Generalized Lorentz invariance with an invariant energy scale

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The hypothesis that the Lorentz transformations may be modified at Planck scale energies is further explored. We present a general formalism for theories which preserve the relativity of inertial frames with a nonlinear action of the Lorentz transformations on momentum space. Several examples are discussed in which the speed of light varies with energy and elementary particles have a maximum momenta and/or energy. Energy and momentum conservation are suitably generalized and a proposal is made for how the new transformation laws apply to composite systems. We then use these results to explain the ultrahigh-energy cosmic ray anomaly and we find a form of the theory that explains the anomaly, and leads also to a maximum momentum and a speed of light that diverges with energy. We finally propose that the spatial coordinates be identified as the generators of translation in Minkowski spacetime. In some examples this leads to a commutative geometry, but with an energy dependent Planck constant.

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I. INTRODUCTION

Several experimental and theoretical developments point to the possibility that the usual relation between energy and momentum that characterizes the special theory of relativity,

$$E^2=p^2+m^2, \tag{1}$$

may be modified at Planck scales. Among these is the observed threshold anomalies in ultrahigh-energy cosmic ray (UHECR) protons [1,2], and possibly also TeV photons [3]. As pointed out first by Amelino-Camelia and Piran [4], these can be explained by modifications of the energy momentum relations of the form

$$E^2=p^2+m^2+\lambda E^3+\cdots, \tag{2}$$

where λ is of the order of the Planck length.

Such a modified energy-momentum relationship leads to further predictions which are falsifiable with planned experiments. Among these is an energy dependent speed of light, observable (with λ of the order of the Planck length) in planned gamma ray observations [5]. An energy dependent speed of light may also imply that the speed of light was faster in the very early universe, when the average energy was comparable to Planck energies [6]. As pointed out by Moffat [7], and Albrecht and Magueijo [8], such an effect could provide an alternative solution to the horizon problem and other problems addressed by inflation. Such modified dispersion relations also may lead to corrections to the predictions of inflationary cosmology, observable in future high precision measurements of the cosmic microwave background (CMB) spectrum [9]. Finally a modified dispersion relation may lead to an explanation of the dark energy, in terms of energy trapped very high momentum and low-energy quanta, as pointed out by Mersini and collaborators [10].

These effects open up the possibility of testing hypotheses about Planck scale physics by more than one window, in the present and near future. Indeed, the fact that the hypothesis that the energy-momentum relations receive corrections of the form of Eq. (2) is experimentally testable, with λ on the form of the Planck scale, is alone sufficient reason to consider it.

However there are also theoretical issues which motivate such a modification. Several calculations [11] in loop quantum gravity in fact predict modified dispersion relations of the form (2). That they do so is not surprising for, from the point of view of the quantum theory of gravity, global Lorentz invariance is no more than an accidental symmetry of the ground state of the classical limit of the theory.[1] Thus, it is to be expected that corrections to consequences of Lorentz invariance appear as quantum gravitational effects, which is to say as corrections to the laws of special relativity which are suppressed by $l_{\rm Planck}$.

At the same time, there is a simple reason to be skeptical that the energy momentum relations may receive modifications of the form of Eq. (2). Such a modification contradicts the transformation laws of special relativity, according to which energy and momentum transform according to the Lorentz transformations, so as to preserve the Minkowski metric. Lorentz invariance is generally assumed to be a consequence of the oldest and most reliable principle in all of dynamics, which is the relativity of inertial frame. One may then worry that were the hypothesis (2) confirmed experimentally, that event would signal that after all there is a preferred frame of reference in nature, in contradiction to the last 400 years of progress in science.

Very recently several people have realized that this worry is unnecessary [12–14]. It is possible to keep the principle of

[1] Neglecting the cosmological constant.

the relativity of inertial frames, and simply modify the laws by which energy and momenta measured by different inertial observers are related to each other. By adding nonlinear terms to the action of the Lorentz transformations on momentum space, one can maintain the relativity of inertial frames. The quadratic invariant is replaced by a nonlinear invariant, which in turn leads to modified dispersion relations of the form of Eq. (2).

There is indeed a simple argument that suggests that such a modification may be necessary. In quantum theories of gravity such as loop quantum gravity, the Planck length plays the role of a threshold below which the classical picture of smooth spacetime geometry gives way to a discrete quantum geometry. This suggests that the Planck length plays a role analogous to the atomic spacing in condensed matter physics. Below that length there is no concept of a smooth metric. It is then not surprising if quantities involving the metric, such as the quadratic invariant, receive corrections of order of the Planck length.

However this raises a problem. Lengths are not invariant under Lorentz transformations, so one observer's threshold will be perceived to be set in at a different length scale than another's.[2]

Alternatively, various hypotheses concerning quantum gravity [16,17] and string theory [18,19] suggest that the geometry of spacetime is in fact noncommutative. In such a modification the spacetime coordinates may no longer commute and there may be modified energy-momentum relations. This in turn suggests a deformation of the Poincaré symmetry of flat spacetime, one example of which is given by the κ-Poincaré symmetry discussed, for example, in [20–22]. In all these proposals, the noncommutativity is measured by a parameter which has the dimensions of a length. Again, we can ask how it is that all observers agree on the scale at which noncommutativity appears, given that lengths normally are not invariant under Lorentz transformations.

These paradoxes may be resolved if the Lorentz transformations may be modified so as to preserve a single energy or momentum scale. Then all observers will agree that there is an invariant energy or momentum above which the picture of spacetime as a smooth manifold breaks down. Because there are then two constants which are preserved, this proposal has been called "doubly special relativity [12,14]."

To summarize, the various experimental and theoretical issues we have mentioned lead us to ask whether it is possible to modify the principles of physics so that all of the following requirements are met.

(1) The relativity of inertial frames, as proposed by Galileo, Descartes, Newton and Einstein, is preserved.

(2) There is nevertheless an invariant energy scale $E_P = \lambda^{-1}$, which is of the order of the Planck scale.

(3) The threshold for UHECRs should be increased as suggested by experiment.[3]

(4) The theory should exhibit a varying, preferably diverging, speed of light, at high energies.

(5) The theory should have a maximal momentum, corresponding to the granular nature of space.

The main question this paper raises is whether it is possible to modify the principles of special relativity so that all five of these requirements are met. The main result of this paper is that the answer is affirmative.

To explain the viewpoint we take, we can start by emphasizing that no matter how quantum mechanical, noncommutative or deformed the geometry of spacetime may become, if the principle of the relativity of inertial frames is to hold, the transformations between measurements made by inertial observers must satisfy the group property. Furthermore, the group must be a six parameter extension of the rotation group, with the three additional parameters going over into the boosts of special relativity whenever quantities of the order of the Planck scale can be ignored. However, as we argued in [13] the only group with these properties is the Lorentz group itself. Hence, the only possibility to achieve all of these conditions is through a nonlinear action of the ordinary Lorentz group on the states of the theory.

In a recent paper, we proposed this viewpoint, and we proposed a class of theories in which the Lorentz transformations act nonlinearly on momentum space. We studied in some detail a simple example [13] of a theory in which the action of the Lorentz group on momentum space was made nonlinear, in such a way that the Planck energy became an invariant. However, as we will demonstrate below, this example does not satisfy all of the conditions just mentioned. Furthermore, studies by other authors suggested that it was not obvious how to modify the action of the Lorentz transformations so as to achieve all these conditions, in particular, it appeared that theories in which the relativity of inertial frames is preserved may not be able to account for the observed threshold anomalies.

Before going further, we want to emphasize that the proposal to modify the action of the Lorentz transformations on momentum space was not originated by us. Modifications of special relativity in which the action of the transformations are nonlinear have been considered by a number of authors [5,12,14,20–23]. To our knowledge the earliest such proposal is by Fock [23] and related proposals have been considered earlier also in [5,12,14,20–22]. We consider that our contribution is mainly to take a phenomenological point of view in which we insist that the modifications of special relativity are to be treated in the most general way possible, so as to allow nature to teach us if and how the relativity of inertial frames is realized in a fundamental theory. While it may in the future happen that a fundamental quantum theory of gravity makes predictions for the exact form of a modified dispersion relation or action of the Lorentz transformations, we want to avoid too hastily following any particular mathematical hypothesis about the structure of spacetime to the exclusion of others.

Recently a number of authors have contributed to the study of such theories, discussing many aspects which we are not able to consider here [24–26]. However, we are able

[2] For another view of this apparent paradox, however, see [15].

[3] The situation for the TeV photon threshold is not as convincing, so we do not yet impose it as a requirement.

to address a number of issues which concern the whole class of theories in which the Lorentz transformations act nonlinearly on momentum space.

It is not hard to see that if one adds momenta and energy linearly, as we normally do in physics, the conservation of momentum is inconsistent with the new nonlinear action of the Lorentz group on momentum space. Is it then possible that energy and momentum conservation is maintained, but with these laws also becoming nonlinear?

There is no experimental reason why the energy and momenta of elementary particles cannot be bounded by the Planck energy, but this is certainly not the case for macroscopic systems. Thus, the transformation laws must distinguish elementary from composite systems in such a way that the macroscopic bodies can have Planck energies and momenta while transforming and moving according to the usual laws of special relativity.

If a modified energy momentum relation is the explanation for the observed threshold anomalies, there appears to be a necessity that these modifications are significant already at the scales of the thresholds themselves, which are on scales of TeV's for the photons and 10^{11} GeV for the protons, i.e. small compared to the Planck energy. At the same time, there is no observed energy dependence of the speed of light, seen in gamma ray observations, to scales up to $10^{-3}E_{\text{Planck}}$, which is in fact much higher than the scale of the problematic thresholds. So how could a theory resolve the problem of the threshold anomalies, without at the same time causing an energy dependent speed of light at scales which are already ruled out by experiment?

Before the present viewpoint was formulated a number of authors arrived at modified Lorentz transformations by investigating the hypothesis that geometry becomes noncommutative, so that spacetime coordinates no longer commute. A beautiful example of such a theory is the κ-Poincaré symmetry of κ-deformed Minkowski spacetime, which is indeed a noncommutative geometry [21,22]. However, it is important to ask whether the noncommutativity of spacetime coordinates is a necessary consequence of modifying the action of the Lorentz transformations on momentum space, so as to have an invariant Planck energy and/or momentum, or whether one can achieve such a modification in the context of a commutative spacetime geometry.

In this paper we will resolve all of these issues, in the course of showing that all 5 of our conditions can be met.

The plan of this paper is as follows. In Sec. II we start by noting that the procedure defined in [13] is not unique: there are other possible nonlinear realizations of the Lorentz group in momentum space, associated with different operators in place of $U(p_0)$ (defined in [13]). Each of these leads to different modified invariants and hence to different dispersion relations for massive and massless particles. We also write down the necessary and sufficient conditions for a general nonlinear action to display an invariant energy scale, and a maximal momentum. We place conditions upon U so that the group property of the action is preserved and highlight the emergence of a preferred frame should these conditions be violated.

In Sec. III we discuss several examples, and show how the variable speed of light theories discussed in cosmology [6–8] and the κ-Poincaré group fit into the general framework introduced here.

Then, in Sec. IV we explain how energy and momentum conservation are maintained in these theories, a matter closely related to the definition of momenta addition. This allows us in Sec. V to examine the kinematics of UHECRs and gamma rays, placing constraints upon the possible realizations of our theory which can explain current observations. We find that standard deformations can only explain the threshold anomalies with a *negative* Planck energy, of the order of 10^{11} GeV. However, we exhibit one class of dispersion relations (and an associated nonlinear Lorentz action) where this problem does not exist.

Finally in Sec. VI, we point out that because the Lorentz group acts nonlinearly on momentum space, the action on spacetime coordinates is also nontrivial. We propose that in quantum theory the space and time coordinates are to be defined to be generators of translations in momentum space. In contrast to other definitions, this leads to a commutative spacetime geometry. But the commutation relations between position and momentum become energy dependent, leading to a new energy dependent modification of the uncertainty relations.

II. GENERALIZED NONLINEAR ACTIONS AND DEFORMED DISPERSION RELATIONS

In our first paper [13] we proposed a nonlinear modification of the action of the Lorentz group in momentum space which contains an observer independent length scale (denoted here by λ), and reduces to the usual linear action at low energies. For the new proposal the concept of metric (a quadratic invariant) collapses at high energies, being replaced by the nonquadratic invariant

$$||p||^2 \equiv \frac{\eta^{ab}p_a p_b}{(1-\lambda p_0)^2}. \tag{3}$$

The group algebra, however, is left unchanged, suggesting that the spin connection formulation of general relativity may still be valid (as the connection takes values in the algebra). This remarkable feature may be traced to the fact that our modified boost generators (and likewise for the rotation generators) may be written in the form

$$K^i = U^{-1}[p_0]L_0^i U[p_0], \tag{4}$$

where $L_{ab}=p_a(\partial/\partial p^b)-p_b(\partial/\partial p^a)$ are the standard Lorentz generators. For the particular boosts we have proposed we have:

$$U[p_0]\equiv \exp(\lambda p_0 D) \tag{5}$$

(where $D=p_a(\partial/\partial p_a)$ is a dilatation), or more specifically:

$$U[p_0](p_a)=\frac{p_a}{1-\lambda p_0} \tag{6}$$

[where $U[p_0](p_a)$ denotes the a component of the image]. We note that λ may have either sign. We expect that it will be proportional to plus or minus the Planck length. We also note that this is not a unitary equivalence, because, as may be easily checked, in general U is not unitary. In fact we shall be mainly interested in singular expressions for U, for reasons to be explained shortly. In addition the transformation depends on parameter λ, and we shall at times recall this dependence with the notation $U[p_0;\lambda]$. When there is no risk of confusion we shall drop one or both of the variables in square brackets. Note that the transformation induced by U coincides with the linearization procedure proposed by Judes [27] *in the single particle sector*.

The choice of U made in Eq. (5) is dictated by nothing but simplicity and the fact that it leads to the Fock-Lorentz group acting in momentum space [23,28,29]. Any other nonunitary, nonlinear U leads to a nontrivial alternative representation of the Lorentz group, with algebra:

$$[J^i,K^j]=\epsilon^{ijk}K_k;\ [K^i,K^j]=\epsilon^{ijk}J_k \tag{7}$$

(with $[J^i,J^j]=\epsilon^{ijk}J_k$ trivially preserved). Indeed any generators produced via Eq. (4) (whatever the choice of U) have the same commutators as the original L_0^i, since we are merely changing the representation. Hence any group action generated via Eq. (4), but with a different form for U, generalizes our formalism, even if the resulting action does not preserve $E_P=\lambda^{-1}$ [as is the case with Eq. (5) [13]]. Thus one has to face the issue of how to decide which U is the correct one. Our view is that such a matter should be decided by experiment. To this end we note that the most general invariant associated with the new group action is

$$||p||^2\equiv\eta^{ab}U(p_a)U(p_b) \tag{8}$$

from which follows a deformed dispersion relation. Dispersion relations may then act as the experimental input into the formalism.

Alternatively, a dispersion relation may be derived from calculations in a theory such as loop quantum gravity [11]. If we have reason to believe that the theory maintains the relativity of inertial frames, in spite of the appearance of modified terms in the energy-momentum relations, this implies that the symmetry of the ground state, corresponding in the classical limit to Minkowski spacetime, is a nonlinear representation of the Lorentz group such as that considered here. In this case we may read off the U from the calculated modifications to the energy-momentum relations.

Ideally, then, the formalism we are discussing can be used to compare experiment and theory, as well as to extrapolate between predictions of different experimental results. Thus, we see this formalism as being part of a phenomenology of quantum gravity effects, as opposed to directly having fundamental significance.

A. Building a general *U*-map

Following this philosophy, we start from a hypothetical measurement of a set of dispersion relations and from this we infer the group action. Any isotropic dispersion relation may be written as

$$E^2f_1^2(E;\lambda)-p^2f_2^2(E;\lambda)=m^2 \tag{9}$$

implying

$$U\circ(E,\mathbf{p})=(Ef_1,\mathbf{p}f_2). \tag{10}$$

U defined in this way maps energy-momentum space, $\mathcal{P}$, onto itself. In general such a map is not invertible. For the action of the Lorentz group to be modified according to Eq. (4) we must have an invertible map. It must then be that there is a sector of $\mathcal{P}$, which we will call $\mathcal{P}^{phys}$, such that U, with range restricted to $\mathcal{P}^{phys}$, is invertible. Typical examples of the restriction which defines $\mathcal{P}^{phys}$ are $E<E_{\rm Planck}$ and/or $|\mathbf{p}|<E_{\rm Planck}$.

A further condition is that the image of U must include the range $[0,\infty]$ for both energy and momentum. This is because the ordinary Lorentz boosts L span this interval, and so $U^{-1}LU$ would not always exist otherwise. If this condition is not satisfied the group property of the modified Lorentz action is destroyed. If for instance Ef_1 does not span $[0,\infty]$ then there is a limiting γ factor for each energy, a feature which not only destroys the group property but also selects a preferred frame, thereby violating the principle of relativity. [In contrast, the condition that the algebra (7) be preserved does not restrict in any way f_1 and f_2.]

With these two assumptions, we can then use Eq. (4) to construct the modified boosts and translations on $\mathcal{P}^{phys}$ and obtain the particular realization of our theory incorporating the new results. In particular, all such theories will have a unmodified Lorentz algebra, realized generally nonlinearly on momentum space. The restriction to $\mathcal{P}^{phys}$ will also become part of the new theory.

Unless $f_1=f_2$ we obtain a theory displaying a frequency dependent speed of light. More precisely, defining $f_3=f_2/f_1$ we have

$$c=\frac{dE}{dp}=\frac{f_3}{1-\frac{Ef_3'}{f_3}}. \tag{11}$$

Hence our formalism may be readily adapted to varying speed of light (VSL) theories, justifying some of the assumptions in [6].

B. A general action which preserves an energy scale or has a maximal momentum

We mentioned in the Introduction that several theoretical arguments suggest that in nature E_P should be an invariant under the action of the Lorentz group. We discuss here the conditions on U such that this will be the case.

Given Eq. (4) we know that the invariants of the new theory are the inverse images via U of the invariants of standard special relativity. But the only invariant energy in linear relativity is the infinite energy.[4] Hence the condition we are looking for is

[4]Another possibility is the zero energy, so that a condition for invariance is $U(E_P)=E_pf_1(E_P)=0$ [30]. This condition, however, is inconsistent with the other requirements discussed in this section.

$$U(E_P)=E_p f_1(E_P)=\infty \tag{12}$$

that is, U should be singular at E_P. In addition note that in special relativity there are three situations in which $E=\infty$:

A photon ($||p||^2=0$), for which $E=p=\infty$.
A particle with infinite rest mass.
A particle with finite rest mass moving at the speed of light.

These are mapped by U^{-1} into 3 distinct types of objects that can have the (invariant) Planck energy: those with zero, infinite, and finite mass, respectively, photons, particles, and something we may call infinitons. The latter have the property that, like photons, they cannot be boosted to a rest frame; however, they are not zero mass objects. The second have the property that their momentum can only have two values: zero or the Planck momentum. These objects will generally mark the boundaries of $\mathcal{P}^{phys}$. As in the case of the limiting velocity of the speed of light in ordinary special relativity, whether they are limiting idealizations, or real physical cases, depends on the dynamics of the particular theory.

The condition for the existence of a maximal momentum is simply that $Ef_1/f_2=E/f_3$ has a maximum. Hence we note that the conditions for a varying speed of light and for the existence of a maximum momentum are related, and indeed one may show that existence of a maximum momentum implies that the speed of light must diverge at some energy.

III. SOME EXAMPLES

We now turn to discuss several examples of theories that meet the various requirements we posited. In Secs. III A and III B we discuss examples of VSL theories. Another interesting exercise (performed in Sec. III C) consists of using the dispersion relations associated with the κ-Poincaré group to build a realization of our theory.

A. A VSL dispersion relation

The varying speed of light scenario [7,8] is an interesting alternative to cosmological inflation. It was found in [6,31] that some deformed dispersion relations (such as the ones in [21,32,33]) might lead to a realization of VSL (and even inflation).

The dispersion relations for massless particles were written in [6,31] in the form $E^2-p^2f^2(E)=0$, failing to define fully $f_1(E)$ and $f_2(E)$. However, as an example let us consider the case of [33] with $f_2=f=1+\lambda E$, and $f_1=1$. Then, from Eq. (10), we have

$$U\circ(E,\mathbf{p})=[E,\mathbf{p}(1+\lambda E)]. \tag{13}$$

This model is known to have an energy dependent speed of light $c(E)=dE/dp=(1+\lambda E)^2$; also all momenta must be smaller than the maximum momentum $p=\lambda^{-1}$, which can only be reached by photons with infinite energy. We note that if the theory is to provide a solution to the horizon problem, independent of inflation, we require $\lambda>0$. Even though the image of U does not span $[0,\infty]$ if we restrict ourselves to positive energies, it does so in $E\in[-\infty,\infty]$, so the group property is preserved for this proposal.

Following the procedures described above we arrive at the following transformation laws for photons:

$$E'=\gamma(1-v)E \tag{14}$$

$$p'=\frac{\gamma(1-v)p}{1+\lambda p[\gamma(1-v)-1]}. \tag{15}$$

Although no energy remains invariant, the Planck momentum $p=\lambda^{-1}$ is an invariant and is also the maximal momentum. The gravitational redshift formula is unmodified in this theory, but expressions for phenomena involving exchange of momentum will be different.

For massive particles we find that we still have that $E_0=m_0c^2$, that the mass still transforms like $m=m_0\gamma$, and that in any frame $E=mc^2$. However, the general expression for the momentum is now

$$p=\frac{mv}{1+\lambda m} \tag{16}$$

showing that for massive particles we must have $p<p_{max}=\lambda^{-1}$.

Similar expressions may be derived for other VSL models considered in the literature.

B. A second VSL theory

A second VSL theory is obtained by choosing

$$U=e^{-\lambda E^2\partial/\partial E}. \tag{17}$$

leading to the energy momentum relation

$$\frac{E^2}{(1+\lambda E)^2}-p^2=m^2. \tag{18}$$

This results in the same modification of the speed of light for photons as the first example, but differs in the energy-momentum relation for massive particles. Note that in this case there is an invariant energy scale, which with the notation used is $E=-\lambda^{-1}$, that is, it is negative for $\lambda>0$.

A more general set of isotropic energy momentum relations may be derived from the choice,

$$U=e^{g_1(E)E\partial/\partial E+g_2(E)p_i\partial/\partial p_i}. \tag{19}$$

We thus see the need for experiment, or further theoretical considerations, to fix the high energy behavior of the action of the Lorentz transformations on momentum space.

C. The κ-Poincaré group

The κ-Poincaré group is a quantum deformation of the usual Poincaré group [20,22,34], which we now show can be reinterpreted in our formalism. It leads to dispersion relations of the form (9) with

$$f_1 = \frac{\sinh(\lambda E)}{\lambda E} \tag{20}$$

$$f_2 = \exp(\lambda E) \tag{21}$$

from which a U can be read off according to our prescription. It leads to modified boost generators:

$$F_i = \frac{e^{\lambda E}}{\cosh \lambda E}[p_i \partial_E - \lambda p_i D] + \frac{\sinh \lambda E}{\lambda e^{\lambda E}} \partial_{p_i} \tag{22}$$

and it can be checked that these satisfy the standard Lorentz algebra.

Exponentiation reveals the finite Lorentz transformations:

$$E' = \lambda^{-1} \sinh^{-1}(F) \tag{23}$$

$$p_z' = \frac{p_z e^{\lambda E} - v \lambda^{-1} \sinh(\lambda E)}{F + \sqrt{F^2 + 1}} \tag{24}$$

$$p_x' = \frac{p_x}{F + \sqrt{F^2 + 1}} \tag{25}$$

$$p_y' = \frac{p_y}{F + \sqrt{F^2 + 1}} \tag{26}$$

$$F(E, p_z) = \gamma[\sinh(\lambda E) - v p_z \lambda e^{\lambda E}]. \tag{27}$$

For photons these reduce to the Doppler shift formula:

$$E' = \lambda^{-1} \sinh^{-1}[\gamma(1 - v) \sinh \lambda E]. \tag{28}$$

For massive particles we have the relation

$$E = \lambda^{-1} \log[(\lambda m) + \sqrt{(\lambda m)^2 + 1}] \tag{29}$$

with $m = \gamma m_0$.

Note that although the theory we have written down has the same dispersion relations as those of the κ-Poincaré group, this may not necessarily imply that the structure of spacetime must be assumed to be noncommutative. Our theory is not based on a quantum deformation of the Poincaré group, but merely a nonlinear realization of the undeformed Lorentz group. This is true even in the case in which the dispersion relations are the same as derived from the κ-Poincaré group. We will see below how the spacetime coordinates may be introduced, in a way that does not require the introduction of noncommutative spacetime geometry.

IV. COMPOSITE SYSTEMS AND CONSERVATION LAWS

Once we accept the possibility of nonlinear transformation laws, we soon discover that kinematic relations valid for single particles need not be true for composite systems (this is certainly the case with the transformation laws themselves). In fact we are left with an ambiguity concerning how momenta are added and how composite quantities transform. To some extent this is a desirable feature: nonlinearity appears to build into the theory the concept of elementary particle, clearly differentiating between them and composites. In any case, as we mentioned in the Introduction, such a distinction is necessary for theories in which energy or momentum of elementary particles are bounded.

A. Composite systems

One has to tread gingerly when defining the multiparticle sector, as theories predicting deformed dispersion relations often have ill-behaved multiparticle sectors.[5] For instance, a straightforward addition rule is the following:

$$p_a^{(12)} = U^{-1}[U(p_a^{(1)}) + U(p_a^{(2)})] \tag{30}$$

and likewise for larger collections of particles, $p^{(1 \ldots n)}$. This is the simplest composition map, and it does lead to the conservation of energy and momentum, as may be easily checked.

However, there is a problem with this law. As may be checked, with this definition the composite momenta transform according to the same nonlinear equations as the momenta of the constituents. This quickly leads to inconsistencies, for instance, it implies that composite momenta satisfy the same deformed dispersion relations as single particles (with $m_{(1 \ldots n)} = m_1 + \ldots + m_n$). For the choice (5) this implies that a set of particles satisfying $E \ll E_P$ can never have a collective energy larger than E_p. This is blatantly in contradiction with observation: macroscopic collections of objects with $E \ll E_P$ quite often have energies far in excess of E_P, and in fact satisfy to good approximation undeformed dispersion relations. A severe inconsistency has arisen, traceable to definition (30).

There are several ways around this problem. One was proposed in [27], here we describe another. We have noted before that the transformation U depends on p_0 but also on parameter λ, and we restore the latter dependence with notation $U[p_0;\lambda]$. The idea is now that a system of n elementary particles should satisfy kinematical relations obtained from a map $U[p_0;\lambda/n]$, that is, a map for which the Planck energy $E_P = \lambda^{-1}$ is replaced by nE_P. We can therefore define:

$$\begin{aligned} p^{(12)} &\equiv p^{(1)} \oplus p^{(2)} \\ &= U^{-1}[p_0;\lambda/2]((U[p_0;\lambda](p^{(1)}) + U[p_0;\lambda](p^{(2)})). \end{aligned} \tag{31}$$

This defines a new, generally nonlinear, composition law for energy and momenta, which we denote by $\oplus$ to indicate that it is not ordinary addition. In general

$$\begin{aligned} p^{(1 \ldots n)} = U^{-1}[p_0;\lambda/n](U[p_0;\lambda](p_1) + \ldots \\ + U[p_0;\lambda](p_n)). \end{aligned} \tag{32}$$

[5] We thank G. Amelino-Camelia for bringing this point to our attention.

With this definition a system of n particles satisfies a system of transformations obtained from $U[p_0;\lambda/n]$ via Eq. (4), equivalent to the usual ones but replacing λ with λ/n. As a result, the collective momentum $P^{(N)}=p^{(1\dots n)}$ satisfies deformed dispersion relations with λ replaced by λ/n. This can never lead to inconsistencies because if all n particles of a system have sub-Planckian energies then the total will still be sub-Planckian, in the sense that $E_{tot}\ll nE_P$. We have therefore circumvented the paradox described above.

For instance, in the case where U is given by Eq. (5) the addition rule for N particles becomes:

$$\frac{p_a^{(N)}}{1-\frac{\lambda p_0^{(N)}}{N}}=\sum_i \frac{p_a^{(i)}}{1-\lambda p_0^{(i)}}. \tag{33}$$

Interestingly, for sets of particles with identical energies and momenta, this reduces to plain additivity.

The composite momentum satisfies the dispersion relations:

$$||p^{(N)}||^2\equiv\frac{\eta^{ab}p_a^{(N)}p_b^{(N)}}{\left(1-\frac{\lambda p_0^{(N)}}{N}\right)^2}=M^2 \tag{34}$$

and the transformation laws are now:

$$p_0^{(N)\prime}=\frac{\gamma(p_0^{(N)}-vp_z^{(N)})}{1+\frac{\lambda}{N}(\gamma-1)p_0^{(N)}-\frac{\lambda}{N}\gamma v p_z^{(N)}} \tag{35}$$

$$p_z^{(N)\prime}=\frac{\gamma(p_z^{(N)}-vp_0^{(N)})}{1+\frac{\lambda}{N}(\gamma-1)p_0^{(N)}-\frac{\lambda}{N}\gamma v p_z^{(N)}} \tag{36}$$

$$p_x^{(N)\prime}=\frac{p_x^{(N)}}{1+\frac{\lambda}{N}(\gamma-1)p_0^{(N)}-\frac{\lambda}{N}\gamma v p_z^{(N)}} \tag{37}$$

$$p_y^{(N)\prime}=\frac{p_y^{(N)}}{1+\frac{\lambda}{N}(\gamma-1)p_0^{(N)}-\frac{\lambda}{N}\gamma v p_z^{(N)}}. \tag{38}$$

If $p_0^{(i)}\ll E_P=\lambda^{-1}$ for all i, we find that Eq. (33) reduces to standard addition and $p^{(N)}$ transforms according to the usual Lorentz transformations and satisfies quadratic dispersion relations.

More generally the proposal (32) for momenta addition has the following U-independent properties. It is commutative but nonassociative, as expected from any addition law incorporating the concept of elementary particle. If each elementary particle satisfies $E\ll\lambda^{-1}$, then energy and momentum are approximately additive and the composite momenta approximately satisfy all relations and laws of linear special relativity. Hence our definition avoids the pathologies of the choice (30). However, if a single particle within a collection is Planckian (so that its invariant is $m=\infty$), then the full collection (say, of N particles) is Planckian (it has energy $E_N=NE_P$.) This can be proved by noting that the collection also has infinite total mass. This feature is physically acceptable.

The addition law (32) can be generalized to

$$p^{(1\dots n)}=U^{-1}[p_0;f(n,\lambda)](U[p_0;\lambda](p^{(1)}+\dots+U[p_0;\lambda](p^{(n)}) \tag{39}$$

where $f(n,\lambda)$ can be any function leading to suitable limiting properties. Then Eq. (32) is a particular case of Eq. (39) with $f(n,\lambda)=\lambda/n$.

This proposal solves the problem of how macroscopic bodies transform at a cost, which is that for an observer to transform the energy and momenta of a system from their measurements to those made by an observer moving with respect to them, they must know if the system is elementary or composite and, if composite, how many quanta it contains. The idea that kinematics should make a distinction between elementary and composite systems is new to physics, but we would like to suggest that this is not necessarily a reason to abandon it. Instead, it is possible that this is a feature of a classical or quantum mechanics of fundamental particles. For example, a theory that makes such a distinction may be able to resolve the measurement problem because it has an objective way to distinguish macroscopic bodies from fundamental particles.

B. Energy and momentum conservation

Our theory conserves energy and momentum because it is spacetime translation invariant. However, the theory is nonlinear, and the p_a of a system of two particles is not the sum $p_a^1+p_a^2$, a matter which filters into the definition of energy-momentum conservation. This point was made in [27], and is closely related the definition of momenta in the multiparticle sector discussed in the previous section. We find that in the same way that the transition from Galilean to special relativity destroys the additivity of speeds, the transition from linear to nonlinear relativity destroys additivity of energy and momentum themselves. Hence energy-momentum conservation, say for a 2-body collision, can now be written as

$$p_a\oplus q_a=p'_a\oplus q'_a \tag{40}$$

where unprimed/primed variables refer to energy-momenta before/after the collision. For instance for the choice (5) we have:

$$\frac{p_a}{1-\lambda p_0}+\frac{q_a}{1-\lambda q_0}=\frac{p'_a}{1-\lambda p'_0}+\frac{q'_a}{1-\lambda q'_0}. \tag{41}$$

This leads to a number of kinematic novelties at high energies. For instance (if $\lambda>0$) a particle close to Planck energy becomes more and more unreceptive to receiving energy in collisions, no matter how hard one might hit it. This may be thought of as a novel kind of asymptotic freedom.

C. Modified Fock space

We finally note that even though the nonlinearities of our formalism distinguish between elementary particles and composites, its setup is purely classical. Elementary particles, however, are quantum particles, i.e. excitations of quantum fields for which a Fock space can be defined. It is therefore reassuring that one can easily adapt our construction to quantum particles. Let the Fock space be defined in the usual way, i.e. by means of standard creation and annihilation operators $a^{\dagger}(p)$ and $a(p)$, so that the vacuum satisfies $a(p)|0\rangle=0$ for all p, single particle states take the form $|p\rangle=a^{\dagger}(p)|0\rangle$, and multiple particle states are defined by iteration. We can therefore write the quantum *free* Hamiltonian as

$$\hat{H}_0=\sum_k \hbar E_k|k\rangle\left\langle k\right|+\sum_{k,k'}(E_k\oplus E_{k'})\left|kk'\right\rangle\langle kk'|+\dots. \tag{42}$$

Quantum interactions can be written likewise. For instance a ϕ^4 interaction leads to the interacting Hamiltonian:

$$\hat{H}_{int}=:\phi^4: \approx\sum_{kk'}\sum_{pp'}\delta(p\oplus p'-k\oplus k')a^{\dagger}(p)a^{\dagger}(p')a(k)a(k'). \tag{43}$$

Thus we have incorporated our proposal for classical momentum addition into a quantum framework.

V. MODIFICATIONS OF THE THRESHOLD ANOMALIES

Now that we know that our theory is consistent with energy-momentum conservation and is not obviously in contradiction with the observed properties of macroscopic bodies, we may attempt to apply it to the real world. The first application we would like to consider is to follow the suggestion of Amelino-Camelia and Piran [4] that a modified dispersion relation may resolve the problem of the observed threshold anomalies. We study first the gamma ray anomaly because it is a bit simpler than the cosmic ray anomaly. We will see that while it was essential to establish that energy and momentum are conserved in the theory, the analysis is actually simpler than might have been expected.

A. Gamma ray threshold anomalies

The issue of the gamma ray threshold anomaly arises because one expects a cutoff at around 10 TeV in the flux of gamma rays, due to their interaction with the infrared background. At these energies it becomes kinematically possible to produce an electron positron pair by scattering of a gamma ray from a photon of the infrared background, leading to a prediction for an upper limit to observed energies. However, while the experimental situation is still somewhat controversial, there are indications that the predicted threshold is not observed (see, e.g. [3]).

For a threshold reaction, in the center of mass frame the electron and positron have no momentum. Hence, due to momentum conservation, in this frame the gamma ray and the infrared photon have the same energy. Energy conservation then implies that their energy equals the rest energy of an electron m_e. We can draw these conclusions because $m_e \ll E_P$, and so all corrections imposed by our theory are negligible.

We then need to perform a boost transformation from the center of mass frame to the cosmological frame. This can be pinned down by the condition that one of the photons be redshifted to the infrared background energy. Since in this process all energies involved are again sub-Planckian we can use plain special relativistic formulas to conclude that $E_{\rm IR}=(1-v)\gamma m_e$, and since $\gamma\gg1$ [implying $1-v\approx1/(2\gamma^2)$] we have:

$$\gamma=\frac{m_e}{2E_{\rm IR}}. \tag{44}$$

The same boost transformation blueshifts the other photon to our predicted value for the gamma ray threshold energy. This operation, however, may have to be performed with the corrected boost. The uncorrected threshold energy is

$$E_{th0}=\gamma(1+v)m_e\approx2\gamma m_e=\frac{m_e^2}{E_{\rm IR}}. \tag{45}$$

This is now corrected to

$$E_{th}=U^{-1}(E_{th0}) \tag{46}$$

since the full boost is now $U(E_{th})=\gamma(1+v)U(m_e)$ and $U(m_e)\approx m_e$.

We may now obtain exact threshold formulas for the various proposals in the literature. For [13] we have

$$E_{th}=\frac{\dfrac{m_e^2}{E_{\rm IR}}}{1+\dfrac{\lambda m_e^2}{E_{\rm IR}}} \tag{47}$$

and for the κ-Poincaré group:

$$E_{th}=\lambda^{-1}\sinh^{-1}\frac{\lambda m_e^2}{E_{\rm IR}}. \tag{48}$$

In both cases we note that with $\lambda>0$ the threshold is *lowered* rather than raised. Hence if the observations have anything to do with these dispersion relations the implication seems to be that $\lambda<0$. In this case the invariant Planck energy is negative $E_P=-\lambda^{-1}$, a situation already discussed in [13].

The dispersion relation (18) on the other hand is obtained from a U acting on energy like the U but with $\lambda\rightarrow-\lambda$. It is therefore not surprising that the threshold formula in this theory is

$$E_{th}=\frac{\frac{m_e^2}{E_{\mathrm{IR}}}}{1-\frac{\lambda m_e^2}{E_{\mathrm{IR}}}} \tag{49}$$

which is raised with $\lambda>0$. However, the invariant in this case is also a negative Planck energy $E_P=-\lambda^{-1}$, so the previous conclusion remains—threshold anomalies imply a negative Planck energy for dispersion relations proposed in literature. This example is interesting as it tells us that the modifications necessary to raise the speed of light, if the theory is to serve as a VSL theory and explain the horizon problem, are of the same sign as those required to explain the absence of the thresholds, at least in these classes of models.

Regardless of the issue of the sign of E_P there remains its order of magnitude. From Eq. (47) we get

$$\lambda=\frac{1}{E_{th}}-\frac{1}{E_{th0}} \tag{50}$$

so we find that we would need $|E_P|\sim 10$ TeV to explain the gamma ray anomaly. In addition fine tuning is required: how close $|\lambda^{-1}|$ lies to E_{th0} determines the actual threshold energy E_{th}.

This example teaches us an important lesson. So long as the modified transformation law has a single dimensional parameter, $\mu\approx\lambda^{-1}$, then from Eq. (50) we see that if the usual and new threshold are the same rough order of magnitude, then μ must be of the same order of magnitude as well. This problem is a direct consequence of Eq. (46), from which we can see that so long as there are no small dimensionless parameters in U then the result is a formula with three-dimensional parameters; so long as two are of the same order of magnitude, so must be the third. This tells us that the mechanism for moving the threshold used by Amelino-Camelia and Piran in [4] cannot work in a relativistic theory because it relies on a coincidence of small ratios in the cosmological frame. However, this coincidence does not exist in all frames of reference, hence it cannot be part of the solution of the problem in a relativistic theory. This is then not a problem with our example, but a general issue with theories of the kind we are considering, which preserve the relativity of inertial frames.

B. UHECR threshold anomalies

A similar anomaly also seems to plague ultrahigh-energy cosmic rays (UHECRs). These are rare showers derived from a primary cosmic ray, probably a proton, with energy above 10^{11} GeV. At these energies there are no known cosmic ray sources within our own galaxy, so it is expected that in their travels, the extra-galactic UHECRs interact significantly with the cosmic microwave background (CMB). These interactions should impose a hard cutoff above $E_{th0}\approx 10^{11}$ GeV, the energy at which it becomes kinematically possible to produce a pion. This is the so-called Greisen-Zatsepin-Kuzmin (GZK) cutoff; however, UHECRs have been observed beyond the threshold [1,2].

An argument identical to the one just made for gamma rays leads to the conclusion that any corrections imposed in our theory appear at the level of boosting the proton in a threshold reaction from the center of mass to the cosmological frame. Hence the corrected threshold formula is simply $E_{th}=U^{-1}(E_{th0})$. However, a novelty appears at this stage because the proton is not an elementary particle, so that in the boost transformation we should replace λ by λ/N_p, where N_p is the "number of quanta living inside a proton." Hence the correct formula is

$$E_{th}=U^{-1}[E_{th0};\lambda/N_p](E_{th0}). \tag{51}$$

All formulas presented above for gamma rays may now be adapted to UHECRs. The conclusion is now that $N_pE_P\sim -10^{11}$ GeV for previously proposed dispersion relations. Using the more general definition (39) we have $f(E_P^{-1},N_p)\sim -10^{11}$ GeV.

The question is then what is the right value of N_p in the case of a confined state such as the proton. An answer to this problem may require the application of this theory to the full quantum field theory of QCD. From the point of view of phenomenology, the suggestion in any case is that the parameters that modify the boost properties of the proton may differ from those of the electron and photon. One may then adjust the free function $f(\lambda,n)$ used in defining the multiparticle sector to reconcile the difference in energy scales of the cosmic ray and gamma ray thresholds. However, given that N_p for the proton is likely of order 3, it is difficult to see how this could be accomplished for any simple function f, unless it contains small dimensionless parameters.

C. How to resolve all five issues

From the preceeding discussion we see that there is a basic problem with using a modified form of special relativity such as we are considering here to solve the problem of the threshold anomolies. The problem is that so long as the function f_1 has a single dimensional parameter, λ, and no small dimensionless parameters, then λ^{-1} must not be too many orders of magnitude away from the threshold predicted by the usual linear theory. This prevents a single kinematical effect from solving both the gamma ray and UHECR anomalies, as they occur at very different energies, it also prevents a theory with λ on the order of the Planck length from solving either.

One might conclude from this that in the event that observations do in the end support the hypothesis that modified dispersion relations with the approximate form of Eq. (2), when applied in the cosmological rest frame, do resolve the threshold anomolies, this would be inconsistent with the relativity of inertial frames. Indeed, our argument shows that no simple form of f_1 depending on only one scale could resolve the problem in a relativistic theory, so long as that scale were the Planck scale.

However, before reaching such a drastic conclusion, there is a simpler possibility to consider, which is that the function

f_1 has more than one scale in it.[6] To see that this is sufficient to resolve the UHECR anomaly, while preserving an invariant energy scale of the order of the Planck energy, note that we can multiply the previous f_1 used in fitting the threshold anomalies by a function that is approximately 1 for $E \ll E_P = 10^{19}$ GeV, but which diverges at E_P. For instance we may take the function

$$f_1 = \frac{1}{(1+\lambda_1 E)(1-\lambda E)} \tag{52}$$

with $\lambda_1 \gg \lambda$. It is easy to see that if $N_p \lambda_1^{-1} \approx 10^{11}$ GeV then the UHECR threshold is raised. This theory makes the prediction that the actual UHCR threshold should lie somewhere between its special relativity value E_{th0} and $N_p E_P$ (as λ_1^{-1} can never be as high as $E_P = \lambda^{-1}$).

In addition such a theory displays an invariant positive energy, $E_P = \lambda^{-1}$ which may be of order 10^{19} GeV. Also the image of U associated with this f_1 is $[0,\infty]$, so that in this theory the threshold anomalies are consistent with the group property of the action, and the principle of relativity.

Finally, can we pick a theory that satisfies the other criteria we set out in the introduction? To see that this is straightforward, note that the function f_2 does not enter the discussion of threshold anomalies, and so the issues of VSL and of the existence of a maximal momentum are decoupled from threshold anomalies. Instead as we see from Eq. (11) and the discussion at the end of Sec. II B, both of these properties are governed by $f_3 = f_2 / f_1$.

To avoid an energy dependent speed of light that so far would have been detected in observations of gamma ray bursts, f_3 should differ from unity only on the Planck scale. For example, consider $f_3 = e^{E/E_P}$. It is easy to see that this gives a maximum momentum, equal to E_P and a diverging speed of light.

VI. REAL SPACE FORMULATION

If Lorentz transformations are nonlinear they take a different aspect in real and momentum space. The choice of momentum space in [13] is tied to the use of the Fock-Lorentz representation, which has a large time-like invariant suitable for identification with the Planck energy E_P but not the Planck time t_P. Once in momentum space one may ask how to recover a real space formulation.

One prescription is to define space coordinates as the generators of shifts in momentum space (this seems to be at odds with the proposal in [34]). Because the theory is nonlinear, shifts are not pure additive constants, and may be read off from standard shifts subject to a U transformation. For [13], small energy shifts take the form:

$$\delta E = (1-\lambda E)^2 \epsilon \tag{53}$$

$$\delta p_i = -\lambda(1-\lambda E) p_i \epsilon \tag{54}$$

whereas momentum shifts are

$$\delta E = 0 \tag{55}$$

$$\delta p_i = (1-\lambda E)\epsilon. \tag{56}$$

Hence the corresponding spatial coordinates are

$$t = (1-\lambda p_0)\left((1-\lambda p_0)\frac{\partial}{\partial p_0} - \lambda D\right) \tag{57}$$

$$x^i = (1-\lambda p_0)\frac{\partial}{\partial p_i}. \tag{58}$$

This bears some resemblance to Snyder's noncommutative geometry [36], which has

$$x^\mu = \frac{\partial}{\partial p_\mu} - \lambda p^\mu D \tag{59}$$

[see Eq. (9) in [36]]. However, there is an important difference. As may be easily checked, the space and time coordinates all commute with each other.

$$[x^a, x^b] = 0. \tag{60}$$

The price to pay for this is that there are now novelties in the commutators of the spacetime coordinates with energy and momenta. Indeed:

$$[x^i, p_j] = \delta^i_j (1-\lambda p_0) \tag{61}$$

$$[x^0, p_i] = -\lambda(1-\lambda p_0) p_i \tag{62}$$

$$[x^0, p_0] = (1-\lambda p_0)^2. \tag{63}$$

This suggests that we have now an energy dependent Planck's "constant" since Eq. (61) implies $\hbar = 1 - \lambda p_0$. As a result for Planck energies there is no uncertainty principle—the Planck energy is not only an invariant but it is also apparently perfectly classical. We are currently investigating further the implications of this proposal.

VII. CONCLUSIONS

In this paper we have presented a general method for implementing nonlinear actions of the Lorentz group based upon knowledge of the dispersion relations. Our results complement those of other authors who have studied the possibility that the action of the Lorentz transformations is modified at high energies. The approach taken in this paper generalizes that in [13] by considering different maps U, which we identified with functions f_1 and f_2 in Eq. (9). We found that the group property is preserved if U is invertible and its image contains $[0,\infty]$. If $E f_1$ diverges at some finite energy E_P, this takes the place of an invariant Planck energy. Careful design of U may also explain the threshold

[6] As pointed out in [35], yet another solution is to allow for a nonuniversal U. Different particles could then have a different U, or U could depend on the rest mass m of the particle it acts on. One can then use the proton mass as an automatic extra scale in the problem.

anomalies. The function f_2 may then be used to implement a maximal momentum and a diverging speed of light. Using the freedom to choose f_1 and f_2 we found that all five requirements we listed in the Introduction may be achieved in one theory. One, among many, examples that do so is the following:

$$f_1 = \frac{1}{(1+\lambda_1 E)(1-\lambda E)}, \quad f_2 = e^{E/E_{\mathrm{Planck}}}, \tag{64}$$

with $\lambda_1 \gg E_{\mathrm{Planck}}^{-1}$.

We also discussed the extension of the transformation laws to real space. We found that there is no general need for the coordinates of space to become noncommutative. Instead, by defining the coordinates of space to be generators of translations in momentum space, we arrived at a commutative spacetime geometry. While it remains for experiment to decide, we note that this approach is closer to the spirit of general relativity, in which the local properties of spacetime arise from the tangent space of a manifold. It then may be close to that expected from the classical limit of quantum gravity, according to which the Poincare invariance of Minkowski spacetime has no fundamental significance, but is only an accidental symmetry of the ground state of the classical limit. Furthermore, by taking this point of view we discovered a novel feature of the theory, which is that the effective Planck's constant appears to become energy dependent.

Of course there are many things still to do to investigate whether theories of the kind discussed here have a chance to be true. It is important to understand whether the modifications of the energy momentum relations predicted by loop quantum gravity in [11] are necessary consequences of that theory and, if so, whether that theory predicts the existence of a preferred frame or a modification of Lorentz transformations preserving the relativity of inertial frames. Of equal interest is the question of whether critical string theory[7] can be made consistent with deformed dispersion relations and modifications of the action of Lorentz transformations, or whether observations of such effects would disprove critical string theory [37]. Indeed, the general question of how to incorporate the kinds of modifications of kinematics contemplated here and in related papers into a fully interacting quantum field theory remains open, as does the question of how these modifications may be incorporated into classical general relativity [24–26,38].

Of course, the main motivation for studying this class of theories is the hope that in the not too distant future astrophysical and cosmological observations of the kind considered here will teach us whether and how Lorentz invariance is realized at the scales relevent for quantum gravity.

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[7]There are also related results concerning noncritical string theory [5].

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