

Null energy conditions in quantum field theory

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For the quantized, massless, minimally coupled real scalar field in four-dimensional Minkowski space, we show (by an explicit construction) that weighted averages of the null-contracted stress-energy tensor along null geodesics are unbounded from below on the class of Hadamard states. Thus there are no quantum inequalities along null geodesics in four-dimensional Minkowski spacetime. This is in contrast with the case for two-dimensional flat spacetime, where such inequalities do exist. We discuss in detail the properties of the quantum states used in our analysis, and also show that the renormalized expectation value of the stress energy tensor evaluated in these states satisfies the averaged null energy condition (as expected), despite the nonexistence of a null-averaged quantum inequality. However, we also show that in any globally hyperbolic spacetime the null-contracted stress energy averaged over a *timelike* worldline *does* satisfy a quantum inequality bound (for both massive and massless fields). We comment briefly on the implications of our results for singularity theorems.

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I. INTRODUCTION

The field equations of general relativity have little or no predictive power in the absence of some notion of what metrics or stress-energy tensors are to be regarded as physically reasonable. In classical general relativity it has proved profitable to require the stress-energy tensor to satisfy one or more of the so-called energy conditions: in particular, the Hawking-Penrose singularity theorems [1,2] and the positive mass theorem [3,4] are proved under such assumptions.

The present paper addresses the status of the null energy condition (NEC) in quantum field theory. The classical NEC is the requirement that the stress-energy tensor T_{ab} should obey $T_{ab}\ell^a\ell^b \geq 0$ for all null vectors ℓ^a and at every spacetime point. Although this condition is satisfied by many (but not all [5,6]) classical matter models, including the minimally coupled scalar field and the electromagnetic field, it is known, however, that this condition is violated by quantum fields. In fact, the expectation $\langle T_{ab}^{\text{ren}}\ell^a\ell^b \rangle_\omega$ of the renormalized null-contracted stress-energy is unbounded from below as a function of the quantum state ω . Exactly the same phenomenon afflicts the weak energy condition (WEC) which, classically, requires that $T_{ab}v^av^b \geq 0$ for all timelike vectors v^a . In this case, it is known that the renormalized energy density is still subject to constraints on its averages along timelike curves. For example, the massless real scalar field in n -dimensional Minkowski space obeys [7,8]

$$\int dt \langle :T_{00}: \rangle_\omega(t, \mathbf{0}) g(t)^2 \geq -c_n \int du u^n |\hat{g}(u)|^2 \quad (1.1)$$

for all Hadamard states [9] ω and any smooth, real-valued compactly supported function g , where \hat{g} is the Fourier trans-

form of g [see Eq. (2.3)], and the c_n are explicitly known constants independent of ω and g . Such constraints are known as *quantum weak energy inequalities* (QWEIs) and appear to be the vestiges of the weak energy condition in quantum field theory [10]. Over the past decade, QWEIs have been developed in a variety of circumstances [7,8,11–17] and are known to hold for the minimally coupled scalar field, the Dirac field and the electromagnetic and Proca fields in great generality [18–20]. The QWEIs also imply that the averaged weak energy condition (AWEC)

$$\int dt \langle :T_{00}: \rangle_\omega(t, \mathbf{0}) \geq 0 \quad (1.2)$$

holds at least for Hadamard states ω for which the integral converges absolutely [21].

It is natural to enquire whether similar vestiges of the NEC persist in quantum field theory. This is particularly relevant to attempts to generalize the singularity theorems to quantized matter fields as it is the NEC which is assumed in the Penrose theorem [1]. While the final stages of gravitational collapse presumably require a full theory of quantum gravitation for their description, the early stages can certainly be treated within quantum field theory on a fixed curved spacetime. The question to be addressed is whether an initially contracting matter distribution will continue to do so and it is here that the NEC (or its variants) appears in the classical arguments. Accordingly, it is important to understand whether the NEC has an analogue in quantum field theory. For massless fields in two-dimensional Minkowski space, this question was answered affirmatively in Ref. [11], using a Lorentzian sampling function. The bound has the form

$$\frac{\lambda_0}{\pi} \int_{-\infty}^{\infty} d\lambda \frac{\langle :T_{ab}: l^a l^b \rangle_\omega(\gamma(\lambda))}{\lambda^2 + \lambda_0^2} \geq -\frac{1}{16\pi\lambda_0^2}, \quad (1.3)$$

for all $\lambda_0 > 0$ and a large class of states [22] ω , where γ is an affinely parametrized null geodesic with tangent vector ℓ^a

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$= (d\gamma/d\lambda)^a$. It can be easily seen that this bound is invariant under a rescaling of the affine parameter. If we now take the limit of Eq. (1.3) as $\lambda_0 \rightarrow \infty$, which corresponds to sampling the entire null geodesic, we get the ANEC [23]:

$$\int_{-\infty}^{\infty} d\lambda \langle :T_{ab} : \ell^a \ell^b \rangle_{\omega} \geq 0. \quad (1.4)$$

Reference [11] left open the question of whether an analogous QNEI exists in spacetime dimensions other than two. The techniques used there to obtain a timelike world line QI in four dimensions could not be employed to derive a similar QNEI, starting with null geodesics *ab initio*, because the former derivation was based upon a mode expansion in the timelike observer's rest frame. There are also technical problems which obstruct the adaptation of the arguments of Ref. [18] to null world lines (see the remark following Theorem III.1). In addition, Ref. [11] noted a potential problem: any such inequality involving an average along a null geodesic would have to be invariant under rescaling of the affine parameter [amounting to the replacements $\lambda \mapsto \lambda/\sigma$, $\lambda_0 \mapsto \lambda_0/\sigma$ and $\ell^a \mapsto \sigma \ell^a$ in Eq. (1.3)] to be physically meaningful. While the left-hand side of Eq. (1.3) scales as σ^2 , one might expect (on dimensional grounds) that the right-hand side of such a bound would behave like λ_0^{-d} , where d is the spacetime dimension, and therefore scale as σ^d . This hints that the extension of QNEIs to spacetime dimensions $d > 2$ might be problematic. (Of course, these arguments would not apply in the presence of a mass or some other geometrical length scale—see Ref. [24].)

In this paper, we consider world line averages of the null-contracted stress-energy tensor of the form

$$\langle \rho(f) \rangle_{\omega} = \int d\lambda \langle :T_{ab} : \ell^a \ell^b \rangle_{\omega}(\gamma(\lambda)), \quad (1.5)$$

where $\gamma(\lambda)$ is a smooth causal curve and ℓ^a is a smooth null vector field defined on γ . First, in Sec. II we study the case in which γ is an affinely parametrized null geodesic in four-dimensional Minkowski space with tangent vector $\ell^a = (d\gamma/d\lambda)^a$. By an explicit construction, we show that $\langle \rho(f) \rangle_{\omega}$ is unbounded from below as ω varies among the class of Hadamard states of the massless minimally coupled scalar field. Thus there are no null-world-line QNEIs in four-dimensional Minkowski space. Although we consider only the massless field, we comment that our results generalize directly to the massive case. Our construction involves a sequence of states, each of which is a superposition of the vacuum with a multimode two-particle state. A closely related construction has recently been used in Ref. [25] to prove the nonexistence of spatially averaged quantum inequalities in four-dimensional Minkowski space.

It would be incorrect, however, to conclude from the above result that the null-contracted stress-energy tensor is completely unconstrained in quantum field theory. In Sec. III we consider the averages $\langle \rho(f) \rangle_{\omega}$ for smooth timelike γ in an arbitrary globally hyperbolic spacetime and for any smooth null vector field ℓ^a . For both massive and massless fields, these quantities do obey lower bounds—which we call

timelike world line QNEIs—as a direct consequence of the arguments used in Ref. [18]. We evaluate our bound explicitly for the case of four-dimensional Minkowski space. Taken together with the results of Sec. II, we see that large negative values of the null-contracted stress-energy tensor on one null geodesic must be compensated by positive values on neighboring geodesics, because the transverse extent of the negative values is constrained by timelike world line QNEIs. In the conclusion, we briefly speculate on the significance of these results for attempts to derive singularity theorems for quantized matter.

II. NONEXISTENCE OF NULL-WORLD-LINE QNEIs

A. Nonexistence result

We consider a massless minimally coupled real scalar field in (1+3)-dimensional Minkowski space, with signature $+- - -$. We employ units with $\hbar = c = 1$. The quantum field is given by

$$\Phi(x) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3 (2\omega)^{1/2}} (a(\mathbf{k}) e^{-ik_a x^a} + a^\dagger(\mathbf{k}) e^{ik_a x^a}), \quad (2.1)$$

in which $k^a = (\omega, \mathbf{k})$ with $\omega = \|\mathbf{k}\|$, the magnitude of \mathbf{k} . The canonical commutation relations are

$$\begin{aligned} [a(\mathbf{k}), a(\mathbf{k}')] &= 0, \\ [a(\mathbf{k}), a^\dagger(\mathbf{k}')] &= (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), \end{aligned} \quad (2.2)$$

and our convention for Fourier transformation is

$$\hat{f}(u) = \int dt e^{-iut} f(t). \quad (2.3)$$

Now let f be any smooth nonnegative function of compact support, normalized so that

$$\int d\lambda f(\lambda) = 1, \quad (2.4)$$

and, for some fixed future-pointing null vector ℓ^a , let $\gamma(\lambda)$ be the null geodesic $\gamma(\lambda)^a = \lambda \ell^a$. For simplicity, we will assume that the three-vector part of ℓ^a has unit length (in our frame of reference), so $\ell^a = (1, \ell)$ with $\|\ell\| = 1$. We will consider the averaged quantity

$$\langle \rho(f) \rangle_{\omega} = \int d\lambda f(\lambda) \langle :T_{ab} : \ell^a \ell^b \rangle_{\omega}(\gamma(\lambda)), \quad (2.5)$$

which corresponds to a weighted average of the null-contracted stress energy tensor along γ . If ω is a Hadamard state, the renormalized contracted stress tensor is a smooth function on spacetime, so the above integral will certainly converge. In order to establish a quantum null energy inequality, one would need to bound $\langle \rho(f) \rangle_{\omega}$ from below as ω ranges over the class of Hadamard states; however, this is not possible, as we now show.

Theorem II.1. *The quantity $\langle \rho(f) \rangle_{\omega}$ is unbounded from below as ω varies over the class of Hadamard states.*

Proof. We will construct a family of vector Hadamard states ω_α [$\alpha \in (0,1)$] with the property that $\langle \rho(f) \rangle_{\omega_\alpha} \rightarrow -\infty$ as $\alpha \rightarrow 0$. We begin by choosing a fixed $\Lambda_0 > 0$ such that $\text{Re} \hat{f}$ is non-negative on the interval $[-2\Lambda_0, 2\Lambda_0]$. To see that this is possible, we observe that

$$\hat{f}(0) = \int d\lambda f(\lambda) = 1, \quad (2.6)$$

which, by continuity, implies that $\text{Re} \hat{f}$ is positive in some neighborhood of the origin.

Next, let σ and ν be fixed positive numbers with $2\nu + 3/2 < \sigma < 2\nu + 2$. For each $\alpha \in (0,1)$, we define a “vacuum-plus-two-particle” vector

$$\psi_\alpha = N_\alpha \left[|0\rangle + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} b_\alpha(\mathbf{k}, \mathbf{k}') |\mathbf{k}, \mathbf{k}'\rangle \right], \quad (2.7)$$

where N_α is a normalization constant ensuring $\|\psi_\alpha\| = 1$ and

$$b_\alpha(\mathbf{k}, \mathbf{k}') = \alpha^\sigma \vartheta(\Lambda - k) \vartheta(\Lambda - k') \\ \times \chi_\alpha(\theta) \chi_\alpha(\theta') B(k_a \ell^a, k'_a \ell^a) (kk')^{\nu-1/2}. \quad (2.8)$$

Here, ϑ is the usual Heaviside step function, $\Lambda = \Lambda_0/\alpha$ will be called the momentum cutoff and

$$\chi_\alpha(\theta) = \begin{cases} 1, & \cos \theta > 1 - \alpha, \\ 0, & \text{otherwise,} \end{cases} \quad (2.9)$$

where θ (respectively, θ') is the angle between \mathbf{k} (respectively, \mathbf{k}') and ℓ . We choose $B: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ to be (a) symmetric [i.e., $B(u, u') = B(u', u)$]; (b) jointly continuous in u and u' ; (c) everywhere non-negative [26] and strictly positive near $u = u' = 0$; and (d) normalized so that

$$\Lambda_0^{4\nu+2} \int_0^\infty du \int_0^\infty du' |B(u, u')|^2 = 1. \quad (2.10)$$

(The prefactor ensures dimensional consistency.) An example of a function meeting these requirements is $B(u, u') = \Lambda_0^{-(2\nu+1)} e^{-(u+u')/2}$. We wish to emphasize, however, that there are many functions (and hence many vectors ψ_α) with the properties we require. We will use ω_α to denote the state induced by ψ_α so that $\langle A \rangle_{\omega_\alpha} = \langle \psi_\alpha | A | \psi_\alpha \rangle$.

Let us note various features of this family of states. First, the momentum cutoff ensures that no modes of momentum greater than $\Lambda = \Lambda_0/\alpha$ are excited. Second, the effect of the χ_α factors is to ensure that modes can only be excited if their three-momenta make an angle less than $\cos^{-1}(1-\alpha)$ with the direction ℓ . The excited mode three-momenta therefore lie in the solid sector formed by the intersection of a ball of radius Λ_0/α (center the origin) with a cone of opening angle $\cos^{-1}(1-\alpha)$ about ℓ (with apex at the origin). As $\alpha \rightarrow 0$, this solid sector lengthens and tightens up along the direction ℓ , so the four-momenta of excited modes become more and more parallel to ℓ^a , the tangent vector to the null line along which we are averaging. See Fig. 1.

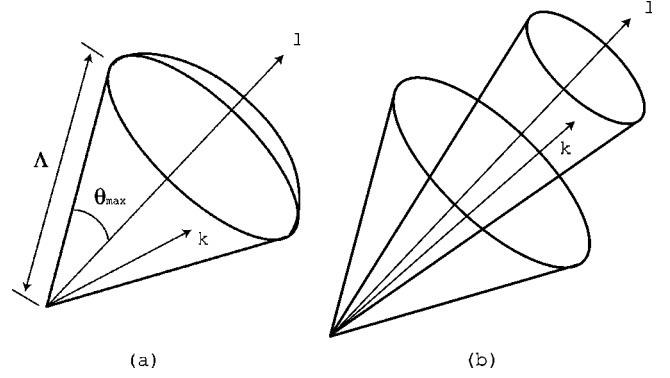


FIG. 1. (a) Only modes lying “inside the cone” [i.e., those whose three-momenta make an angle less than $\theta_{\max} = \cos^{-1}(1-\alpha)$ with ℓ] are excited; (b) the cones lengthen and tighten around ℓ as $\alpha \rightarrow 0$.

The third feature of interest concerns the amplitude of the two-particle contribution. Choosing the normalization constant to be

$$N_\alpha = \left[1 + 2 \int \frac{d^2\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} |b_\alpha(\mathbf{k}, \mathbf{k}')|^2 \right]^{-1/2}, \quad (2.11)$$

we note that, for a null vector $k^a = (k, \mathbf{k})$, the quantity $\ell^a k_a$ appearing in Eq. (2.8) is equal to $k(1 - \cos \theta)$, where θ is the angle between ℓ and \mathbf{k} . We therefore perform the \mathbf{k} and \mathbf{k}' integrals in Eq. (2.11) by adopting spherical polar coordinates about ℓ , integrating out the trivial azimuthal dependence and then changing variables to $\beta = 1 - \cos \theta$, $\beta' = 1 - \cos \theta'$. This yields

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} |b_\alpha(\mathbf{k}, \mathbf{k}')|^2 \\ = \frac{\alpha^{2\sigma}}{(2\pi)^4} \int_0^\Lambda dk \int_0^\Lambda dk' (kk')^{2\nu+1} \\ \times \int_0^\alpha d\beta \int_0^\alpha d\beta' |B(k\beta, k'\beta')|^2 \\ = \frac{\alpha^{2\sigma-2-4\nu}}{(2\pi)^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' (vv')^{2\nu} \\ \times \int_0^v du \int_0^{v'} du' |B(u, u')|^2 \\ \leq \frac{\alpha^{2(\sigma-2\nu-1)}}{(2\pi)^4 (2\nu+1)^2}, \quad (2.12)$$

where we have made the further changes of variable $u = k\beta$, $u' = k'\beta'$, $v = k\alpha$, $v' = k'\alpha$ and used the normalization property Eq. (2.10) of B . Because $\sigma > 2\nu + 3/2 > 2\nu + 1$, we see that the right-hand side of Eq. (2.12) tends to zero as $\alpha \rightarrow 0$. By Eq. (2.11) we now have $N_\alpha \rightarrow 1$ as $\alpha \rightarrow 0$; since the left-hand side of Eq. (2.12) is equal to $\|N_\alpha^{-1} \psi_\alpha - |0\rangle\|^2$, we also see that the states ψ_α are in fact converging to the vacuum vector $|0\rangle$. As we shall see, this

does not entail that the normal-ordered energy density is converging to zero. (See also the discussion in Sec. II C.)

The remaining properties of our family of states concern the corresponding normal ordered two-point functions, given by

$$\begin{aligned} \langle : \Phi(x) \Phi(x') : \rangle_{\omega_\alpha} &= 2N_\alpha^2 \operatorname{Re} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{1}{\sqrt{\omega \omega'}} \\ &\quad \times [c_\alpha(\mathbf{k}, \mathbf{k}') e^{i(x^a k_a - x'^a k'_a)} \\ &\quad + b_\alpha(\mathbf{k}, \mathbf{k}') e^{-i(x^a k_a + x'^a k'_a)}], \end{aligned} \quad (2.13)$$

where

$$c_\alpha(\mathbf{k}, \mathbf{k}') = 2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \overline{b_\alpha(\mathbf{k}_1, \mathbf{k})} b_\alpha(\mathbf{k}_1, \mathbf{k}'). \quad (2.14)$$

Using the same changes of variable as above, we find

$$\begin{aligned} c_\alpha(\mathbf{k}, \mathbf{k}') &= \alpha^{2\sigma - (2\nu+1)} \vartheta(\Lambda - k) \vartheta(\Lambda - k') \\ &\quad \times \chi_\alpha(\theta) \chi_\alpha(\theta') C(k_\alpha \ell^a, k'_\alpha \ell^a) (kk')^{\nu-1/2}, \end{aligned} \quad (2.15)$$

where $C(u, u') = C(u', u)$ is given by

$$\begin{aligned} C(u, u') &= 2\alpha^{2\nu+1} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \vartheta(\Lambda - k_1) \chi_\alpha(\theta_1) \\ &\quad \times (k_1^2)^{\nu-1/2} \overline{B(k_1^a \ell_a, u)} B(k_1^a \ell_a, u') \\ &= \frac{\alpha^{2\nu+1}}{2\pi^2} \int_0^\Lambda dk_1 k_1^{2\nu+1} \int_0^\alpha d\beta B(k_1 \beta_1, u) \\ &\quad \times B(k_1 \beta_1, u') \\ &= \frac{1}{2\pi^2} \int_0^{\Lambda_0} dv v^{2\nu} \int_0^v du_1 \overline{B(u_1, u)} B(u_1, u'). \end{aligned} \quad (2.16)$$

Rearranging the order of integration, this becomes

$$C(u, u') = \frac{1}{2\pi^2(2\nu+1)} \int_0^{\Lambda_0} du_1 u_1^{2\nu+1} \overline{B(u_1, u)} B(u_1, u') \quad (2.17)$$

and we may conclude that (i) C is jointly continuous in u and u' by joint continuity of B and compactness of $[0, \Lambda_0]$; (ii) C has the same engineering dimension as B ; (iii) $C(u, u') \geq 0$ for all u, u' and, crucially, (iv) that the exponent of α in $c_\alpha(\mathbf{k}, \mathbf{k}')$ differs from that in the corresponding expression for $b_\alpha(\mathbf{k}, \mathbf{k}')$. Furthermore, since both b_α and c_α have momentum cutoffs, it is evident that the normal ordered two-point function is smooth (because one may differentiate under the integral sign as often as required to obtain finite derivatives). Accordingly, each ω_α is a Hadamard state.

The null-contracted normal ordered energy density $\langle \rho(f) \rangle_{\omega_\alpha}$ is obtained by differentiating the normal ordered two-point function

$$\begin{aligned} \langle : T_{ab}(x) : \ell^a \ell^b \rangle_{\omega_\alpha} &= \langle : \ell^a \nabla_a \Phi(x) \ell^b \nabla_b \Phi(x) : \rangle_{\omega_\alpha} \\ &= 2N_\alpha^2 \operatorname{Re} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{\ell^a k_a \ell^b k'_b}{\sqrt{\omega \omega'}} \\ &\quad \times [c_\alpha(\mathbf{k}, \mathbf{k}') e^{ix^a(k_a - k'_a)} \\ &\quad - b_\alpha(\mathbf{k}, \mathbf{k}') e^{-ix^a(k_a + k'_a)}], \end{aligned} \quad (2.18)$$

and substituting into Eq. (2.5). Noting, for any K_a , that

$$\int d\lambda f(\lambda) e^{-i\gamma(\lambda)^a K_a} = \hat{f}(\ell^a K_a), \quad (2.19)$$

we may write $\langle \rho(f) \rangle_{\omega_\alpha} = \rho_1(f) + \rho_2(f)$, where

$$\begin{aligned} \rho_1(f) &= 2N_\alpha^2 \operatorname{Re} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{\ell^a k_a \ell^b k'_b}{\sqrt{\omega \omega'}} \\ &\quad \times \hat{f}(\ell^a k'_a - \ell^a k_a) c_\alpha(\mathbf{k}, \mathbf{k}') \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} \rho_2(f) &= -2N_\alpha^2 \operatorname{Re} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{d^3 \mathbf{k}'}{(2\pi)^3} \frac{\ell^a k_a \ell^b k'_b}{\sqrt{\omega \omega'}} \\ &\quad \times \hat{f}(\ell^a k_a + \ell^a k'_a) b_\alpha(\mathbf{k}, \mathbf{k}'). \end{aligned} \quad (2.21)$$

Our aim is now to show that $\rho_1(f) \rightarrow 0$ and $\rho_2(f) \rightarrow -\infty$ in the limit $\alpha \rightarrow 0$. Taking the dominant contribution $\rho_2(f)$ first, and making the same changes of variable as before we may calculate

$$\begin{aligned} \rho_2(f) &= -\frac{N_\alpha^2 \alpha^\sigma}{(2\pi)^4} \operatorname{Re} \int_0^\Lambda dk \int_0^\Lambda dk' (kk')^{\nu+2} \int_0^\alpha d\beta \\ &\quad \times \int_0^\alpha d\beta' \beta \beta' B(k\beta, k'\beta') \hat{f}(k\beta + k'\beta') \\ &= -\frac{N_\alpha^2 \alpha^\sigma}{(2\pi)^4} \int_0^\Lambda dk \int_0^\Lambda dk' (kk')^\nu \varphi(k\alpha, k'\alpha) \\ &= -\frac{N_\alpha^2 \alpha^{\sigma-2(\nu+1)}}{(2\pi)^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' (vv')^\nu \varphi(v, v'), \end{aligned} \quad (2.22)$$

where

$$\varphi(v, v') = \operatorname{Re} \int_0^v du \int_0^{v'} du' uu' B(u, u') \hat{f}(u+u'). \quad (2.23)$$

Recalling that $\operatorname{Re} \hat{f}$ is strictly positive on the interval $[-2\Lambda_0, 2\Lambda_0]$, and that $B(u, u')$ is non-negative and strictly positive for small u, u' , it follows that $\varphi(v, v')$ is non-negative for $v, v' \in [0, \Lambda_0]$ and that the right-hand side of Eq. (2.22) is strictly negative. We therefore have $\rho_2(f) \rightarrow -\infty$ in the limit $\alpha \rightarrow 0$, because $\sigma < 2\nu + 2$ and $N_\alpha \rightarrow 1$.

Turning to the remaining contribution $\rho_1(f)$, we may use a similar analysis to obtain

$$\rho_1(f) = \frac{N_\alpha^2 \alpha^{2\sigma - (4\nu + 3)}}{(2\pi)^4} \int_0^{\Lambda_0} dv \int_0^{\Lambda_0} dv' (vv')^\nu \psi(v, v'), \quad (2.24)$$

where

$$\psi(v, v') = \text{Re} \int_0^v du \int_0^{v'} du' uu' C(u, u') \hat{f}(u' - u). \quad (2.25)$$

Since C is jointly continuous, the double integrals in Eqs. (2.24) and (2.25) exist and are finite; we may now conclude that $\rho_1(f) \rightarrow 0$ in the limit $\alpha \rightarrow 0$, because $\sigma > 2\nu + 3/2$ and $N_\alpha \rightarrow 1$.

Summarizing, we have shown that $\langle \rho(f) \rangle_{\omega_\alpha} \rightarrow -\infty$ as $\alpha \rightarrow 0$. ■

At this point it is worth considering the difference between the present situation and that studied in Ref. [11], in which a QNEI was obtained for massless fields in two-dimensional Minkowski space. The crucial difference is that, in two-dimensional spacetime, the one-momenta of any two field modes are either parallel or antiparallel. The modes which propagate in the same direction as the chosen null geodesic contribute nothing to the integral (due to factors of the form $\ell^a k_a$). The only contribution comes from modes moving in the opposite direction, and turns out to be bounded. By contrast, the proof of Theorem II.1 makes essential use of field modes which are almost, but not exactly, parallel to ℓ^a .

B. An explicit calculation

To make the foregoing result more explicit, we consider a simple example: setting $\nu = 1$, we define

$$B(u, u') = \Lambda_0^{-4} \vartheta(\Lambda_0 - u) \vartheta(\Lambda_0 - u'), \quad (2.26)$$

which satisfies properties (a), (c) and (d) required in the proof of Theorem II.1, but not the joint continuity property (b). Inspection of the proof reveals, however, that this property was only used to establish the existence of certain integrals arising in the derivation, all of which may easily be seen to exist in this case.

The calculations are simplified by the fact that $B(u, u')$ factorizes into functions of u and u' . In particular, one may calculate

$$C(u, u') = \frac{1}{24\pi^2 \Lambda_0^4} \vartheta(\Lambda_0 - u) \vartheta(\Lambda_0 - u'), \quad (2.27)$$

and

$$N_\alpha = \left(1 + \frac{\alpha^{2\sigma - 6}}{128\pi^4} \right)^{-1/2}. \quad (2.28)$$

Furthermore, because B is also real-valued,

$$\langle \rho(x) \rangle_{\omega_\alpha} = \frac{2N_\alpha^2}{\Lambda_0^4} \text{Re} \left(|F_\alpha(x)|^2 \frac{\alpha^{2\sigma - 3}}{24\pi^2} - F_\alpha(x)^2 \alpha^\sigma \right), \quad (2.29)$$

where here $\rho(x)$ represents the *unsampled* energy density at position x , and

$$F_\alpha(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{\ell^a k_a}{k^{1/2}} \vartheta(\Lambda - k) \chi_\alpha(\theta) k^{\nu - 1/2} e^{-ik_\alpha x^a}. \quad (2.30)$$

In Fig. 2, we plot $\langle \rho(t, 0, 0, z) \rangle_{\omega_\alpha}$ for the case $\ell^a = (1, 0, 0, 1)$, in which Eq. (2.29) simplifies to

$$\langle \rho(t, 0, 0, z) \rangle_{\omega_\alpha} = \frac{2N_\alpha^2}{16\pi^4 \Lambda_0^4} \text{Re} \left(|\xi_\alpha(t, z)|^2 \frac{\alpha^{2\sigma - 7}}{24\pi^2} - \xi_\alpha(t, z)^2 \alpha^{\sigma - 4} \right), \quad (2.31)$$

where

$$\xi_\alpha(t, z) = \int_0^{\Lambda_0} dv e^{-iv(t-z)/\alpha} \left[\frac{iv^2}{z} e^{-ivz} + \frac{v}{z^2} (e^{-ivz} - 1) \right]. \quad (2.32)$$

These plots share the common feature of an oscillatory fringe pattern, with dark regions representing negative values for $\langle \rho(t, 0, 0, z) \rangle_{\omega_\alpha}$ and light regions representing positive values. It is no coincidence that these plots resemble interference patterns: the dominant contribution arises precisely from interference between the vacuum and two-particle components of ψ_α . The fringes are centered near the null ray parallel to ℓ^a running from the lower left to upper right corners of the figures, and in fact point along spacelike directions which become more parallel to ℓ^a as α is decreased (moving from the left-hand to the right-hand figure in each row). That these directions cannot be timelike follows from the existence of the timelike world line QNEIs discussed in Sec. III—an observer cannot “surf” along a negative energy trough for an indefinite length of time. See Ref. [27] for similar examples and discussion. The decreasing fringe separation (as either α decreases or Λ_0 increases) indicates a more highly oscillatory energy density.

Further insight may be gained from Fig. 3, in which we plot $\langle :T_{ab} \ell^a \ell^b : \rangle_{\omega_\alpha}$ along (a) the null line $(\lambda, 0, 0, \lambda)$ and (b) the timelike line $(t, 0, 0, 0)$. Along the null line, the effect of decreasing α is essentially to modify the amplitude of the curve while leaving its shape substantially unaltered. For a sampling function f supported within the central trough, it is clear that $\langle \rho(f) \rangle_{\omega_\alpha} \rightarrow -\infty$ as $\alpha \rightarrow 0$, in accordance with Theorem II.1. Along the timelike curve, however, decreasing α increases both the amplitude and frequency of the oscillations. Averaged against a fixed sampling function, one might expect that the rapid oscillations would tend to cancel, so that the averages $\int f(t) \langle \rho(t, 0, 0, 0) \rangle_{\omega_\alpha}$ could be bounded below. This is borne out by the results of Sec. III below.

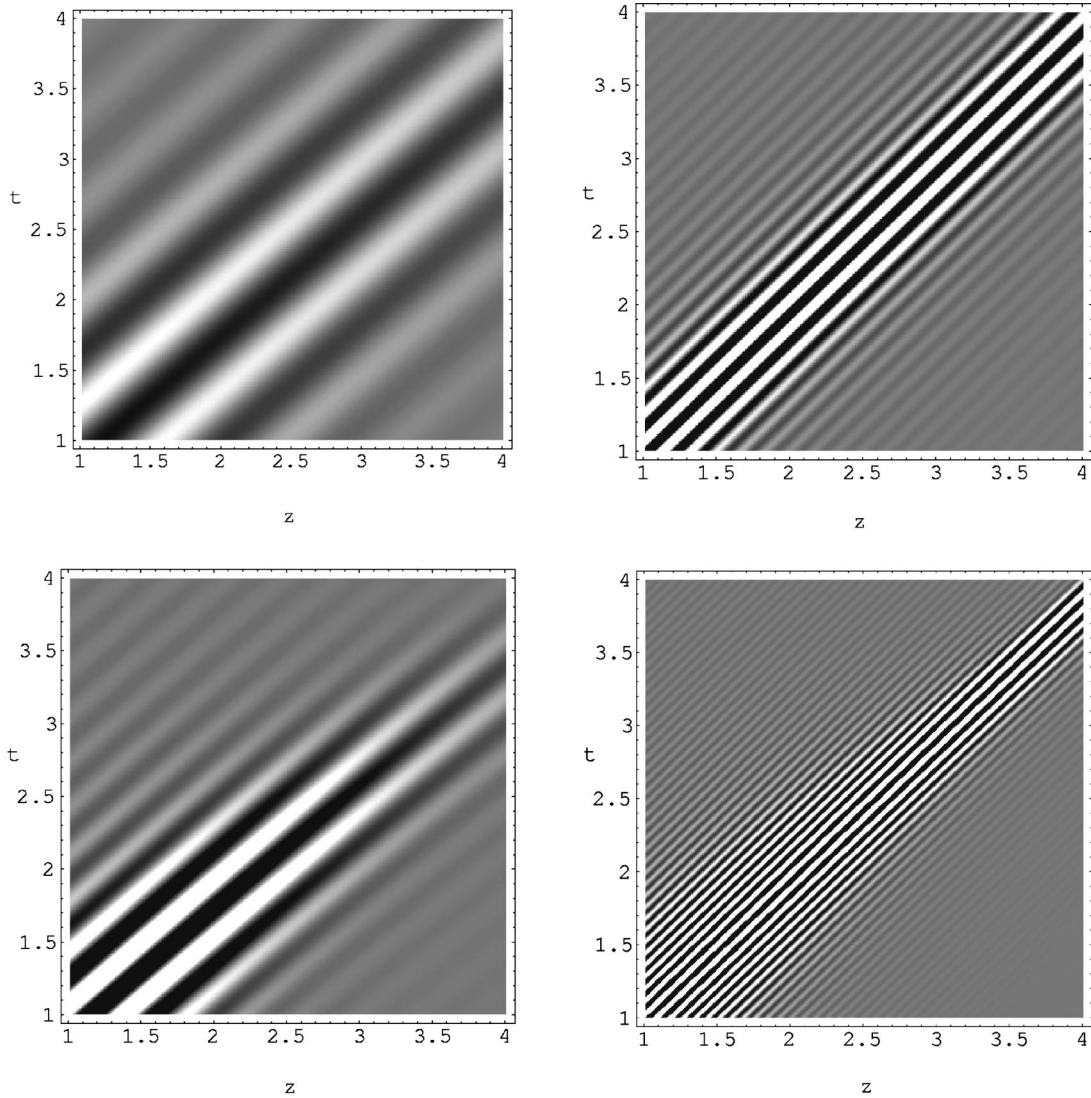


FIG. 2. Density plots of $\langle \rho(t,0,0,z) \rangle_{\omega_\alpha}$ for four different parameter choices. The top row corresponds to $\Lambda_0=1$ while the lower row has $\Lambda_0=2$; the left-hand column has $\alpha=0.2$, while the right-hand column has $\alpha=0.05$. In all four plots, $\sigma=3.75$. Dark and light areas represent negative and positive values, respectively.

The behavior of the energy density in the vicinity of our chosen null geodesic, e.g., as exhibited in Fig. 2, is almost exactly analogous to that found in the analysis of Ref. [25] for spatially averaged QIs, in the following sense. There, it was shown that the sampled energy density could be unboundedly negative in a spatially compact region on a $t = \text{const}$ surface, in four-dimensional Minkowski spacetime. However, for the ordinary world line QIs to hold, the energy density must fluctuate wildly as one moves off the $t = \text{const}$ surface. In the present paper, we find a similar result for null rays. For example, the central trough in Fig. 3(a) is a compactly supported region of the null geodesic, analogous to the compactly supported spatially sampled region considered in Ref. [25], where the energy density can be made unboundedly negative. (Of course, in the null case we are considering a one-dimensional average along a line, as opposed to a three-dimensional spatial average.) As we move off the null geodesic, as shown in Fig. 3(b) and Fig. 2, the energy density oscillates rapidly in sign, which must happen if the world

line QIs are to be satisfied. We speculate that a rotation of the plots in Fig. 2, which makes the white and dark lines horizontal, would yield a representative picture of the behavior in the spatial case. The null and spatial cases seem intuitively to be very similar.

C. Convergence to the vacuum state

We have seen that the massless scalar field in four-dimensional Minkowski spacetime does not satisfy nontrivial null world line quantum inequalities. As described above, this was shown by considering a sequence of vacuum-plus-two-particle states in which the three-momenta of excited modes become more and more parallel to the spatial part ℓ of the null vector ℓ^a as we take the momentum cutoff to infinity. A perhaps puzzling feature of our sequence is that it converges in Fock space to the vacuum vector. How, then, can the energy density diverge?

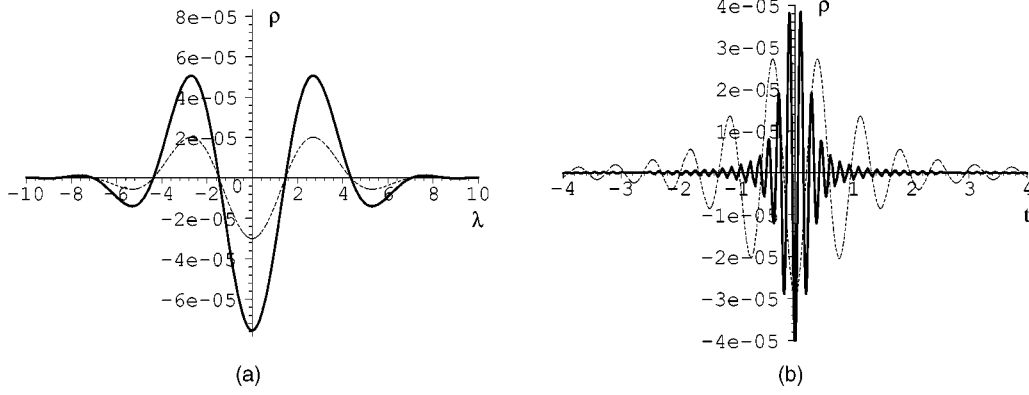


FIG. 3. Plots of the null-contracted stress-energy along (a) the null geodesic $\lambda \mapsto \langle \rho(\lambda, 0, 0, \lambda) \rangle_{\omega_\alpha}$ for $\alpha=0.2$ (dotted) and $\alpha=0.005$ (solid); (b) the timelike curve $t \mapsto \langle \rho(t, 0, 0, 0) \rangle_{\omega_\alpha}$ for $\alpha=0.2$ (dotted) and $\alpha=0.05$ (solid). In all cases, $\Lambda_0=1$, $\sigma=3.75$.

The answer to this question resides in the fact that the averaged energy density is an unbounded quadratic form, so the convergence of a sequence of states in the Hilbert space norm does not imply the convergence of the corresponding expectation values. As a more familiar example, consider the quantum mechanics of a single harmonic oscillator with angular frequency ω . Let

$$\phi_n = |0\rangle + n^{-1/4}|n\rangle, \quad (2.33)$$

where $n=1,2,3,\dots$ and $|n\rangle$ is a normalized eigenstate of energy $\hbar\omega(n+1/2)$. Noting that $\|\phi_n\|^2 = 1+n^{-1/2}$, the expected energy is

$$\langle H \rangle_{\phi_n} = \frac{1}{2} \hbar\omega \frac{1+n^{-1/2}(2n+1)}{1+n^{-1/2}} = \left[n^{1/2} - \frac{1}{2} + O(n^{-1/2}) \right] \hbar\omega \quad (2.34)$$

and therefore diverges as $n \rightarrow \infty$, while ϕ_n manifestly converges to the ground state $|0\rangle$, because $\|\phi_n - |0\rangle\| = n^{-1/4} \rightarrow 0$.

D. Consistency with the ANEC

Although, as we have seen, null world line QNEIs do not exist, there is nonetheless a nontrivial restriction on the null-contracted stress energy, namely the averaged null energy condition (ANEC)

$$\int d\lambda \langle :T_{ab} : \ell^a \ell^b \rangle_{\omega}(\gamma(\lambda)) \geq 0 \quad (2.35)$$

established by Klinkhammer [28] at least for a dense set of states in the Fock space of the Minkowski vacuum, and by Wald and Yurtsever [29] for a large subclass of Hadamard states [30]. As a consistency check, we now show explicitly that each state ω_α obeys the ANEC, regarded as the requirement that

$$\liminf_{\lambda_0 \rightarrow +\infty} \frac{1}{f(0)} \int d\lambda f(\lambda/\lambda_0) \langle :T_{ab} : \ell^a \ell^b \rangle_{\omega_\alpha}(\gamma(\lambda)) \geq 0 \quad (2.36)$$

for any f satisfying the hypotheses stated above Theorem II.1, and $f(0) \neq 0$. Now $\lambda \mapsto f(\lambda/\lambda_0)$ has Fourier transform $v \mapsto \lambda_0 \hat{f}(\lambda_0 v)$, which converges to $2\pi f(0) \delta(v)$ as $\lambda_0 \rightarrow \infty$. Replacing \hat{f} by this distribution in Eqs. (2.23) and (2.25), we see that, in the limit $\lambda_0 \rightarrow \infty$,

$$\begin{aligned} \varphi(v, v') &\rightarrow 2\pi f(0) \operatorname{Re} \int_0^v du \int_0^{v'} du' uu' B(u, u') \delta(u+u') \\ &= 0, \end{aligned} \quad (2.37)$$

while

$$\begin{aligned} \psi(v, v') &\rightarrow 2\pi f(0) \operatorname{Re} \int_0^v du \int_0^{v'} du' uu' C(u, u') \delta(u-u') \\ &= 2\pi f(0) \int_0^{\min\{v, v'\}} du u^2 C(u, u) \geq 0, \end{aligned} \quad (2.38)$$

from which Eq. (2.36) follows. This may also be confirmed by a more careful analysis.

At this point, we take the opportunity to clarify an issue relating to the derivation of the ANEC given in Ref. [11], in which it was suggested that (in Minkowski space) the ANEC could be derived by first taking the infinite sampling time limit of the QWEI to obtain the AWEC, and then taking the null limit to conclude that the ANEC holds. However, the following example shows that the second step cannot be accomplished without further assumptions: define a function $h(z)$ such that $h(z)$ equals $+1$ for $|z| > 2$, and -1 for $|z| < 1$ with $h(z)$ otherwise smooth and bounded between ± 1 . Setting $t^a = (1, \mathbf{0})$,

$$T_{ab}(x) = t_a t_b h(x^c x_c) \quad (2.39)$$

is a symmetric tensor which satisfies the AWEC along any timelike geodesic, but fails to satisfy the ANEC along any null generator of the light cone at the origin. See Fig. 4. Although it is not clear to us whether a conserved tensor field could display this behavior, our example shows—even in Minkowski space—that the ANEC cannot be obtained from the AWEC without more assumptions than used in Ref. [11].

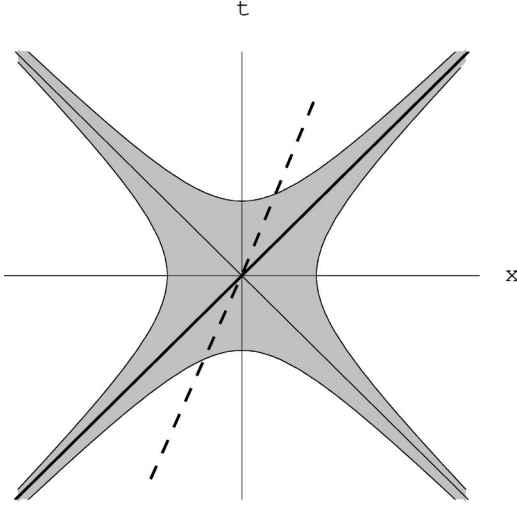


FIG. 4. An example to illustrate the distinction between the AWEC and the ANEC. The shaded region consists of spacetime points x with $h(x^a x_a) < 0$. (Only one spatial dimension is shown.) The tensor T_{ab} obeys the AWEC along any timelike geodesic (e.g., the dotted line) but fails to obey the ANEC on any null geodesic through the origin (e.g., the solid line).

III. TIMELIKE WORLD LINE QNEIs

Theorem II.1 may appear to suggest that null-contracted stress energy tensors are not subject to any constraints in quantum field theory. This is by no means the case. Let (M, \mathbf{g}) be any globally hyperbolic spacetime and ℓ^a a smooth null vector field defined on a tubular neighborhood of a smooth timelike curve γ , parametrized by its proper time τ . Let ω_0 be any Hadamard state of the Klein-Gordon field Φ of mass $m \geq 0$.

Theorem III.1. For any smooth, real-valued, compactly supported function g , the inequality [31]

$$\int d\tau \langle T_{ab} : \ell^a \ell^b \rangle_{\omega}(\gamma(\tau)) g(\tau)^2 \geq - \int_0^{\infty} \frac{d\alpha}{\pi} \hat{F}(\alpha, -\alpha) \quad (3.1)$$

holds for all Hadamard states ω of the Klein-Gordon field of mass m , where normal ordering is performed relative to the state ω_0 and

$$F(\tau, \tau') = g(\tau)g(\tau') \times \langle (\ell^a \nabla_a \Phi)(\gamma(\tau)) (\ell^{b'} \nabla_{b'} \Phi)(\gamma(\tau')) \rangle_{\omega_0}. \quad (3.2)$$

Remark. Because the differentiated two-point function

$$H(x, x') = \langle (\ell^a \nabla_a \Phi)(x) (\ell^{b'} \nabla_{b'} \Phi)(x') \rangle_{\omega_0} \quad (3.3)$$

is a distribution it is not clear *a priori* that one can restrict it to the curve γ as we have done in Eq. (3.2) [32]. Techniques drawn from microlocal analysis provide sufficient conditions for this to be accomplished, which are satisfied for timelike γ owing to the singularity properties of Hadamard states—see Ref. [18] for more details on this point. However, the suffi-

cient conditions would not be satisfied if γ was null, which explains why one cannot derive null world line QNEIs using the arguments of Ref. [18] (although this does not in itself demonstrate the nonexistence of such bounds).

Proof. The argument is identical to that used for the QWEI derived in Ref. [18], in which the averaged quantity was $\langle :T_{ab} : v^a v^b \rangle_{\omega}$ (where v^a is the tangent vector to γ). We refer to Ref. [18] for the details. ■

The above bound can be made more quantitative if we return to four-dimensional Minkowski space, with ω_0 chosen to be the Poincaré invariant vacuum, γ chosen to be the world line of an inertial observer with four-velocity v^a and with ℓ^a some constant null vector field. By Poincaré invariance we may write $\gamma(\tau) = (\tau, 0, 0, 0)$ without loss of generality; in this frame of reference, we write $\ell^a = (\ell^0, \ell)$, with $\ell^0 = v^a \ell_a$. We have

$$H(x, x') = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{(\ell^a k_a)^2}{2\omega} e^{-ik_a(x^a - x'^a)}, \quad (3.4)$$

from which it follows that

$$F(\tau, \tau') = g(\tau)g(\tau') \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{(\ell^a k_a)^2}{2\omega} e^{-i\omega(\tau - \tau')} \quad (3.5)$$

and

$$\begin{aligned} \hat{F}(\alpha, -\alpha) &= \int d\tau d\tau' \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{(\ell^a k_a)^2}{2\omega} e^{-i(\omega + \alpha)(\tau - \tau')} g(\tau)g(\tau') \\ &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{(\ell^a k_a)^2}{2\omega} \hat{g}(\omega + \alpha) \hat{g}(-\omega - \alpha) \\ &= \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{(\ell^a k_a)^2}{2\omega} |\hat{g}(\alpha + \omega)|^2, \end{aligned} \quad (3.6)$$

where we have used the fact that $\hat{g}(-u) = \overline{\hat{g}(u)}$ since g is real. Introducing polar coordinates about ℓ and changing variables from k to $\omega = \sqrt{k^2 + m^2}$, we have

$$\begin{aligned} \hat{F}(\alpha, -\alpha) &= \frac{(\ell^0)^2}{8\pi^2} \int_0^{\infty} dk \frac{k^2}{\omega} |\hat{g}(\alpha + \omega)|^2 \int_{-1}^1 d(\cos \theta) \\ &\quad \times (\omega - k \cos \theta)^2 \\ &= \frac{(\ell^0)^2}{12\pi^2} \int_0^{\infty} dk \frac{k^2}{\omega} |\hat{g}(\alpha + \omega)|^2 (3\omega^2 + k^2) \\ &= \frac{(\ell^0)^2}{12\pi^2} \int_m^{\infty} d\omega (\omega^2 - m^2)^{1/2} (4\omega^2 - m^2) \\ &\quad \times |\hat{g}(\alpha + \omega)|^2. \end{aligned} \quad (3.7)$$

The right-hand side of the bound (3.1) is thus

$$\begin{aligned}
 -\int_0^\infty \frac{d\alpha}{\pi} \hat{F}(\alpha, -\alpha) &= -\frac{(v^a \ell_a)^2}{12\pi^3} \int_m^\infty du |\hat{g}(u)|^2 \\
 &\quad \times \int_m^u d\omega (\omega^2 - m^2)^{1/2} (4\omega^2 - m^2),
 \end{aligned} \tag{3.8}$$

so the quantum inequality is

$$\begin{aligned}
 \int d\tau \langle :T_{ab} : \ell^a \ell^b \rangle_\omega(\gamma(\tau)) g(\tau)^2 \\
 \geq -\frac{(v^a \ell_a)^2}{12\pi^3} \int_m^\infty du |\hat{g}(u)|^2 u (u^2 - m^2)^{3/2}.
 \end{aligned} \tag{3.9}$$

In the massless case, we have the simpler expression

$$\begin{aligned}
 \int d\tau \langle :T_{ab} : \ell^a \ell^b \rangle_\omega(\gamma(\tau)) g(\tau)^2 \\
 \geq -\frac{(v^a \ell_a)^2}{12\pi^3} \int_0^\infty du u^4 |\hat{g}(u)|^2 = -\frac{(v^a \ell_a)^2}{12\pi^2} \int_{-\infty}^\infty d\tau g''(\tau)^2,
 \end{aligned} \tag{3.10}$$

where we have used Parseval's theorem and the fact that $|\hat{g}(u)|$ is even. This takes the same form as the corresponding QWEI derived in [8] which reads

$$\int d\tau \langle :T_{ab} v^a v^b : \rangle_\omega(\gamma(\tau)) g(\tau)^2 \geq -\frac{1}{16\pi^2} \int_{-\infty}^\infty d\tau g''(\tau)^2 \tag{3.11}$$

in our present notation. For nonzero mass, the two bounds differ by more than just an overall factor.

To give a specific example, suppose that

$$g(\tau) = (2\pi\tau_0^2)^{-1/4} e^{-(1/4)(\tau/\tau_0)^2}, \tag{3.12}$$

so that $g(\tau)^2$ is a normalized Gaussian with mean zero and variance $\tau_0 > 0$. For massless fields, we obtain

$$\int d\tau \langle :T_{ab} : \ell^a \ell^b \rangle_\omega(\gamma(\tau)) g(\tau)^2 \geq -\frac{(v^a \ell_a)^2}{64\pi^2 \tau_0^4}, \tag{3.13}$$

at least for Hadamard states for which the integral on the left-hand side converges absolutely [33]. We note that both sides of this expression scale by a factor of σ^2 under $\ell^a \mapsto \sigma \ell^a$.

IV. CONCLUSION

We have shown, by explicitly constructing a counterexample, that quantum inequalities along null geodesics do not exist in four-dimensional Minkowski spacetime, for the massless minimally coupled scalar field. By contrast, it was shown in Ref. [11] that such bounds do exist in two-dimensional flat spacetime. The quantum states used in our analysis are superpositions of the vacuum and multimode

two-particle states in which the excited modes are those whose three-momenta lie in a cone centered around our chosen null vector. We considered the limit of a sequence of such states in which the three-momenta become arbitrarily large while the radius of the cone shrinks to zero. Because the dominant contribution arises from modes with large three-momenta, we expect this result to hold for massive fields as well.

An interesting feature of our example is that the sampled energy density along the null geodesic becomes unbounded from below while the sequence of quantum states converges to the vacuum state. We demonstrated how such behavior is possible by considering an analogous example involving the simple harmonic oscillator in ordinary quantum mechanics. It was also shown that, as expected, the renormalized stress energy in our class of states satisfies the ANEC.

As we have learned from Verch (private communication, based on a remark of Buchholz) our result may be understood as a consequence of the fact that, in any algebraic quantum field theory in Minkowski space of dimension $d > 2$ obeying a minimal set of reasonable conditions [34] there are no nontrivial observables localized on any bounded null line segment [35]. Given further reasonable conditions (cf. [36]) this could provide a general argument for the non-existence of null world line QNEIs even for interacting field theories.

Our results imply that it is not possible to prove a singularity theorem, such as Penrose's theorem [1], by using a null world line QNEI instead of, say, the NEC or the ANEC. Although our results have been proven only for flat spacetime, we have no reason to believe that a null world line QNEI is any more likely to exist in curved spacetime.

Singularity theorems such as Penrose's theorem involve the focusing of null geodesics which generate the boundary of the future of a closed trapped surface. The latter initiates convergence of a bundle of null rays, and then some bound on the stress-tensor, such as the NEC, is required to maintain the focusing. Various sufficient conditions for focusing have been suggested in the literature [37–42]. However, it should be pointed out that *no* bound on the stress energy tensor which is strong enough to ensure sufficient focusing to guarantee the existence of conjugate points on half-complete null geodesics (as required in the Penrose theorem) could hold everywhere in an evaporating black hole spacetime. Such a bound would be inconsistent with the existence of Hawking evaporation [43]. (See Sec. IV of Ref. [24] for a more detailed discussion of this point.) To obtain a singularity, however, it is only necessary that the required focusing condition hold for *at least one* trapped surface. It is somewhat difficult to see how one would prove that such a trapped surface would always exist. As suggested in Refs. [24], [44], in regions of evaporating black hole spacetimes where the ANEC is violated, it may be possible to get a (more limited) QI-type bound that measures the degree of ANEC violation and which is also invariant under rescaling of the affine parameter. Alternatively, one might argue that on dimensional grounds, the curvature which promotes focusing scales as

l_c^{-2} , where l_c is the local proper radius of curvature, while the energy densities produced by quantum fields typically scale only like l_c^{-4} . If this line of reasoning is correct, then one might expect the breakdown of the energy conditions to only affect the validity of the singularity theorems when $l_c = l_{Planck}$ [43].

In this paper, we also showed that—in general globally hyperbolic spacetimes—averages of null-contracted stress-energy of massive and massless fields along *timelike* curves are constrained by quantum inequalities. Large negative energy densities concentrated along a null geodesic must therefore be compensated by large positive energy densities on neighboring null geodesics. This is reminiscent of the transverse smearing employed by Flanagan and Wald [45] in their study of the ANEC in semiclassical quantum gravity. Such transversely smeared observables also evade the Buchholz-Verch argument mentioned above. Whether physically interesting global results, such as singularity theorems, can be

proved using inequalities such as Eq. (3.1) is an open question, which is currently under investigation.

ACKNOWLEDGMENTS

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 [31] Note that Ref. [18] used a different convention for the Fourier transform in which Eq. (3.1) would involve $\hat{F}(-\alpha, \alpha)$ rather than $\hat{F}(\alpha, -\alpha)$.
 [32] As an example, consider the distribution $u(x, y) = \delta(x)$, which has a sensible restriction $u(x, y_0) = \delta(x)$ to lines of the form $y = y_0$, but no well-defined restriction to the line $x = 0$.
 [33] This qualification is required in order to extend the result of Theorem III.1 to noncompactly supported g . To obtain a statement valid for all Hadamard states, one could alternatively replace the left-hand side by $\liminf_{\lambda \rightarrow +\infty} \int d\tau \langle :T_{ab}: \ell^a \ell^b \rangle_\omega(\gamma(\tau)) [g(\tau)\varphi(\tau/\lambda)]^2$ where $\varphi(s)$ is smooth, equal to 1 for $|s| < 1$, vanishing for $|s| > 2$ and monotone decreasing as $|s|$ increases.
 [34] In more detail, we consider theories described by von Neumann algebras $R(\mathcal{O})$, indexed by open sets \mathcal{O} of Minkowski space and consisting of bounded operators on a Hilbert space \mathcal{H} . The assignment $\mathcal{O} \rightarrow R(\mathcal{O})$ is assumed to be isotonus [i.e., $\mathcal{O}_1 \subset \mathcal{O}_2$ implies that $R(\mathcal{O}_1)$ is a subalgebra of $R(\mathcal{O}_2)$] and local [i.e., if \mathcal{O}_1 and \mathcal{O}_2 are spacelike separated then any element of $R(\mathcal{O}_1)$ commutes with every element of $R(\mathcal{O}_2)$]. In addition, the group of translations is implemented on \mathcal{H} by unitary operators $U(a)$ such that $U(a)R(\mathcal{O})U(a)^{-1} = R(\mathcal{O} + a)$ and which obey the spectral condition: $U(a)$

$= \int_{\mathbb{R}^d} dE(p) e^{-ia \cdot p}$ where the spectral measure $dE(\cdot)$ is supported in the forward light cone $\{p: p \cdot p \geq 0, p^0 \geq 0\}$. It is also assumed that there is a unique translationally invariant vacuum state $\Omega \in \mathcal{H}$ and that Ω is cyclic [i.e., $R(\mathcal{O})\Omega$ is dense in \mathcal{H} for each \mathcal{O}].

[35] Under the assumptions in the previous reference, let K be a compact subset of Minkowski space such that $(p-q) \cdot (p-q) \leq 0$ for all $p^a, q^a \in K$. Suppose that there exist vectors w_1^a and w_2^a such that $w_1 \cdot w_1 = w_2 \cdot w_2 = -1$, $(w_1 - w_2) \cdot (w_1 - w_2) \geq 0$ and with the property that, for all $p^a, q^a \in K$, $|(p-q) \cdot w_i| \leq \sqrt{-(p-q) \cdot (p-q)}$ ($i=1,2$). Then a theorem of Woronowicz [46] entails that the algebra $\cap_{\mathcal{O} \supset K} R(\mathcal{O})$ of observables localized on K consists only of scalar multiples of the identity. In the case of a bounded null line segment $K = \{\lambda \ell^a: \lambda \in [\alpha, \beta]\}$ where ℓ^a is null, it is readily verified that

$w_1^a = \ell^a + s^a$, $w_2^a = -\ell^a + s^a$ satisfy these conditions for any spacelike vector s^a with $s \cdot s = -1$ and $\ell \cdot s = 0$. Such vectors s^a exist in dimensions $d > 2$, thus justifying the claim made in the text.

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