Can structure formation influence the cosmological evolution?

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The back reaction of structure formation influences the cosmological evolution equation for the homogenous and isotropic average metric. In a cold dark matter universe this effect leads only to small corrections unless a substantial fraction of matter is located in regions where strong gravitational fields evolve in time. A ''cosmic virial theorem'' states that the sum of gravitational and matter pressures vanishes and, therefore relates the average kinetic energy to a suitable average of the Newtonian potential. In the presence of a scalar ''cosmon'' field mediating quintessence, however, cosmology could be modified if local cosmon fluctuations grow large. We speculate that this may trigger the accelerated expansion of the universe after the formation of structure.

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I. INTRODUCTION

Recent observations of the Hubble diagram for supernova of type 1a indicate that the expansion of the universe may be accelerating in the present epoch [1]. In this case the previous decrease of the Hubble parameter $H \sim t^{-1}$ must have slowed down just in the last 2×10^9 years and an obvious question asks: Why has this slowdown happened[1] ''just now''? A possible explanation would be a cosmological constant which sets a mass scale $\lambda \approx (10^{-3}\ \text{eV})^4$ and, therefore, also a corresponding time scale $\sim(\lambda/M_p^2)^{-1/2}$, with $M_p \approx 10^{19}$ GeV the Planck mass. Since it seems to be very hard to understand the origin of the tiny mass scale λ theoretically, one is tempted to look for alternatives. A possible scenario is the model of quintessence [2,3]. It is based on the time evolution of a scalar field—the cosmon—which is of cosmological relevance today. In the simplest viable models, however, the characteristic time for the onset of acceleration is put in by hand in the form of the effective scalar potential or kinetic term.[2] This does not always need tremendous fine tuning of the order of 100 digits as for the case of the cosmological constant. Indeed, there are models where it is sufficient to tune parameters on the level of percent to permille. We feel, nevertheless, that these ideas would become much more credible if a natural solution of the ''why now'' problem could be given. As one possibility, the relevant time scale may be linked to a natural small number arising from a fundamental theory. A recent proposal in this direction involves the properties of a conformal fixed point [7]. As an alternative, some event in the more recent cosmological evolution could have ''set the clock'' to trigger the acceleration at present. An idea in this direction [8]—''k essence''—tries to use the transition from a radiation-dominated to a matter-dominated universe in order to set the clock. Here we explore whether structure formation could have induced a change of the pace of expansion.

One of the most striking qualitative changes in the recent history of the universe is the formation of structure. For most of the cosmological evolution the universe was homogenous to a high degree. Looking at the sky today we see, however, strong inhomogeneities in the form of stars, galaxies, and clusters on length scales sufficiently small as compared to the horizon. Could the emergence of the inhomogeneities set the clock [9] for the present acceleration? In order to answer this question, we have to understand how inhomogeneities on the scales of clusters or smaller ''act back'' on the evolution of the homogenous ''average metric.'' (For the purpose of this paper we consider formally an average over the present horizon. More accurately, the supernova results concern an average over a volume corresponding to $z \approx 1$.) After all, the universe is not homogenous at present and the Einstein equations determine the metric in the presence of these inhomogeneities. One can still formulate a type of ''macroscopic Einstein equation'' for the average metric, which, by definition, can be considered as homogenous. The macroscopic equation is simply obtained by averaging the *microscopic* Einstein equation. In this averaging procedure the ''back reaction'' of the inhomogeneities appears in the form of ''correction terms'' in the macroscopic Einstein equation [10]. In models of quintessence this holds also for the macroscopic evolution equation for the scalar field.

It is the aim of the present paper to estimate the size and, therefore, the relevance of the back-reaction effects. For this purpose, we express in Sec. II the back-reaction effects in terms of a gravitational energy density ρ_g and corresponding pressure p_g. It is obvious that ρ_g and p_g are relevant only if they are not tiny as compared to the energy density $\bar{\rho}$ in radiation or matter. A very rough estimate shows that in early cosmology the effects of ρ_g and p_g are indeed completely negligible. One the other hand, once stars and galaxies have formed, the ratio $\rho_g/\bar{\rho}$ is not many orders below 1 any more, and a more detailed investigation becomes necessary. In Sec. III we evaluate ρ_g and p_g in terms of the correlation function for the local energy momentum tensor of matter and radiation. This form exhibits clearly the relation of these quantities to the inhomogeneities.

In Sec. IV we attempt a quantitative estimate for a standard cold dark matter universe (without quintessence, but possibly in the presence of a cosmological constant). We find that the effects of inhomogeneities on the scales of stars and

[1] On a logarithmic scale as relevant for cosmology, the last 2×10^9 years are more or less the ''present'' epoch.
[2] For more recent examples, see [4–6].

galaxies are small; they contribute typically $\rho_g/\overline{\rho}\approx 10^{-6}$. A typical contribution from inhomogeneities on the scales of clusters is $\rho_g/\overline{\rho}\approx 10^{-4}$. These estimates hold, however, only if the fraction of matter in regions of strong gravitational fields, such as black holes or the center of galaxies, is small. In Sec. V we address the back-reaction effects from black holes and similar objects. The gravitational energy density ρ_g can indeed be large. Nevertheless, the combined energy momentum tensor for gravitational and matter contributions behaves as for a nonrelativistic gas if the objects are static. We conclude that for a cold dark matter universe, the back-reaction effect could play a significant role only if a substantial fraction of matter is found in regions where strong gravitational fields evolve in time. This does not seem to be very likely.

In models with quintessence the situation could change dramatically, but only if the inhomogeneities in the cosmon field are substantial. The *gravitational* back-reaction ρ_g, p_g is then supplemented by a *cosmon* back reaction ρ_c, p_c due to the cosmon fluctuations. We discuss a simple collection of static and isotropic cosmon lumps in Sec. VI. This would behave similar to black holes. We argue that for more general, in particular, nonstatic, cosmon fluctuations the fine cancellation between p_c and p_g observed in the static isotropic solutions may not be maintained. In particular, it seems conceivable for a cosmon dark matter scenario [11] that ρ_c/ρ and ρ_g/ρ are of order unity. Under this condition it would become quite likely that the formation of structure would lead to a qualitative change in the evolution equation for the average metric. One would expect deviations from $H\sim t^{-1}$ once structure has formed. We summarize our conclusions in Sec. VII.

II. THE INFLUENCE OF STRUCTURE ON THE COSMOLOGICAL EQUATIONS

After the formation of structure the universe does not remain homogenous on small scales. Nevertheless, we believe that homogeneity and isotropy are realized on large scales and describe the cosmological evolution by a Robertson-Walker metric. The true metric $g_{\mu\nu}$ of the universe has to reflect the inhomogeneities due to stars, galaxies, and clusters. Therefore the homogenous cosmological metric can at best be interpreted as some type of average metric[3] $\overline{g}_{\mu\nu}=\langle g_{\mu\nu}\rangle$. This situation introduces a mismatch in the standard treatment of the cosmological Einstein equations. On the right-hand side one uses the average of the energy momentum $\langle t_{\mu\nu}\rangle$, whereas for the left-hand side one employs the Einstein tensor formed from the average metric $\overline{g}_{\mu\nu}$. The correct averaged Einstein equation involves,[4] however, the averaged value of the Einstein tensor

$$\left\langle R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right\rangle=\frac{1}{2M^2}\langle t_{\mu\nu}\rangle. \tag{1}$$

The difference between the averaged Einstein tensor and the Einstein tensor formed from the average metric, i.e., $\overline{R}_{\mu\nu}-\frac{1}{2}\overline{Rg}_{\mu\nu}$, introduces a correction term in the cosmological equation for the average metric

$$\overline{R}_{\mu\nu}-\frac{1}{2}\overline{Rg}_{\mu\nu}=\frac{1}{2M^2}T_{\mu\nu}=\frac{1}{2M^2}(\langle t_{\mu\nu}\rangle+T^g_{\mu\nu}). \tag{2}$$

Here the gravitational correction to the energy momentum tensor

$$T^g_{\mu\nu}=-2M^2\langle\delta G_{\mu\nu}\rangle, \tag{3}$$

$$\delta G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\left(\overline{R}_{\mu\nu}-\frac{1}{2}\overline{Rg}_{\mu\nu}\right) \tag{4}$$

reflects the influence of the inhomogeneities. It accounts for the back reaction of structure formation on the evolution of the homogenous background metric $\overline{g}_{\mu\nu}$. Homogeneity and isotropy of all averaged quantities imply that the only nonvanishing components of $T^g_{\mu\nu}$ are given by

$$T^g_{00}=\rho_g=-2M^2\langle\delta G_{00}\rangle$$

$$T^g_{ij}=p_g\overline{g}_{ij}=-2M^2\langle\delta G_{ij}\rangle. \tag{5}$$

The cosmological equation therefore preserves its form, but T_{00} is not given solely by the average of the energy density in matter and radiation. It also contains a gravitational contribution which reflects the imprint of structure formation on inhomogeneities of the metric. We observe that for a flat background metric $\overline{g}_{\mu\nu}=\eta_{\mu\nu}$, the quantity $T^g_{\mu\nu}$ represents precisely the definition of the gravitational energy momentum densities [13]. Our setting is therefore a straightforward generalization to cosmology.

At this point some comments about our averaging procedure seem in order. Assume that in a given suitable gauge—we will later specify a particular one—the detailed inhomogeneous geometry of the universe is described by the microscopic metric $g_{\mu\nu}(\vec{x},t)$. It is related to the microscopic energy momentum tensor $t_{\mu\nu}(\vec{x},t)$ by the microscopic Einstein equation

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{1}{2M^2}t_{\mu\nu}. \tag{6}$$

We next take a reference metric $\tilde{g}_{\mu\nu}$ of the homogeneous and isotropic Robertson-Walker form $ds^2=-dt^2+a^2(t)d\vec{x}^2$. The microscopic metric can be written as $g_{\mu\nu}=\tilde{g}_{\mu\nu}+h_{\mu\nu}$. Our reference metric defines surfaces of fixed t. At any time t we define the average by

[3]The averaging is done here with respect to the background metric. See Ref. [12] for a recent discussion of averaging procedures in a more general context.

[4]We use signature $(-,+,+,+)$, $R^{\lambda}_{\mu\nu\rho}=-\partial_\mu\Gamma^{\lambda}_{\nu\rho}+\cdots$, and $M^2=M_p^2/(16\pi)=1/(16\pi G_N)$.

$$\langle A\rangle(\vec{x},t)=\frac{1}{V}\int_V d^3y A(\vec{y}-\vec{x},t), \tag{7}$$

with V a very large volume (typically of the horizon size) in the comoving coordinates.[5] We can then fix the scale factor $a(t)$ in $\tilde{g}_{\mu\nu}$ self-consistently by requiring[6] $\langle h_{\mu\nu}\rangle=0$. This finally identifies $\tilde{g}_{\mu\nu}$ and $\langle g_{\mu\nu}\rangle$.

Spatial averaging at fixed t has been proposed by Futamase [14]. We are free to use such an averaged description—the only physical assumption in our paper concerns averaged homogeneity and isotropy of the real universe, namely in our paper $\langle h_{\mu\nu}\rangle=0$ and $\langle t_{\mu\nu}\rangle$ depend only on t. A much more subtle point is the question as to what extent a real observer actually observes the spatially averaged quantities in the way introduced here. Detailed studies conclude [15] that this may actually be the case—we will not address this topic in the present paper. For a particular picture of the cold dark matter scenario Futamase concludes that back-reaction effects are small, whereas Buchert speculates [12] that the influence on the cosmological evolution could be substantial, nevertheless.

Let us next discuss the general structure of ρ_g and p_g [Eq. 5]. As long as gravity remains weak, one can expand in the small inhomogeneities of the metric

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}, \tag{8}$$

such that

$$\delta G_{\mu\nu}=D^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}+E^{\alpha\beta\gamma\delta}_{\mu\nu}h_{\alpha\beta}h_{\gamma\delta}. \tag{9}$$

Here the differential operators D and E involve two derivatives acting on h or $\bar{g}$. They will be computed more explicitly in Sec. III. From $\langle h_{\alpha\beta}\rangle\equiv 0$ one concludes that ρ_g and p_g are quadratic in h,

$$\rho_g=-2M^2\langle E^{\alpha\beta\gamma\delta}_{00}h_{\alpha\beta}h_{\gamma\delta}\rangle$$

$$p_g=-\frac{2M^2}{3a^2}\langle E^{\alpha\beta\gamma\delta}_{ii}h_{\alpha\beta}h_{\gamma\delta}\rangle. \tag{10}$$

Thus ρ_g and p_g involve the correlation function for the metric and do not vanish, in general.

The local variation of the metric reflects the local variations of the energy momentum tensor according to the "microscopic" Einstein equation (6).[7] Within the linear approximation to Eq. (6), namely,

$$D^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}=\frac{1}{2M^2}(t_{\mu\nu}-\bar{T}_{\mu\nu})=\frac{1}{2M^2}\delta t_{\mu\nu}, \tag{11}$$

the metric fluctuations $h_{\mu\nu}$ are linear in the fluctuations of the energy momentum tensor $\delta t_{\mu\nu}$. In consequence, ρ_g can also be viewed as the effect of a nonvanishing correlation function for the fluctuations of the energy density. This correlation function can be observed as a galaxy—or cluster—correlation function on appropriate length scales. In particular, we know that on small scales the universe is far from homogenous. As an example, on the length scale of the size of stars, very dense regions (stars) contrast with an almost empty environment. This is equivalent to a huge correlation function and brings us back to the question: Can the formation of stars or galaxies influence the evolution of the universe as a whole?

In order to get a first rough estimate of the magnitude[8] of this *back reaction of structure formation* let us assume for a moment that the universe consists of randomly distributed[9] "stars" with radius L or volume $v_L=(4\pi/3)L^3$ and density ρ_L. Consider our horizon volume V with N_V stars. Since $T_{00}\equiv\bar{\rho}$ is of the same order as $\langle t_{00}\rangle=N_V v_L\rho_L/V$ and, on the other side, $\langle t_{00}^2\rangle=N_V v_L\rho_L^2/V$, one has

$$\frac{\langle\delta\rho^2\rangle}{\bar{\rho}^2}=\frac{\langle(t_{00}-\bar{\rho})^2\rangle}{\bar{\rho}^2}\approx\frac{1}{f},\quad f=\frac{N_V v_L}{V}\approx\frac{\bar{\rho}}{\rho_L}. \tag{12}$$

The fraction of volume occupied by stars, f, is indeed a tiny number and one concludes that the relative density fluctuations are huge. On the other hand, the weak gravitational coupling enters such that the relative size of ρ_g as compared to $\bar{\rho}$ could still be small. Since the operator D in Eq. (11) contains two derivatives, a rough estimate assumes

$$\langle h^2\rangle\approx\frac{L^4}{M^4}\langle\delta\rho^2\rangle \tag{13}$$

and, with a similar dimension argument,

$$\rho_g\approx\frac{L^2}{M^2}\langle\delta\rho^2\rangle\approx\frac{L^2}{M^2}\rho_L\bar{\rho}. \tag{14}$$

The critical quantity for the relevance of the back reaction is therefore given by the ratio

$$R=\frac{\rho_g}{\bar{\rho}}\approx\frac{L^2}{M^2}\rho_L\approx\frac{m_L^{2/3}\rho_L^{1/3}}{M^2}\approx\frac{m_L}{LM^2} \tag{15}$$

[5]Since we average at fixed t, it does actually not matter if we average over coordinates $a(t)\vec{x}$ or $\vec{x}$ if the physical averaging volume grows $\sim a^3$.

[6]In order to achieve this task, we may use the freedom of selecting a suitable gauge. Of course, $\langle h_{\mu\nu}\rangle=0$ is only possible if the universe is really homogeneous and isotropic on average. In particular, $\langle h_{\mu\nu}\rangle$ should not depend on $\vec{x}$ (after using the gauge freedom).

[7]"Microscopic" means here the scales of stars or galaxies. These scales are to be compared with the "macroscopic" scale of the order of the horizon.

[8]Note that this estimate does not account for the total back reaction. Since we want to study here the effects of structure formation we can concentrate on a typical wavelength well within the horizon. Discussions of the back reaction from modes outside the horizon can be found in [16,11].

[9]Stars may be replaced by galaxies or other extended objects.

with m_L the mass of the stars. It is suppressed by two powers of the Planck mass as expected for a gravitational fluctuation effect. On the other hand, the mass m_L of the star and its size L are huge in microphysical units. Inserting values typical for the sun, $m_L = 2\times 10^{33}$ g$=1.1\times 10^{57}$ GeV, $L = 7\times 10^{8}$ m $=3.5\times 10^{24}$ GeV^{-1}, and using $M = 1.72\times 10^{18}$ GeV, one finds for main sequence stars

$$\frac{m_L}{LM^2} \approx 10^{-4}. \tag{16}$$

Another estimate relates the gravitational back-reaction effect to typical values of the Newtonian gravitational potential $\phi = -h_{00}/2$ in extended objects. Indeed, we note that R is proportional to the gravitational potential at the surface of the star, $m_L G/L$, with $G^{-1} = 16\pi M^2$. Its value for the sun is

$$-\phi = \frac{m_L G}{L} = 2.12\times 10^{-6}. \tag{17}$$

Similarly, for idealized neutron stars with masses at the Oppenheimer-Volkoff limit, $m_L = 1.4\times 10^{33}$ g, $L = 9.6$ km, one has

$$\frac{m_L}{LM^2} = 5.5, \quad -\phi = \frac{G m_L}{L} = 0.11. \tag{18}$$

These first estimates are, perhaps surprisingly, not much below 1 (as could have been expected from the factor M^{-2}). A more detailed investigation, including the various proportionality constants and the distribution of objects with different L and M_L, becomes necessary.

Before doing so, it is instructive to discuss a few qualitative aspects of the dependence of the ratio R on L and ρ_L.

(1) The ratio $\rho_g/\bar{\rho}$ is independent of $\bar{\rho}$. It therefore shows essentially no time dependence once the objects have condensed with a stationary density and size.

(2) For a fixed density ρ_L the contribution from smaller objects vanishes rapidly. For example, the condensation to dust particles or planets is many orders of magnitude too small to be relevant.

(3) Microphysical objects such as nuclei play no role for ρ_g (i.e., $R \approx 10^{-36}$ for a gas of nuclei). In early cosmology the contribution of ρ_g is therefore completely negligible.

(4) For an (elliptical) galaxy consisting of ν_G roughly uniformly distributed stars within a radius L_G, the density scales as $\rho_G \approx \nu_G (L/L_G)^3 \rho_L$. (There may be some moderate enhancement from dark matter.) For a uniform mass distribution in a galaxy, the combination $L_G^2 \rho_G = \nu_G (L/L_G) L^2 \rho_L$ is changed by a factor $\nu_G L/L_G$ as compared to stars.

III. GRAVITATIONAL ENERGY MOMENTUM TENSOR IN COSMOLOGY

We next turn to a more quantitative discussion of Eqs. (9) and (10) for the metric inhomogeneities. Since the relevant length scale for the dominant fluctuations is much smaller than the horizon, we can neglect derivatives acting on $\bar{g}_{\mu\nu}$ as compared to those acting on $h_{\mu\nu}$. This yields the microscopic field equation up to quadratic order in the metric fluctuations

$$\begin{aligned}\delta G_{\mu\nu} = {} & -\tfrac{1}{2}\{\partial^\rho\partial_\rho h_{\mu\nu} + \partial_\mu\partial_\nu h^\rho{}_\rho - \partial_\mu\partial_\rho h^\rho{}_\nu - \partial_\nu\partial_\rho h^\rho{}_\mu - \partial^\rho\partial_\rho h^\alpha{}_\alpha \bar{g}_{\mu\nu} + \partial_\alpha\partial_\rho h^{\alpha\rho}\bar{g}_{\mu\nu}\} + \tfrac{1}{2} h^{\alpha\rho}\{\partial_\mu\partial_\nu h_{\alpha\rho} + \partial_\alpha\partial_\rho h_{\mu\nu} - \partial_\rho\partial_\mu h_{\alpha\nu} \\ & - \partial_\rho\partial_\nu h_{\alpha\mu}\} + \tfrac{1}{2} h_{\mu\nu}\{\partial^\rho\partial_\rho h^\alpha{}_\alpha - \partial_\rho\partial_\alpha h^{\alpha\rho}\} - \tfrac{1}{2}\bar{g}_{\mu\nu} h^{\alpha\rho}\{\partial^\beta\partial_\beta h_{\alpha\rho} + \partial_\alpha\partial_\rho h^\beta{}_\beta - 2\partial_\rho\partial_\beta h^\beta{}_\alpha\} + \tfrac{1}{4}\{\partial_\mu h^{\alpha\rho}\partial_\nu h_{\alpha\rho} \\ & + 2\partial_\alpha h^{\alpha\rho}\partial_\rho h_{\mu\nu} - \partial^\rho h^\alpha_\alpha \partial_\rho h_{\mu\nu} + 2\partial^\alpha h^\rho_\mu \partial_\alpha h_{\rho\nu} - 2\partial^\alpha h_{\rho\mu}\partial^\rho h_{\alpha\nu} - 2\partial_\alpha h^{\alpha\rho}\partial_\mu h_{\nu\rho} - 2\partial_\alpha h^{\alpha\rho}\partial_\nu h_{\mu\rho} + \partial^\rho h^\alpha_\alpha \partial_\mu h_{\nu\rho} \\ & + \partial^\rho h^\alpha{}_\alpha \partial_\nu h_{\mu\rho}\} - \tfrac{1}{8}\bar{g}_{\mu\nu}\{3\partial^\alpha h_{\rho\beta}\partial_\alpha h^{\rho\beta} + 4\partial_\alpha h^{\alpha\rho}\partial_\rho h^\beta{}_\beta - \partial^\alpha h^\rho{}_\rho \partial_\alpha h^\beta{}_\beta - 4\partial_\alpha h^{\alpha\beta}\partial_\rho h^\rho{}_\beta - 2\partial_\alpha h^{\rho\beta}\partial_\rho h^\alpha{}_\beta\} = \frac{1}{2M^2}\delta t_{\mu\nu}.\end{aligned} \tag{19}$$

Here the indices are raised and lowered with the homogenous background metric $\bar{g}_{\mu\nu}$. The average $\langle \delta G_{\mu\nu}\rangle$ concerns only the part quadratic in $h_{\mu\nu}$, since $\langle h_{\mu\nu}\rangle$ vanishes by definition. It is homogenous (it involves a volume integral) and we can therefore perform integration by parts for the space derivatives. On time scales of the order of the characteristic length scales of the fluctuations, $\langle \delta G_{\mu\nu}\rangle$ is also essentially static. This allows us to perform integration by parts for the time derivatives as well, and we obtain

$$\begin{aligned}\langle \delta G_{\mu\nu}\rangle = {} & \tfrac{1}{4}\langle h^{\alpha\rho}\partial_\mu\partial_\nu h_{\alpha\rho} + 3h^\alpha{}_\alpha \partial^\rho\partial_\rho h_{\mu\nu} - 2h^{\alpha\rho}\partial_\rho\partial_\alpha h_{\mu\nu} \\ & - 2h^\rho{}_\mu \partial^\alpha\partial_\alpha h_{\rho\nu} + 2h_{\rho\mu}\partial^\rho\partial_\alpha h^\alpha{}_\nu - h^\alpha{}_\alpha \partial_\mu\partial_\rho h^\rho{}_\nu \\ & - h^\alpha{}_\alpha \partial_\nu\partial_\rho h^\rho{}_\mu\rangle - \tfrac{1}{8}\bar{g}_{\mu\nu}\langle h^{\alpha\rho}\partial^\beta\partial_\beta h_{\alpha\rho} \\ & + h^\alpha{}_\alpha \partial^\rho\partial_\rho h^\beta{}_\beta - 2h^{\alpha\rho}\partial_\rho\partial_\beta h^\beta{}_\alpha\rangle.\end{aligned} \tag{20}$$

On the other hand, the linear part of the field equation relates the metric perturbations to the perturbations in $t_{\mu\nu}$,

$$\partial^2 h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial_\rho h^\rho{}_\nu - \partial_\nu \partial_\rho h^\rho{}_\mu = -\frac{1}{M^2}\left(\delta t_{\mu\nu} - \frac{1}{2}\delta t^\rho{}_\rho \bar{g}_{\mu\nu}\right),$$

$$\partial^2 h - \partial_\mu \partial_\nu h^{\mu\nu} = \frac{1}{2M^2}\delta t^\rho{}_\rho, \tag{21}$$

where we use $\partial^2 = \partial^\rho \partial_\rho$ and $h = h^\rho{}_\rho$.

Equations (20) and (21) simplify considerably in the harmonic gauge, which we adopt from now on,

$$\partial_\mu h^\mu{}_\nu = \tfrac{1}{2}\partial_\nu h. \tag{22}$$

The linear field equation becomes

$$\partial^2 h_{\mu\nu} = -\frac{1}{M^2}(\delta t_{\mu\nu} - \tfrac{1}{2}\delta t^\rho{}_\rho \bar{g}_{\mu\nu}) = -\frac{1}{M^2}\delta s_{\mu\nu} \tag{23}$$

and the quadratic metric fluctuations read

$$\langle \delta G_{\mu\nu}\rangle = \tfrac{1}{4}\langle h^{\alpha\rho}\partial_\mu\partial_\nu h_{\alpha\rho} - \tfrac{1}{2}h\partial_\mu\partial_\nu h - 2h^\rho{}_\mu \partial^2 h_{\rho\nu} + 2h\partial^2 h_{\mu\nu}\rangle - \tfrac{1}{8}\bar{g}_{\mu\nu}\langle h^{\alpha\rho}\partial^2 h_{\alpha\rho} + \tfrac{1}{2}h\partial^2 h\rangle. \tag{24}$$

Neglecting graviational waves, Eq. (23) has the retarded solution

$$h_{\mu\nu}(\vec{x},\tau) = \frac{a^2}{4\pi M^2}\int d^3\vec{x}'\,\frac{\delta s_{\mu\nu}(\vec{x}',\tau - |\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}, \tag{25}$$

where τ obeys $d\tau = dt/a$. We recover Newton's law for the graviational potential $\phi = -h_{00}/2$ for static point sources.

For a distribution of starlike objects, the time derivatives of $h_{\mu\nu}$ involve the peculiar comoving velocities of these objects. Since the peculiar velocities are small as compared to the speed of light, we can neglect the time derivatives in Eq. (24) as compared to the space derivatives. Furthermore, by virtue of rotation symmetry, the expectation values involving only one derivative in a given space direction vanish, and one infers

$$\rho_g = -2M^2\langle \delta G_{00}\rangle = -2M^2\langle \tfrac{1}{8}h^{\alpha\rho}\Delta h_{\alpha\rho} + \tfrac{1}{16}h\Delta h - \tfrac{1}{2}h_0^\rho \Delta h_{\rho 0} + \tfrac{1}{2}h\Delta h_{00}\rangle,$$

$$p_g = -\frac{2M^2}{3}\langle \delta G^i_i\rangle = 2M^2\langle \tfrac{1}{24}h^{\alpha\rho}\Delta h_{\alpha\rho} + \tfrac{5}{48}h\Delta h + \tfrac{1}{6}h^{\rho i}\Delta h_{\rho i} - \tfrac{1}{6}h\Delta h^i_i\rangle \tag{26}$$

with $\Delta = \bar{g}^{ij}\partial_i\partial_j$. Sums over all double indices are implied, with latin indices running from 1 to 3. To leading order we only need to keep δt_{00} such that $\delta s_{00} = \frac{1}{2}\delta t_{00} = \frac{1}{2}\delta\rho, \delta s_{ij} = \frac{1}{2}\delta\rho\bar{g}_{ij}$ and $h_{ij} = h_{00}\bar{g}_{ij}$, $h_{0i} = 0$, $h = 2h_{00}$. This results in

$$\rho_g = -\tfrac{9}{2}M^2\langle h_{00}\Delta h_{00}\rangle, \quad p_g = \tfrac{1}{6}M^2\langle h_{00}\Delta h_{00}\rangle \tag{27}$$

and we infer the equation of state for the gravitational energy momentum tensors of starlike objects,

$$p_g = -\tfrac{1}{27}\rho_g \tag{28}$$

Using Eq. (25) we can also express ρ_g in terms of the correlation function for the energy density fluctuations ($V = a^3\int d^3x$)

$$\rho_g = \frac{9a^5}{32\pi M^2 V}\int d^3x \int d^3y \frac{1}{|\vec{x}-\vec{y}|}\delta\rho(\vec{x})\,\delta\rho(\vec{y}). \tag{29}$$

[The "retardation" in Eq. (25) can be neglected since it involves again the peculiar velocities.] It is instructive to employ a comoving Fourier basis

$$\delta\rho(x) = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\vec{x}}\delta\rho(\vec{k}) \tag{30}$$

where

$$\langle \delta\rho(\vec{k})\,\delta\rho(\vec{k}')\rangle = G(k)(2\pi)^3\delta(\vec{k}+\vec{k}'). \tag{31}$$

The two-point density correlation function $G(k)$ depends only on the invariant $k^2 \equiv \vec{k}^2$. One finds

$$\rho_g = \frac{9a^2}{8M^2}\int \frac{d^3k}{(2\pi)^3}k^{-2}G(k). \tag{32}$$

From $\rho(-k) = \rho^*(k)$, one infers that $G(k)$ is a real positive quantity. This implies that ρ_g is positive whereas p_g is negative. For small k or long distances, $G(k)$ decreases rapidly and the k integral is infrared finite. The interesting part comes from large k, where condensed objects such as stars contribute. On these scales it is convenient to switch to physical momenta $\vec{q} = \vec{k}/a$ such that

$$\rho_g = \frac{9}{8M^2}\int \frac{d^3q}{(2\pi)^3}q^{-2}\tilde{G}(q). \tag{33}$$

Here we employ $\delta\rho(x) = \int[d^3q/(2\pi)^3]e^{ia\vec{q}\vec{x}}\delta\rho(q)$ and $\langle \delta\rho(\vec{q})\,\delta\rho(\vec{q}')\rangle = \tilde{G}(q)(2\pi)^3\delta(\vec{q}-\vec{q}')$ is the correlation function as a function of physical (not comoving) momenta. For a given static $\tilde{G}(q)$, the gravitational incoherent energy density ρ_g would not depend on the scale factor. However, the condensed objects are diluted by the cosmological expansion, and $G(q) \sim a^{-3}$ implies $\rho_g \sim a^{-3}$, similar to the energy density in dark or baryonic matter. We conclude that ρ_g is essentially a fixed fraction of the energy density of matter: $R=\rho_g/\bar{\rho}>0$ is independent of time.

Together with the gravitational equation of state (28) this can actually be used for an estimate of corrections to the equation of state of matter, $\bar{p}=w_m\bar{\rho}$. Indeed, the matter and gravitational energy momentum tensor are not separately conserved. Gravitational potentials lead to pecular velocities and, therefore, to nonzero $\bar{p}$. In other words, the equation of state $w_m=0$ holds only for "free particles" (ideal dust), i.e., if the gravitational interactions are neglected. As for all interacting systems, we expect corrections. Conservation of the total energy momentum tensor is, of course, exact and implies for $dR/dt=0$,

$$\dot{\bar{\rho}}(1+R)+3H(\bar{\rho}+\bar{p}+\rho_g+p_g)=0. \tag{34}$$

If $\bar{\rho}$ is dominated by massive objects or massive nonrelativistic particles, we can approximate $\bar{\rho}=\bar{\rho}_M+3\bar{p}/2$ where $\bar{\rho}_M$ is the contribution of the particle masses. Assuming that no masses are added or changed during the relevant period in the cosmological evolution, we infer $\bar{\rho}_M\sim a^{-3},\dot{\bar{\rho}}_M=-3H\bar{\rho}_M$. Furthermore, if $\bar{p}/\bar{\rho}$ is approximately constant, this extends to $\dot{\bar{\rho}}=-3H\bar{\rho}$. Equation (34) therefore yields the simple relation

$$\bar{p}+\bar{p}_g=0 \tag{35}$$

and we infer an estimate for the pressure of matter, which is due to the gravitational interactions,

$$\bar{p}=w_m\bar{\rho}=-\frac{p_g}{\rho_g}R\bar{\rho}=\frac{R}{27}\bar{\rho}. \tag{36}$$

Since R is small (see the next section for an estimate) this amounts only to a tiny correction, justifying our neglection of pecular velocities. We note that the estimate (35), (28) plays the role of a "cosmic virial theorem" since it relates the average kinetic energy[10] $\bar{p}$ to the average gravitational potential ρ_g.

Turning our argument around we emphasize that the relation (35) implies the cold dark matter expansion law $\bar{\rho}\sim a^{-3}$ provided $\dot{R}=0$. If we would neglect p_g, a nonzero pressure $\bar{p}$ would correct the expansion according to $\bar{\rho}\sim a^{-3(1+w_m)}$. This correction is cancelled by the presence of the gravitational pressure p_g. Thus back-reaction effects play a role in evolution—their role being to ensure the validity of the cold dark matter expansion law even in the presence of pecular velocities or nonzero $\bar{p}$. Corrections arise only for periods where $\dot{R}$ or $\dot{w}_m$ do not vanish.

[10] The value of ρ_g depends on the precise definition of this quantity, e.g., $\tilde{\rho}_g=2M^2\langle G_{0\rho}g^{\rho 0}\rangle=\rho_g-2M^2\langle G^{(1)}_{0\rho}h^{\rho 0}\rangle=(5/9)\rho_g$. This does not affect p_g and the relation $\bar{p}+p_g=0$, whereas the ratios p_g/ρ_g and $\rho_g/\bar{\rho}$ get modified.

IV. DO STARS AND GALAXIES MODIFY THE EXPANSION OF THE UNIVERSE?

In this section we estimate the size of the back-reaction effect quantitatively for a standard cold dark matter universe. We can use Eq. (23) in order to express Δh_{00} in terms of $\delta\rho$, and obtain from Eq. (27)

$$\rho_g=\tfrac{9}{4}\langle h_{00}\delta\rho\rangle. \tag{37}$$

For small h_{00} this "weighs" the energy contrast $\delta\rho$ with the Newtonian potential ϕ,

$$\rho_g=-\tfrac{9}{2}\langle\phi\,\delta\rho\rangle. \tag{38}$$

For compact objects, $\delta\rho$ is almost equal to the local value of ρ. For starlike extended objects the size of their own gravitational potential is maximal at the surface, $\phi_{max}=-mG/L$. For small ϕ_{max}, the contribution of isolated stars to ρ_g is therefore suppressed by a factor $\sim 10^{-6}$ as compared to their contribution to $\bar{\rho}$, in accordance with Eqs. (15), (17), (18). We need, however, also the contribution of other stars to ϕ. This becomes particularly simple in the language (37) or (38). As long as gravity remains weak, we only have to fold any mass concentration with the gravitational potential at the same location. Incidentally, this shows that our previous association of the relevant ratio $R=\rho_g/\bar{\rho}$ with the Newtonian potential can be made quantitative

$$R=-\frac{9}{2}\langle\!\langle\phi\rangle\!\rangle=-\frac{9\langle\phi\,\delta\rho\rangle}{2\bar{\rho}} \tag{39}$$

where $\langle\!\langle\phi\rangle\!\rangle$ means an appropriately weighted value of ϕ. This also yields a quantitative value for the pressure of matter (and therefore the kinetic energy or pecular velocities) according to the cosmic viral theorem,

$$\bar{p}=-\tfrac{1}{6}\langle\!\langle\phi\rangle\!\rangle. \tag{40}$$

Note that equilibration has not been invoked for this estimate. The cosmic virial theorem follows directly from $p_g/\bar{\rho}=\langle\!\langle\phi\rangle\!\rangle/6, \dot{R}=0$, and $\bar{p}/\bar{\rho}=$const. The average kinetic energy density $\langle T\rangle/V=3\bar{p}/2=-\langle\!\langle\phi\rangle\!\rangle/4$ may be compared with a virialized gravitationally bound system, where $\langle T\rangle/V=-\langle\phi\rangle/2$.

For cold dark matter galaxies, the value of the galactic gravitational potential in the outer regions, in particular, the halo, can be estimated from the rotation velocities

$$v_{rot}^2(r)=r\frac{\partial}{\partial r}\phi. \tag{41}$$

Within the halo ($r\leq r_H$) the dependence of ϕ on r is approximately logarithmic,

$$\phi=-\overline{v_{rot}^2}\ln\frac{r_H}{r}. \tag{42}$$

With $v_{rot}=O(10^{-3})$ we conclude that the galactic potential is of a size similar to the local potential on the surface of a typical star (17).

Clusters of galaxies, however, have a deeper potential well. A typical value for a cluster is

$$\phi_{cl}=-10^{-4}. \tag{43}$$

If most matter is found within clusters, this gives an approximate lower bound for the gravitational energy density

$$\rho_g \lesssim \tfrac{9}{2}|\phi_{cl}|\bar{\rho}. \tag{44}$$

We observe that this effect results from the mutual coherent correlations between all the stars in a cluster. The dominant length scale of this contribution to the correlation function (33) is related to the size of the cluster.

There may still be sizeable contributions arising from correlations on smaller scales. The center of the galaxy typically contains a region with large gravitational field. In this region, however, our linearized analysis does not apply any more. A similar statement holds for individual black holes outside the center of the galaxy. The precise evaluation of these contributions to ρ_g needs a nonlinear analysis and depends crucially on the question of how much of the matter in the universe is found in regions with a strong gravitational field. We will briefly turn to this question in the next section. Only if such strong field contributions are substantially above the bound (44), the gravitational energy density could be relevant for the evolution of the universe. On the other hand, for a moderate contribution from strong field regions, the back-reaction effect remains small for conventional dark matter galaxies and clusters. A value $\rho_g/\bar{\rho}\lesssim 10^{-3}$ seems to be too small to substantially modify the evolution of the universe after structure formation.

V. CONTRIBUTION OF BLACK HOLES

For black holes and other regions with strong gravitational fields, the linear analysis of the preceding sections does not remain valid. For an individual black hole—or any other static and isotropic object—in a flat space-time background, the sum of matter and gravitational energy density is fixed, however, by a conservation law. This also holds for the pressure. These laws can be expressed in terms of linearized gravity [13] and are the analogues of charge conservation in electromagnetism. We parametrize the static and isotropic metric outside a mass concentration in ''isotropic coordinates'' as $ds^2=-B(u)dt^2+C(u)d\vec{x}d\vec{x}$ with $u^2=\vec{x}\vec{x}$. The sums of the energy densities and the total pressures are related to the functions B and C by the linearized Einstein equations, with $C'=\partial C/\partial u$, etc., as

$$\rho(u)+\rho_g(u)=-2M^2\left(C''+\frac{2}{u}C'\right),$$

$$p(u)+p_g(u)=-\frac{2M^2}{3}\left(C''+\frac{2}{u}C'+B''+\frac{2}{u}B'\right). \tag{45}$$

Also, using Gauss's law, one finds for the integrals over a volume with $u'<u$,

$$m(u)=4\pi\int du'u'^2[\rho(u')+\rho_g(u')]=-8\pi M^2u^2C'(u)$$

$$\hat{P}(u)=4\pi\int du'u'^2[p(u')+p_g(u')]$$

$$=\frac{1}{3}m(u)-\frac{8\pi}{3}M^2u^2B'(u). \tag{46}$$

For the Schwarzschild metric, the functions $B(u)$ and $C(u)$ are given by ($G^{-1}=16\pi M^2$)

$$B(u)=\left(1-\frac{mG}{2u}\right)^2\left(1+\frac{mG}{2u}\right)^{-2},$$

$$C(u)=\left(1+\frac{mG}{2u}\right)^4. \tag{47}$$

This yields, in particular, $m(u\to\infty)=m$ with m the total mass of the object related to the Schwarzschild radius $R_S=m/(8\pi M^2)$. Similarly, we observe that the integrated pressure vanishes, $\hat{P}(u\to\infty)=0$.

On a length scale which is large as compared to the characteristic size of the objects, a collection of static isotropic objects—including black holes—can be viewed as a collection of point particles with masses m_l. The total energy momentum tensor $T_{\mu\nu}$ in Eq. (2) averages both the matter and gravitational contributions. If the objects are sufficiently distant from each other, this amounts to summing $m_l(u\to\infty)$ and $\hat{P}_l(u\to\infty)$. A collection of static isotropic objects behaves therefore like a nonrelativistic gas with zero pressure.[11] In particular, black holes that have already formed before structure formation—this is the meaning of ''static'' in a cosmological context—behave just as a contribution to cold dark matter. Irrespective of the fact that their gravitational energy density ρ_g can be substantial, the back-reaction effect from condensed black holes during or after structure formation would not lead to a deviation from the usual equation of state.

The only loophole in the argument that back-reaction effects can be neglected in a cold dark matter universe remains the hypothesis of a substantial contribution from black holes forming during or after structure formation. For such objects we cannot use the static approximation (46). At this stage we cannot exclude that a nonzero, perhaps even negative, pressure could play a role for nonstatic regions with strong gravitational fields.

For a cold dark matter universe we conclude that a sizeable influence of back-reaction effects is only possible if a substantial fraction of the energy density is found in regions

[11] The vanishing of the pressure including the gravitational contribution p_g is actually even better obeyed, as if p_g had been neglected. Note that $\langle t_{\mu\nu}\rangle+T^g_{\mu\nu}$ is covariantly conserved with respect to the background metric $\bar{g}_{\mu\nu}$ by virtue of Eq. (2).

of strong gravitational fields which evolve in time and cannot be described effectively as static objects. For being important at the time relevant to the supernova Hubble diagram such a hypothetical evolution would have to persist at a redshift $z \approx 1$. Discarding this—perhaps rather unlikely—possibility we find no relevant back-reaction effect from structure formation in a cold dark matter universe.

VI. COSMON FLUCTUATIONS

Recent cosmological observations suggest the presence of a homogenous dark energy component. It has been proposed that the dark energy density is time dependent and can be described by the dynamics of a scalar field, the cosmon [2]. If this quintessence scenario [2,3] is true, one may also suspect that inhomogeneities in the cosmon field could be associated with extended structures [17]. In this section we argue that the back-reaction effect of structure formation is much stronger in a "cosmon dark matter universe" [11] than in the standard cold dark matter universe. One main reason is the direct contribution of cosmon fluctuations to the averaged energy momentum tensor. One also observes large spacelike components of the gravitational field in cosmon lumps, contributing to large ρ_g and p_g. We underline that the material of this section is only relevant if the present local fluctuations of the cosmon field are really substantial—a possibility that remains speculative as long as no consistent picture of a cosmon dark matter universe has been developed. If the scalar field mediating quintessence remains homogenous to a high degree in the present epoch, its back-reaction effects are small and can be neglected.

There are three new ingredients for the back reaction in presence of an inhomogenous scalar field:

(1) Local fluctuations of the scalar field around its homogenous background value induce a new contribution to the total energy momentum tensor (2). The scalar contribution to the local energy momentum tensor

$$t^{\varphi}_{\mu\nu} = -V(\varphi)g_{\mu\nu} + \partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}\partial^\rho\varphi\partial_\rho\varphi g_{\mu\nu} \tag{48}$$

yields, after averaging in Eq. (1) or (2), both a contribution from the homogenous background field $\bar{\varphi}$ (the average of φ) and from the inhomogenous local fluctuations of the cosmon field $\delta\varphi = \varphi - \bar{\varphi}$, namely,

$$\langle t^{\varphi}_{\mu\nu}\rangle = T^h_{\mu\nu} + T^c_{\mu\nu}. \tag{49}$$

Here $T^h_{\mu\nu}$ stands for the time variable dark energy or homogenous quintessence and corresponds to Eq. (48), with φ replaced by $\bar{\varphi}$:

$$T^h_{00} = \rho_h = V(\bar{\varphi}) + \tfrac{1}{2}\dot{\bar{\varphi}}^2,$$

$$T^h_{ij} = p_h \bar{g}_{ij}, \quad p_h = -V(\bar{\varphi}) + \tfrac{1}{2}\dot{\bar{\varphi}}^2. \tag{50}$$

The difference $T^c_{\mu\nu} = \langle t^{\varphi}_{\mu\nu}\rangle - T^h_{\mu\nu}$ is due to the cosmon fluctuations [similar to Eq. (4)] and can again be written in the form

$$T^c_{00} = \rho_c, \quad T^c_{ij} = p_c \bar{g}_{ij}. \tag{51}$$

It has been discussed in [11].

(2) The evolution equation for the background scalar field also obtains a contribution q^φ from back-reaction effects [10,19]

$$\ddot{\bar{\varphi}} + 3H\dot{\bar{\varphi}} + \frac{\partial V}{\partial\varphi}(\bar{\varphi}) = q^\varphi. \tag{52}$$

For cosmon dark matter, the "incoherence force" q^φ has been discussed in [11]. We note that q^φ can also receive a contribution if the cosmon couples to "standard" cold dark matter [19]. Such a contribution would not be affected by structure formation.

(3) The gravitational energy density ρ_g and pressure p_g can be enhanced as compared to cold dark matter. We discuss this possible effect in a simple model of a collection of cosmon lumps [17]. We expect that the most important features could be present also for more general nonlinear cosmon field configurations beyond the specific model considered here.

Let us consider a collection of cosmon lumps (some of them could be associated [17] to some of the galaxies[12]), which can be described in comoving coordinates as

$$\varphi = \bar{\varphi}(t) + \sum_\ell \delta\varphi_\ell(u_\ell),$$

$$g_{00} = -\left\{1 + \sum_\ell [B_\ell(u_\ell) - 1]\right\},$$

$$g_{ij} = a^2(t)\delta_{ij}\left\{1 + \sum_\ell [C_\ell(u_\ell) - 1]\right\},$$

$$g_{0i} = 0. \tag{53}$$

Here $\vec{x}_\ell$ is the comoving coordinate of the center of the lump ℓ,

$$u^2_\ell = a^2(\vec{x} - \vec{x}_\ell)^2, \tag{54}$$

and $\bar{\varphi}(t)$ is the cosmological background value of the cosmon field φ which leads to homogenous quintessence. We assume that the lumps are well separated such that $\delta\varphi_\ell, B_\ell$, and C_ℓ can be determined from the coupled gravity-scalar field equation for a single (spherically symmetric) lump. This system has been discussed in [17] and we concentrate on the "halo region," which may give an important contribution to the energy momentum tensor. In this region we can approximate[13] ($R^2_{H,\ell}/e^2 \langle u^2_\ell \leq R^2_{H,\ell}$)

[12]For our own galaxy a large "cosmon halo" seems unlikely in view of the strong distortion of light trajectories [18].

[13]Note that the singularities at $u_\ell = R_{H,\ell}/e$ correspond to pointlike singularities at $r_\ell = 0$ in Schwarzschild coordinates.

$$C_\ell = \frac{R_{H,\ell}^2}{u^2\ell}\ln^2\left(\frac{eu_\ell}{R_{H,\ell}}\right)$$

$$B_\ell = 1+\frac{1}{|\gamma_\ell|}\ln\left(\frac{C_\ell u_\ell^2}{R_{H,\ell}^2}\right) = 1+\frac{1}{|\gamma_\ell|}\ln\left[\ln^2\left(\frac{eu_\ell}{R_{H,\ell}}\right)\right],$$

$$\delta\varphi_\ell = \gamma_\ell M \ln B_\ell, \tag{55}$$

where the scale $R_{H,\ell}$ can be associated with the radius of the halo and $1/|\gamma_\ell| = v_{rot}^2$ is associated with the rotation velocity of objects in circular orbits within the halo. The spherically symmetric solution of the coupled gravity-scalar system in empty space has indeed two integration constants $(R_{H,\ell}, \gamma_\ell)$. The total mass of the object can be expressed in terms of these constants [17]. We consider here small values of $1/|\gamma_\ell|$ which correspond to realistic rotation velocities and halo extensions of galaxies [17]. While B_ℓ is close to 1 except for in the vicinity of the singularity, we observe that C_ℓ deviates substantially from 1 within the halo. As a consequence, h_{ij}/a^2 is of the order one within the halo region and we expect substantial contributions to the back-reaction effects.

For cosmon lumps, the spacelike components of the energy momentum tensor are important. This contrasts with stars. For a static lump the time derivative of the scalar field vanishes, and one finds for a single cosmon lump

$$\rho_\varphi = -t_0^0 = V(\varphi) + \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi = V(\varphi) + \frac{1}{2C}\left(\frac{\partial\varphi}{\partial u}\right)^2,$$

$$p_\varphi = \tfrac{1}{3}t_i^i = -\tfrac{1}{3}[\rho_\varphi + 2V(\varphi)]. \tag{56}$$

These relations are easily generalized to a collection of well separated lumps. The cosmon part of the energy momentum tensor obeys the equation of state

$$p_c = w_c\rho_c, \quad w_c = -\tfrac{1}{3}(1+2\langle\Delta V\rangle/\rho_c). \tag{57}$$

Here we have kept only the contribution from the inhomogenous fluctuations, and ΔV is the difference between the local value of the cosmon potential and the homogenous cosmological value.[14] For simplicity we will later concentrate on the case where $\langle\Delta V\rangle$ can be neglected such that $w_c = -1/3$. Within the halo the potential contribution is indeed small. More generally, the background potential $V(\overline{\varphi})$ is small as compared to the local energy densities such that effectively $\Delta V \geqslant 0$. This implies that static lumps lead to a negative cosmon equation of state, $w_c \leqslant -1/3$.

We next turn to the gravitational contribution. For a spherically symmetric static lump, we can again use the relations (45) and (46) for the total energy density and pressure. In particular, far away from the lump, the solution approaches the standard Schwarzschild solution [17] and we infer that the total integrated pressure vanishes ($u_c = R_H/e$),

$$\hat{P}(u\to\infty) = \int dV(p_c + p_g + p_M) = 0. \tag{58}$$

This implies a cancellation between a negative cosmon and positive gravitational contribution. [We also have included possible matter (p_M, ρ_M) in the "bulk" of the object.] A partial cancellation also happens for the energy density

$$m(u\to\infty) = \int dV(\rho_c + \rho_g + \rho_M) = m. \tag{59}$$

Indeed, if γ is large, the Schwarzschild radius $R_s = m/(8\pi M^2) \approx 2R_H/|\gamma|$ is small compared to the halo radius R_H and this is equivalent to a substantial cancellation between ρ_c and ρ_g.

For an understanding of these cancellations in greater detail, it is instructive to study the gravitational contribution in the linear approximation. Since the coordinates used for the metric (53) are not harmonic, the formulas (23) and (26) receive corrections.[15] Being only interested in the qualitative features, we neglect these corrections here. This yields for the gravitational energy density and pressure

$$\rho_g = \langle \tfrac{1}{4}h^{\mu\nu}\delta s_{\mu\nu} + \tfrac{1}{8}h\overline{g}^{\mu\nu}\delta s_{\mu\nu} + h^{\mu 0}\delta s_{\mu 0} + h\,\delta s_{00}\rangle,$$

$$p_g = -\langle \tfrac{1}{12}h^{\mu\nu}\delta s_{\mu\nu} + \tfrac{5}{24}h\overline{g}^{\mu\nu}\delta s_{\mu\nu} + \tfrac{1}{3}h^{\mu i}\delta s_{\mu i} - \tfrac{1}{3}h\overline{g}^{ij}\delta s_{ij}\rangle. \tag{60}$$

If the potential term can be neglected, the cosmon lumps obey $\delta s_{\mu 0} = 0$ such that

$$\rho_g = \tfrac{1}{8}\langle(2h^{ij} + h\overline{g}^{ij})\delta s_{ij}\rangle,$$

$$p_g = \tfrac{1}{8}\langle(-\tfrac{10}{3}h^{ij} + h\overline{g}^{ij})\delta s_{ij}\rangle. \tag{61}$$

We evaluate the above expression in coordinates adapted to the present cosmological time with $a=1, \overline{g}^{ij} = \delta^{ij}$ and assume first that only the halo region of the lumps contributes effectively to ρ_g, p_g, ρ_c and p_c. We can therefore evaluate the ratios p_g/ρ_g and ρ_g/ρ_c for a single cosmon lump. In the coordinate system (53), one has [neglecting again $V(\varphi)$]

$$\delta s_{ij} = \partial_i\varphi\partial_j\varphi = \frac{x_i x_j}{u^2}\left(\frac{\partial\varphi}{\partial u}\right)^2. \tag{62}$$

With $h_{ij} = (C-1)\delta_{ij} = \tfrac{1}{3}h\,\delta_{ij}$, one finds

$$p_g = -\tfrac{1}{15}\rho_g \tag{63}$$

and the gravitational energy density ρ_g reads

$$\rho_g = \frac{5}{8}\left\langle [C(u)-1]\left(\frac{\partial\varphi}{\partial u}\right)^2\right\rangle. \tag{64}$$

[14] For a single cosmon lump, the sign ΔV may be positive or negative, depending on the sign of $\partial\varphi/\partial u$.

[15] Alternatively, one may translate the metric (53) into harmonic coordinates.

This is to be compared with the energy density in the cosmon field

$$\rho_c = \frac{1}{2}\left\langle C^{-1}(u)\left(\frac{\partial\varphi}{\partial u}\right)^2\right\rangle, \quad p_c = -\frac{1}{3}\rho_c . \tag{65}$$

The average in Eqs. (64) and (65) is given as an integration over the volume of the lump,

$$\langle T(u)\rangle = 3\int_{u_b}^{R_H} du\, u^2 T(u)/(\rho_H^3 - u_b^3), \tag{66}$$

where u_b corresponds to the radius of the "bulk" of the galaxy and must be larger than the critical value $u_c = R_H/e$ for the central singularity. (We recall here that the averaging needs to be done with respect to the background metric $\bar{g}_{\mu\nu}$ such that the volume is just the Cartesian volume in the coordinates $\vec{x}$. It does not involve the "microscopic volume" which would have an additional factor $\sqrt{g} = B^{1/2}C^{3/2}$.)

We note that $C(u)$ becomes smaller than 1 inside the halo such that ρ_g is indeed negative and p_g positive. [The numerical prefactors will be altered if Eqs. (21) and (20) are used instead of Eqs. (23) and (26).] This demonstrates how the cancellation between positive ρ_c and negative ρ_g becomes visible already in the linear approximation. With $B \approx 1$ and

$$\left(\frac{\partial\varphi}{\partial u}\right)^2 \approx \frac{4M^2}{u^2 \ln^2(u/u_c)}, \tag{67}$$

we find that the integrands relevant for ρ_g and ρ_c, respectively, can be characterized by

$$I_{\rho_g} = u^3[C(u)-1]\left(\frac{\partial\varphi}{\partial u}\right)^2 = 4M^2\left\{\frac{R_H^2}{u} - \frac{u}{\ln^2(u/u_c)}\right\},$$

$$I_{\rho_c} = u^3 C^{-1}(u)\left(\frac{\partial\varphi}{\partial u}\right)^2 = \frac{4M^2u^3}{R_H^2 \ln^4(u/u_c)}. \tag{68}$$

They are both dominated by the region $u \rightarrow u_b$. We conclude that the energy density and pressure are actually dominated by the interior of the halo and/or by the bulk. An assumption about a halo domination is actually not justified, nor is the linear approximation for the computation of ρ_g and p_g.

Nevertheless, the need of a large cancellation between a positive cosmon energy denstiy and a negative gravitational energy density remains true for large $|\gamma|$, irrespective of the shortcomings of the above calculation. Already the integration of the cosmon energy density over the halo exceeds the total mass by a large factor $> R_H/R_s \approx |\gamma|$. The total sum (46) can only be balanced by a negative gravitational energy density of almost equal (averaged) size. We may summarize our discussion by extracting the following general features for large cosmon fluctuations: The cosmon energy density ρ_c is positive and the pressure p_c negative, typically with $p_c \approx -\rho_c/3$. This is accompanied by a negative gravitational energy density ρ_g and positive gravitational pressure p_g. For static isotropic configurations, large cancellations occur both for $\rho_c + \rho_g$ and $p_c + p_g$, implying $p_g \approx -\rho_g/3$. [This differs from Eq. (63) which involves unjustified approximations.] For more general, in particular, nonstatic, large cosmon fluctuations the detailed balance between gravitational and cosmon contributions may not occur anymore. It is plausible, however, that the above findings about the sign of the various contributions remain valid.

A very simple, but perhaps important, observation states that the cosmon pressure p_c is likely to be negative. Indeed, for large fluctuations we may neglect the subtraction of the potential and kinetic energy of the background field $\bar{\varphi}$. The cosmon pressure is then given by

$$p_c = \langle -V(\varphi) - \tfrac{1}{6} g^{ij}\partial_i\varphi\partial_j\varphi + \tfrac{1}{2}\dot{\varphi}^2\rangle. \tag{69}$$

We observe a negative contribution from the gradient term reflecting the spatial inhomogeneities of φ. Also the contribution of the potential is negative and only a fast time variation could cancel these two negative contributions.

Imagine now a period in the cosmological evolution where the cosmon fluctuations become substantial and their negative pressure is not (or only partially) cancelled by the pressure of metric fluctuations. The cosmological evolution would then be substantially affected by the negative pressure of cosmon dark matter. Furthermore, the cancellation between cosmon and gravitational energy density could be more effective than for the pressure. This could lead to a situation where the total energy momentum tensor is dominated by cosmon dark matter and quintessence with a substantially negative equation of state w,

$$w = p/\rho \approx \frac{p_c + p_g + p_h}{\rho_c + \rho_g + \rho_h}. \tag{70}$$

In fact, the pressure of dark energy, p_h, could also turn negative if the potential dominates over the kinetic energy during such an epoch. If w becomes smaller than $-1/3$, the expansion of the universe accelerates

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{\rho}{12M^2}(1+3w). \tag{71}$$

It is tempting to speculate that such a situation may occur towards the end of structure formation. The dominant contribution to $p_c + p_g$ may arise from cosmon inhomogeneities on the scales of clusters or larger. It is even conceivable that the present acceleration occurs only effectively for the metric averaged over a volume corresponding to a redshift z of order 1. At earlier times it may have been "visible" in the averaged metric relating to a smaller effective volume.

VII. CONCLUSION

As a conclusion, let us turn back to the question asked in the title: can structure formation influence the cosmological evolution? We have presented in this paper a few estimates and simple model calculations within a formalism which describes the back reaction of fluctuations. We find it unlikely that standard cold dark matter fluctuations lead to a substantial effect, even though these fluctuations are today large and strongly nonlinear. The basic reason is that the averaged Einstein equations are linear in the energy momentum tensor. The direct contribution of fluctuations in the energy density and pressure therefore cancels by virtue of the averaging. An indirect effect of these fluctuations shows up in the form of induced metric fluctuations. This effect is related to the gravitational energy density and pressure. We have seen, however, that the size of this induced fluctuation effect is small unless a substantial part of the matter is in regions with strong and time-varying gravitational fields. Furthermore, we have seen that by a ''cosmic virial theorem'' the gravitational pressure cancels the effect of the pressure of cold dark matter.

The situation can change drastically in the presence of a scalar cosmon field mediating quintessence. If the cosmon fluctuations grow large, their contribution to the back reaction becomes typically quite sizeable. The averaged Einstein equation as well as the averaged scalar evolution equation are not linear in the cosmon fluctuations. In contrast to standard cold dark matter, large fluctuations make therefore directly a large contribution to the averaged equations. Our computation for cosmon lumps has revealed that, typically the induced gravitational energy density and pressure are also large. For a collection of static and isotropic cosmon lumps this *gravitational* back reaction cancels the *cosmon* back reaction to a high degree. For the pressure, one observes a matching of a negative cosmon and a positive gravitational contribution. For more general large cosmon fluctuations, in particular if they are not static, this cancellation may not be perfect. A large back-reaction effect would then be expected for large cosmon fluctuations. We conclude that the back reaction could substantially influence the cosmological evolution after the time when large cosmon fluctuations have developed.

We have also argued that the equation of state of the combined cosmon and gravitational fluctuations may be substantially negative. In this event, a growth of fluctuations in the cosmon field towards the end of structure formation could trigger an acceleration of the expansion of the universe and provide an answer to the question why such an acceleration happened ''just now.'' Many pieces of the scenario outlined here are, however, fairly speculative. In particular, it remains to be seen whether a realistic effective action for the cosmon field can be found such that the cosmon fluctuations indeed grow large in consistency with the present observational information.

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