

Chaos and preheating

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We study the semiclassical particle creation in the preheating phase after inflation. We work in the long-wavelength limit, in which all fields are considered homogeneous. The particle creation is shown to be intrinsically connected to the existence of chaos in the system.

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I. INTRODUCTION

One of the most successful ideas to describe the early universe is inflation [1]. Its main problem may be how to leave behind an exponential scale factor and reach a radiation-dominated universe in thermal equilibrium. A possible way to solve this question is to turn on the interaction of the inflaton with another scalar field, typically radiation, when the former reaches the bottom of its potential. Initially, this interaction was introduced via a somewhat *ad hoc* dissipation term characterizing the so-called *reheating* process [2]. Later on, it was assumed a quadratic coupling between two scalar fields, which allowed for the transfer of energy from the inflaton to the radiation by parametric resonance: this is the *preheating* phase [3–6]. This model yields a much more effective amplification of particular modes of the radiation, at the same time raising some questions about the thermal state of the universe after the process is over [10,11].

In the usual approach to preheating models, one assumes a fixed evolution for the inflaton, and calculates the equation for a given mode of the coupled field χ , which depends on the particular inflaton potential chosen. In any case, the inflaton behaves as an infinite reservoir of energy, driving the exponential amplification

$$\chi_k \propto \exp(\mu_k t) \quad (1)$$

characterized by the Floquet exponent μ_k , of particular modes k of the radiation field χ indefinitely.

In this paper we consider both fields as a coupled system in a given background and investigate their properties; of particular interest are the issues of how chaotic the dynamics is and what is the relation between the resonance effects and its chaotic character.

Indeed, as we will shortly show, chaos arises precisely at the time when exponential amplification occurs. We further propose a relationship between the metric entropy, usually defined for chaotic systems, and the entropy corresponding to the particles produced after inflation by parametric resonance.

As will be seen below, in our simple model there is no interaction among different modes; thus, as in any system with a finite number of degrees of freedom, the energy will keep oscillating from one field to the other indefinitely. In practice, we expect that it will eventually be transferred into other fields. We conjecture this can be taken into account introducing by hand a viscosity term in the equation of motion for the radiation—this analysis will be accomplished elsewhere [7].

Note that there are fundamental differences between this work and previous papers in similar subjects. In Ref. [8], the existence of chaos on the evolution of the *scale factor* was investigated; here, the background evolution is given *a priori*. We stress that we are interested in the beginning of the preheating phase, when back reaction—the effect of the amplified field on the evolution of the scale factor—is not strong enough yet. Actually, we will consider a *static* universe, since the parametric resonance happens in a much shorter time scale than the expansion of a radiation-dominated universe [3]. The authors of Ref. [9] also discussed the chaotic behavior in the case of two-field inflation, but they used a symmetry breaking potential and investigated the enhancement on the production of topological defects. We have chosen a single-well potential for the sake of simplicity; a double-well potential would eclipse our point somehow. Reference [10] studies the approach to equilibrium for a couple of different potentials, but being interested in the turbulent phase right *after* preheating, the authors introduce a “normalized distance” in the phase space, according to which chaos sets in *after* the preheating is over. Here we apply the usual recipe [12,13] for computing the largest Lyapunov exponent, as explained below. As we mentioned above, we will show that parametric resonance and chaos seem to be fundamentally related.

In spite of the aforementioned assumptions, our model is able to grasp important qualitative features such as the energy threshold above which the dynamics becomes chaotic. The next section illustrates the relation between the Lyapunov (LE) and the Floquet (FE) exponents for two well known problems: the parametrically excited pendulum and the Mathieu equation. In Sec. III we introduce the analogous question for the coupled scalar fields in cosmology, followed by a brief description of the classical chaotic properties of the system. In Sec. IV, we follow a simple and transparent

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method [14], consistent with Heisenberg's principle, to obtain a semiclassical approximation for our model. We then study the onset of chaos on the effective equations of motion which describe the particle production process. Then we discuss the relationship between (*a priori*) different entropy definitions in Sec. V.

II. PARAMETRIC RESONANCE

A positive Lyapunov Exponent (LE) is the main characteristic of chaotic motion, as it indicates a strong sensitivity to small changes in the initial conditions. The LE measures the mean separation of two initially neighboring trajectories in the phase space, in logarithmic scale:

$$\lambda_i = \lim_{t \rightarrow \infty} \left\{ \frac{1}{t} \ln \left[\frac{L_i(t)}{L_i(0)} \right] \right\}. \quad (2)$$

Because such separation soon approaches the size of the attractor, a naive computation using the expression above would fail to detect the local rates of expansion. Thus the distance between the trajectories must be periodically normalized; the LE will then be the average of the exponential rates obtained this way.

The parametrically excited pendulum

$$\ddot{\theta} + 2\eta\dot{\theta} + [1 + p \cos(\omega t)] \sin(\theta) = 0 \quad (3)$$

is known to be chaotic [15]. Indeed, one can calculate its largest Lyapunov exponent and find a positive quantity. On the other hand, the Floquet theorem assures that the solution of Eq. (3) is given by

$$\theta(t) = f_p(t) \exp(\pm \mu t) \quad (4)$$

where μ is the Floquet exponent. It is easy to see that both exponents must actually be the same, since the only difference between them is the normalization procedure in the calculation of the LE. In the parametrically excited pendulum such procedure is not needed, since the independent variable θ is cyclic: it is never too large. The same reasoning applies to the study of metric perturbations in a chaotic background: the rate of growth of perturbations—valid only in the *linear* regime—is given by the LE, as shown in Ref. [16].

The relation between both exponents is also clearly seen in the typical parametric resonance phenomenon, described by the Mathieu equation

$$\ddot{R}(t) = -[\Omega^2 + g^2 x^2(t)] R(t) \quad (5)$$

where $x(t) = \sin(\omega t)$. We used $\omega = 10$, $\Omega = \sqrt{4/5}\omega$, and $g = 2\omega/\sqrt{10}$. The FE for the above equation can be exactly calculated [17], and for the used values one obtains $\mu = 0.5$. For the sake of completeness and as a test of our numerical code, we calculated it by plotting $(1/2t) \ln(n_R)$ versus t , where $n_R = \Omega/2(|R(t)|^2 + |\dot{R}(t)|^2/\Omega^2)$ can be interpreted as the energy. Figure 1 shows that the LE and both the calculated and theoretical values for the FE converge to the same value.

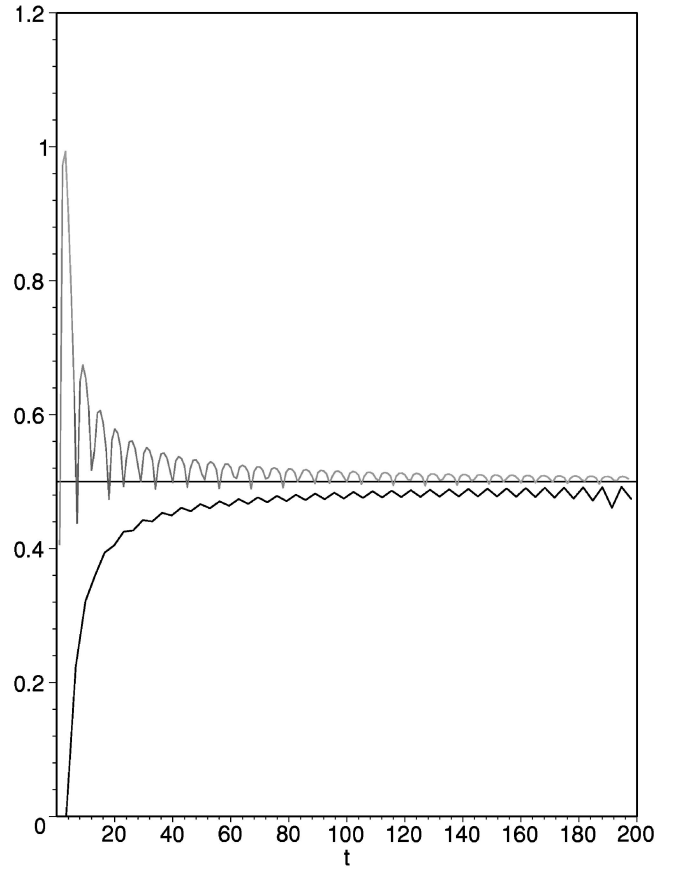


FIG. 1. Plot of the LE for the exact parametric resonance (upper curve). The straight line is the theoretical value for the FE (for the parameters indicated in the text); the lower curve is the FE as calculated by us.

If the phase space is not limited—as in the case of exact parametric resonance described by the Mathieu equation (5)—one cannot rely on the LE to tell if the system is chaotic or not [23]. Nevertheless, actual physical systems will have only a finite amount of energy available. Then, the available phase space will be finite and the LE criterium for the existence of chaos will hold.

III. COUPLED FIELDS

Most of the work in preheating has been made assuming a biquadratic coupling between the inflaton ϕ and the secondary field χ :

$$V_{\text{int}} = g^2 \phi^2 \chi^2. \quad (6)$$

We assume a flat Friedmann-Robertson-Walker (FRW) universe background whose line element is written as

$$\begin{aligned} ds^2 &= dt^2 - a^2(t) d\mathbf{x}^2 \\ &= a(\eta)(d\eta^2 - d\mathbf{x}^2) \end{aligned} \quad (7)$$

where $a(\eta)$ is the scale factor in conformal time. In this metric the equations of motions are

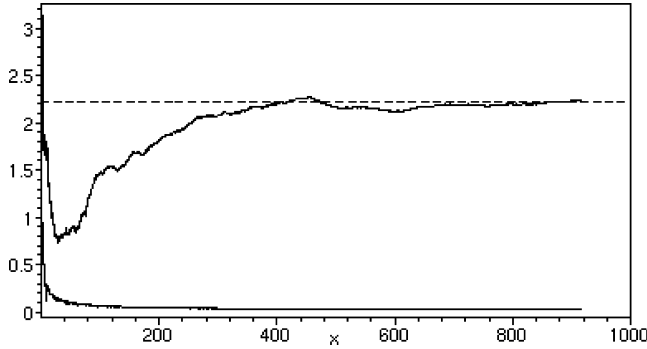


FIG. 2. Plot of the Lyapunov exponent for two different initial conditions. For both trajectories, $M=10$, $m=1$ and $g=1$. The dashed line indicates the limiting value for the nonvanishing LE.

$$Y''(\eta) + \Omega^2(\eta)Y(\eta) = 0, \quad (8)$$

$$X''(\eta) + \omega^2(\eta)X(\eta) = 0, \quad (9)$$

where $\Omega^2 = a^2 M^2 + g^2 X^2 - a''/a$ and $\omega^2 = a^2 m^2 + g^2 Y^2 - a''/a$, $X \equiv a\chi$ and $Y \equiv a\phi$. If we restrict ourselves to the beginning of the preheating phase, the back reaction can be safely neglected. In this case, the evolution of the universe is nearly a radiation dominated phase, with $a(\eta) = \eta/2$ and thus $a''/a = 0$. Actually, for the sake of simplicity, we rely on the much faster dynamics of preheating to assume a static universe, and, for convenience we adopt $a(\eta) = 1$ (and thus $t \equiv \eta$). By doing so we are neglecting the time dependence of the instability bands; we expect the qualitative aspects of the results presented here to remain unaltered in a more refined analysis [7]. We work in the long-wavelength limit and assume all fields as homogeneous. Indeed, the main contribution to the parametric resonance is due to the zero mode [18].

At the classical level, the system of Eqs. (8), (9) describes a well know chaotic system [19]. Indeed, if we use the standard technique [12,13] we can calculate the largest LE for two different trajectories. Figure 2 shows two such plots, one of which is nonchaotic. Since this system presents a finite number of degrees of freedom, the energy will oscillate back and forth between them. Thus a given variable will increase exponentially only during the energy transfer. The naive way to determine the FE is to look at the plot of $1/2 \eta \ln[Y^2(\eta)/Y^2(0)]$ but, because of the oscillation just mentioned, we would have to reset the time variable at the beginning of each phase, just as for the chaotic pendulum. Instead, we decided to plot $1/2 \ln[Y^2(\eta)/Y^2(0)]$ vs η and looked at the angular coefficients of the straight lines. Figure 3 shows such a plot for the same two initial conditions used in the previous figure.

The last two figures seem to show that the LE and the FE are numerically equal. For the parameters chosen the resonance period, although clearly noticeable, is too short for a statistical analysis that would allow for a measure of the small discrepancy between the obtained values for the FE and LE.

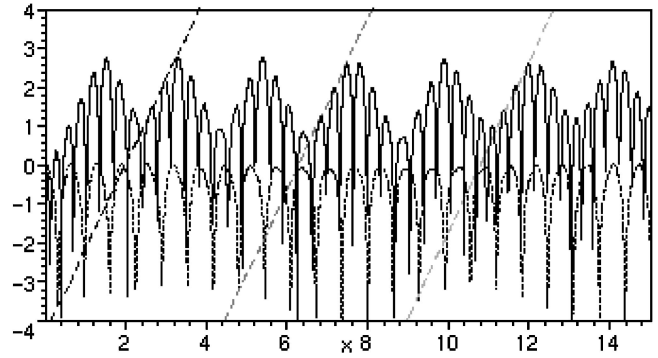


FIG. 3. Plot of $\ln(\mu t)$ vs t for the same initial conditions used for the previous graph. The angular coefficient of the straight dashed lines are given by the nonvanishing LE from the previous graph; their horizontal positions are arbitrary.

Energy threshold

We were able to numerically determine the existence of an energy threshold below which there is no amplification of the secondary field. Figures 4 and 5 show a plot of the LE and FE, respectively, for ten randomly chosen initial conditions. One can see that the trajectories with energies below a critical value are restricted to a certain region, mainly below the horizontal axis of the latter graph, while the ones above that limit oscillate with a much larger amplitude. Note that

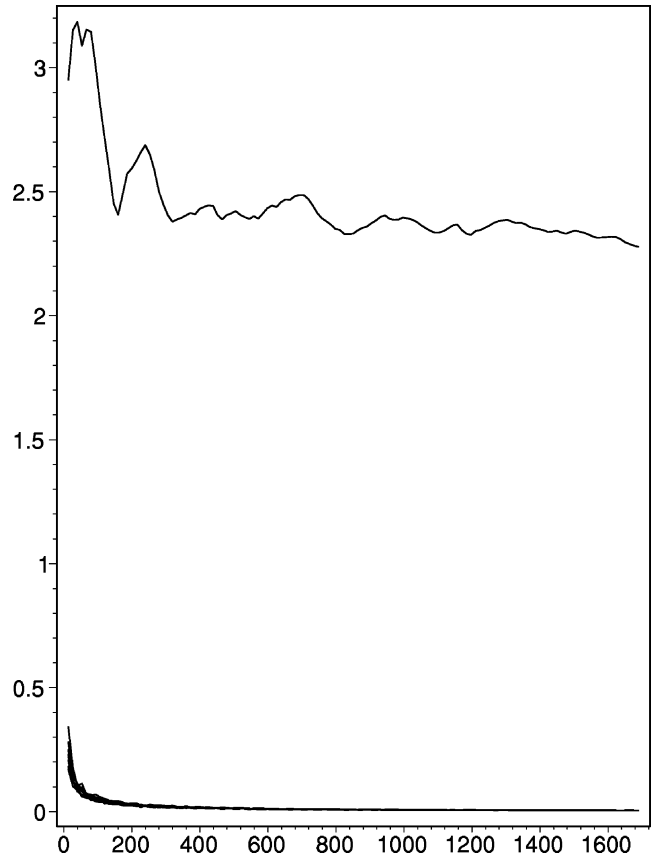


FIG. 4. Plot of the LE for ten different initial conditions. The upper line corresponds to a trajectory above a critical value for the energy.

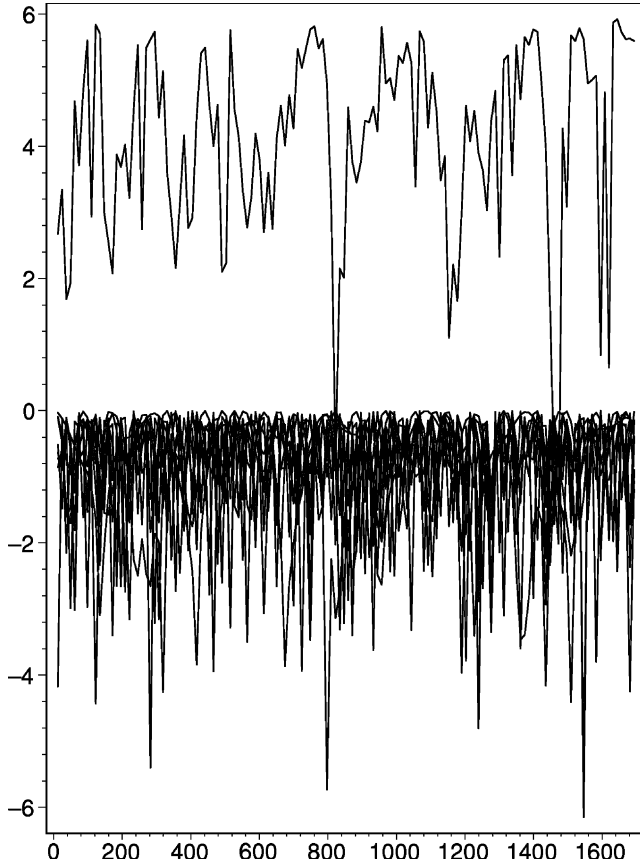


FIG. 5. Plot of the FE for the same initial conditions used in the previous graph.

the trajectory presenting a nonvanishing LE corresponds to the upper curve in Fig. 5, precisely because it indicates a nonvanishing net value for its angular coefficient. An exact calculation of the energy threshold can be done by plotting the fraction of the phase space covered with invariant tori [20]; this analysis will be presented elsewhere [7].

In order to explore this statement, we shall study the stability properties of the potential $V(X, Y)$. From Eq. (6) we know the potential (both mass and interaction terms) is

$$V(X, Y) = \frac{1}{2}(M^2 Y^2 + m^2 X^2 + g^2 Y^2 X^2). \quad (10)$$

Using the Toda-Brumer-Duff test for instabilities [21], we find that the system develops instabilities for energies larger than

$$E^* \simeq \frac{m^2 M^2}{g^2}, \quad (11)$$

which indeed lies between the two energy ranges presented in the two previous plots. We stress that this result does not mean that chaos must not happen for energies below that critical value.

IV. PARTICLE PRODUCTION

As we saw in the last section, even the classical-field-theory version of our model exhibits chaos. Note that we do not have the right to use words like “preheating” and “particle creation” because they only have meaning in the quantum version of the model. In this section, we use the method presented in Ref. [14] in order to take into account the quantum effects. We will show that most of the results of the last section can be used to describe the semiclassical system. We called this system semiclassical because we are assuming that $Y(\eta)$ is a *classical* field and $X(\eta)$ is treated as a *quantum* operator. We have to stress here that this procedure is equivalent to taking the large N approximation (see Ref. [22]).

Let us expand the quantum field X in the Heisenberg representation

$$X(\eta) = f(\eta)a + f^*(\eta)a^\dagger, \quad (12)$$

where a and a^\dagger are annihilation and creation operators satisfying the standard commutator $[a, a^\dagger] = 1$. From this result, the mode function $f(\eta)$ has to satisfy the Wronskian condition

$$f'^* - f^* f' = -i, \quad (13)$$

and from Eq. (9)

$$f''(\eta) + \omega^2(\eta)f(\eta) = 0. \quad (14)$$

In order to satisfy Eqs. (13) and (14) we use the ansatz

$$f(\eta) = \frac{1}{\sqrt{2W(\eta)}} \exp\left(-i \int^\eta d\eta' W(\eta')\right). \quad (15)$$

Choosing the vacuum $|0\rangle$ of the number operator $n = aa^\dagger$, defined by $a|0\rangle = 0$, to compute averages $\langle \dots \rangle$, we obtain an effective Lagrangian $L_{eff} = \langle L \rangle$

$$L_{eff} = \frac{a^2}{2} \left[Y'^2 + R'^2 - \frac{1}{2R^2} - a^2 m^2 Y^2 - \omega^2 R^2 \right], \quad (16)$$

where we have defined a new field $R^2(\eta) = 1/2W(\eta)$. In this case, the equations of motion are

$$Y''(\eta) + \Omega^2(\eta)Y(\eta) = 0, \quad (17)$$

$$R''(\eta) + \omega^2(\eta)R(\eta) - \frac{1}{4R^3(\eta)} = 0, \quad (18)$$

where the averaging process redefined $\Omega^2(\eta) = a^2 M^2 + g^2 R^2 - a''/a$. Because $R^2 = \langle X^2 \rangle$ the centrifugal term, $1/4R^3$, keeps the quantum expectation value away from zero, consistently with the Heisenberg's uncertainty principle and splits in two the phase space of $R(\eta)$. Note that now the amplification of R can actually be interpreted as particle production if also the number

$$n_R = \frac{\omega}{2} \left[\frac{\dot{R}^2}{2\omega} + R^2 \right] - \frac{1}{2}, \quad (19)$$

grows.

The question is: Is the system described by Eqs. (17), (18) chaotic? As we have described in Sec. II, the main feature of chaotic motion is a positive Lyapunov exponent, which shows a sensitivity to small changes in the initial conditions. Another way to reveal chaotic behavior is the study of Poincaré sections. A Poincaré section or a surface of section is a two-dimensional map of the phase space obtained by intercepting the Hamiltonian flow at a fixed position during the motion. For integrable systems, the map is a collection of lines and regions of stability. As soon as the system becomes chaotic, the lines become distorted and the stability regions disappeared. Our investigation of this issue for the system in Eqs. (17), (18) shows that it is indeed chaotic.

Our task is made somewhat easier if we separate the analysis between the $R \gg 1$ and $R \sim 1$ regions. When $R \gg 1$, the system (17), (18) behaves similarly as described in the last section, i.e. Eqs. (8), (9), but constrained to one of the two halves of the phase space. This last statement is not rigorously required, because we can leave the system evolving through the barrier at $R=0$ without affecting the general properties of the chaotic system, e.g. both have the same LE.

In the small R region, where the centrifugal term is important, we find an interesting saturation effect. Because we assume that initially the fluctuation R is not big (its minimum is around $1/\sqrt{2\omega}$), there is a transient period, where although a resonance condition is fulfilled, the R field grows until the nonlinear term breaks the resonant tuning. This fact is indeed connected with the existence of a critical energy under which the system does not amplify the field, as we see later. Unfortunately, we were not able to calculate the LE here because of the very phenomenon we are studying, the parametric resonance itself. As the R oscillation amplitude is increased, R gets closer and closer to the origin, and then the centrifugal barrier just explodes. Therefore, one cannot follow the evolution of the system for a time long enough to allow the graph to reach the constant value which would correspond to the largest LE.

Nevertheless, we can show the existence of chaos, and even conjecture about a threshold, by plotting Poincaré sections valid for both regions ($R \gg 1$ and $R \sim 1$) for different values of energy, as shown in Fig. 6.

By using the Toda-Brumer-Duff instability condition we can estimate a critical energy under which the system does not amplify efficiently the field R . For the case $R \gg 1$ the result is Eq. (11). In Fig. 6 we can see the gradual destruction of the tori as the energy increases. For the values in the numerical example, $M=10$, $m=1$, $g=1$ we obtain $E^* \sim 100$. The upper panel shows the $E=50$ case well inside the integrable region, shows clearly continuous lines with a single stability region, the central one. The central panel shows the $E=150$ case, beyond our estimation for stability, where is possible to see that the original continuous exterior

lines have disappeared, and in their place there is a band of scattered points. The case $E=200$ shows the same effect even more dramatically.

V. ENTROPY PRODUCTION

In the previous sections we have shown evidence for a relation between classical chaos and particle production via a correspondence between the FE, which characterizes exponential growth during preheating, and the maximal positive LE for the associated chaotic system. In this section we discuss further consequences of this relation.

On one hand, we know that for a chaotic dynamical system we can define a *metric* or *Kolmogorov entropy* \mathcal{K} [13] in terms of the LEs λ_i as

$$\mathcal{K} \leq \sum_{\{\lambda_i\} > 0} \lambda_i, \quad (20)$$

—where the equality holds for *typical* Hamiltonian systems [23]—which gives the rate of change of the available information. Local trajectories get stretched in the direction in which the eigenvalues λ_i are positive, and get compressed in the directions in which they are negative. If there are no positive LEs then there is no change in the amount of information available, and the Kolmogorov entropy vanishes. In our problem, there are at least two directions in phase space with comparable, actually equal, LE: R and \dot{R} . Thus

$$\mathcal{K} \leq 2\lambda. \quad (21)$$

On the other hand, the process of particle creation can also be described as a period of entropy production. The problem is then to try to find a formula for entropy valid during the transfer of energy between the oscillators, i.e., in a nonequilibrium system. A lot of work has been done in this context (see Ref. [24] for a review), each one arguing in favor of a different definition for the entropy and its physical reasoning as such. In spite of their different conceptual foundations, all of them agree that, in the high squeezing limit,

$$S \approx \ln(n_k), \quad (22)$$

where n_k is the number of particles in the mode k . In our case, it is given by $n_k \propto \exp(2\mu t)$, and then

$$S \approx 2\mu t, \quad (23)$$

showing the equivalence between the Lyapunov and the Floquet exponents once more.

VI. DISCUSSION

We notice that the reheating process after inflation, in particular the initial stage called preheating, could be driven by a different dynamical behavior, more complicated than the currently believed parametric resonant picture. We investigated the preheating phase by using a system comprised by two interacting background fields. We showed that the study of this model simplifies to that of two coupled harmonic oscillators. Our main results suggest a strong correspondence

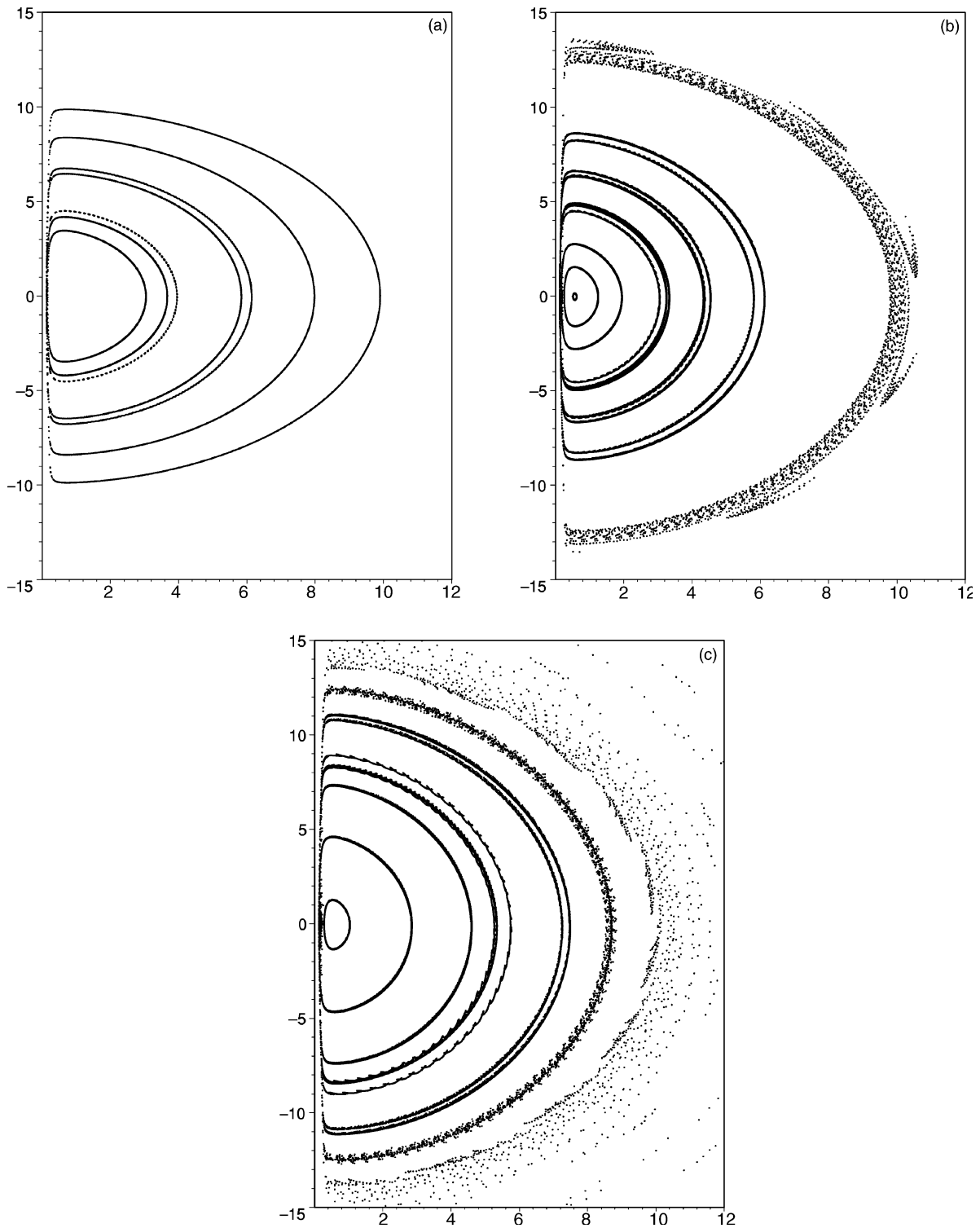


FIG. 6. Poincaré sections (\dot{R} vs R) for $E=50$, 150 and 200. One can clearly see the gradual destruction of the tori as the energy increases.

between the parametric resonance phenomenon and the chaotic properties of the system, namely the numerical equivalence of its Floquet and Lyapunov exponents.

We study the onset of chaos on the effective equations of motion which describes the particle production process. Nevertheless, we can only say the system is *strongly dependent*

on the initial conditions, since the LE are not unambiguously characterizing chaos in General Relativity. We can talk about chaos with additional information, for example in our case by computing Poincaré sections.

In fact, we showed that chaos arises precisely when exponential amplification occurs, showing that the real source

for these amplifications is not a particular resonance condition, but it is a consequence of the dynamical chaos of the background fields.

The evidence for an increasing destruction of the invariant tori shows, together with instability analysis, the existence of an energy threshold above which the dynamics becomes chaotic, and the amplifications become important.

We also address the subtle issue concerning the precise relationship between chaos and parametric resonance. In this context we showed a relationship between the metric entropy, which is a measure of chaos, and the thermodynamic entropy, computed in the high squeezing limit, which is also another way to see the equivalence between the exponents. It may seem natural that both FE and LE are equal, since both show exponential behavior of the system. Nevertheless, the authors of Ref. [10] take the exponential behavior as “rather formal,” and, being interested only in the turbulent phase *after* preheating, define a normalized distance in the phase space

$$\Delta(t) = \sum_a \left(\frac{f'_a - f_a}{f'_a + f_a} \right)^2 + \left(\frac{\dot{f}'_a - \dot{f}_a}{\dot{f}'_a + \dot{f}_a} \right)^2, \quad (24)$$

according to which chaos sets in only when the preheating period is over. The fundamental difference between this formula and the one we used [12,13] is the normalization factors in the denominators. However, as we showed in Sec. III, the parametric resonance makes it chaotic from the very beginning, by construction. Of course, we restrict ourselves to the preheating period. Therefore, our results do not concern the turbulent phase, i.e. interaction with other modes, which would account for the thermalization process.

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