Quasinormal modes for massless topological black holes

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An exact expression for the quasinormal modes of scalar perturbation on a massless topological black hole in four and higher dimensions is presented. The massive scalar field is nonminimally coupled to the curvature, and the horizon geometry is assumed to have a negative constant curvature.

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I. INTRODUCTION

In a black hole geometry, the event horizon acts as a sink that drains the linearized perturbations of the geometry or matter fields, damping the oscillations. These are the so-called quasinormal modes, which are typically characterized by a spectrum that is independent of the initial conditions. The quasinormal modes are a sort of fingerprint of the black hole depending only on its parameters and on the fundamental constants of the system.

Quasinormal modes have been extensively studied in asymptotically flat spacetimes (see, e.g., Ref. [1], and references therein). The inclusion of a negative cosmological constant adds a new angle of interest to the problem [2–18]. Through the AdS conformal field theory (CFT) correspondence, the quasinormal modes can be related to the relaxation time scale of the associated thermal states [2–4]. Recently, a connection has also been conjectured between the quasinormal modes and critical phenomena of black hole formation in an asymptotically AdS background [2]. In three dimensions analytic results supporting these conjectures have been established recently [5–10]. In four dimensions, however, the quasinormal frequencies on black holes have been obtained by numerical methods only.

A negative cosmological constant allows the existence of black holes with a topology $\mathbb{R}^2\times\Sigma$, where Σ is a two-dimensional manifold of constant curvature [19–21]. The simplest solution of this kind when Σ has negative constant curvature, reads

$$ds^2=-\left(-1+\frac{r^2}{l^2}-\frac{2\mu}{r}\right)dt^2+\frac{dr^2}{\left(-1+\frac{r^2}{l^2}-\frac{2\mu}{r}\right)}+r^2d\sigma^2, \tag{1}$$

where the constant μ is proportional to the mass and is bounded from below as $\mu\geqslant-l/3\sqrt{3}$. Here l is the AdS radius, and $d\sigma^2$ is the line element of Σ, which must be locally isomorphic to the hyperbolic manifold H^2. By virtue of the Killing-Hopf theorem, Σ must be of the form

$$\Sigma=H^2/\Gamma \quad \text{with} \quad \Gamma\subset O(2,1),$$

where Γ is a freely acting discrete subgroup (i.e., without fixed points).

The configurations (1) are asymptotically locally AdS spacetimes. These spacetimes can admit Killing spinors for $\mu=0$ provided Σ is a noncompact surface [19]. In this case Eq. (1) describes a warped black string with a supersymmetric ground state for $\mu=0$, and therefore expected to be stable. On the other hand, it has been recently shown in Ref. [23] that the massless configurations where Σ has negative constant curvature are stable under gravitational perturbations.

In this paper, the exact expression for the quasinormal modes of a massive scalar field in the geometry (1) with $\mu=0$ is presented. Since this geometry has constant curvature, the inclusion of a conformal coupling amounts just to a shift in the mass parameter of the scalar field. The generalization of the expression for the quasinormal modes in higher dimensions is also obtained.

II. QUASINORMAL MODES IN FOUR DIMENSIONS

Consider the exterior region of the black hole (1) in the massless case

$$ds^2=-\left(\frac{r^2}{l^2}-1\right)dt^2+\frac{dr^2}{\left(\frac{r^2}{l^2}-1\right)}+r^2d\sigma^2.$$

This is a manifold of negative constant curvature possessing an event horizon at $r=l$. A massive scalar field with a nonminimal coupling satisfies

$$\left(\Box-m^2-\frac{\gamma}{6}R\right)\phi=0, \tag{2}$$

which becomes conformally invariant for $m=0$ and $\gamma=1$. Here $\Box$ stands for the Laplace-Beltrami operator. Since the scalar curvature is $R=-12l^{-2}$, this equation reduces to

$$(\Box-m_{\mathrm{eff}}^2)\phi=0, \tag{3}$$

with an effective mass given by $m_{\mathrm{eff}}^2=m^2-2\gamma l^{-2}$.

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The solution of Eq. (3) can be readily found making the following coordinate transformation $z=1-l^2/r^2$ and $t\rightarrow lt$, so that the metric reads

$$ds^2=\frac{l^2}{(1-z)}\left[-zdt^2+\frac{dz^2}{4z(1-z)}+d\sigma^2\right], \tag{4}$$

where $0\leqslant z<1$, and adopting the following ansatz

$$\phi=R(z)e^{-i\omega t}Y(\Sigma). \tag{5}$$

Here Y is a normalizable harmonic function on Σ, i.e., it satisfies $\nabla^2Y=-QY$, where ∇^2 is Laplace operator on Σ. The eigenvalues for the hyperbolic manifold H^2 are

$$Q=\frac{1}{4}+\xi^2, \tag{6}$$

where ξ is any real number, see, e.g., Ref. [24]. Since Σ is a quotient of the form H^2/Γ, the spectrum has the form (6), but the parameter ξ becomes restricted depending on Γ. Indeed, if Σ is a closed manifold the spectrum is discrete. Note that the zero mode, $Q=0$, is not in the spectrum.

The radial function $R(z)$ satisfies

$$\left[z(1-z)\partial_z^2+\left(1-\frac{z}{2}\right)\partial_z+\left(\frac{\omega^2}{4z}-\frac{Q}{4}-\frac{m_{\rm eff}^2l^2}{4(1-z)}\right)\right]R(z)=0. \tag{7}$$

Under the decomposition $R(z)=z^\alpha(1-z)^\beta K(z)$, Eq. (7) becomes the hypergeometric equation for K,

$$z(1-z)K''+[c-(1+a+b)z]K'-abK=0, \tag{8}$$

provided[1]

$$\alpha=-\frac{i\omega}{2}, \tag{9}$$

$$\beta=\beta_\pm=\frac{3}{4}\pm\frac{1}{4}\sqrt{9+4m_{\rm eff}^2l^2}. \tag{10}$$

The solution of Eq. (8) takes the form

$$K=C_1F(a,b,c,z)+C_2z^{1-c}F(a-c+1,b-c+1,2-c,z), \tag{11}$$

where the coefficients are defined as

$$a=-\frac{1}{4}+\alpha+\beta_\pm+\frac{i\xi}{2},$$

$$b=-\frac{1}{4}+\alpha+\beta_\pm-\frac{i\xi}{2}, \tag{12}$$

$$c=1+2\alpha,$$

and c cannot be an integer.

On physical grounds, ϕ must be restricted to be a purely ingoing wave at the horizon. Furthermore, since the spacetime is locally AdS, the energy-momentum flux density at the asymptotic region should vanish.

The behavior of the scalar field near the horizon $(z=0)$ is given by

$$\phi\sim C_1e^{-i\omega[t+\ln(z)/2]}+C_2e^{-i\omega[t-\ln(z)/2]}.$$

Then, ϕ is purely ingoing at the horizon for $C_2=0$, and therefore the radial function is

$$R(z)=z^\alpha(1-z)^\beta F(a,b,c,z). \tag{13}$$

The energy-momentum tensor for the scalar field is given by

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi-\frac{m^2}{2}g_{\mu\nu}\phi^2 + \frac{\gamma}{6}[g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+G_{\mu\nu}]\phi^2,$$

where $G_{\mu\nu}$ is the Einstein tensor. The current $j^\mu=\sqrt{-g}\xi^\nu T_\nu^\mu$, which is conserved if ξ^μ is a Killing vector, allows to define the energy choosing $\xi^\mu=\delta_0^\mu$. Thus, the energy-momentum flux density $\sqrt{-g}T_{z0}g^{zz}$ at the asymptotic region vanishes if

$$\lim_{z\rightarrow1}\frac{1}{\sqrt{1-z}}\left[\left(1-\frac{2\gamma}{3}\right)z\partial_z+\frac{\gamma}{3}\frac{1}{(1-z)}\right]\phi^2\rightarrow0.$$

A detailed analysis (see the Appendix) shows that this last condition is satisfied only if

$$m_{\rm eff}^2\geqslant-\frac{9}{4l^2},$$

which agrees with the Breitenlohner-Freedman bound for the positivity of energy in global AdS_4 [25,26]. In this case, the expression for the scalar field is given by Eqs. (5) and (13), with $\beta=\beta_+$ as defined in Eq. (10). The quasinormal frequencies are determined by the conditions $a|_{\beta_+}=-n$ or $b|_{\beta_+}=-n$, with $n=0,1,2,\ldots$, in Eq. (12) yielding

$$\omega=\pm\xi-i\left(2n+1+\sqrt{\frac{9}{4}+m_{\rm eff}^2l^2}\right). \tag{14}$$

As shown in the Appendix, in analogy with the normal modes for AdS_4, if the mass and the coupling constant γ satisfy

$$\sqrt{\frac{9}{4}+m_{\rm eff}^2l^2}=\frac{3}{2}-\frac{\gamma}{3-2\gamma}, \tag{15}$$

there exists an alternative set of modes for the range of effective mass

[1]Actually $\alpha=\pm i\omega/2$, and without loss of generality the negative sign can be chosen.

$$-\frac{9}{4}<m_{\rm eff}^2 l^2<-\frac{5}{4}, \tag{16}$$

for which the scalar field is obtained from Eqs. (5),(13) but with $\beta=\beta_-$. In this case, the quasinormal frequencies are found analogously by $a|_{\beta_-}=-n$ or $b|_{\beta_-}=-n$, so that

$$\omega=\pm\xi-i\left(2n+1-\sqrt{\frac{9}{4}+m_{\rm eff}^2 l^2}\right). \tag{17}$$

Note that a massless scalar field ($m=0$) satisfies the condition (15) only for conformal coupling ($\gamma=1$). Actually, there are two other values of γ which satisfy Eq. (15) for $m=0$, namely, $\gamma=0$, 9/8. These two roots, however, yield an effective mass which lies outside the range (16).

Remarkably, the damping time scale is independent of the parameter ξ, which determines the eigenvalue of the Laplacian in Σ. This is contrary to the observation in the Schwarzschild-AdS case, where surprisingly, the damping time scale increases with the angular momentum of the mode [2]. In the next section, the generalization of these results for higher dimensions is discussed.

III. HIGHER DIMENSIONS

Black holes with topologically nontrivial transverse sections of negative constant curvature exist for $d>4$ dimensions [27–29] and also for gravity theories containing higher powers of the curvature [30,31]. Following the procedure of Refs. [32,33], the mass can be obtained from a surface integral at infinity. From this, it can be seen that, for all cases, the massless solution is described by a metric of the same general form as in Eq. (4), but now $d\sigma^2$ stands for the line element of a $(d-2)$-dimensional surface Σ_{d-2} of negative constant curvature. In this case, the configuration (4) is a locally AdS spacetime

Spacetimes of the form (4) in d dimensions admitting Killing spinors were classified [22], where it was shown that global supersymmetry can be attained only if Σ_{d-2} is a noncompact surface.

Topological black holes have scalar curvature given by $R=-d(d-1)l^{-2}$ and therefore the massive scalar field with a nonminimal coupling satisfies the Klein-Gordon equation with an effective mass given by

$$m_{\rm eff}^2=m^2-\gamma\frac{d(d-2)}{4l^2}. \tag{18}$$

This equation can be solved for the massless background with the same ansatz as for the four-dimensional case (5), but now, Y is a harmonic function of finite norm with eigenvalue $-Q$ on Σ_{d-2}. Since Σ_{d-2} is surface of constant curvature it must be H^{d-2} or a quotient thereof, and hence, the spectrum of the Laplace operator takes the form [34]

$$Q=\left(\frac{d-3}{2}\right)^2+\xi^2, \tag{19}$$

where for H^{d-2} the parameter ξ takes all real values, and upon identifications, the parameter ξ is generically restricted, becoming a discrete set if Σ_{d-2} is a closed surface.

Now, the radial function $R(z)$ satisfies

$$\left\{z(1-z)\partial_z^2+\left[1+\left(\frac{d-5}{2}\right)z\right]\partial_z+\left(\frac{\omega^2}{4z}-\frac{Q}{4}-\frac{m_{\rm eff}^2 l^2}{4(1-z)}\right)\right\}R(z)=0,$$

and if we choose $R=z^\alpha(1-z)^\beta K(z)$ the function K satisfies the hypergeometric equation (8) whose solution is also given by Eq. (11) where hypergeometric parameters are now

$$a=-\left(\frac{d-3}{4}\right)+\alpha+\beta_\pm+\frac{i\xi}{2},$$
$$b=-\left(\frac{d-3}{4}\right)+\alpha+\beta_\pm-\frac{i\xi}{2}, \tag{20}$$
$$c=1+2\alpha,$$

where c is not an integer and

$$\alpha=-\frac{i\omega}{2},$$
$$\beta=\beta_\pm=\frac{d-1}{4}\pm\frac{1}{2}\sqrt{\left(\frac{d-1}{2}\right)^2+m_{\rm eff}^2 l^2}. \tag{21}$$

In analogy with the four-dimensional case, requiring ϕ to be purely ingoing at the horizon, fixes the form of the radial function as

$$R(z)=z^\alpha(1-z)^\beta F(a,b,c,z). \tag{22}$$

As shown in the Appendix, the vanishing of the energy-momentum flux density implies that if the effective mass satisfies the bound

$$m_{\rm eff}^2 l^2\geqslant-\left(\frac{d-1}{2}\right)^2, \tag{23}$$

the scalar field is given by Eqs. (5), (22) with $\beta=\beta_+$ in Eq. (21). The quasinormal frequencies are determined by $a|_{\beta_+}=-n$ or $b|_{\beta_+}=-n$, with $n=0,1,2,\ldots$, in Eq. (20), which yields

$$\omega=\pm\xi-i\left[2n+1+\sqrt{\left(\frac{d-1}{2}\right)^2+m_{\rm eff}^2 l^2}\right]. \tag{24}$$

The bound (23) coincides with the one obtained by Mezincescu and Townsend for the normal modes in global AdS$_d$ [35].

If the mass and the coupling constant γ satisfy the relation

$$\sqrt{\left(\frac{d-1}{2}\right)^2+m_{\rm eff}^2l^2}=\frac{d-1}{2}-\frac{\gamma}{2}\,\frac{d-2}{d-1-\gamma(d-2)} \quad (25)$$

for the range of effective mass given by

$$-\left(\frac{d-1}{2}\right)^2<m_{\rm eff}^2l^2<1-\left(\frac{d-1}{2}\right)^2, \quad (26)$$

there is another set of modes for which the scalar field is obtained from Eqs. (5),(22) but now for $\beta=\beta_-$. This second set of quasinormal frequencies is given by

$$\omega=\pm\xi-i\left[2n+1-\sqrt{\left(\frac{d-1}{2}\right)^2+m_{\rm eff}^2l^2}\right]. \quad (27)$$

As it occurs in four dimensions, the massless scalar field $(m=0)$ satisfies the condition (25) only for conformal coupling $(\gamma=1)$. Also, the damping time scale is independent of the eigenvalue of the laplacian on Σ.

IV. DISCUSSION AND COMMENTS

The scalar perturbations on the massless black hole geometry (4) have been described in four and higher dimensions. As the transverse section Σ_{d-2} is a quotient of H^{d-2}, the imaginary part of ω is non-negative, and therefore stability is always guaranteed. The expressions for the quasinormal frequencies and eigenfunctions are explicitly found. To these authors' knowledge, this is the only analytic result for a black hole in four and higher dimensions.

The advantage of working with analytic expressions is that it is possible to impose the vanishing of the energy-momentum flux at infinity, which is more general than requiring the vanishing of ϕ at infinity as is usually done in numerical computations. As can be seen from Eq. (A3), the vanishing of the scalar field at infinity leads to the same modes as those found here for $m_{\rm eff}^2\geqslant 0$. However, for the range $-(d-1)^2/4\leqslant m_{\rm eff}^2l^2<0$, the field vanishes identically at infinity so that this condition does not yield information about the modes.

A numerical analysis of the quasinormal modes for topological black holes was done in Ref. [16] assuming the eigenvalue of the Laplacian on Σ to be $Q=0$. This value, however, is not in the spectrum of the Laplacian and, according to our result, would give rise to damping without oscillations, or even to unstable modes for certain values of the effective mass.

According to the AdS/CFT correspondence, the quasinormal modes are related to the relaxation time scale of a perturbation in the associated thermal states at the boundary. In this case, the thermal CFT is defined on $S^1\times\Sigma_{d-2}$, and the characteristic time scale is given by $\tau=(\mathrm{Im}[\omega])^{-1}$, where ω can be given by Eq. (24) or (27). Following Ref. [2], it would be interesting to compare our results against the critical exponents of the formation process for these black holes.

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APPENDIX

Consider a massive real scalar field in d dimensions nonminimally coupled to the background geometry

$$\left(\Box-m^2-\frac{\gamma}{4}\,\frac{d-2}{d-1}R\right)\phi=0, \quad \text{(A1)}$$

which becomes conformally invariant for $\gamma=1$ and $m=0$. The energy-momentum tensor is given by

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi-\frac{m^2}{2}g_{\mu\nu}\phi^2 + \theta[g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+G_{\mu\nu}]\phi^2,$$

with $\theta=(\gamma/4)(d-2)/(d-1)$. The current $j^\mu=\sqrt{-g}\xi^\nu T_\nu^\mu$ is conserved provided ξ^μ is a Killing vector. The vanishing of the energy-momentum flux density $\sqrt{-g}T_0^i d\Sigma_i$ in the asymptotic region of Eq. (4) is expressed as

$$\lim_{z\to 1}\frac{1}{(1-z)^{(d-3)/2}}\left((1-4\theta)z\partial_z+2\theta\frac{1}{(1-z)}\right)\phi^2\to 0. \quad \text{(A2)}$$

The scalar field is given by Eq. (5) with $R(z)$ defined in Eq. (22). The behavior of $R(z)$ in the asymptotic region $(z\to 1)$ is given by

$$R_{z\to 1}\sim(1-z)^\beta A[1+\mathcal{O}(1-z)]+(1-z)^{(d-1)/2-\beta}\times B[1+\mathcal{O}(1-z)], \quad \text{(A3)}$$

where

$$A=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

$$B=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$

where a, b, c, and β are defined in Eqs. (20) and (21), and $c-a-b=(d-1)/2-2\beta$ cannot be an integer. Substituting the asymptotic form of ϕ in Eq. (A2), the following condition is obtained:

$$AB(1-d+4\theta d)+2A^2(\theta-\beta+4\theta\beta)(1-z)^{2\beta-(d-1)/2} +B^2(1-d-2\theta+2\beta-8\theta\beta+4\theta d)(1-z)^{(d-1)/2-2\beta} +A^2C(1-z)^{(3-d)/2+2\beta}+B^2D(1-z)^{(d+1)/2-2\beta} +ABE(1-z)=0, \tag{A4}$$

where C,D,E are of the form $[\text{const} + O(1-z)]$.

If the effective mass does not satisfy the bound (23), the condition (A4) can only be satisfied if the scalar field identically vanishes. Hence, nontrivial solutions require β to be a real number.

Since $(d-1)/2-2\beta$ cannot be an integer, the condition (A4) can only be satisfied for $A=0$ or $B=0$. In the case of $B=0$, the condition (A4) is always met if

$$\beta>\frac{d-1}{4}, \tag{A5}$$

which is only satisfied for the branch $\beta=\beta_+$. This means that the quasinormal frequencies are found through $a|_{\beta_+}=-n$ or $b|_{\beta_+}=-n$, where n is a non-negative integer.

If the condition

$$\theta-\beta+4\theta\beta=0,$$

holds, then Eq. (A4) requires $\beta>(d-3)/4$, which can also be satisfied for the branch $\beta=\beta_-$ in the range

$$\frac{d-3}{4}<\beta_-<\frac{d-1}{4},$$

which gives rise to the bound (26), and the quasinormal frequencies are determined $a|_{\beta_-}=-n$ or $b|_{\beta_-}=-n$.

The case with $A=0$ is equivalent to the former, because of the relations

$$A|_{\beta_\pm}=B|_{\beta_\mp},$$

$$b|_{\beta_\pm}=(c-a)|_{\beta_\mp},$$

$$a|_{\beta_\pm}=(c-b)|_{\beta_\mp},$$

$$(\theta-\beta+4\theta\beta)|_{\beta_\pm}=(1-d-2\theta+2\beta-8\theta\beta+4\theta d)|_{\beta_\mp}.$$

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