

Rotating quantum thermal distribution

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We show that the rigidly rotating quantum thermal distribution on flat space-time suffers from a global pathology which can be cured by introducing a cylindrical mirror if and only if it has a radius smaller than that of the speed-of-light cylinder. When this condition is met, we demonstrate numerically that the renormalized expectation value of the energy-momentum stress tensor corresponds to a rigidly rotating thermal bath up to a finite correction except on the mirror where there are the usual Casimir divergences.

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I. INTRODUCTION

In a recent investigation of the rotating quantum vacuum, Davies *et al.* [1] uncovered a remarkable issue involving the speed-of-light cylinder of flat space-time. A particle detector rotating at the same angular velocity as this vacuum fails to remain inert in unbounded space-time and remains inert when the space-time is bounded by an infinite cylindrical mirror only if the mirror has a radius less than that of the speed-of-light cylinder. It is only in this case that the concept of a rotating vacuum is unambiguously defined. In the present article, we attempt to extend this discussion of rotation to the problem of defining a rigidly rotating quantum thermal distribution. Whereas we might have expected that this distribution would be pathological only on and outside the speed-of-light cylinder, in fact we find that it is pathological almost everywhere on the unbounded space-time and likewise on the space-time bounded by a cylindrical mirror except when the mirror has a radius less than that of the speed-of-light cylinder.

This problem is closely related to the definition of a Hartle-Hawking vacuum on Kerr space-time. In particular, on the asymptotically flat region of the space-time for a black hole which is rotating arbitrarily slowly, the contributions made to the field by the upgoing modes become negligible and when these are discarded the anticommutator function associated with the Hartle-Hawking vacuum of Ref. [2] coincides with the thermal distribution considered here.

II. THE DISTRIBUTION ON UNBOUNDED SPACE-TIME

We first introduce a cylindrical coordinate system $\{t_+, R, \varphi_+, z\}$ rigidly rotating at a fixed angular velocity Ω ; this is related to the usual cylindrical Minkowski coordinate system $\{t, R, \varphi, z\}$ by the transformation

$$t_+ = t, \quad \varphi_+ = \varphi - \Omega t. \quad (2.1)$$

These coordinates are appropriate to the discussion of rigidly rotating observers (RRO's) in flat space-time. These are analogous to observers corotating with the horizon in Kerr

space-time. The scalar wave equation is separable in these coordinates. Indeed the corresponding positive Klein-Gordon norm modes

$$\tilde{u}_{\tilde{\omega}km} = \frac{1}{\sqrt{8\pi^2}} e^{-i\tilde{\omega}t_+ + im\varphi_+ + ikz} J_m[\sqrt{(\tilde{\omega} + m\Omega)^2 - k^2}R], \quad (2.2)$$

may be obtained from the standard modes of Minkowski space-time in cylindrical coordinates

$$u_{\omega km} = \frac{1}{\sqrt{8\pi^2}} e^{-i\omega t + im\varphi + ikz} J_m(\sqrt{\omega^2 - k^2}R) \quad (2.3)$$

by the identification

$$\tilde{\omega} = \omega - m\Omega. \quad (2.4)$$

Since it is the norms not the frequencies of the RRO modes which determine the commutation relations of the associated creation and annihilation operators, the rotating vacuum naively coincides with the conventional vacuum of Minkowski space-time.

The thermal distribution at inverse temperature β and rigidly rotating at angular velocity Ω will be described by the density operator

$$\hat{\rho}_+ = e^{-\beta\hat{H}_+}, \quad (2.5)$$

with the Hamiltonian

$$\hat{H}_+ = i \frac{\partial}{\partial t_+}. \quad (2.6)$$

The thermal anticommutator function $G_{(1)}^\beta$ associated with this distribution is therefore characterized by the condition [3]

$$G_{(1)}^\beta(t_+, \mathbf{x}; t'_+, \mathbf{x}') = G_{(1)}^\beta(t_+ + i\beta, \mathbf{x}; t'_+, \mathbf{x}'). \quad (2.7)$$

It can be checked that at least formally this condition is satisfied by the anticommutator function defined by the mode sum

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$$\begin{aligned}
 G_{(1)}^\beta(x, x') &= \frac{1}{8\pi^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \int_0^\omega dk \coth\left(\frac{\beta\tilde{\omega}}{2}\right) \\
 &\times e^{im(\varphi-\varphi')+ik(z-z')} \text{Re}[e^{-i\omega(t-t')}] \\
 &\times J_m(\sqrt{\omega^2-k^2}R) J_m(\sqrt{\omega^2-k^2}R'). \quad (2.8)
 \end{aligned}$$

As the energy $\tilde{\omega}$ as measured by \hat{H}_+ tends towards zero, the density of states factor $\coth(\beta\tilde{\omega}/2)$ becomes infinite although the modes themselves remain nonzero whenever $m \neq 0$. These modes clearly make divergent contributions to the mode sum (2.8) except when either R or R' is zero. The anticommutator function $G_{(1)}^\beta$ is thus pathological except when at least one of the two points is on the z axis. In Ref. [4], a similar pathology was noted in the mode sum for the anticommutator function of the Hartle-Hawking state considered in Ref. [2].

III. THE DISTRIBUTION WITHIN AN INFINITE CYLINDER

We now introduce a cylindrical mirror of arbitrary radius R_0 . For brevity, we only treat the case of a field which satisfies Dirichlet conditions on this cylinder. Introducing a nondimensional radial variable $\bar{R} = R/R_0$, a complete set of orthonormal solutions to the field equation subject to the boundary conditions is

$$\begin{aligned}
 u_{kmn} &= \frac{1}{2\pi R_0 \sqrt{|\omega_{kmn}|} |J_{m+1}(\xi_{mn})|} \\
 &\times e^{-i\omega_{kmn}t + im\varphi + ikz} J_m(\xi_{mn}\bar{R}), \quad (3.1)
 \end{aligned}$$

where

$$\omega_{kmn} = \pm \sqrt{\frac{\xi_{mn}^2}{R_0^2} + k^2} \quad (3.2)$$

and ξ_{mn} denotes the n th positive zero of J_m . The normalization factor has been calculated by making use of an identity for the Bessel functions (Ref. [5], see p. 765). The modes which have positive norm are precisely those which have positive frequency ω . The vacuum state associated with the field when it is expanded in terms of this set of modes has a anticommutator function which is given by the mode sum

$$\begin{aligned}
 G_{(1)}(x, x') &= \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk e^{im(\varphi-\varphi')+ik(z-z')} \\
 &\times \frac{J_m(\xi_{mn}\bar{R}) J_m(\xi_{mn}\bar{R}')}{4\pi^2 R_0^2 \omega_{kmn} J_{m+1}^2(\xi_{mn})} 2 \text{Re}[e^{-i\omega_{kmn}(t-t')}] . \quad (3.3)
 \end{aligned}$$

The anticommutator function

$$\begin{aligned}
 G_{(1)}^\beta(x, x') &= \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk \coth\left(\frac{\beta\tilde{\omega}_{kmn}}{2}\right) \\
 &\times e^{im(\varphi-\varphi')+ik(z-z')} \frac{J_m(\xi_{mn}\bar{R}) J_m(\xi_{mn}\bar{R}')}{4\pi^2 R_0^2 \omega_{kmn} J_{m+1}^2(\xi_{mn})} \\
 &\times 2 \text{Re}[e^{-i\omega_{kmn}(t-t')}] \quad (3.4)
 \end{aligned}$$

is associated with a thermal distribution described by the density operator (2.5) with Hamiltonian (2.6). This mode sum suffers from a similar pathology to that on the unbounded space-time unless there are no positive frequency modes for which $\tilde{\omega}_{kmn}$ is zero. It is a well known property of the zeros of J_m that $\xi_{m1} > |m|$ (Ref. [6], see Sec. 15.3) and so we see from Eq. (3.2) that if $R_0 < \Omega^{-1}$, no such modes exist. On the other hand, the asymptotic behavior of this first zero is (Ref. [7], see p. xviii)

$$\xi_{m1} \sim m + 1.85575m^{1/3}, \quad (m \rightarrow \infty), \quad (3.5)$$

from which we see that if $R_0 > \Omega^{-1}$, there are modes of this type for all sufficiently large m . It follows that the mode sum given in Eq. (3.4) is well behaved if and only if the mirror lies within the speed-of-light cylinder.

IV. THE MEASUREMENTS OF AN RRO

When the mirror lies inside the speed-of-light cylinder, static observers and RRO's both make measurements with respect to the vacuum state whose anticommutator function is given in Eq. (3.3). These measurements can be calculated from

$$\begin{aligned}
 G_{(1)}^\beta(x, x') - G_{(1)}(x, x') &= \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk \frac{1}{e^{\beta\tilde{\omega}_{kmn}} - 1} e^{im(\varphi-\varphi')+ik(z-z')} \\
 &\times \frac{J_m(\xi_{mn}\bar{R}) J_m(\xi_{mn}\bar{R}')}{4\pi^2 R_0^2 \omega_{kmn} J_{m+1}^2(\xi_{mn})} 2 \text{Re}[e^{-i\omega_{kmn}(t-t')}] . \quad (4.1)
 \end{aligned}$$

We can derive from this a set of expressions for the nonzero components of the energy-momentum stress tensor corresponding to the the conformally invariant field, the details of which can be found in Ref. [8]. The mode by mode cancellation of the high frequency divergences which afflict both anticommutator functions in the coincident limit makes these expressions amenable to numerical analysis and the results are shown in Fig. 1. They are compared with the Planckian forms corresponding to a rigidly rotating thermal distribution at temperature T which are

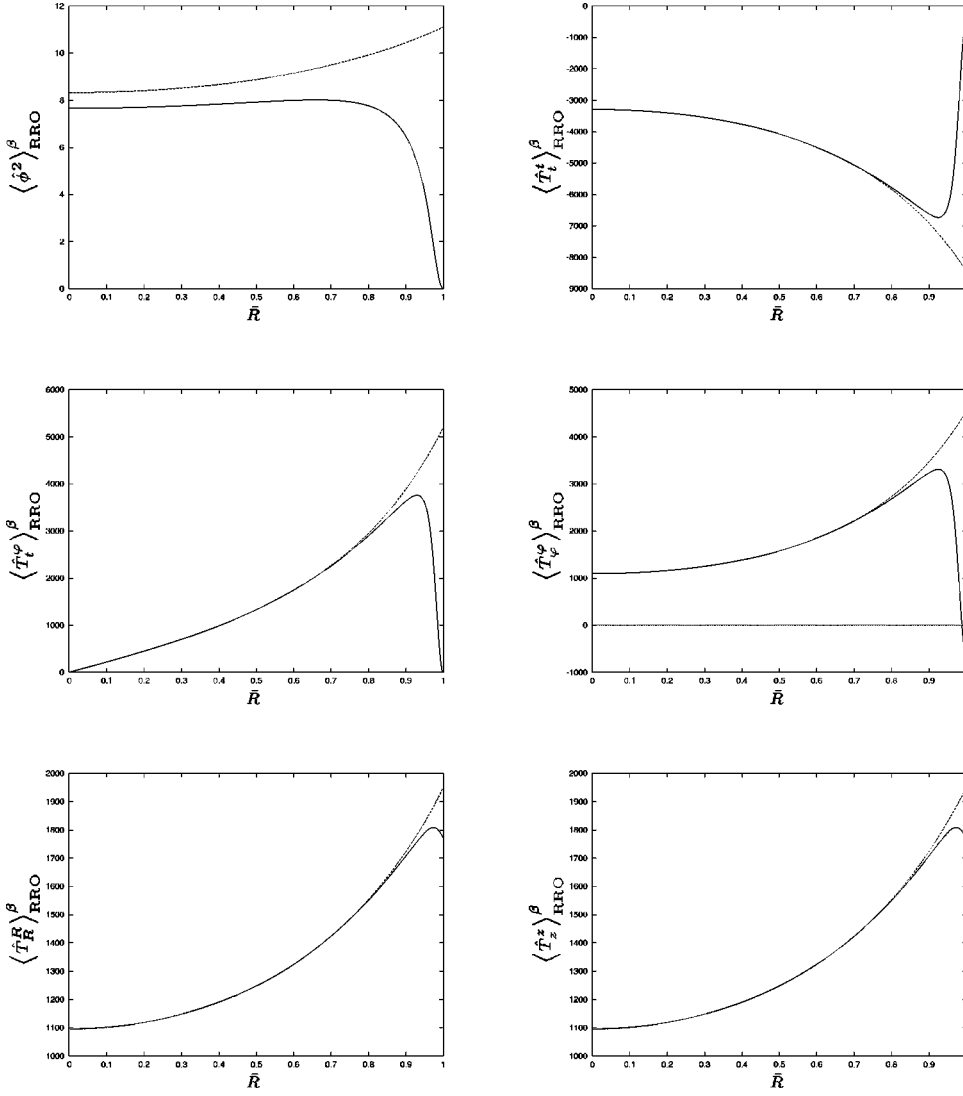


FIG. 1. The graphs are given in units in which R_0 , the radius of the cylinder, is unity. The temperature is $T=10/R_0$ and the angular velocity is $\Omega=0.5/R_0$. The dashed line is a plot of the value for a rigidly rotating thermal distribution (4.2)–(4.7).

$$\langle \phi^2 \rangle_{\text{Planck}}^\beta = \frac{(\gamma T)^2}{12}, \quad (4.2)$$

$$\langle T_t^t \rangle_{\text{Planck}}^\beta = -\frac{\pi^2}{90} (3+v^2) \gamma^2 (\gamma T)^4, \quad (4.3)$$

$$\langle T_t^\varphi \rangle_{\text{Planck}}^\beta = \frac{4\pi^2}{90} v \gamma^2 (\gamma T)^4, \quad (4.4)$$

$$\langle T_\varphi^\varphi \rangle_{\text{Planck}}^\beta = \frac{\pi^2}{90} (1+3v^2) \gamma^2 (\gamma T)^4, \quad (4.5)$$

$$\langle T_R^R \rangle_{\text{Planck}}^\beta = \frac{\pi^2}{90} (\gamma T)^4, \quad (4.6)$$

$$\langle T_z^z \rangle_{\text{Planck}}^\beta = \frac{\pi^2}{90} (\gamma T)^4, \quad (4.7)$$

where v and γ are given by

$$v = R\Omega, \quad \gamma = \frac{1}{\sqrt{1-v^2}} \quad (4.8)$$

and are the speed and Lorentz factor of an RRO at the appropriate space-time point. We find that they are in close agreement everywhere except, as expected, close to the mirror.

V. RENORMALIZED EXPECTATION VALUES

A renormalized expectation value differs from that an RRO measures by a term due to polarization of the vacuum by the mirror. This term can be calculated by making use of the relationship between that Feynman propagator and the Euclidean Green function; on the Euclidean section of the manifold, the analysis becomes essentially identical to that of a uniformly accelerating infinite flat mirror on the Euclidean section of the Rindler manifold and we can proceed along the lines of Ref. [9]. We find that the Euclidean Green function which vanishes on the mirror is

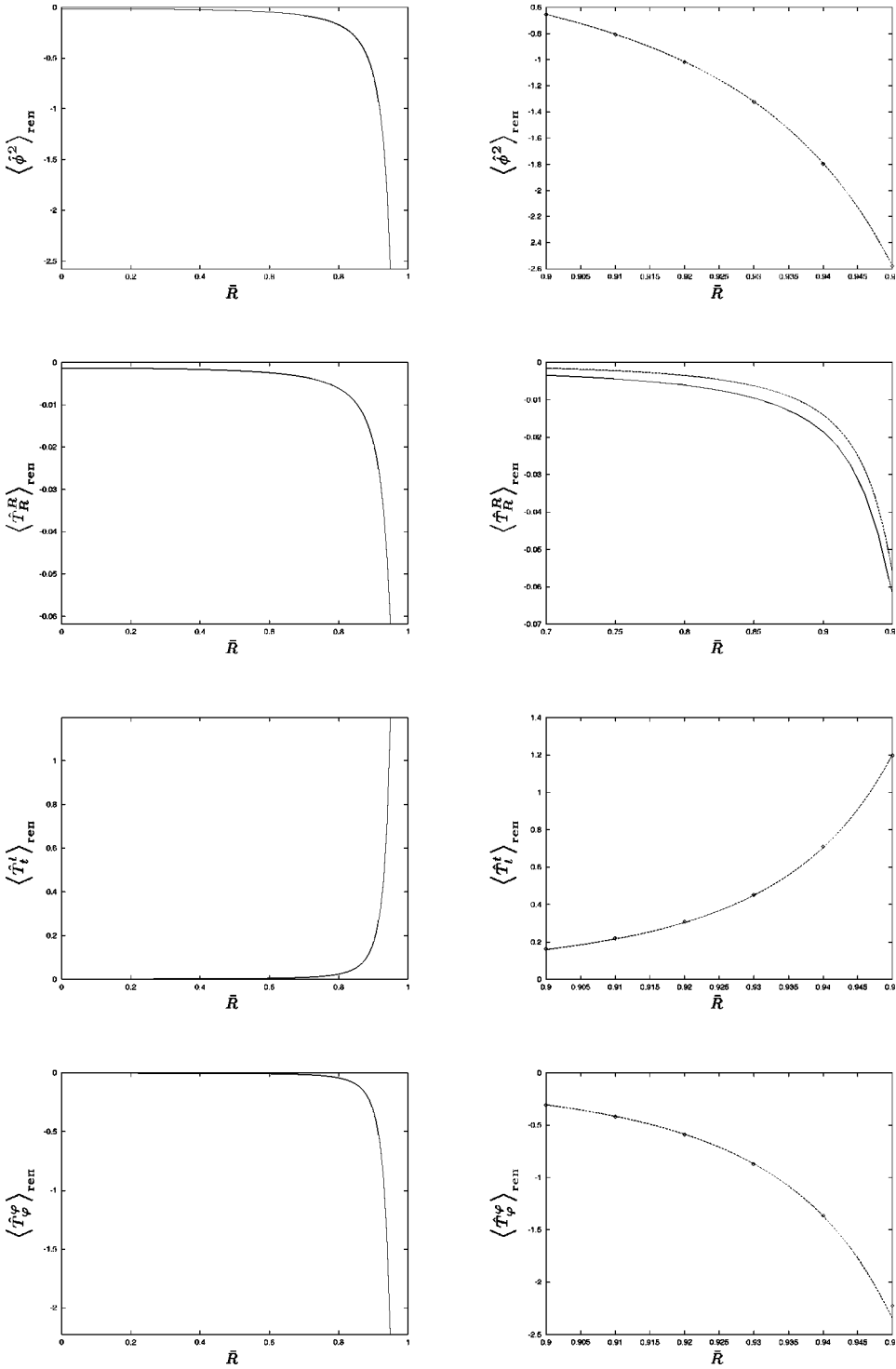


FIG. 2. The graphs are given in units in which R_0 , the radius of the cylinder, is unity. In the right hand graphs, the dashed line gives the analytically calculated Casimir divergence (5.4)–(5.7) while the solid line in the second graph and the points in the others give the numerically calculated values.

$$\begin{aligned}
 G_E(x, x') &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(\tau-\tau')} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} \\
 &\times \sum_{m=-\infty}^{\infty} \frac{e^{im(\varphi-\varphi')}}{2\pi} I_m(\sqrt{\omega^2+k^2}R_<) \\
 &\times \left\{ K_m(\sqrt{\omega^2+k^2}R_>) - I_m(\sqrt{\omega^2+k^2}R_>) \right. \\
 &\quad \left. \times \frac{K_m(\sqrt{\omega^2+k^2}R_0)}{I_m(\sqrt{\omega^2+k^2}R_0)} \right\}, \tag{5.1}
 \end{aligned}$$

where $R_< = \min\{R, R'\}$, $R_> = \max\{R, R'\}$, and $t = i\tau$. The second term in the braces is absent in the case of the Euclidean Green function on the unbounded manifold and so this is the term that remains after renormalization. Now, closing the points and making a transformation to polar variables α and γ defined by

$$kR_0 = \alpha \sin \gamma, \quad \omega R_0 = \alpha \cos \gamma, \quad (5.2)$$

we find that we can perform the integral over γ to obtain

$$\langle \hat{\phi}^2 \rangle_{\text{ren}} = -\frac{1}{4\pi^2 R_0^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\alpha \alpha I_m^2(\alpha \bar{R}) \frac{K_m(\alpha)}{I_m(\alpha)}. \quad (5.3)$$

A similar thing can be done for the components of the energy-momentum stress tensor and the resulting expressions together with Eq. (5.3) are amenable to numerical analysis. Once again, the details can be found in Ref. [8]. The results are presented in Fig. 2. They are compared with the Casimir divergence close to the mirror which can be calculated by an asymptotic analysis following Ref. [9]. The relevant expressions are

$$\langle \hat{\phi}^2 \rangle_{\text{ren}} \sim -\frac{1}{16\pi^2 R_0^2 (1-\bar{R})^2} \left[1 + \frac{1-\bar{R}}{3} \right] \quad (\bar{R} \rightarrow 1), \quad (5.4)$$

$$\langle \hat{T}_R^R \rangle_{\text{ren}} \sim -\frac{1}{720\pi^2 R_0^4 (1-\bar{R})^2} \quad (\bar{R} \rightarrow 1), \quad (5.5)$$

$$\langle \hat{T}_t^t \rangle_{\text{ren}} \sim \frac{1}{720\pi^2 R_0^4 (1-\bar{R})^3} \left[1 + \frac{19(1-\bar{R})}{14} \right] \quad (\bar{R} \rightarrow 1), \quad (5.6)$$

$$\langle \hat{T}_\varphi^\varphi \rangle_{\text{ren}} \sim -\frac{1}{360\pi^2 R_0^4 (1-\bar{R})^3} \left[1 + \frac{6(1-\bar{R})}{7} \right] \quad (\bar{R} \rightarrow 1), \quad (5.7)$$

and are in agreement with the general expressions of Ref. [10]. We have used these to calculate renormalized expectation values in the thermal distribution when $\Omega=0$ and checked that they are well approximated by the general expressions given in Ref. [11].

VI. CONCLUSION

We found that the anticommutator function associated with the rigidly rotating thermal distribution on unbounded Minkowski space-time is pathological almost everywhere. The pathology is caused by the existence of nonzero modes which have zero energy as measured by the Hamiltonian \hat{H}_+ relevant to RRO's. In Ref. [4], a similar pathology was noted in the anticommutator function of the Hartle-Hawking state considered in Ref. [2]. In this case \hat{H}_+ is the Hamiltonian relevant to observers rigidly rotating with the horizon. The corresponding modes are thus at the critical point of superradiant scattering. When Minkowski space-time is bounded by an infinite cylinder of radius larger than the speed-of-light cylinder we found that the anticommutator function is once again pathological almost everywhere because of the existence of these modes for all sufficiently high m . In a future article we will show that when the Kerr black hole is enclosed within a mirror of constant Boyer-Lindquist radius larger than the minimum radius of the speed-of-light surface, for all sufficiently high m there are complex frequency modes whose real parts lie in the regime we associate with superradiance in the absence of the mirror. This set of modes has the critical point of superradiant scattering as an accumulation point.

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