$SU(2)$ loop quantum gravity seen from covariant theory

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Covariant loop gravity comes out of the canonical analysis of the Palatini action and the use of the Dirac brackets arising from dealing with the second class constraints ("simplicity" constraints). Within this framework, we underline a quantization ambiguity due to the existence of a family of possible Lorentz connections. We show the existence of a Lorentz connection generalizing the Ashtekar-Barbero connection and we loop quantize the theory showing that it leads to the usual $SU(2)$ loop quantum gravity and to the area spectrum given by the $SU(2)$ Casimir operator. This covariant point of view allows us to analyze closely the drawbacks of the $SU(2)$ formalism: the quantization based on the (generalized) Ashtekar-Barbero connection breaks time diffeomorphisms and physical outputs depend nontrivially on the embedding of the canonical hypersurface into the space-time manifold. On the other hand, there exists a true space-time connection, transforming properly under all diffeomorphisms. We argue that it is this connection that should be used in the definition of loop variables. However, we are still not able to complete the quantization program for this connection giving a full solution of the second class constraints at the Hilbert space level. Nevertheless, we show how a canonical quantization of the Dirac brackets at a finite number of points leads to the kinematical setting of the Barrett-Crane model, with simple spin networks and an area spectrum given by the $SL(2,C)$ Casimir operator.

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I. INTRODUCTION

Loop quantum gravity as developed today seems to be a promising approach for quantizing general relativity (for reviews see [1,2]). Although it gives some interesting results, such as discrete quanta of area and volume [3,4] and a derivation of the black hole entropy [5], several problems appear. First of all, it is based on the use of a space triad and an $SU(2)$ connection, where $SU(2)$ is the gauge group for three-dimensional space. This particular choice of variables loses the explicit covariance of the theory and a space-time geometrical interpretation [6]. Moreover, there exists an additional puzzle: a free parameter in the theory, the so-called Immirzi parameter [7]. This parameter comes out of a canonical transformation but creates a full one-parameter family of quantizations which are not unitarily equivalent [8]. It was an open problem to understand the physical relevance of the Immirzi parameter and how it effectively influences the dynamics of the quantum theory. It turned out that this problem can be studied from a new point of view in the framework of an explicitly covariant formalism [9]. The obtained results suggest that the Immirzi parameter should disappear from the physical output of a path integral formulation of quantum gravity [9] as well as of its canonical quantization based on this covariant formulation [10,11]. The goal of the present paper is to explain how one can derive the $SU(2)$ Loop quantum gravity (LQG) from the covariant canonical quantization. This will allow us to tackle the issues of LQG from this different point of view, and discuss the drawbacks of LQG.

Loop quantum gravity with the Immirzi parameter was shown to come from a canonical analysis of the generalized Hilbert-Palatini action in the so-called time gauge [12,13]. An explicitly covariant canonical analysis of this action was carried out in [9] and led to a proposal for its quantization in [11,14]. Although in [14] a Hilbert space for the quantum theory has been proposed, it is not clear whether it is the right solution or not. A rigorous construction of such a space of quantum states remains to be done and there are still many questions to be answered within this new formalism. In addition to the issues related to the noncompactness of the Lorentz gauge group [15], the situation is complicated by the nontrivial canonical structure of the theory. Indeed, since the covariant analysis was done through introducing the Dirac brackets taking into account the second class constraints (also called simplicity constraints), the commutation relations of the basic variables have changed. In particular, the connection becomes noncommutative which is a major obstacle to understanding the geometrical meaning of the theory and to building an appropriate Hilbert space.

Nevertheless, a strong result of the formalism is that there exists a unique Lorentz connection in the theory which transforms properly under space-time diffeomorphisms. It is the true space-time connection. Still, its geometrical interpretation in quantum theory is not straightforward since it is noncommutative. However, it is possible to write observables

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and a Hilbert space of quantum geometry states using functionals of the connection and of the foliation. This brings the theory close in its formulation to the spin foam setting [16]. Moreover, a nice feature of the resulting theory is that the physical output of the theory does not depend on the Immirzi parameter. This quantization seems the most natural since it respects all the classical symmetry and does not break the space-time diffeomorphism invariance.

Now, is there a place for the usual $SU(2)$ loop quantum gravity in the framework of the covariant theory? The answer is affirmative. It turns out there exists a natural covariant generalization of the Ashtekar-Barbero connection that makes it possible to derive LQG starting from the covariant quantization. Moreover, this connection is the only commutative one. This last feature simplifies a lot the quantization process. It yields exactly the same Hilbert space as $SU(2)$ LQG and reproduces the area spectrum of the $SU(2)$ approach. This derivation establishes an exact correspondence between the covariant formalism and the usual one. The study of this case is interesting because it can be a guide for the "correct" diffeomorphism-preserving quantization since it is possible to solve explicitly the second class constraints at the quantum level using this generalized Ashtekar-Barbero connection. It also helps to look at the issues of $SU(2)$ LQG from a new point of view since the problems encountered in this new (covariant) approach are unavoidable in $SU(2)$ LQG. In particular, the scalar (Hamiltonian) constraint is hard to understand and the $SU(2)$ theory definitively breaks the space-time diffeomorphism invariance, as it was foreseen in [6].

The paper is organized as follows. We begin in Sec. II by considering the basic features of possible ways to construct the canonical formulation of general relativity with the Lorentz gauge group. We introduce several objects, generalizing the Ashtekar-Barbero connection in different ways, which are shown to be all related to each other. Then we introduce the covariant Ashtekar-Barbero connection and give a precise account of its properties. Using this connection, we quantize the theory following the usual techniques of the loop approach. Namely, we construct the corresponding Hilbert space and show that, in a particular gauge, it reproduces the Hilbert space of the $SU(2)$ approach. In other words, we derive $SU(2)$ LQG from covariant loop gravity at the level of the Hilbert space. Then we explain different drawbacks of the $SU(2)$ formalism; in particular, we point out that it breaks the diffeomorphism invariance at the quantum level. We argue that a correct quantization should be based on the covariant space-time connection described in Sec. III. We also discuss the link of the canonical formalism with the spin foam approach. Spin foams should arise as models of the space-time resulting from LQG [17]. The current model, the Barrett-Crane model, is shown to be closely related to the present covariant approach and to share the same kinematical Hilbert space of quantum states. In Sec. IV, we comment on the role of the Immirzi parameter in loop quantum gravity. Section V is devoted to conclusions and discussions.

II. DERIVING THE $SU(2)$ FORMALISM FROM THE COVARIANT ONE

A. Canonical formulations and the Ashtekar-Barbero connection

The action for general relativity that we study here is the generalized Hilbert-Palatini action:

$$S_{(\beta)}=\frac{1}{2}\int \varepsilon_{\alpha\beta\gamma\delta}e^{\alpha}\wedge e^{\beta}\wedge\left(\Omega^{\gamma\delta}+\frac{1}{\beta}\star\Omega^{\gamma\delta}\right), \qquad (1)$$

where e^{α} (α is an internal Lorentz index) is the tetrad field, $\Omega^{\alpha\beta}$ is the curvature of the spin-connection $\omega^{\alpha\beta}$ and the star operator is the Hodge operator defined as $\star\Omega^{\alpha\beta}=\frac{1}{2}\varepsilon^{\alpha\beta}{}_{\gamma\delta}\Omega^{\gamma\delta}$. Under the restriction that the tetrad e is not degenerate, the equations of motion of this action lead to the usual Einstein equations and thus do not modify general relativity. Nevertheless, the addition of an extra term compared to the original Palatini action leads to the introduction of a new coupling constant β. As was shown by Holst [12], in the so-called *time gauge*, this additional term leads to loop quantum gravity with β as the Immirzi parameter in the quantum theory. Therefore, it was suggested that β gives rise to a new fundamental physical constant [8].

A summary of the canonical analysis of the action (1) without any gauge fixing can be found in Appendix A. There are second class constraints in the theory. There are two main ways to deal with such a system. We can either solve them or take them into account in the symplectic structure by modifying the Poisson bracket to the Dirac bracket.

The first alternative has been worked out by Barros e Sá in [13]. After solving the second class constraints, the natural configuration variables parametrizing the system are the field $\chi^{a}=-e^{ta}/e^{t0}$ [a being an $su(2)$ index and t the 0 space-time index], which is the space components of the time normal or internal time direction, and a generalization of the Ashtekar-Barbero connection of LQG (i being a space index):

$$A_{ia}^{(\mathrm{bar})}=\omega_{i0a}^{(\beta)}+\omega_{iab}^{(\beta)}\chi^{b}, \qquad (2)$$

where we use the notation $\omega_{\alpha\beta}^{(\beta)}=\omega_{\alpha\beta}-1/\beta*\omega_{\alpha\beta}$. To show the relation of $A^{(\mathrm{bar})}$ with the Ashtekar-Barbero connection, one imposes the time gauge. In these variables, it is described by the choice $\chi=0$, which can be achieved by using the Gauss law constraints generating the internal Lorentz boost transformations. In this gauge, one finds the exact setup of LQG with the Immirzi parameter β, reproducing Holst's result [12]. In particular, $\beta A^{(\mathrm{bar})}$ coincides with the usual Ashtekar-Barbero connection, when expressing ω_{i}^{ab} through the triad by means of (a half of) the Gauss constraints.

The disadvantage of this formalism is that it breaks the covariance of the theory when solving explicitly the second class constraints. As a result, it becomes rather complicated and awkward for making calculations. To simplify the calculations, one imposes the time gauge, which breaks the boost part of the Lorentz symmetry and returns us to the usual

$SU(2)$ formulation. Thus, this way can add nothing new to our understanding. Besides, the canonical variable $A^{(\mathrm{bar})}$ is in fact neither a Lorentz nor an $SU(2)$ connection. Therefore, it turns out to be inappropriate for loop quantization, especially if one seeks a quantization preserving the Lorentz symmetry.

The other alternative, used by one of the authors, is to use the second class constraints to induce a Dirac bracket [9]. This allows us to leave the Lorentz covariance untouched and thus to construct an explicitly covariant theory. Doing so, we can keep an $SL(2,C)$ connection as a canonical variable. However, due to the presence of the Dirac bracket, the canonical variables do not play a preferable role anymore and there exists many other Lorentz connections which can be constructed from the canonical variables. This gives rise to a quantization ambiguity in the loop approach, since each of them can be used in the definition of loop variables. Then, following the philosophy of LQG, if we require that the area operator be diagonal on the Wilson loops defined by the connection, we end up with a two-parameter family of possible $SL(2,C)$ connections. Following the methods of LQG, one can derive the corresponding family of area spectra (A23), which now contain the Casimir operators of $SL(2,C)$ [10,11]. The main technical difficulty of this method is the resulting noncommutativity of the $SL(2,C)$ connection. An expression of $\{\mathcal{A},\mathcal{A}\}_D$ computed in [14] can be found in Appendix C.

From this covariant approach, it is easy to reconstruct the variables of the $SU(2)$ approach. Indeed, a suitable projection of the canonical $SL(2,C)$ connection A_i^X[X is an $sl(2,C)$ index] gives an equivalent of the Ashtekar-Barbero connection:

$$A_i^{(\mathrm{ash})X}=I^X_{(Q)Y}(\delta^Y_Z-\beta\Pi^Y_Z)A^Z_i\,. \tag{3}$$

Namely, as is shown in Appendix B, its three nonvanishing components $A_i^{(\mathrm{ash})a}$ coincide (up to β) with the quantity (2) and thus coincide with the Ashtekar-Barbero connection in the time gauge $\chi=0$. Further, they form a connection of the "boosted" subgroup $SU_\chi(2)$, constructed explicitly in Appendix B, when χ is constant over the canonical hypersurface. Moreover, despite the fact that the canonical connection A is noncommutative, $A^{(\mathrm{ash})}$ commutes with itself (see Appendix C). Thus, taking $A^{(\mathrm{ash})}$ as a canonical variable, in any gauge $\chi=$const, the phase space has the same structure as in the usual $SU(2)$ approach.

This consideration allows us to reproduce the phase space of the $SU(2)$ approach at the classical level. However, it simply amounts to breaking the covariance of the theory and translating the $SU(2)$ connection variables into the new formalism: it is not equivalent to deriving the $SU(2)$ setting from a covariant quantum theory. The reason is that $A^{(\mathrm{ash})}$ can be considered as an $SU(2)$ connection, whereas the covariant loop quantization should be based on a Lorentz connection. Remarkably, there exists such a Lorentz connection which is a natural generalization of the Ashtekar-Barbero one [11]. We describe its properties and the results, which it leads to, in the next paragraph.

B. The covariant Ashtekar-Barbero connection

The canonical analysis done in preserving covariance, as described in Appendix A, leads to a two-parameter family of Lorentz connections (A15) with such commutation relations that the corresponding Wilson lines are eigenstates of the area operator. As it had been noted in [11], if we choose the parameters of the connection, a and b in Eq. (A15), to be as follows:

$$a=-\beta,\quad b=1, \tag{4}$$

we obtain a connection, which reproduces the results of the $SU(2)$ approach. Indeed, it takes the form

$$\begin{aligned}\mathbf{A}_i^X&=I^X_{(Q)Y}(\delta^Y_Z-\beta\Pi^Y_Z)A^Z_i-\beta R^X_Y\Lambda^Y_i(\tilde{Q})\\&=A_i^{(\mathrm{ash})X}-\beta R^X_Y\Lambda^Y_i(\tilde{Q}),\end{aligned} \tag{5}$$

where $\Lambda^X_i(\tilde{Q})$ is a function of χ only:[1]

$$\Lambda^X_i(\tilde{Q})=\left(-\frac{\varepsilon^{abc}\chi_b\partial_i\chi_c}{1-\chi^2},\frac{\partial_i\chi^a}{1-\chi^2}\right). \tag{6}$$

This particular connection possesses the following properties:

For χ constant on the hypersurface it coincides with $A^{(\mathrm{ash})}$ from Eq. (3) and, in particular, for the "time gauge" $\chi=0$, it coincides with the Ashtekar-Barbero $SU(2)$ connection, thus being its Lorentz generalization:

$$\mathbf{A}_i^X\underset{\chi=0}{=}\left(0,\frac{1}{2}\varepsilon^a{}_{bc}\omega_i^{bc}-\beta\omega_i^{0a}\right). \tag{7}$$

As the Ashtekar-Barbero connection, it is commutative (see Appendix C)

$$\{\mathbf{A}_i^X,\mathbf{A}_j^Y\}_D=0. \tag{8}$$

Its commutator with the triad multiplet is [see Eq. (A18)]

$$\{\mathbf{A}_i^X,\tilde{Q}^j_Y\}_D=\beta\delta_i^jI^X_{(Q)Y}, \tag{9}$$

where $I_{(Q)}$ is the projector on the $SU(2)_\chi$ part of the Lorentz group.

Due to this last relation, the area spectrum given by this $SL(2,C)$ connection coincides exactly with the one coming from loop quantum gravity given by the Casimir operator of $SU(2)$:

$$\mathcal{S}\sim\hbar\beta\sqrt{C(su(2))}. \tag{10}$$

C. SU(2) Hilbert space from covariant quantization

Having in hand a Lorentz connection reproducing classically the properties of the $SU(2)$ Ashtekar-Barbero connec-

[1]Let us note that the same expression arises naturally when one extends the Ashtekar-Barbero connection by technics of differential geometry (see Appendix E).

tion, we can ask if this relation is maintained at the quantum level. It turns out that choosing the connection (5) as a basic variable, the covariant theory can be relatively easy quantized. In particular, the second class constraints can be properly imposed at the quantum level. Then, in a particular gauge, we actually recover the usual $SU(2)$ spin network Hilbert space of LQG. Thus, we are able to derive the $SU(2)$ LQG from the quantized covariant formulation.

Having a Lorentz connection, the natural objects to consider are the holonomies or the Wilson lines

$$U_\alpha[\mathbf{A}]=\mathcal{P}\exp\left(\int_\alpha dx^i\mathbf{A}_i^X T_X\right), \tag{11}$$

where α is an oriented path and T_X are the $SL(2,C)$ generators. Taking their trace for a closed loop α in a given representation of $SL(2,C)$, we obtain gauge invariant objects. We usually look at the representations $R^{(n,\rho)}$ from the principal series of unitary irreducible representations of $SL(2,C)$ (see Appendix D for details), since they are the ones entering the Plancherel formula. Such observables can be generalized to an arbitrary oriented graph and give rise to spin networks [15]. However, this construction is not enough in our case because these functionals are not eigenvectors of the area operator and do not have a direct physical interpretation. Indeed, at a given point x of intersection of the loop and a small surface whose area we are computing, we need to decompose the representation $R^{(n,\rho)}$ of $SL(2,C)$ into representations $V^j_{\chi(x)}$ of the subgroup $SU_{\chi(x)}(2)$, which leaves the vector $\chi(x)$ invariant. According to Eq. (10), each subspace V^j will contribute $\beta\sqrt{j(j+1)}$ and the overall area operator will not be simple multiplication on the $SL(2,C)$ Wilson line.

In order to get an eigenvector, we need to select a particular subspace $V^j_{\chi(x)}$. Since this subspace depends on the field χ, this leads us to consider gauge invariant functionals of both the connection $\mathbf{A}$ and the time normal field χ. Notice, that it is consistent due to the relation (A21). Gauge invariance will then read

$$f(\mathbf{A},\chi)=f({}^g\mathbf{A}=g\mathbf{A}g^{-1}+g\partial g^{-1},{}^g\chi=g\cdot\chi). \tag{12}$$

Such invariant functions are in fact entirely given by the functions $f_{\chi_0}(\mathbf{A})=f(\mathbf{A},\chi=\chi_0)$ taken for χ constant on the hypersurface equal to χ_0. The remaining gauge symmetry of f_{χ_0} is only an $SU(2)$ gauge symmetry and is compact. The choice of section "$\chi=\chi_0$" will be called the time gauge.

Following the ideas of loop quantum gravity where one considers cylindrical functions of the Ashtekar-Barbero connection, we introduce cylindrical functions $f_\Gamma(\mathbf{A},\chi)$ which will be constructed on an oriented graph Γ. They will depend only on the holonomies $U_1,\dots,U_E$ of $\mathbf{A}$ along the edges of Γ and on the values $\chi_1,\dots,\chi_V$ of the field χ at the vertices of Γ. The gauge invariance will then read[2]

$$f_\Gamma(U_1,\dots,U_E,\chi_1,\dots,\chi_V)=f_\Gamma(g_{t(e)}U_e g^{-1}_{s(e)},g_v\chi_v)\quad \text{with}\quad g_v\in SL(2,C), \tag{13}$$

where $s(e)[t(e)]$ is the source (target) vertex of edge e. One can use the Haar measure on $[SL(2,C)]^E$ to introduce a (kinematical) scalar product on the space of L^2 gauge invariant functions:

$$\langle f|g\rangle=\int_{[SL(2,C)]^E}dU_e\overline{f_\Gamma(U_e,\chi_v)}g_\Gamma(U_e,\chi_v). \tag{14}$$

This scalar product does not depend on the choice of $(\chi_1,\dots,\chi_V)$ due to the Lorentz invariance. We denote $\mathcal{H}_0$ the resulting Hilbert space. Let us emphasize that this will not be the physical Hilbert space since it is likely that we will need to modify the scalar product (14) to take into account the second class constraints. Nevertheless, exhibiting a basis of $\mathcal{H}_0$ sheds light on the structure of the theory.

To construct it, we take the usual $SL(2,C)$ spin networks, and insert a projector $I^{(j)}_{(\chi_v)}:R^{(n,\rho)}\to V^j_{\chi_v}$ at each edge around every vertex v. This procedure is equivalent to the change of the Wilson lines (11) by Wilson lines *projected at the ends*

$$\mathcal{U}_e^{(j_{s(e)},j_{t(e)})}[\mathbf{A},\chi]=I^{(j_{t(e)})}_{(\chi_{t(e)})}U_e[\mathbf{A}]I^{(j_{s(e)})}_{(\chi_{s(e)})}. \tag{15}$$

The projector can be written as

$$I^{(j)}_{(\chi)}=(2j+1)\int_{SU_\chi(2)}dh\overline{C^j(h)}D(h), \tag{16}$$

where C^j is the character of the $SU(2)$ representation j and $D(h)$ is the representation matrix of the group element h. It is important that the projector depends on χ. Due to this dependence it transforms homogeneously under Lorentz boosts

$$I^{(j)}_{({}^g\chi)}=D(g)I^{(j)}_{(\chi)}D^{-1}(g), \tag{17}$$

[2]In fact, the χ field is a vector field $(1,\chi^a)$, with $\chi^2\leqslant 1$. One can normalize the time normal so it is represented by a vector living on the (upper) hyperboloid of the Minkovski space, as in the spin foam context [16]. This defines a vector field

$$x=\left(\frac{1}{\sqrt{1-\chi^2}},\frac{\chi^a}{\sqrt{1-\chi^2}}\right).$$

Then, the transformation law of x is simply the usual Lorentz transformation in the Minkovski space. This is what is implicit in the definition of the new cylindrical functions and the projected spin networks. Moreover, using this new field can simplify expressions of some functions of χ such as

$$I^Y_{(P)X}=\begin{pmatrix}\delta^b_a x_0^2-x_a x^b & \varepsilon_a{}^{bc}x_0x_c\\ \varepsilon_a{}^{bc}x_0x_c & -(\delta^b_a\vec{x}^2-x_a x^b)\end{pmatrix}.$$

as well as the Wilson lines (15). Therefore, the resulting *projected spin networks* are still gauge invariant and belong to our Hilbert space $\mathcal{H}_0$. Due to the projections, they are labeled by one $SL(2,C)$ representation (n_e,ρ_e) for each edge e, two $SU(2)$ representations for each edge $(j_{s(e)},j_{t(e)})$ (for the source and target vertices of e) and $SU(2)$ intertwiners at all vertices [16]. It is straightforward to check that two projected spin networks with different labels are orthogonal with respect to the scalar product (14). Their completeness is also evident. Thus, the projected spin networks realize an orthonormal basis in $\mathcal{H}_0$.

Moreover, such states are eigenvectors of the area operators of surfaces intersecting the spin networks at vertices, and the area attached to one edge at a vertex is given by the $SU(2)$ representation j attached to the corresponding end of the edge:

$$\mathcal{S}\sim\beta\sqrt{j(j+1)}. \tag{18}$$

Interestingly, this does not depend at all on the $SL(2,C)$ representations. What are they here for? For the moment, they give the way the projected spin networks change under $SL(2,C)$ gauge transformations. We can say that the $SU(2)$ representations define the space geometry while the $SL(2,C)$ representations give its space-time embedding and define how it gets modified under boosts. However, we have not finished the job yet and we still need to take into account the second class constraints.

The second class constraints now correspond to the constraints satisfied by the connection (5):

$$I^X_{(P)Y}\mathbf{A}^Y_i=\Pi^X_Y\Lambda^Y_i(\tilde{Q}). \tag{19}$$

Through this relation, $\mathbf{A}$ depends explicitly on χ, and this reduces the number of independent components of $\mathbf{A}$ from 18 to 9. The physical meaning of the constraints becomes obvious in the time gauge, when one rotates χ to χ_0 on all the hypersurface. Then we have $I_{(P)}\mathbf{A}=0$. As a result $\mathbf{A}$ is reduced to its $SU_{\chi_0}(2)$ part and becomes simply an $SU(2)$ connection. Computing the holonomies of $\mathbf{A}$, we get group elements belonging to the $SU_{\chi_0}(2)$ subgroup. This has an immediate consequence that the projected Wilson lines (15) are nonvanishing only for $j_{s(e)}=j_{t(e)}$ and produce the usual $SU(2)$ Wilson lines:

$$\mathcal{U}_e^{(j_{s(e)},j_{t(e)})}[\mathbf{A},\chi_0]=\delta_{j_{s(e)}j_{t(e)}}\iota(U_e[A^{(\mathrm{ash})}]), \tag{20}$$

where ι denotes the embedding of an $SU(2)$ group element into a representation $R^{(n,\rho)}$ of $SL(2,C)$. In fact, since the result does not depend on the $SL(2,C)$ representation, we can omit this embedding provided it was chosen so that $j\geqslant n$. Otherwise the representation j does not enter the decomposition of $R^{(n,\rho)}$ over the subgroup and the projection (15) vanishes. Therefore, it is enough to restrict ourselves from the very beginning to one arbitrary simple representation of type $R^{(0,\rho)}$ since each of them contains in its decomposition the entire spectrum of $SU(2)$ representations.

Thus, from Eq. (20) we obtain that each edge is labeled by only one $SU(2)$ representation j_e and associated with an $SU(2)$ group element. Then, the projected spin networks, evaluated in the time gauge, reduce to the usual $SU(2)$ spin networks, which are actually the natural $SU(2)$ gauge invariant objects. The scalar product, which takes into account the reduction of the configuration space induced by the second class constraints, is not anymore the kinematical one but

$$\langle f|g\rangle=\int_{[SU_{\chi_0}(2)]^E}dU_e\overline{f_{\chi_0}(U_e)}g_{\chi_0}(U_e). \tag{21}$$

We have actually recovered the full (kinematical) structure of $SU(2)$ loop quantum gravity at the level of the Hilbert space.

Up to now, we have described how covariant functions of the connection and the time normal field, which are solutions to the second class constraints, look like in the time gauge. However, we would like to be more ambitious and describe the physical Hilbert space out of the time gauge, i.e., characterize the space of all gauge invariant functionals of the Lorentz connection and χ, which are nontrivial only for the solutions of the constraints. This will complete the implementation of the second class constraints at the quantum level.

Let us have a closer look at the situation in the time gauge. One can note that for the Lorentz connection satisfying Eq. (19) the insertion of the projector on a representation j in the middle of an edge has a trivial effect: if $j_{s(e)}=j_{t(e)}=j$ we get identity, otherwise the result vanishes. Therefore, we can infinitely refine each edge of the initial graph by adding an infinite number of bivalent vertices. Each of them introduces the corresponding projector so that the refinement is equivalent to consider the following *fully projected Wilson lines* [14]:

$$U^{(j)}_\alpha[\mathbf{A},\chi]=\lim_{N\to\infty}\mathcal{P}\left\{\prod_{n=1}^{N}I^{(j)}_{(\chi_{v_{n+1}})}U_{\alpha_n}[\mathbf{A}]I^{(j)}_{(\chi_{v_n})}\right\}, \tag{22}$$

where $\alpha=\cup_{n=1}^N\alpha_n$ is a partition of the line into small pieces. As we just showed, this procedure does not change the projected spin network for the connection $\mathbf{A}$:

$$\mathcal{U}^{(j,j)}_\alpha[\mathbf{A},\chi]=U^{(j)}_\alpha[\mathbf{A},\chi]. \tag{23}$$

Now we prove that provided each edge is associated with a simple representation $R^{(0,\rho)}$ of $SL(2,C)$, such infinitely refined projected spin networks depend only on the solution of the second class constraints (19). This statement follows from the property of the Lorentz generators:

$$I^{(j)}_{(\chi)}F_aI^{(j)}_{(\chi)}=\beta_{(j)}I^{(j)}_{(\chi)}H_aI^{(j)}_{(\chi)},\quad \beta_{(j)}=\frac{n\rho}{j(j+1)}, \tag{24}$$

where H_a are generators of the $SU_\chi(2)$ subgroup, F_a are the corresponding boost generators. For the simple representations $\beta_{(j)}=0$, which implies that the projected boost generators then vanish. On the other hand, the infinite refinement

(22) is equivalent to such projection in the exponent of Wilson lines (see [14]), since for sufficiently fine partition, we can write

$$U_{\alpha}^{(j)}[A,\chi]=\mathcal{P}\left\{\prod_{n} I^{(j)}_{(\chi_{v_{n+1}})}\left(1+\int_{\alpha_n} dx^i A_i^X T_X\right) I^{(j)}_{(\chi_{v_n})}\right\} \tag{25}$$

and the projectors act directly on the connection $A_i^X T_X$. Therefore, in the time gauge it has an effect to change a projected Wilson line (15) with $j_{s(e)}=j_{t(e)}=j$ by an $SU(2)$ Wilson line in the representation j dependent only on $SU_{\chi_0}(2)$ components of the connection. This is just the same result as the connection **A** gives. To leave the time gauge it is enough to make a gauge transformation. Due to the Lorentz invariance the infinitely refined projected spin networks still give a solution of our problem representing a basis of the physical Hilbert space.

It is important to note that the constructed states are eigenvectors of all area operators [14]. This result is provided by the infinite refinement: due to this, each point of an edge can be considered as a vertex. Thus, our states possess all properties of the $SU(2)$ spin networks and are their Lorentz generalization.

D. Drawbacks of the $SU(2)$ formalism

Thus the SU(2) LQG can be rigorously derived from the quantization of the covariant formalism based on the connection (5). This allows us to look at its status from the point of view of the covariant quantization. First of all, let us elucidate which problems the SU(2) LQG possesses and whether they can be solved in the covariant formalism.

Apart from the problem of implementing the right Hamiltonian operator, there are two main issues. The first one is the Immirzi parameter β [7]. It appears in the classical theory parametrizing different Ashtekar–Barbero connections of the $SU(2)$ approach, which all are related by a canonical transformation. However, this parameter enters the area spectrum. Therefore, that canonical transformation cannot be implemented by a unitary operator and the resulting quantum theories are inequivalent. We discuss the physical relevance of the Immirzi parameter in Sec. IV. Nevertheless, let us note that there does not exist any canonical transformation, relating theories with different values of β, within our covariant formalism.

The next problem is the loss of the space-time interpretation for the SU(2) connection [6]. Although it has not been taken into account seriously, it has deep consequences. In particular, this fact is probably the reason why the quantum constraint algebra does not reproduce the classical one and contains an anomaly [18,19,17].

It turns out that the covariant formalism is very convenient to address this second problem. In terms of transformation properties it means that the Ashtekar-Barbero connection cannot be extended in such a way that it transforms as a true space-time connection under four-dimensional diffeomorphisms. And indeed, it was shown that its Lorentz extension (5) does not tranform correctly under the time diffeomorphisms [11]. As a result, the implementation of this symmetry in quantum theory in the framework of loop quantization fails. The reason is that loop operators are not mapped to the time translated loop operators after symmetry transformation.[3] Hence the quantum diffeomorphism algebra contains an anomaly.

However, may it not be a problem but an unavoidable property of quantum gravity? The answer depends on whether one can find a quantization preserving the full diffeomorphism invariance. If such a quantization exists, of course, it should be considered as more preferable, since the whole history of quantum theory tells that one should try to preserve classical symmetries as much as possible, especially, when they are as fundamental as the diffeomorphism invariance is believed to be.

To answer this question, let us recall that the covariant Ashtekar-Barbero connection (5) is only one among the two-parameter family of Lorentz connections found in the covariant approach. In principle, each of the connections could be used in loop quantization and each would lead to different physics (for example, different area spectra). Thus, they represent a real quantization ambiguity of the loop approach. Could this ambiguity be resolved? Is it possible to find a criterion which allows us to choose the right connection?

The answer is affirmative and the corresponding criterion is actually simply that it transforms properly under the time diffeomorphisms. Indeed, it turns out that if we impose this additional restriction, there is only one connection satisfying it [11], i.e., possessing a genuine space-time interpretation. This means that there is a *unique* loop quantization preserving all classical symmetries of general relativity. Moreover, for this choice of connection the area spectrum does not depend on the Immirzi parameter. This gives an additional evidence in favor of such quantization and shows that all the problems appearing in SU(2) LQG are likely to find their solutions in the covariant approach.

III. QUANTIZATION-PRESERVING DIFFEOMORPHISM INVARIANCE

A. Canonical structure and area spectrum

In this section we describe the unique space-time Lorentz connection diagonalizing the area operator and the resulting quantum picture. This connection corresponds to the choice $a=b=0$ in Eq. (A15) that leads to the following shifted connection [10,11]:

$$\mathcal{A}_i^X=A_i^X+\frac{1}{2\left(1+\frac{1}{\beta^2}\right)}R_S^X I_{(Q)}^{ST} R_T^Z f_{ZW}^Y \underset{\sim}{P}{}_i^W \mathcal{G}_Y . \tag{26}$$

In this case the Dirac brackets can be given in the simple form

[3]The situation is essentially the same as it would be if the connection does not transform correctly under the space diffeomorphisms. Then there would not be an easy way to realize this symmetry on the space of loop states.

$$\{\tilde{P}^i_X,\tilde{P}^j_Y\}_D=0, \tag{27}$$

$$\{\mathcal{A}^X_i,\tilde{P}^j_Y\}_D=\delta^j_i I^X_{(P)Y}, \tag{28}$$

whereas the commutator of two connections is much more complicated (see Appendix C) except from the relation

$$\{I_{(P)}R^{-1}\mathcal{A},I_{(P)}R^{-1}\mathcal{A}\}_D=0. \tag{29}$$

The area operator following from the commutation relations (28) is expressed as a combination of two Casimir operators:

$$\mathcal{S}\sim\hbar\sqrt{C(su(2))-C_1(so(3,1))}. \tag{30}$$

As the connection (5), $\mathcal{A}$ satisfies some constraints reducing the number of its independent components. There are three such constraints [10]. However, we can use additional ambiguity to add to any quantity a combination of the second class constraints (A14) in order to remove six more components, without modifying any commutation relations. The most natural choice is

$$\begin{aligned}\tilde{\mathcal{A}}^X_i&=\mathcal{A}^X_i-\frac{1}{2}R^X_Y\left(\underset{\sim}{Q}^Y_l(\underset{\sim}{Q}\underset{\sim}{Q})_{ik}-\frac{1}{2}\underset{\sim}{Q}^Y_i(\underset{\sim}{Q}\underset{\sim}{Q})_{lk}\right)\psi^{lk}\\&=\left(1+\frac{1}{\beta^2}\right)I^X_{(P)Y}(R^{-1})^Y_Z A^Z_i+R^X_Y\Gamma^Y_i,\end{aligned} \tag{31}$$

where

$$\begin{aligned}\Gamma^X_i=&\frac{1}{2}f^W_{YZ}I^{XY}_{(Q)}\underset{\sim}{Q}^Z_i\partial_l\tilde{Q}^l_W+\frac{1}{2}f^{ZW}_Y[(\underset{\sim}{Q}\underset{\sim}{Q})_{ij}I^{XY}_{(Q)}+\underset{\sim}{Q}^X_j\underset{\sim}{Q}^Y_i\\&-\underset{\sim}{Q}^X_i\underset{\sim}{Q}^Y_j]\tilde{Q}^l_Z\partial_l\tilde{Q}^j_W.\end{aligned} \tag{32}$$

It is clear then that the connection (31) satisfies the constraints

$$I^X_{(Q)Y}\tilde{\mathcal{A}}^Y_i=\Gamma^X_i(\tilde{Q}). \tag{33}$$

Let us note the most important differences in comparison with the case of the Lorentz generalization of the Ashtekar-Barbero connection described in Sec. II B.

The nontrivial part of the connection is contained in the boost rather than $SU(2)$ components [see Eqs. (28) and (33)].

The nondynamical part of the connection given by $\Gamma^X_i(\tilde{Q})$ does not vanish in the time gauge. It gives actually a generalization of the Christoffel connection and it is defined by the triad field.

The connection remains noncommutative.

The commutation relations (28) and the area spectrum does not depend on the Immirzi parameter.

All these differences have deep consequences for quantization. First of all, the noncommutativity of the connection makes it harder to deal with the connection representation and the loop functionals as configuration variables[4] on both the mathematical and the physical interpretation levels. Nevertheless, we can try to ignore this difficulty for a moment with a hope to resolve it in the end of the way.[5] Then we try to carry out the same program which was realized in Sec. II C. And we do not encounter any problems in the first part. The construction of the projected spin networks based on Wilson lines (15) does not refer to particular properties of the used connection and it is still valid for any Lorentz connection. In this way we end up with the same kinematical Hilbert space $\mathcal{H}_0$.

However, we are not able to carry out the second part of the program and solve the second class constraints on this Hilbert space as we had done for the connection **A**. Indeed, we should somehow take into account at the quantum level that we fix the $SU(2)$ components of the connection. Moreover, this fixed value should be determined by the triad $\tilde{E}$, which is difficult to realize using only functionals of connection. Maybe this problem can be solved in a triad representation as done with the reality conditions for the self-dual (complex) Ashtekar formulation corresponding to the case $\beta=i$ [21]. This will be investigated in future work. Thus, the problem of a solution for the second class constraints for the space-time connection at the level of Hilbert space looks quite nontrivial and remains to be done.

Nevertheless, it is possible to sidestep this problem and impose the second class constraints at a finite number of points. Indeed, as explained in the next paragraph, it turns out that projected spin networks projected onto the trivial $SU(2)$ representation $j=0$ solve the second class constraints at their vertices and give the same Hilbert space of quantum states as obtained from the spin foam approach.

B. Recovering the spin foam basis

Spin foam models are the space-time models corresponding to the evolving spin networks from LQG. Up to now, there has been lacking an explicit link between the existing spin foam models, which are based on $SL(2,C)$, and the canonical framework, which is based on the $SU(2)$ symmetry group. The present covariant canonical framework builds a bridge between these two pictures and this may help to build a consistent quantum space-time picture.

The most promising spin foam model for both Euclidean and Lorentzian gravity is the Barrett-Crane model [22,23]. Its construction relies on methods from geometrical quantization [22,24], but can also be related to the generalized

[4]In fact, the loop functionals can still be good configuration variables even at the presence of the noncommutativity. An example of such a situation can be found in quantization of the Chern-Simons theory [20]. However, the noncommutativity in our case is different and much more complicated than the one appearing in the analysis of that theory. Nevertheless, the possibility of applying the ideas of [20] to the present problem should be investigated in a future work.

[5]An example of such a sitation, when we end up with commutative spin networks, can be found in [14]. Another way would be to look for a triad representation.

Palatini action (with the Immirzi parameter) through a generalized constrained BF theory [25–27]. From the point of view of canonical quantization, it can be interpreted as quantizing the system without imposing the second class constraints and imposing them only afterwards at the quantum level. They can then be translated into the so-called simplicity constraints restricting the used $SL(2,C)$ representations to simple ones, which have a vanishing quadratic Casimir operator [28,26,27].

More precisely, when carrying out the canonical analysis [9], we get second class constraints of two types $\phi^{ij}=\Pi^{XY}\tilde{Q}_X^i\tilde{Q}_Y^j$ (A13) and ψ^{ij} [see (A14) for an explicit expression]. These constraints give rise to Dirac brackets. Now, an interesting property of the Dirac brackets is that it is equal to the initial Poisson brackets when both the considered quantities commute with only the ϕ^{ij} constraints:

$$\{K,\phi\}=\{L,\phi\}=0\Rightarrow\{K,L\}_D=\{K,L\}. \tag{34}$$

This leads us to think that considering only quantities that commute with the ϕ constraints allows us to ignore the ψ constraints. The advantage of such a viewpoint is that the ϕ constraints seem much easier to implement than the ψ constraints. Moreover, it coincides exactly with the simplicity constraint used in spin foam models. More precisely, let us consider a spin network based on the initial $SL(2,C)$ connection A and pick a point on a given link of the graph. Imposing that it commutes with ϕ leads to an equation on the Casimir operators of the representation living on the chosen link:

$$\frac{2}{\beta}C_1(SL(2,C))=\left(1-\frac{1}{\beta^2}\right)C_2(SL(2,C)), \tag{35}$$

where $C_1=g^{XY}T_XT_Y$ and $C_2=\Pi^{XY}T_XT_Y$ are the two Casimir operators of $SO(3,1)$. This equation is exactly the same as the one arising in the construction of spin foam models from the generalized Palatini action [26]. Within the spin foam context, it was argued that such an equation is meaningless and it turned out that there exists an ambiguity in the quantization procedure which allows us to rotate the constraint to the usual simplicity one $C_2=0$ [27]. This ambiguity is to be compared to the two-parameter ambiguity $\mathcal{A}(a,b)$ in the choice of a $SL(2,C)$ connection in the covariant canonical frame (A15): does taking full account of the second class constraints through the Dirac brackets and choosing the space-time connection $\mathcal{A}=\mathcal{A}(0,0)$ cancel the rotation introduced by β and lead to the same result obtained for spin foams?

The answer to this question is the affirmative. Indeed, by using the Dirac brackets obtained with the full second class constraints and the space-time connection described previously, we are able to quantize the theory and solve the second class constraints (at a finite number of points) and we obtain the exact same Hilbert space of quantum states as the Barrett-Crane model. This link, which we explicitate below, is a much stronger statement that the simpler one consisting of noticing (as above) that the ϕ constraints looks like the simplicity constraints.

To start with, let us look at the case $\beta\to\infty$. This corresponds to the usual Palatini action without the extra term introduced by the Immirzi parameter, and we have the following Poisson bracket relations:

$$\{I_{(P)}\mathcal{A},I_{(P)}\mathcal{A}\}_D=0,$$
$$\{\tilde{P},\tilde{P}\}_D=0,$$
$$\{\mathcal{A}_i^X,\tilde{P}_Y^j\}_D=\delta_i^jI_{(P)Y}^X. \tag{36}$$

We would like to ignore $I_{(Q)}\mathcal{A}$ (which does not come in the commutation relation with the P field, and could be completely constructed as an operator from the P operator at the quantum level) and construct functionals depending only on $I_{(P)}\mathcal{A}$. To this purpose, we consider simple spin networks, which are the $j=0$ case of the projected spin networks introduced in II C [16]. We choose a fixed (oriented) graph, whose edges we label with $SL(2,C)$ representations (n_e,ρ_e). We construct the holonomies U_e of the connection $\mathcal{A}$ along these edges. We consider their trace on the $SU(2)$ invariant subspace of the edge representation. This is the $j=0$ subspace, and its existence selects out the simple representation $(n_e=0,\rho_e)$ which we will note simply as ρ_e. Finally, the simple spin network functionals are

$$s^{\{\rho_e\}}(U_e)=\prod_e\langle\rho_e\chi_{s(e)}j=0|U_e|\rho_e\chi_{t(e)}j=0\rangle, \tag{37}$$

where χ_v is the value of the field χ at the vertex v and $|\rho\chi_vj=0\rangle$ is the vector in the ρ representation which is invariant under $SU(2)_{\chi_v}$. These are cylindrical functionals of the connection $\mathcal{A}$ and depend only on $I_{(P)}\mathcal{A}$ at the vertices. Therefore, the Poisson brackets of two such functionals whose graphs intersect only at some common vertices vanishes. This is understandable within the spin foam context, where we have a complete discrete view of space-time. Only the vertices are (space-time) points, then an edge is a relation between two points and is not considered as a continuous line of points. Indeed only at the vertices, we do know the time normal. From this point of view, considering two simple spin networks, if their graphs intersect, then the intersection point is to be defined, and so it should be a vertex of the two graphs. Thus, either it is already a common vertex, or we should refine the graph (add a bivalent vertex in the middle of the edges) so that it becomes one. Moreover, we can compute the action of the area operator of a surface intersecting the graph at the end of an edge (at the vertex). The simple spin network is its eigenvector with eigenvalue:

$$\mathcal{S}\sim\sqrt{-C_1(SL(2,C))}=\sqrt{\rho_e^2+1}. \tag{38}$$

This spectrum is always well defined, corresponds truly to a space-like surface and is, in fact, exactly the same area spectrum as obtained through the spin foam approach. To sum up, we can choose an initial set of points on the manifold which will be the vertices of all the considered graphs, then we consider the simple spin networks based on such graphs and we obtain a representation of the initial commutation relations and quantum structure which reproduces exactly the kinematical setting and the boundary states of the Barrett-Crane model [16]. This representation takes into account only the commutation relations (36) at the chosen points. In principle, we could build a delicate ladder of operators taking into account the necessity of considering the commutator of two simple spin networks only when their graphs intersect at common points, then it would be possible to get a complete representation of the Poisson brackets (36).

This shows that the second class constraints build a theory based on the coset $SO(3,1)/SO(3)$ or $SL(2,C)/SU(2)$ just as in the spin foam scheme. This result opens the door to a generalization to higher dimensions: it seems possible to reproduce, within the canonical approach, the spin foam result that simplicity conditions impose a $SO(D)/SO(D-1)$ coset structure to the quantum theory [29].

Now let us look at the general case with β arbitrary. The Poisson algebra reads

$$\{I_{(P)}R^{-1}\mathcal{A},I_{(P)}R^{-1}\mathcal{A}\}_D=0, \tag{39}$$

$$\{\tilde{P},\tilde{P}\}_D=0, \tag{40}$$

$$\{\mathcal{A}_i^X,\tilde{P}_Y^j\}_D=\delta_i^j I_{(P)Y}^X. \tag{41}$$

The situation is complicated by the R change of basis. And it is not obvious how the above quantization procedure generalizes in this case. Indeed the commutation relations between $\mathcal{A}$ and $\tilde{P}$ invite us to consider the same functionals as previously. Nevertheless, the commutation relations $\mathcal{A}$ and itself tell us that the operator for $I_{(P)}\mathcal{A}$ will then not be trivial. The solution to the problem is to use the (second class) constraint $I_{(Q)Y}^X\tilde{\mathcal{A}}_i^Y=\Gamma_i^X(\tilde{Q})$.

First, we can change $\mathcal{A}$ to $\tilde{\mathcal{A}}$ without modifying anything in the Poisson brackets. Then, as $I_{(Q)}\tilde{\mathcal{A}}$ commutes with $\tilde{P}$, we get

$$\{I_{(P)}R^{-1}\tilde{\mathcal{A}},\tilde{P}\}_D=\left\{I_{(P)}\tilde{\mathcal{A}}-\frac{1}{\beta}\Pi I_{(Q)}\tilde{\mathcal{A}},\tilde{P}\right\}_D=I_{(P)}. \tag{42}$$

Then, one has the same structure as previously replacing $I_{(P)}\mathcal{A}$ of the case $\beta\to\infty$ by $I_{(P)}R^{-1}\tilde{\mathcal{A}}$. One considers simple spin networks constructed with the connection $\tilde{\mathcal{A}}$. The operator $\hat{\tilde{P}}$ is still the derivation with respect to $\tilde{\mathcal{A}}$ and its action is the insertion of the generators T of $SL(2,C)$. The operator $\widehat{I_{(Q)}}\tilde{\mathcal{A}}$ can be deduced as the Christofel symbol $\Gamma(\tilde{Q})$ constructed with the $\hat{\tilde{Q}}$ operator. Then, we can choose the operator $I_{(P)}\widehat{R^{-1}}\tilde{\mathcal{A}}$ to be the multiplication by $I_{(P)}\tilde{\mathcal{A}}$. This operator commutes with itself as any multiplication, which realizes (39). Moreover, the commutator of the multiplication $I_{(P)}\tilde{\mathcal{A}}\times$ with the derivation operator $\hat{\tilde{P}}$ gives the identity, so that this choice of quantizaion realises the Dirac bracket (42) and thus the bracket (41). Therefore, we have a complete realization of the Dirac brackets of the connection and the triad. Finally, we can deduce the operator for $I_{(P)}\tilde{\mathcal{A}}$ from the operator for $I_{(P)}R^{-1}\tilde{\mathcal{A}}$:

$$I_{(P)}\widehat{R^{-1}}\tilde{\mathcal{A}}=\widehat{I_{(P)}}\tilde{\mathcal{A}}-\frac{1}{\beta}\Pi\widehat{I_{(Q)}}\tilde{\mathcal{A}}\Rightarrow\widehat{I_{(P)}}\tilde{\mathcal{A}}=I_{(P)}\tilde{\mathcal{A}}+\frac{1}{\beta}\Pi\Gamma(\hat{\tilde{Q}}). \tag{43}$$

This concludes the spin foam quantization in the canonical framework and shows that the Immirzi parameter β is not an obstacle to the quantization as the modified simplicity condition (35) had suggested.

IV. PHYSICAL RELEVANCE OF THE IMMIRZI PARAMETER

From the very beginning we introduced the parameter β in the theory. It was identified as the Immirzi parameter of $SU(2)$ LQG through the canonical analysis in the time gauge [12]. At the classical level it does not change the equations of motion and, therefore, it does not influence the dynamics. Does it play any role at the quantum level? The SU(2) LQG says it does. The main reason is that the physical spectra of geometrical operators, like the area spectrum (10), depend on it. Besides, it comes in the Hamiltonian constraint and thus modifies *a priori* the dynamics of the theory. As a result, the Immirzi parameter should become physical and it should be considered as a new fundamental constant [8]. It is usually argued that it can be fixed by looking at the black hole entropy.

But we argued that the SU(2) approach is based on a wrong choice of the connection and there is another choice which seems to be the only correct one. Does the Immirzi parameter become physical in this second quantization? The results of Sec. III say it does not. The main reason is that it does not appear in the commutation relations of the connection and triad mutliplet (28). It is this commutator from which we derive the spectrum of area and other geometrical operators. Of course, the Immirzi parameter appears in the commutator of two connections (C1), but only in the universal form in the prefactors. Therefore, even if this commutator contributes to some physical results, it is very likely that the Immirzi parameter will be cancelled.

One more evidence that the Immirzi parameter remains unphysical is given by the path integral quantization. It was shown that the formal path integral constructed for the generalized Hilbert-Palatini action does not depend on β [9]. Note that the path integral does not refer to any choice of connection but relies only on the Becchi-Rouet-Stora-Tyutin (BRST) analysis based on the classical symmetry algebra. Similarly, the spin foam quantization, which can be understood as a discrete path integral, of the generalized Hilbert-Palatini action was shown not to depend on the Immirzi parameter [26,27]. These two results point to the nonrelevance of the Immirzi parameter in the space-time dynamics.

All this allows us to conclude that the appearence of the Immirzi parameter in the results of the SU(2) LQG is a consequence of the quantum anomaly in the four-dimensional diffeomorphism invariance. Instead, once the quantization preserving this symmetry is chosen, it remains unphysical as it was at the classical level.

Moreover, the covariant formalism reveals that actually there are two ambiguities in the theory: the choice of β due to the additional term in the generalized Palatini action, and the choice of connection [the parameters (a,b)] which is used in the definition of loop variables. (This is also true in the spin foam context [26,27], where we have an ambiguity at the level of the BF action and an ambiguity when quantizing bivectors.) The first ambiguity is a classical one, whereas the latter is trully quantum. It is explicily seen from the commutation relations (A18) and the area spectrum (A23) found for the arbitrary connection of the family (A15). They depend only on the perimeters a and b from the definition of the connection but not on β. The dependence on the Immirzi parameter in the $SU(2)$ case appears only after the identification $(a,b)=(-\beta,1)$ (4).[6] In any case, this quantum ambiguity is fixed by the requirement to retain the classical symmetry.

V. CONCLUSIONS AND OUTLOOKS

In the present work we have derived the framework of $SU(2)$ loop quantum gravity from the explicitly Lorentz covariant formalism. It was done not only at the classical level, but quantizing the covariant theory, so that we were able to reproduce the Hilbert space structure of $SU(2)$ LQG from the Hilbert space of covariant quantization. This was possible due to the existance of a Lorentz generalization of the Ashtekar-Barbero connection. Choosing this connection as a basic quantum variable, the quantization program can be easily carried out. In particular, one can accomlish the most important step: to implement the second class constraints at the Hilbert space level.

This derivation allows a new viewpoint on the dynamics of LQG. Indeed, the covariant approach makes easy to study the space-time symmetries and whether or not they are preserved through the quantization process. Because the covariant Ashtekar-Barbero connection does not transform correctly under time diffeomorphisms, it is not a space-time connection. Therefore, there seems to be a preferred frame defined by the time gauge and the theory seems to break diffeomorphism invariance.

We have also underlined the existence of a true space-time connection, the unique Lorentz connection which transforms properly under space-time diffeomorphisms. We described its properties, how to quantize the theory with this connection, and the problems which appear. In particular, we suggested a quantization procedure which leads to the resulting picture of quantum geometry states similar to one of a particular spin foam model—the Barrett-Crane model. This hints toward an explicit link between the space-time formalism given by spin foams and the canonical frame given by LQG and spin network states. Moreover, our analysis reveals that the right canonical theory for the Barrett-Crane model is the presented covariant LQG and not the $SU(2)$ LQG. One may hope to better understand the geometry defined by spin foams and find the right dynamics of the theory.

Therefore, it would be very interesting to study the implementation of the Hamiltonian constraint within this covariant theory, an issue which we will investigate in future work. Nevertheless, this quantization procedure linking the canonical theory to the spin foam setting is based on solutions at a finite number of points to the second class constraints, relating the (space-like part of the) connection to the triad. It should be investigated whether it would be possible to impose them entirely, maybe using a triad representation as done in the self-dual complex Ashtekar formulation to deal with the reality conditions.

The covariant approach also opens the door to the study of Lorentz boosts and related covariance issues in loop quantum gravity, as already discussed in [30]. Finally, the present analysis should be generalized to higher dimensions and it should be possible to prove the same link at the kinematical level between the canonical approach and the spin foam model as described in [29].

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APPENDIX A: LORENTZ COVARIANT CANONICAL FORMULATION

In this appendix we review the covariant formalism developed in [9]. It is a canonical formulation of general relativity based on the generalized Hilbert-Palatini action (1). We use the following $3+1$ decomposition of the fields

$$e^0 = N dt + \chi_a E_i^a dx^i, \quad e^a = E_i^a dx^i + E_i^a N^i dt,$$
$$\tilde{E}_a^i = h^{1/2} E_a^i, \quad \underset{\sim}{N} = h^{-1/2} N, \quad \sqrt{h} = \det E_i^a. \tag{A1}$$

Here E_a^i is the inverse of E_i^a. The field χ_a describes deviation of the normal to the hypersurface $\{t=0\}$ from the time direction. Let us change the lapse and shift variables as follows:

$$N^i = \mathcal{N}_D^i + \tilde{E}_a^i \chi^a \underset{\sim}{\mathcal{N}}, \quad \underset{\sim}{N} = \underset{\sim}{\mathcal{N}} + \underset{\sim}{E}_i^a \chi_a \mathcal{N}_D^i \tag{A2}$$

and introduce the multiplets which play the role of canonical variables: the connection multiplet,

$$A_i^X = \left(\omega_i^{0a}, \frac{1}{2} \varepsilon^a{}_{bc} \omega_i^{bc} \right), \tag{A3a}$$

the first triad multiplet,

[6] In fact, we can obtain the $SU(2)$ area spectrum even in the limit $\beta \to \infty$. In this limit the covariant formalism still exists and possesses the same (a,b) ambiguity in the choice of connection. Choosing $b=1$, we get the spectrum (10) given by the Casimir operator of $SU(2)$ with β replaced by a. The only difference of this formulation from the previous one is that the connection remains noncommutative. We need to have a finite β to get commutativity. From this point of view, the introduction of β and the extra term in the Palatini action appears like a regularization.

$$\tilde{P}_X^i = (\tilde{E}_a^i, \varepsilon_a{}^{bc}\tilde{E}_b^i\chi_c), \tag{A3b}$$

the second triad multiplet,

$$\tilde{Q}_X^i = (-\varepsilon_a{}^{bc}\tilde{E}_b^i\chi_c, \tilde{E}_a^i), \tag{A3c}$$

and the canonical triad multiplet,

$$\tilde{P}_{(\beta)}{}^i_X = \tilde{P}_X^i - \frac{1}{\beta}\tilde{Q}_X^i. \tag{A3d}$$

All triad multiplets are related by numerical matrices

$$\tilde{P}_X^i = \Pi_X^Y \tilde{Q}_Y^i, \quad \tilde{P}_X^i = \frac{R_X^Y}{1+\frac{1}{\beta^2}}\tilde{P}_{(\beta)}{}^i_Y, \tag{A4}$$

$$g^{XY} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\delta_a^b, \quad \Pi^{XY} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\delta_a^b,$$

$$R^{XY} = g^{XY} - \frac{1}{\beta}\Pi^{XY} = \begin{pmatrix} 1 & -\frac{1}{\beta} \\ -\frac{1}{\beta} & -1 \end{pmatrix}\delta_a^b. \tag{A5}$$

Also one can introduce the *inverse triad multiplets* $\underset{\sim}{P}_i^X$ and $\underset{\sim}{Q}_i^X$ and *projectors*

$$I_{(P)X}^Y \equiv \tilde{P}_X^i \underset{\sim}{P}_i^Y = \begin{pmatrix} \frac{\delta_a^b - \chi_a\chi^b}{1-\chi^2} & \frac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2} \\ \frac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2} & -\frac{\delta_a^b\chi^2 - \chi_a\chi^b}{1-\chi^2} \end{pmatrix}, \tag{A6}$$

$$I_{(Q)X}^Y \equiv \tilde{Q}_X^i \underset{\sim}{Q}_i^Y = \begin{pmatrix} -\frac{\delta_a^b\chi^2 - \chi_a\chi^b}{1-\chi^2} & -\frac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2} \\ -\frac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2} & \frac{\delta_a^b - \chi_a\chi^b}{1-\chi^2} \end{pmatrix}. \tag{A7}$$

These projectors are functions of the field χ only and can be considered, respectively, as projectors in the Lorentz algebra on the $su(2)$ subalgebra defined by χ and its orthogonal boost part (see Appendix B for more details).

The decomposed action reads

$$S_{(\beta)} = \int dt d^3x (\tilde{P}_{(\beta)}{}^i_X \partial_t A_i^X + A_0^X \mathcal{G}_X + \mathcal{N}_D^i H_i + \underset{\sim}{\mathcal{N}} H), \tag{A8}$$

$$\mathcal{G}_X = \partial_i \tilde{P}_{(\beta)}{}^i_X + f_{XY}^Z A_i^Y \tilde{P}_{(\beta)}{}^i_Z, \tag{A9}$$

$$H_i = -\tilde{P}_{(\beta)}{}^j_X F_{ij}^X, \tag{A10}$$

$$H = -\frac{1}{2\left(1+\frac{1}{\beta^2}\right)}\tilde{P}_{(\beta)}{}^i_X \tilde{P}_{(\beta)}{}^j_Y f_Z^{XY} R_W^Z F_{ij}^W, \tag{A11}$$

$$F_{ij}^X = \partial_i A_j^X - \partial_j A_i^X + f_{YZ}^X A_i^Y A_j^Z. \tag{A12}$$

The action (A8) gives rise to ten first class constraints $\mathcal{G}_X$, H_i, H and also two sets of second class constraints

$$\phi^{ij} = \Pi^{XY}\tilde{Q}_X^i\tilde{Q}_Y^j = 0, \tag{A13}$$

$$\psi^{ij} = 2f^{XYZ}\tilde{Q}_X^l\tilde{Q}_Y^{\{j}\partial_l\tilde{Q}_Z^{i\}} - 2(\tilde{Q}\tilde{Q})^{ij}\tilde{Q}_Z^l A_l^Z + 2(\tilde{Q}\tilde{Q})^{l\{i}\tilde{Q}_Z^{j\}}A_l^Z = 0, \tag{A14}$$

where

$$(\tilde{Q}\tilde{Q})^{ij} = g^{XY}\tilde{Q}_X^i\tilde{Q}_Y^j,$$

which lead to Dirac brackets. They change the commutation relations of the canonical variables so that the connection becomes noncommutative. Another consequence is that the area operator is not diagonal in the basis of Wilson lines defined with the connection A_i^X, since the bracket $\{A_i^X, \tilde{P}_Y^j\}_D$ is not diagonal in spatial indices. The action of the area operator on such a Wilson line depends on the details of embedding of the surface and line into three-dimensional space [10].

However, one can redefine the connection in such a way that the area operator would be diagonal on Wilson lines defined with the new connection. It has been shown [11] that there is a two-parameter family of such Lorentz connections, i.e., transforming correctly under the Gauss and diffeomorphism constraints. They are written as

$$\begin{aligned}\mathcal{A}_i^X(a,b) = A_i^X + \frac{1}{2}\Bigg[\left(1+\frac{a}{\beta}\right)g^{XX'} - \frac{1}{\beta}(1-b)\Pi^{XX'}\Bigg] \\ \times I_{(Q)X'}^T \frac{R_T^Y}{1+\frac{1}{\beta^2}} f_{YZ}^W \underset{\sim}{P}_i^Z \mathcal{G}_W + (a\,\delta_{X'}^X + b\,\Pi_{X'}^X) \\ \times[I_{(Q)}^{X'W}\Pi_{WZ}A_i^Z + \Lambda_i^{X'}(\tilde{Q})],\end{aligned} \tag{A15}$$

where we introduced

$$\Lambda_i^X(\tilde{Q}) = g^{XX'}\Pi_R^Z\Bigg(I_{(Q)X'}^R f_{YZ}^W \underset{\sim}{Q}_i^Y \partial_l \tilde{Q}_W^l + \frac{1}{2}f_{YZ}^W\tilde{Q}_{X'}^k \underset{\sim}{Q}_i^Y \tilde{Q}_W^l \partial_k \underset{\sim}{Q}_l^R\Bigg). \tag{A16}$$

Using the explicit expressions (A3a)–(A3d), one can show that actually $\Lambda_i^X(\tilde{Q})$ depends only on the field χ:

$$\Lambda_i^X(\tilde{Q}) = -g^{XY}\frac{\tilde{Q}_X^j \underset{\sim}{E}_j^a \partial_i \chi_a}{1-\chi^2} = \left(-\frac{\varepsilon^{abc}\chi_b \partial_i \chi_c}{1-\chi^2}, \frac{\partial_i \chi^a}{1-\chi^2}\right). \tag{A17}$$

The connections (A15) possess the following properties:

$$\{\mathcal{A}_i^X(a,b), \tilde{P}_Y^j\}_D = \delta_i^j[(1-b)\delta_{X'}^X - a\Pi_{X'}^X] I_{(P)Y}^{X'}, \tag{A18}$$

$$\{\mathcal{A}_i^X(a,b), I_{(P)}^{YZ}\}_D = 0, \tag{A19}$$

$$\{\{\mathcal{A}_i^X(a,b), \mathcal{A}_j^Y(a,b)\}_D, \tilde{P}_Z^k\}_D = 0. \tag{A20}$$

From (A19) it follows that

$$\{\mathcal{A}_i^X(a,b), \chi^a\}_D = 0. \tag{A21}$$

Note also that they can be represented as follows:

$$\mathcal{A}_i^X(a,b) = \mathcal{A}_i^X + (a\delta_{X'}^X + b\Pi_{X'}^X)[I_{(Q)}^{X'W}\Pi_{WZ}\mathcal{A}_i^Z + \Lambda_i^{X'}(\tilde{Q})], \tag{A22}$$

where $\mathcal{A}_i^X \equiv \mathcal{A}_i^X(0,0)$ is the unique true space-time connection diagonalizing the area operator [11]. More precisely, it is the only connection from the family (A15), which transforms correctly under the time diffeomorphisms.

The resulting area spectrum for generic (a,b) is

$$\mathcal{S} \sim \hbar\sqrt{[a^2+(1-b)^2]C(su(2)) - (1-b)^2 C_1(sl(2,C)) + a(1-b)C_2(sl(2,C))} \tag{A23}$$

where $C_1 = g^{XY}T_X T_Y$ and $C_2 = \Pi^{XY}T_X T_Y$ are the Casimir operators of $SL(2,C)$ and $C = I_{(Q)}^{XY}T_X T_Y$ is the Casimir operator of the $SU(2)$ subgroup obtained from the canonically embedded one by a boost transformation with the parameter χ^a.

APPENDIX B: REDUCING $SL(2,C)$ TO THE SUBGROUP $SU_\chi(2)$

Let us introduce mixed tensors:

$$q_X^a = (\varepsilon^a{}_{bc}\chi^c, \delta_b^a), \quad p_X^a = (\delta_b^a, -\varepsilon^a{}_{bc}\chi^c), \tag{B1}$$

$$q_a^X = \left(-\frac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2}, \frac{\delta_a^b - \chi_a\chi^b}{1-\chi^2}\right),$$

$$p_a^X = \left(\frac{\delta_a^b - \chi_a\chi^b}{1-\chi^2}, \frac{\varepsilon_a{}^{bc}\chi_c}{1-\chi^2}\right). \tag{B2}$$

They relate the triad and the triad multiplets

$$\tilde{Q}_X^i = q_X^a \tilde{E}_a^i, \quad \tilde{P}_X^i = p_X^a \tilde{E}_a^i, \tag{B3}$$

$$\underset{\sim}{Q}_i^X = q_a^X \underset{\sim}{E}_i^a, \quad \underset{\sim}{P}_i^X = p_a^X \underset{\sim}{E}_i^a \tag{B4}$$

and possess the following properties:

$$q_X^a p_b^X = p_X^a q_b^X = 0, \tag{B5}$$

$$q_X^a q_b^X = p_X^a p_b^X = \delta_b^a, \tag{B6}$$

$$q_X^a q_a^Y = I_{(Q)X}^Y, \quad p_X^a p_a^Y = I_{(P)X}^Y. \tag{B7}$$

One can say that the set of six-dimensional vectors (p^a, q^a) describes a basis in the Lorentz algebra obtained from the standard one by a Lorentz boost with parameter χ.

It is trivial to show that the quantity (3) satisfies $p_X^a A_i^{(\mathrm{ash})X} = 0$. Then one can show that the remaining components of the connection coincide with the quantity (2):

$$A_i^{(\mathrm{ash})a} \equiv q_X^a A_i^{(\mathrm{ash})X} = \beta A_i^{(\mathrm{bar})a}. \tag{B8}$$

Thus, $A_i^{(\mathrm{ash})X}$ gives an embedding of $A_i^{(\mathrm{bar})a}$ into the Lorentz algebra.

To proceed further, we introduce the "boosted" $SU(2)$ subgroup [let us call it $SU_\chi(2)$] with generators

$$\mathcal{G}_a^{(\chi)} = q_a^X \mathcal{G}_X. \tag{B9}$$

They form $su(2)$ algebra with the structure constants $f_{ab}^c = -\varepsilon_{abd}\{\delta^{dc} + [\chi^d\chi^c/(1-\chi^2)]\}$: $\{\mathcal{G}_a^{(\chi)}, \mathcal{G}_b^{(\chi)}\}_D = f_{ab}^c \mathcal{G}_c^{(\chi)}$.[7] It turns out that if $\chi = \mathrm{const}$, $A_i^{(\mathrm{ash})a}$ transforms as a connection with respect to the transformations generated by $\mathcal{G}_a^{(\chi)}$:

$$\{\mathcal{G}^{(\chi)}(n), A_i^{(\mathrm{ash})a}\}_D = \partial_i n^a + f_{bc}^a n^b A_i^{(\mathrm{ash})c}. \tag{B10}$$

As a result, in this restricted situation, $A_i^{(\mathrm{ash})a}$ is a connection of the gauge group $SU_\chi(2)$ and generalizes the Ashtekar-Barbero connection. Similarly, the restriction of the Lorentz covariant connection $\mathbf{A}_i^X$ (5) to $SU_\chi(2)$ given by

$$\mathbf{A}_i^{(\mathrm{ash})a} = q_X^a \mathbf{A}_i^X \tag{B11}$$

transforms according to Eq. (B10) since $\mathcal{G}_a^{(\chi)}$ commute with χ.

[7] In the case of constant χ, one can redefine the generators by a matrix constructed from χ to get the algebra with the usual structure constants $\varepsilon_{ab}{}^c$.

APPENDIX C: COMMUTATOR OF TWO CONNECTIONS

The commutator of the space-time connection $\mathcal{A}_i^a$ with itself can be calculated and is given by a horrible expression [14]:

$$\left\{\int d^3x f(x)\mathcal{A}_i^X(x), \int d^3y g(y)\mathcal{A}_j^Y(y)\right\}_D = \frac{1}{2\left(1+\frac{1}{\beta^2}\right)} R_S^X R_T^Y \int d^3z[(K_{ij}^{ST,l} g\partial_l f - K_{ji}^{TS,l} f\partial_l g) + fg(L_{ij}^{ST} - L_{ji}^{TS})], \tag{C1}$$

where

$$K_{ij}^{ST,l} = \Pi^{SS'} f_{S'}^{PQ}\{\tilde{Q}_P^l[(\underset{\sim}{Q}\underset{\sim}{Q})_{ij} I_{(Q)}{}_Q^T + \underset{\sim}{Q}_i^T \underset{\sim}{Q}_j^Q - \underset{\sim}{Q}_j^T \underset{\sim}{Q}_i^Q] + \delta_i^l I_{(Q)}{}_Q^T \underset{\sim}{Q}_j^P\},$$

$$\begin{aligned}
L_{ij}^{ST} = {} & \Pi_{S'}^S f_Z^{PQ}[\underset{\sim}{Q}_j^{S'} \underset{\sim}{Q}_n^T \underset{\sim}{Q}_i^Z + (\underset{\sim}{Q}\underset{\sim}{Q})_{in} \underset{\sim}{Q}_j^{S'} I_{(Q)}{}^{TZ} + \underset{\sim}{Q}_i^T \underset{\sim}{Q}_n^{S'} \underset{\sim}{Q}_j^Z \\
& - \underset{\sim}{Q}_i^T \underset{\sim}{Q}_j^{S'} \underset{\sim}{Q}_n^Z + (\underset{\sim}{Q}\underset{\sim}{Q})_{ij} \underset{\sim}{Q}_n^{S'} I_{(Q)}{}^{TZ} - \underset{\sim}{Q}_j^T \underset{\sim}{Q}_n^{S'} \underset{\sim}{Q}_i^Z]\tilde{Q}_P^l \partial_l \tilde{Q}_Q^n \\
& + \Pi_{S'}^S f_{ZP}^Q[\underset{\sim}{Q}_n^T \underset{\sim}{Q}_j^P + (\underset{\sim}{Q}\underset{\sim}{Q})_{jn} I_{(Q)}{}^{TP} \\
& - \underset{\sim}{Q}_j^T \underset{\sim}{Q}_n^P] I_{(Q)}{}^{ZS'} \partial_i \tilde{Q}_Q^n + \Pi_Z^{Z'} f_{Z'}^{PQ}[(\underset{\sim}{Q}\underset{\sim}{Q})_{in} \underset{\sim}{Q}_j^Z I_{(Q)}{}^{ST} \\
& - (\underset{\sim}{Q}\underset{\sim}{Q})_{in} \underset{\sim}{Q}_j^T I_{(Q)}{}^{SZ} - (\underset{\sim}{Q}\underset{\sim}{Q})_{ij} \underset{\sim}{Q}_n^T I_{(Q)}{}^{SZ}]\tilde{Q}_P^l \partial_l \tilde{Q}_Q^n \\
& + \Pi_{S'}^S f_{PQ}^Z \underset{\sim}{Q}_j^{S'} \underset{\sim}{Q}_i^Q I_{(Q)}{}^{TP} \partial_l \tilde{Q}_Z^l \\
& + f_{PQ}^Z \underset{\sim}{Q}_i^P \underset{\sim}{Q}_j^Q I_{(Q)}{}_Z^T I_{(Q)}{}^{SW} \Pi_W^{W'} \partial_l \tilde{Q}_{W'}^l .
\end{aligned} \tag{C2}$$

Nevertheless, using this expression one can obtain several important properties. First of all, note that from Eq. (C1) it is straightforward to check the following fact:

$$I_{(P)S}^X (R^{-1})_Z^S \{\mathcal{A}_i^Z, \mathcal{A}_j^W\}_D (R^{-1})_W^T I_{(P)T}^Y = 0. \tag{C3}$$

Then, since the quantity (3) can be written as a similar projection of the space-time connection $\mathcal{A}_i^a$,

$$A_i^{(\text{ash})X} = -\beta\left(1+\frac{1}{\beta^2}\right)\Pi_Y^X I_{(P)T}^Y (R^{-1})_Z^T \mathcal{A}_i^Z, \tag{C4}$$

it commutes with itself. Moreover, from Eq. (A21) it follows that any $\mathcal{A}(a,b)$ commutes with χ and, therefore, with the function $\Lambda(\tilde{Q})$ (A17). This allows us to conclude that the covariant generalization of the Ashtekar-Barbero connection **A** given by Eq. (5) is also commuting.

Another property, which follows immediately from the relation (33), is

$$I_{(Q)Z}^X \{\mathcal{A}_i^Z, \mathcal{A}_j^W\}_D I_{(Q)W}^Y = 0. \tag{C5}$$

APPENDIX D: REPRESENTATIONS OF $SL(2,C)$

The generators T_X form the $sl(2,C)$ algebra with the structure constants f_{XY}^Z:

$$[T_X, T_Y] = f_{XY}^Z T_Z. \tag{D1}$$

Let us introduce the notations $T_X = (A_a, -B_a)$ and

$$H_+ = iB_1 - B_2, \quad H_- = iB_1 + B_2, \quad H_3 = iB_3, \tag{D2}$$

$$F_+ = iA_1 - A_2, \quad F_- = iA_1 + A_2, \quad F_3 = iA_3. \tag{D3}$$

These generators commute in the following way:

$$\begin{aligned}
&[H_+, H_3] = -H_+, \quad [H_-, H_3] = H_-, \quad [H_+, H_-] = 2H_3, \\
&[H_+, F_+] = [H_-, F_-] = [H_3, F_3] = 0, \\
&[H_+, F_3] = -F_+, \quad [H_-, F_3] = F_-, \\
&[H_+, F_-] = -[H_-, F_+] = 2F_3, \\
&[F_+, H_3] = -F_+, \quad [F_-, H_3] = F_-, \\
&[F_+, F_3] = H_+, \quad [F_-, F_3] = -H_-, \\
&[F_+, F_-] = -2H_3.
\end{aligned} \tag{D4}$$

An irreducible representation of the Lorentz group is characterized by two numbers (n, μ), where $n \in \mathbf{N}/2$ and $\mu \in \mathbf{C}$. In the space $\mathcal{H}_{n,\mu}$ of this representation one can introduce an orthonormal basis

$$\{\xi_{j,m}\}, \quad m = -j, -j+1, \ldots, j-1, j, \quad l = n, n+1, \ldots \tag{D5}$$

such that the generators introduced above act in the following way [31]:

$$\begin{aligned}
&H_3 \xi_{j,m} = m\xi_{j,m}, \\
&H_+ \xi_{j,m} = \sqrt{(j+m+1)(j-m)}\,\xi_{j,m+1}, \\
&H_- \xi_{j,m} = \sqrt{(j+m)(j-m+1)}\,\xi_{j,m-1},
\end{aligned} \tag{D6}$$

$$\begin{aligned}
F_3 \xi_{j,m} = {} & \gamma_{(j)} \sqrt{j^2 - m^2}\,\xi_{j-1,m} + \beta_{(j)} m \xi_{j,m} \\
& - \gamma_{(j+1)} \sqrt{(j+1)^2 - m^2}\,\xi_{j+1,m}, \\
F_+ \xi_{j,m} = {} & \gamma_{(j)} \sqrt{(j-m)(j-m-1)}\,\xi_{j-1,m+1} \\
& + \beta_{(j)} \sqrt{(j-m)(j+m+1)}\,\xi_{j,m+1} \\
& + \gamma_{(j+1)} \sqrt{(j+m+1)(j+m+2)}\,\xi_{j+1,m+1}, \\
F_- \xi_{j,m} = {} & -\gamma_{(j)} \sqrt{(j+m)(j+m-1)}\,\xi_{j-1,m-1} \\
& + \beta_{(j)} \sqrt{(j+m)(j-m+1)}\,\xi_{j,m-1} \\
& - \gamma_{(j+1)} \sqrt{(j-m+1)(j-m+2)}\,\xi_{j+1,m-1},
\end{aligned} \tag{D7}$$

where

$$\beta_{(j)}=-\frac{in\mu}{j(j+1)}, \quad \gamma_{(j)}=\frac{i}{2j}\sqrt{\frac{(j^2-n^2)(j^2-\mu^2)}{\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)}}. \tag{D8}$$

The unitary representations correspond to two cases: (1) the principal series,

$$(n,\mu)=(n,i\rho), \quad n\in\mathbf{N}/2, \rho\in\mathbf{R} \tag{D9}$$

and (2) the supplementary series,

$$(n,\mu)=(0,\rho), \quad |\rho|<1, \ \rho\in\mathbf{R}. \tag{D10}$$

The principal series representations are the ones intervening in the Plancherel decomposition formula for L^2 functions over $SL(2,C)$. Simple representations are the representations of the principal series with the vanishing Casimir operator $C_2(sl(2,C))$ [22,23]. There are two types of such representations: $(n,0)$ and $(0,i\rho)$. In both cases we have $\beta_{(j)}=0$ for all j. However, the representations $(0,i\rho)$ have the particularity that they possess an $SU(2)$ invariant vector $\xi_{0,0}$.

APPENDIX E: CREATING A LORENTZ CONNECTION FROM AN $SU(2)$ CONNECTION

Let us suppose that we have a hypersurface with an $SU(2)$ connection a. Now, we would like to extend the $SU(2)$ structure to a $SL(2,C)$ one and the connection a to a $SL(2,C)$ connection A. For this purpose, we introduce on the hypersurface a time normal field $\chi\in\mathcal{H}_+=SL(2,C)/SU(2)$ valued on the (upper part of the two-sheet) hyperboloid. We choose a reference vector χ_0 on the hyperboloid and define $x\in SL(2,C)$ rotating from χ_0 to χ. Let us define the isomorphism

$$i_\chi : su(2)\to su_\chi(2) \tag{E1}$$

and the projection

$$I_\chi : sl(2,C)\to su_\chi(2). \tag{E2}$$

Then, we would like to define the Lorentz connection so that $I_\chi A=i_\chi a$. Let $A=i_\chi a+b$. Action of a gauge transformation $g\in SL(2,C)$ is defined by splitting g into its $SU_\chi(2)$ part acting with the initial $SU(2)$ action on a and its pure boost part acting on the field χ. Good behavior of A under gauge transformation gives the following conditions on b:

$$\forall g\in SU_\chi(2), {}^g b=gbg^{-1}+x(\partial x^{-1})+g(\partial x)x^{-1}g^{-1} \tag{E3}$$

and

$$\forall g \quad \text{pure boost}, \quad {}^g b=gbg^{-1}+g(\partial g^{-1}). \tag{E4}$$

A priori, there are many solutions to these equations. The simplest solution is $b=x(\partial x^{-1})$. Thus, this procedure allows us to create an $SL(2,C)$ connection from an $SU(2)$ one by introducing the vector field χ. It can be used to construct the covariant generalization of the Ashtekar-Barbero connection (5). Indeed, one can start from the $SU(2)$ loop gravity, with a hypersurface provided with an $SU(2)$ connection a (the Ashtekar-Barbero connection), and, following this procedure, one can try to make the formalism explicitly covariant. First, we introduce a time normal field χ (which allows us to go out of the time gauge) and then we construct $A=i_\chi a+x(\partial x^{-1})$, which gives rise to an $SL(2,C)$ structure with a covariant connection on the canonical hypersurface. It turns out that this result matches exactly what comes out of the direct covariant canonical analysis as described in Sec. II. Indeed, looking at Eq. (5), $i_\chi a$ is $A_i^{(\mathrm{ash})X}$ [and a is $A_i^{(\mathrm{ash})a}$ as in Eq. (B8)] and $b=x(\partial x^{-1})$ is to be identified with the function $\Lambda(\tilde{Q})$. However, it is easy to realize that this does not exactly match Eq. (5) since the latter involves a rotation from Λ to $R\Lambda$. This can be explained from an ambiguity in our embedding procdure, more precisely in the definition of boosts: if the boosts are taken to be $K+(1/\beta)J$ acting on both χ and a, then it is straightforward to check that the right solution to the gauge transformation equations is $b=R(x\partial x^{-1})$ and that we retrieve the correct R factor.

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