T then the energy density e can be calculated from the basic thermodynamic identities. In particular, if we want to know the energy density for a specific value of the condensate field, including out of equilibrium configurations also, then we must hold it fixed during the temperature differentiation:

$$\begin{aligned} e(\phi,T)&=f(\phi,T)-T\frac{\partial}{\partial T}f(\phi,T)\\ &=e_h(T)-\frac{\gamma}{2}(T^2+T_0^2)\phi^2+\frac{\lambda}{4}\phi^4. \end{aligned} \tag{6}$$

The first line above is the general thermodynamic identity; the second line applies to the specific parametrization under discussion. If one is at a local minimum of the effective potential, either $\phi_h(T)=0$ or $\phi_l(T)>0$ corresponding to the high and low temperature phases, respectively, then it doesn't matter whether the order parameter is held fixed during the differentiation or not because $\partial f(\phi,T)/\partial\phi=0$ at those points. These two points correspond to thermodynamical equilibrium.

If the system undergoes a first order phase transition at a critical temperature T_c then at that temperature f will have two degenerate minima. For a finite range of temperature above T_c there will persist a higher, local minimum at $\phi_l(T)>0$. For a finite range of temperature below T_c there will persist a higher, local minimum at $\phi_h(T)=0$. The most important physical quantities are the latent heat $L=\Delta e(T_c)=e_h(T_c)-e_l(T_c)$, the correlation lengths $\xi^{-2}=\partial^2 f(0,T_c)/\partial\phi^2=\partial^2 f(\phi_l(T_c),T_c)/\partial\phi^2$, which are equal for the fourth order effective potential at T_c, and the surface energy σ. At the critical temperature the planar interfacial profile has a nice analytical solution on account of the fact that the effective potential becomes symmetric. In the usual way one finds that the profile field $\overline{\phi}(x)$ is

$$\overline{\phi}(x)=\frac{\phi_l(T_c)}{2}[1-\tanh(x/2\xi)]. \tag{7}$$

This interpolates through intermediate nonequilibrium states from $\phi_l(T_c)$ for $x\ll-\xi$ to $\phi_h(T_c)=0$ for $x\gg\xi$. The surface energy can then be expressed as

$$\sigma=\int_{-\infty}^{\infty}dx\left(\frac{d\overline{\phi}}{dx}\right)^2. \tag{8}$$

The parameters in the effective potential can be expressed in terms of these physical parameters as follows:

$$\gamma=\left(1+\frac{L\xi}{6\sigma}\right)\frac{1}{T_c^2\xi^2}$$

$$T_o^2 = \frac{L\xi}{L\xi + 6\sigma} T_c^2$$

$$\lambda = \frac{1}{3\sigma\xi^3}$$

$$\alpha = \frac{1}{T_c\xi}\sqrt{\frac{3}{2\sigma\xi^3}}. \tag{9}$$

For purposes of illustration, and with the QCD phase transition in mind, we choose the following numerical values:

$$\xi = \frac{1}{T_c}$$

$$\sigma = \frac{17}{6}\frac{\pi^2}{45} T_c^3$$

$$L = \frac{68\pi^2}{45} T_c^4. \tag{10}$$

This choice of B results in a critical temperature T_c which is identical to the one obtained in a very simple model for a phase transition between an ideal massless pion gas (3 internal bosonic degrees of freedom) and an ideal two flavor quark-gluon plasma (37 effective bosonic degrees of freedom). The resulting latent heat is large, useful mainly for display purposes. Other choices of parameters or a more sophisticated equation of state can be used instead. The resulting simple equation of state in the quark-gluon plasma phase is

$$p_h(T) = \frac{37\pi^2}{90} T^4 B. \tag{11}$$

The numerical value for the latent heat L corresponds to a transition from the 37 degrees of freedom mentioned above to 3 degrees of freedom for massless pions. Compared to the results of lattice simulations of QCD the values for the latent heat and surface free energy are too large by about an order of magnitude [7]. This is actually a subtle issue to some extent because there are as yet are no reliable values for these quantities in full dynamical QCD with the physical values of the up, down, and strange quark masses. Current lattice simulations show either a weakly first order phase transition or no true phase transition, only a very rapid crossover from hadronic gas to quark-gluon plasma. Homogeneous nucleation with a well-defined bubble surface and radius requires a first order transition, but the value of the latent heat is not crucial. These assumptions must be kept in mind when applying nucleation theory to QCD or to any other theory. It is convenient to make everything dimensionless by measuring energies in units of T_c and lengths in units of $1/T_c$, which practice we adhere to in the rest of this section.

The effective potential (4) as a function of ϕ is plotted in Fig. 1 for three different temperatures: the critical one and 1% above and below it. This displays the traditional behavior of a strong first order phase transition. The plot includes negative values of ϕ which may or may not be allowable, depending on the origin of ϕ and its physical interpretation.

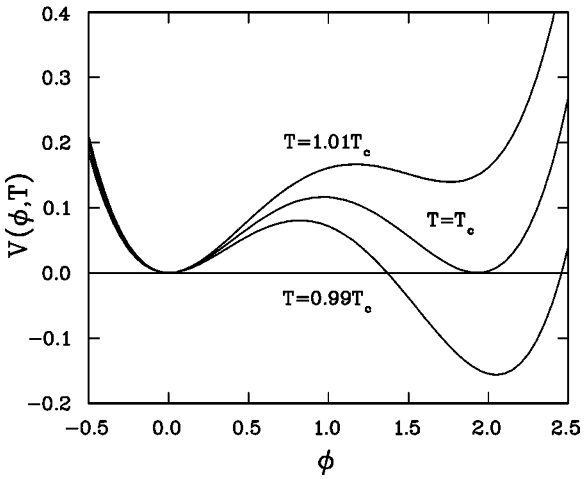

FIG. 1. The effective potential as a function of the field.

The free energy density (5) as a function of ϕ is plotted in Fig. 2 for the same temperatures. The only difference between this and Fig. 1 is the addition of the pressure as evaluated in the high temperature symmetric phase.

The energy density (6) as a function of ϕ is plotted in Fig. 3. It has a maximum at $\phi = 0$. This follows directly from the expression for e given earlier: For small values of ϕ the deviation from $e_h(T)$ is quadratic in ϕ with negative curvature. Note that the energy density goes negative when ϕ is greater than about 2.0.

Rather than using the field ϕ as the order parameter one might consider other choices more appropriate for the problem at hand. For example, one could take as the order parameter the deviation of the energy density from its equilibrium value and expand the free energy in a power series in this difference. This is the Landau approach to the description of fluctuations and to departures from equilibrium states [8]. Such an approach to the nuclear liquid-gas phase transi-

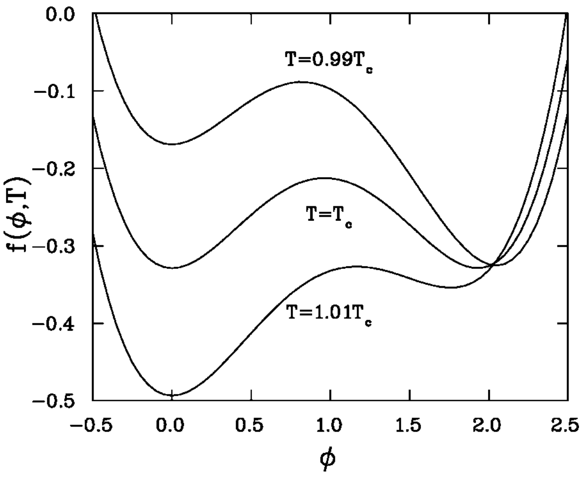

FIG. 2. The free energy density as a function of the field.

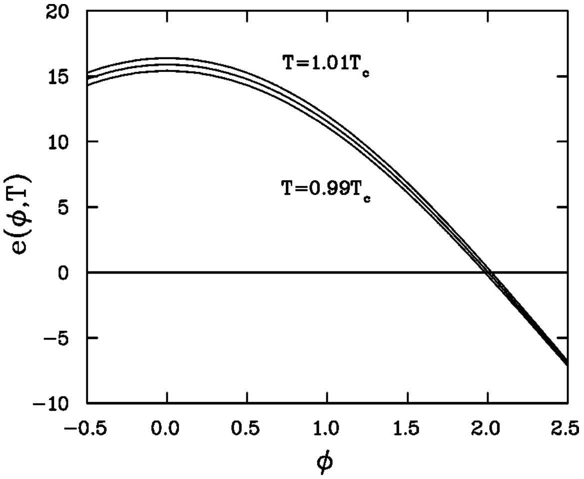

FIG. 3. The energy density as a function of the field.

tion was implemented by Goodman *et al.* [9]. It is also this same function that was used in working out the nucleation of relativistic first order phase transitions by two of us [1]. With these motivations we plot the free energy versus the energy in Fig. 4 by using ϕ as a parameter. There are a number of points to be made concerning this figure. First, there are two minima at each of the chosen temperatures. The minima occur at the energy densities $e_l(T)$ and $e_h(T)$. The lower of these two represents the equilibrium state while the other represents a metastable state except at T_c when they are degenerate. The value of f at a minimum is equal to the negative of the pressure in that particular phase: $p_l(T) = -f(e_l(T),T)$ and $p_h(T) = -f(e_h(T),T)$. Second, the value of $e_l(0.99T_c) < 0$! This is an indication of the inadequacy of the specific parametrization of the effective potential at this temperature, and probably at lower temperatures too. This is an important point to always be aware of when writing down any formula for an effective potential. Third, the free energy has a cusp at the location of the high density symmetric phase. This is an unavoidable consequence of the inference of the free energy from the effective potential. From Fig. 3 or from the corresponding formula it is clear that the energy density is a maximum at $\phi = 0$. This means that the plot of the free energy ends at the equilibrium density $e_h(T)$ of the symmetric phase with a cusp. In case ϕ is defined for positive values only then the effective potential V, the free energy density f, and the energy density e end abruptly at $\phi = 0$ while $f(e,T)$ would end abruptly at the finite value $e_h(T)$. Physically this must be a restriction of fluctuations to those associated with ϕ and only ϕ. This is too restrictive by far; there are certainly other physical processes not taken into account. For example, placing particles in a box in contact with a heat reservoir allows for the exchange of energy between the particles and the reservoir, resulting in fluctuations in energy and pressure. These processes are always present in systems at fixed temperature and are not accounted for, or associated with, fluctuations in an order parameter.

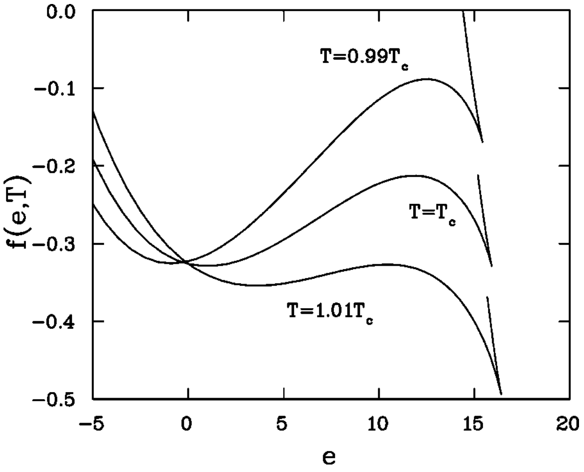

FIG. 4. The free energy as a function of the energy density as derived from Figs. 1–3.

Now let us follow the Landau construction of the free energy away from equilibrium states using the energy density as the order parameter with no reference to ϕ whatsoever. This is the construction made in [1] in the context of constructing spherical surfaces separating two phases at noncritical temperatures. The total free energy of interaction is the sum of a gradient energy and a free energy density integrated over space.

$$F_I\{e(\mathbf{x})\} = \int d^3x \left[\frac{1}{2}K(\nabla e(\mathbf{x}))^2 + f(e(\mathbf{x}),T)\right]. \tag{12}$$

The f is most economically written as a fourth order polynomial in e:

$$f(e,T) = \sum_{n=0}^{4} b_n(T) e^n. \tag{13}$$

The coefficients are functions of temperature. They are determined by several requirements. The first is that f has minima located at $e_l(T)$ and $e_h(T)$, such that f at those points be equal to the negative of the corresponding equilibrium pressure:

$$f(e_h(T),T) = -p_h(T)$$
$$f(e_l(T),T) = -p_l(T). \tag{14}$$

This results in four equations involving the five b_n. Two more equations result from fixing the correlation length (which is the same in the two phases only at T_c) and the surface energy σ (which, strictly speaking, is only well defined at T_c). This set of six equations is not inconsistent because the coefficient of the gradient energy, K, must also be determined:

$$\partial^2 f/\partial e^2|_{e=e_h(T_c)} = \partial^2 f/\partial e^2|_{e=e_l(T_c)} = \frac{6\sigma}{\xi L^2}$$

$$K = \frac{6\sigma\xi}{L^2}. \tag{15}$$

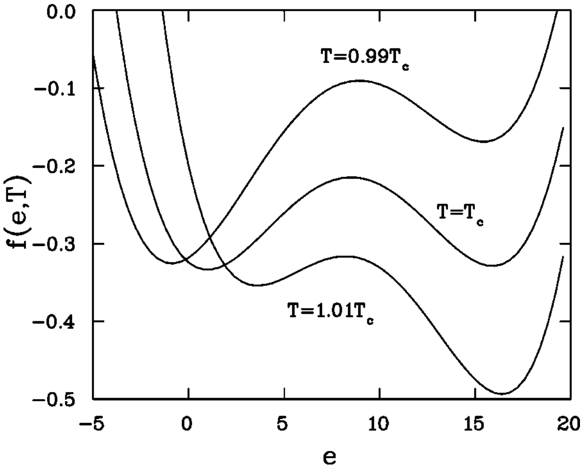

FIG. 5. The free energy as a function of the energy density from a Landau construction.

We insist upon the same equilibrium energy densities and pressures as used in the effective potential approach, and use the same correlation length and surface energy at T_c too:

$$\sigma = K\int_{-\infty}^{\infty} dx \left(\frac{d\overline{e}}{dx}\right)^2 \tag{16}$$

$$\overline{e}(x) = \frac{1}{2}[e_h(T_c) + e_l(T_c) + \Delta e(T_c)\tanh(x/2\xi)]. \tag{17}$$

Then the Landau expansion of the free energy as a function of the energy density is obtained and is plotted in Fig. 5. The $f(e,T)$ is now a smooth and well-behaved function. Its only failing is that the equilibrium energy density at $0.99T_c$ is negative, but that is a consequence of insisting that it be the same as for the effective potential, which was negative. (Actually, fluctuations into states with negative energy can be avoided if one uses the Laurent expansion instead of the fourth order polynomial approximation [10], but only if the lower minimum is at positive energy density.) In fact, with the Landau approach we have a closer connection between the physical observables and the free energy. For example, we can easily specify the equilibrium energy density and pressure in each phase, making it simple to avoid such unwelcome behavior as a negative energy density in equilibrium. Of course the direct connection with the condensate field ϕ is lost.

The main difference between the two approaches is that the field theoretical approach parametrizes the out-of-equilibrium configurations in terms of an effective potential as a function of ϕ that shows smooth, quadratic minima around the two equilibrium states, whereas the Landau type of thermodynamical approach does the same in terms of the free energy as a function of the energy density e. These two approaches are not fully equivalent: the highly nonlinear $f(e)$ dependence obtained from this particular field theoretical model does not yield a smooth quadratic minimum in terms of e at e_h (where $\phi=0$). This leads to unphysical estimates for energy density fluctuations, in our opinion. Which approach one takes depends on the physical situation being addressed.

Finally we comment on the appearance of negative energy density. This is an unphysical artifact arising from parametrizations being used outside their realm of applicability. There are practical means to alleviate this problem. See, for example, Refs. [6] and [10]. Since it is not our purpose to construct accurate physical models we do not pursue these means here but only make the reader aware of them.

III. DYNAMICS OF FIRST ORDER PHASE TRANSITIONS

A phase transition occurs because of some change in the global properties of the system. Examples of such changes are: an expansion chamber or a sudden quench in temperature in the laboratory, the expansion of hot matter in a high-energy heavy ion collision, and the expansion of the early universe. Generally we need to compare the phase transition rate to the rate of expansion or quench. The speed at which the phase transition proceeds comes into play when the speed at which the external variables, such as the volume, becomes comparable to or exceeds that of phase conversion.

If the rate of change of an external variable is slow there is sufficient time to maintain phase equilibrium on a coarse-grained time scale. This also means that all other equilibration processes are completed as these invariably require less time and less interaction than the conversion of one phase of matter to another.

Thus, in the case of slow external dynamics and rapid phase equilibration the matter is in complete equilibrium, including phase equilibrium, and the equation of state of the matter undergoing a first order phase transition is given by the Maxwell construction. Then we have a fully developed mixed phase, and the phase abundances are typically given by the conservation of energy and entropy. In this limiting case no information on dynamical processes is needed. This type of transition is adiabatic. Even in moderately fast dynamical expansions there are only small deviations from the ideal and complete phase equilibrium (Maxwell construction). This deviation results in some delay in the creation of the new phase, leading to supercooling or superheating and extra entropy production [4]. Even in slow, nearly adiabatic, phase transitions there may exist deviations from full equilibrium. Relevant possibilities include baryon number generation in a cosmological electroweak phase transition and baryon and isospin inhomogeneities generated in a cosmological QCD phase transition. Any departure from full equilibrium will result in some entropy generation, however small it may be.

For heavy ion reactions the first attempt to explicitly evaluate the phase transition speed of homogeneous nucleation was described in [1,11]. The homogeneous nucleation mechanism correctly describes the initial stage of the phase transition when the abundance of the newly created phase is still small and when the rate of phase conversion is the slowest process.

Here a couple of remarks are necessary. There are several requirements to form critical-sized bubbles or droplets of the new phase. Both pressure balance and temperature equality should be established between the phases; this requires the transfer of energy and momentum across the phase boundary. If local equilibrium is assumed both before and after the formation of the new phase then we cannot relax the requirement of pressure and temperature equilibrium.

Langer's modern theory of nucleation yields the following formula for the rate:

$$I=\frac{\kappa}{2\pi}\Omega_0 e^{-\Delta F/T} \tag{18}$$

where ΔF is the change in the free energy of the system due to the formation of the critical droplet. Ω_0 is a statistical prefactor which measures the available phase space volume. κ is a dynamical prefactor which determines the exponential growth rate of critical droplets which are perturbed from their quasiequilibrium radius R_*:

$$\kappa=\frac{d}{dt}\ln[R(t)-R_*]. \tag{19}$$

The dynamical prefactor has been calculated by Langer and Turski [2,3] and by Kawasaki [12] for a liquid-gas phase transition near the critical point, where the gas is not dilute, to be

$$\kappa=\frac{2\lambda\sigma T}{\ell^2 n_\ell^2 R_*^3}. \tag{20}$$

This involves the thermal conductivity λ, the surface free energy σ, the latent heat per molecule ℓ and the density of molecules in the liquid phase n_ℓ. The interesting physics in this expression is the appearance of the thermal conductivity. In order for the droplet to grow beyond the critical size, latent heat must be conducted away from the surface into the gas. For a relativistic system of particles or quantum fields which has no net conserved charge, such as baryon number, the thermal conductivity vanishes. The reason is that there is no rest frame defined by the baryon density to refer to heat transport. Hence this formula obviously cannot be applied to such systems. The dynamical prefactor for such systems was first evaluated in the manner of Langer by two of us [1] to be

$$\kappa=\frac{4\sigma}{(\Delta w)^2 R_*^3}\left[\frac{4}{3}\eta+\zeta\right]. \tag{21}$$

Here η and ζ are the shear and bulk viscosities, respectively, and Δw is the enthalpy density difference between the two phases (equal to the latent heat since the pressures are equal at the critical temperature). This fully relativistic expression was subsequently generalized by Venugopalan and Vischer [13] to include a conserved baryon number:

$$\kappa=\frac{2\sigma}{(\Delta w)^2 R_*^3}\left[\lambda T+2\left(\frac{4}{3}\eta+\zeta\right)\right]. \tag{22}$$

In the nonrelativistic limit and when the viscosities are small compared to the heat conductivity this reduces to Langer's expression. In the relativistic limit with no net baryon number, where effectively $\lambda\to 0$, this reduces to Eq. (21).

Unfortunately there are several erroneous expressions for κ which subsequently appeared in the literature. In [14] one finds that, in the relativistic limit with no net baryon number,

$$\kappa=\sqrt{\frac{2\sigma w_h}{(\Delta w)^2 R_*^3}}. \tag{23}$$

This was "derived" under the assumption of vanishing shear viscosities. An expression for κ in the nonrelativistic limit with nonzero baryon number was found which differed from Langer's result too. By including the viscosities another paper [15] finds that

$$\kappa=\sqrt{\frac{2\sigma w_h}{(\Delta w)^2 R_*^3}}+\frac{1}{c_s^2}\frac{\sigma}{(\Delta w)^2 R_*^3}\left[\frac{4}{3}\eta+\zeta\right]. \tag{24}$$

Here c_s is the speed of sound in the low density phase. In the limit that the viscosities vanish this reduces to the result of [14], and in the limit that they are large this reduces to the result of [1] albeit with a factor of $1/c_s^2$ rather than 4.

The results found by [14] and [15] are suspicious because they predict a dynamical growth factor even in the absence of viscous forces. The analysis of [14] is wrong because of a misinterpretation of a derivative. That paper follows the approach of [1] relatively closely except for one crucial point. At some point in the analysis one encounters the derivative $\partial f(\overline{e},T)/\partial\overline{e}$ evaluated at the radial profile $\overline{e}=\overline{e}(r)$ of the energy density for a spherical bubble or droplet connecting the two phase. Thus $\overline{e}(r)$ varies between e_l and e_h. The free energy density varies accordingly. As explained carefully and in some detail in [1], this derivative is taken at fixed temperature. At the equilibrium points it is zero: see the discussion in Sec. II above. However, the paper [14] equates f with $-p$ irrespective of whether we are at one of the minima or not. It then assumes an equation of state $p=c_s^2 e$ to finally obtain $\partial f(e,T)/\partial e=-c_s^2$. From that point on the analysis diverges, and the erroneous results for κ inevitably follow.

The paper [15] attempts to derive an expression for κ which reduces to that of [1] when the viscosity is large and to that of [14] when the viscosity is small or vanishes. Of course this is impossible if the result of [14] is wrong, as we have pointed out. One error in [15] is to sometimes include [their Eq. (10)] and sometimes exclude [their Eq. (9) with p interpretted as the equilibrium pressure] the variation in the free energy through nonequilibrium states in the surface region in an uncontrolled fashion. In particular, an extra term $c_s^2\nabla^2\nu(r)$ is introduced on the right side of their Eq. (20). This term is not present in their pair of Eqs. (17), which are the ones to be solved so as to determine κ. The authors never commit to exactly what c_s^2 is except to say that it "could be the velocity of sound in the medium around the saddle configuration." In the absence of this ad hoc term, the analysis following their Eq. (20) is otherwise consistent with that following Eq. (71) of [1]. A second error is to assume that the divergence of the velocity vanishes in the surface region, meaning that $r^2 v(r)$ is a constant. This is directly contradicted by their own Eq. (30) since both the energy and enthalpy densities are varying rapidly in the surface. As derived in [1] the quantity $r^2 v(r)$ is a constant only when r exceeds the bubble radius R by more than a few correlation lengths but still less than $2R$. A third error is to drop the left side of their Eq. (26) compared to the right side without justification; in fact, they are the same order of magnitude. If the authors had followed the analysis of [1] without introducing *ad hoc*, undetermined quantities, and had not made incorrect approximations, they would have obtained the same results. Their errors seem to be entirely motivated by the desire to find an interpolating formula between the results of [1] and [14].

It is also possible to argue that the erroneous results for κ are not only mathematically incorrect but physically incorrect too. What is relevant is the slowest required process, not the fastest. Furthermore it is incorrect to add up the rates of all processes to obtain the highest possible phase transition speed. If not all required processes are completed the phase transition is not complete either. For example to establish pressure (momentum) equilibrium [14,15] is not sufficient, because then a subsequent step of establishing thermal equilibrium is required, and only when both processes are over has the phase transition been completed.

In Ref. [1] the slowest process, heat transfer via viscosity, was evaluated. In some cases, when we have non-negligible net baryon charge in our system, the heat conductivity may also contribute to the heat transfer to the new phase, thus speeding up somewhat this slowest of all processes [13]. If the transport coefficients are all vanishing so that thermal balance cannot be achieved, we will never reach both phase and thermal equilibrium, so the rate tends to zero as stated in Ref. [1].

In case other degrees of freedom exist which permit energy transport between the phases leading to a common temperature [16,17], these processes can be combined with other transport processes like viscosity, heat conduction, and particle diffusion (such as neutrino transport) leading to faster temperature balance and a higher rate. However, sound waves do not lead to dissipation as perfect fluid dynamics is adiabatic (in the absence of shock waves). Thus, the sound wave as the sole mechanism cannot characterize dissipative transport processes, as was mistakenly claimed in [14] and [15].

A final remark: If the external dynamical evolution is much faster than the processes involved in the phase and kinetic equilibration then the matter involved in the phase transition loses both phase and thermal equilibrium, and we have to abandon the thermal and fluid dynamical approaches altogether. An example in the context of high energy heavy ion collisions is presented in [18] where an effective field theoretical approach is used. Generally this will be the case for small, rapidly developing systems only.

IV. CONCLUSIONS

In this paper we have examined two subtle issues in the dynamics of domain walls. One of them is the difference between the effective potential as a function of a scalar field and the free energy as a function of energy density, both evaluated near the critical temperature. The other involves a misconception in the literature about the derivation of the dynamical prefactor, or growth rate, in homogeneous nucleation theory. It is important to understand these subtleties because of their importance in such physical environments as cosmology, astrophysics and high energy nuclear collisions.

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