Separability of rotational effects on a gravitational lens

Hideki Asada,* Masumi Kasai,† and Tatsuya Yamamoto
Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan
(Received 5 August 2002; published 28 February 2003)

We derive the deflection angle up to $O(m^2a)$ due to a Kerr gravitational lens with a mass m and specific angular momentum a. It is known that at the linear order in m and a the Kerr lens is observationally equivalent to the Schwarzschild one because of the invariance under the global translation of the center of the lens mass. We show, however, that nonlinear couplings break the degeneracy so that the rotational effect becomes in principle separable for multiple images of a single source. Furthermore, it is distinguishable also for each image of an extended source and/or a point source in orbital motion. In practice, the correction at $O(m^2a)$ becomes $O(10^{-10})$ for the supermassive black hole in our galactic center. Hence, these nonlinear gravitational lensing effects are too small to detect by near-future observations.

DOI: 10.1103/PhysRevD.67.043006 PACS number(s): 98.62.Sb, 04.70.Bw, 95.30.Sf

I. INTRODUCTION

It is of great importance to elucidate the nature of compact objects such as black holes and neutron stars. In particular, general relativity predicts the frame-dragging effect around rotating objects, which has not been detected. A way of studying rotational effect of the curved spacetime is by measuring the light propagation as well as monitoring satellite motion. As for the gravitational lensing caused by rotating objects [1], it is known that at linear order the rotational effect is not distinguishable from the translation of the center of the lens mass [2,3]. In other words, the Kerr lens would be equivalent to the Schwarzschild lens without any knowledge of the precise position of the lens [3]. Can *nonlinear* effects break the degeneracy between the Schwarzschild and Kerr lenses? The main purpose of the present paper is to answer this. We will assume a considerably compact object to take into account a coupling between the angular momentum and the mass. Actually, recent observations [4,5] have suggested that there might be in our universe very compact objects such as a *quark* star whose radius is several kilometers, about half that of a neutron star, though some arguments are still going on [6].

The light propagation in the Kerr spacetime was formulated by using the constants of the null geodesics in polar coordinates [7–10]. However, the approach is not suitable for description of the gravitational lens, which is a mapping between 2-dimensional vectors on lens and source planes [11]. Hence, we follow another approach developed recently for the gravitational lens [3].

II. FORMULATION OF THE STATIONARY GRAVITATIONAL LENS

First, we summarize notations and equations for gravitational lensing. We basically follow the notation of Ref. [11], but the signature is $(-,+,+,+)$. It is convenient to express the metric of a stationary spacetime in the following form:

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-h(c\,dt-w_i dx^i)^2+h^{-1}\gamma_{ij}dx^i dx^j, \tag{1}$$

where

$$h\equiv -g_{00}, \quad w_i\equiv -\frac{g_{0i}}{g_{00}}, \tag{2}$$

and

$$\gamma_{ij}dx^i dx^j\equiv -g_{00}\left(g_{ij}+\frac{g_{0i}g_{0j}}{g_{00}}\right)dx^i dx^j\equiv d\ell^2. \tag{3}$$

This is essentially the same as the Landau-Lifshitz $3+1$ decomposition of a stationary spacetime [12]. One difference is the definition of the spatial metric. They use

$$\tilde{\gamma}_{ij}\equiv\left(g_{ij}+\frac{g_{0i}g_{0j}}{g_{00}}\right)=h^{-1}\gamma_{ij} \tag{4}$$

as the spatial metric. We will hereafter use the conformally rescaled γ_{ij}, since the spatial distance $d\ell$ defined by Eq. (3) behaves as the affine parameter of the null geodesics in this spacetime [3]. The conformal factor h corresponds to the gravitational redshift factor.

For a future-directed light ray, the null condition $ds^2=0$ gives

$$c\,dt=\frac{1}{h}\sqrt{\gamma_{ij}dx^i dx^j}+w_i dx^i. \tag{5}$$

Since the spacetime is stationary, h,γ_{ij}, and w_i are functions only of the spatial coordinates x_i. Then, the arrival time of a light ray is given by the integration from the source to the observer denoted by the subscript S and O, respectively,

$$t\equiv\int_{t_S}^{t_O}dt=\frac{1}{c}\int_S^O\left(\frac{1}{h}\sqrt{\gamma_{ij}e^i e^j}+w_i e^i\right)d\ell, \tag{6}$$

where $e^i=dx^i/d\ell$ is the unit tangent vector along the light ray. Hereafter, lowering and raising the indices of the spatial vectors are done by γ_{ij} and its inverse γ^{ij}.

*Electronic address: asada@phys.hirosaki-u.ac.jp
†Electronic address: kasai@phys.hirosaki-u.ac.jp

Fermat's principle states $\delta t=0$, which provides the Euler-Lagrange equation for the light ray, fully valid in any stationary spacetime

$$\frac{de^i}{d\ell}=-(\gamma^{ij}-e^ie^j)\partial_j\ln h-\gamma^{il}\left(\gamma_{lj,k}-\frac{1}{2}\gamma_{jk,l}\right)e^je^k + h\gamma^{ij}(w_{k,j}-w_{j,k})e^k. \tag{7}$$

The deflection angle $\boldsymbol{\alpha}$ is defined as the difference between the ray directions at the source ($\ell=-\infty$) and the observer ($\ell=\infty$) in the asymptotically flat regions,

$$\boldsymbol{\alpha}\equiv\boldsymbol{e}_S-\boldsymbol{e}_O=-\int_{-\infty}^{\infty}d\ell\frac{d\boldsymbol{e}}{d\ell}. \tag{8}$$

The lens equation relates the angular position of the image $\boldsymbol{\theta}$ to the source angular position $\boldsymbol{\beta}$

$$\boldsymbol{\beta}=\boldsymbol{\theta}-\frac{D_{LS}}{D_{OS}}\boldsymbol{\alpha}(D_{OL}\boldsymbol{\theta}), \tag{9}$$

where D_{OS} is the distance from the observer to the source, D_{OL} is from the observer to the lens, and D_{LS} is from the lens to the source, respectively. The vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ are 2-dimensional vectors in the sense that they are orthogonal to the ray direction $\boldsymbol{e}$ within our approximation. In a cosmological situation, the unlensed position $\boldsymbol{\beta}$ is not an observable, because we cannot remove the lens from the observed position.

We choose the origin of the spatial coordinate as the location of the lens. We use a freedom in choosing the origin of ℓ, so that the closest point of the light ray to the lens, denoted by ξ^i, can be set at $\ell=0$, namely $\xi^i=x^i(\ell=0)$. We denote the tangential vector at the closest point by $\overline{e^i}\equiv e^i(\ell=0)$. The impact parameter $\boldsymbol{b}$ is the distance from the lens to a *fiducial* straight line $\overline{\boldsymbol{x}(\ell)}$ which is the tangent to the light ray at the observer, while the impact parameter is defined usually at the emitter in the standard context of the classical mechanics: This is due to the geometrical configuration from which the lens equation for $\boldsymbol{\theta}=\boldsymbol{b}/D_{OL}$ is derived [11]. Hence, the impact parameter $\boldsymbol{b}$ is defined as

$$\boldsymbol{b}=\overline{\boldsymbol{x}(\ell=0)}. \tag{10}$$

III. GRAVITATIONAL LENSING IN THE KERR SPACETIME

For a slowly rotating case, the Kerr metric is written as

$$ds^2=-\left(1-\frac{2m}{r}\right)dt^2-\frac{4ma\sin^2\theta}{r}dt\,d\phi+\frac{dr^2}{1-\frac{2m}{r}} + r^2(d\theta^2+\sin^2\theta d\phi^2)+O(a^2), \tag{11}$$

where we used the units of $G=c=1$.

In order to change this metric into a spatially *isotropic* form, we perform a coordinate transformation as

$$r=R\left(1+\frac{m}{2R}\right)^2, \tag{12}$$

so that we obtain

$$ds^2=-\left(\frac{1-\frac{m}{2R}}{1+\frac{m}{2R}}\right)^2\left(dt+\frac{2m(\boldsymbol{a}\times\boldsymbol{x})\cdot d\boldsymbol{x}}{R^3\left(1-\frac{m}{2R}\right)^2}\right)^2 + \left(1+\frac{m}{2R}\right)^4 d\boldsymbol{x}\cdot d\boldsymbol{x}+O(a^2), \tag{13}$$

where we introduced a 3-dimensional vector notation

$$\boldsymbol{x}=(x,y,z)=(R\sin\theta\cos\phi,R\sin\theta\sin\phi,R\cos\theta), \tag{14}$$

$$\boldsymbol{a}=(0,0,a). \tag{15}$$

The correspondence between the metric and our (3+1) expression given by Eq. (1) is

$$h=\left(\frac{1-\frac{m}{2R}}{1+\frac{m}{2R}}\right)^2+O(a^2), \tag{16}$$

$$\boldsymbol{w}=-\frac{2m(\boldsymbol{a}\times\boldsymbol{x})}{R^3\left(1-\frac{m}{2R}\right)^2}+O(a^2), \tag{17}$$

$$\gamma_{ij}=\left(1-\frac{m^2}{4R^2}\right)^2\delta_{ij}+O(a^2), \tag{18}$$

where δ_{ij} is Kronecker's delta. It is worthwhile to note

$$\frac{\partial}{\partial x^j}\ln h=\frac{2mx^j}{R^3}+O(a^2,m^3). \tag{19}$$

A condition for the closest point is expressed as

$$\frac{d}{d\ell}(\gamma_{ij}x^ix^j)|_{\ell=0}=0, \tag{20}$$

which means

$$\boldsymbol{\xi}\cdot\overline{\boldsymbol{e}}=O(a^2,m^2), \tag{21}$$

where we used

$$\gamma_{ij}=\delta_{ij}+O(a^2,m^2). \tag{22}$$

A. $O(m^0a^0)$

The metric is expanded as

$$h=1+O(a^2,m), \tag{23}$$

$$w_i = O(a^2, m), \tag{24}$$

$$\gamma_{ij} = \delta_{ij} + O(a^2, m^2). \tag{25}$$

At the lowest order, Eq. (7) is expanded as

$$\frac{de^i}{d\ell} = O(a^2, m), \tag{26}$$

which is integrated immediately as

$$e^i = \overline{e^i} + O(a^2, m). \tag{27}$$

Consequently, we obtain a straight trajectory of the light ray as

$$x^i = \xi^i + \ell \overline{e^i} + O(a^2, m). \tag{28}$$

For later convenience, we define $\xi = |\boldsymbol{\xi}|$ and $R_0 = \sqrt{|\xi|^2 + \ell^2}$.

B. $O(m^1 a^1)$

Using the parametrization of the photon trajectory at the lowest order, we obtain

$$\frac{de^i}{d\ell} = -\frac{2m\xi^i}{R_0^3} + m\left(\frac{4(\boldsymbol{a}\times\overline{\boldsymbol{e}})^i}{R_0^3} + \frac{6\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}(\xi^i + \ell\overline{e^i}) - 6\ell(\boldsymbol{a}\times\boldsymbol{\xi})^i - 6\ell^2(\boldsymbol{a}\times\overline{\boldsymbol{e}})^i}{R_0^5}\right) + O(a^2, m^2), \tag{29}$$

which is integrated as

$$\begin{aligned} e^i = \overline{e^i} - 2m\Bigg[& \frac{\ell\xi^i}{\xi^2 R_0} - (\boldsymbol{a}\times\overline{\boldsymbol{e}})^i\left(\frac{\ell}{\xi^2 R_0} + \frac{\ell}{R_0^3}\right) \\ & - \frac{3\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}\xi^i}{\xi^4}\left(\frac{\ell}{R_0} - \frac{\ell^3}{3R_0^3}\right) \\ & + \{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}\overline{e^i}\left(\frac{1}{R_0^3} - \frac{1}{\xi^3}\right) \\ & - (\boldsymbol{a}\times\boldsymbol{\xi})^i\left(\frac{1}{R_0^3} - \frac{1}{\xi^3}\right)\Bigg] + O(a^2, m^2), \end{aligned} \tag{30}$$

where we used $e^i(\ell=0) = \overline{e^i}$. By integrating this, we obtain the light ray trajectory as

$$\begin{aligned} x^i = \xi^i + \ell\overline{e^i} - 2m\Bigg[& \frac{\xi^i(R_0 - \xi)}{\xi^2} - (\boldsymbol{a}\times\overline{\boldsymbol{e}})^i\left(\frac{R_0 - \xi}{\xi^2} - \frac{1}{R_0} + \frac{1}{\xi}\right) \\ & - \frac{\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}\xi^i}{\xi^4}\left(R_0 - \xi + \frac{\ell^2}{R_0}\right) + \frac{\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}\overline{e^i}}{\xi^2}\left(\frac{\ell}{R_0} - \frac{\ell}{\xi}\right) \\ & - \frac{(\boldsymbol{a}\times\boldsymbol{\xi})^i}{\xi^2}\left(\frac{\ell}{R_0} - \frac{\ell}{\xi}\right)\Bigg] + O(a^2, m^2), \end{aligned} \tag{31}$$

where $x^i(\ell=0) = \xi^i$ was used.

The deflection angle is evaluated as

$$\boldsymbol{\alpha} = \frac{4m\boldsymbol{\xi}}{\xi^2} - \frac{4m}{\xi^4}(2\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}\boldsymbol{\xi} + \xi^2(\boldsymbol{a}\times\overline{\boldsymbol{e}})) + O(a^2, m^2). \tag{32}$$

This angle is found to agree with previous results [1] by noticing an identity

$$\boldsymbol{a}\times\overline{\boldsymbol{e}} = \frac{\boldsymbol{a}\cdot\boldsymbol{\xi}}{\xi^2}(\boldsymbol{\xi}\times\overline{\boldsymbol{e}}) - \frac{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}}{\xi^2}\boldsymbol{\xi}. \tag{33}$$

C. $O(m^2 a^1)$

We substitute Eqs. (30) and (31) into Eq. (8). After lengthy but straightforward calculations, we obtain the deflection angle at $O(m^2 a)$ as

$$\begin{aligned} \boldsymbol{\alpha} = & \frac{4m\boldsymbol{\xi}}{\xi^2} - \frac{4m}{\xi^4}(2\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}\boldsymbol{\xi} + \xi^2(\boldsymbol{a}\times\overline{\boldsymbol{e}})) \\ & + 4m^2\Bigg[\left(\frac{15\pi}{16} - 2\right)\frac{\boldsymbol{\xi}}{\xi^3} - \left(\frac{5\pi}{4} - 4\right)\frac{\boldsymbol{a}\times\overline{\boldsymbol{e}}}{\xi^3} \\ & - \left(\frac{15\pi}{4} - 10\right)\frac{\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\overline{\boldsymbol{e}}\}\boldsymbol{\xi}}{\xi^5}\Bigg] + O(a^2, m^3). \end{aligned} \tag{34}$$

It should be noted that some of the coefficients take a peculiar form like π plus a rational number.

Up to this point, we have used $\boldsymbol{\xi}$ which is the vector for the closest point. We are in a position to consider the impact parameter, which is defined at asymptotic regions by Eq. (10). Asymptotic expansions of Eq. (31) for a large ℓ give us the tangent to the light ray at the observer as

$$\bar{\boldsymbol{x}}=\boldsymbol{\xi}+2m\left[\frac{\boldsymbol{\xi}}{\xi}-\frac{\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\bar{\boldsymbol{e}}\}\boldsymbol{\xi}}{\xi^3}+\frac{\boldsymbol{a}\times\boldsymbol{\xi}-\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\bar{\boldsymbol{e}}\}\bar{\boldsymbol{e}}}{\xi^2}\mathrm{sgn}(\ell)\right]+\ell\left[\bar{\boldsymbol{e}}-2m\left(\frac{\xi^2\boldsymbol{\xi}-2\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\bar{\boldsymbol{e}}\}\boldsymbol{\xi}-\xi^2(\boldsymbol{a}\times\bar{\boldsymbol{e}})}{\xi^4}\mathrm{sgn}(\ell)+\frac{\boldsymbol{a}\times\boldsymbol{\xi}-\{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\bar{\boldsymbol{e}}\}\bar{\boldsymbol{e}}}{\xi^3}\right)\right]+O(a^2,m^2), \tag{35}$$

where we denoted a signature function $\ell/|\ell|$ by $\mathrm{sgn}(\ell)$. Substituting this into Eq. (10), we obtain

$$\boldsymbol{b}=\left[1+2m\left(\frac{1}{\xi}-\frac{(\boldsymbol{a}\times\boldsymbol{\xi})\cdot\bar{\boldsymbol{e}}}{\xi^3}\right)\right]\boldsymbol{\xi}+O(a^2,m^2), \tag{36}$$

where we used $\mathrm{sgn}(0)=0$. Hence, we find

$$\boldsymbol{\xi}=\left[1-2m\left(\frac{1}{b}-\frac{(\boldsymbol{a}\times\boldsymbol{b})\cdot\bar{\boldsymbol{e}}}{b^3}\right)\right]\boldsymbol{b}+O(a^2,m^2), \tag{37}$$

where we defined $b=|\boldsymbol{b}|$. We substitute this into Eq. (34), so that we obtain

$$\boldsymbol{\alpha}=\frac{4m\boldsymbol{b}}{b^2}-\frac{4m}{b^4}(b^2(\boldsymbol{a}\times\bar{\boldsymbol{e}})-2\{(\boldsymbol{a}\times\bar{\boldsymbol{e}})\cdot\boldsymbol{b}\}\boldsymbol{b})+4m^2\left[\frac{15\pi\boldsymbol{b}}{16b^3}-\frac{5\pi(\boldsymbol{a}\times\bar{\boldsymbol{e}})}{4b^3}+\frac{15\pi\{(\boldsymbol{a}\times\bar{\boldsymbol{e}})\cdot\boldsymbol{b}\}\boldsymbol{b}}{4b^5}\right]+O(a^2,m^3). \tag{38}$$

IV. DISCUSSION AND CONCLUSIONS

At $O(ma)$, we find out an infinitesimal translation as [3]

$$\bar{\boldsymbol{b}}=\boldsymbol{b}-\boldsymbol{a}\times\bar{\boldsymbol{e}}, \tag{39}$$

so that the deflection angle given by Eq. (32) can be rewritten as

$$\boldsymbol{\alpha}=\frac{4m\bar{\boldsymbol{b}}}{\bar{b}^2}+O(a^2,m^2). \tag{40}$$

This is a global transformation, under which the lens equation is invariant. As a result, we could not separate the rotational effect without independent knowledge of the location of the lens [3]. Namely, lensing properties caused by a Kerr lens, such as the image positions, magnifications and time delay, could be reproduced by a Schwarzschild lens at the suitable position.

At the next order, we can discover an infinitesimal transformation as

$$\bar{\boldsymbol{b}}=\boldsymbol{b}-\boldsymbol{a}\times\bar{\boldsymbol{e}}-\frac{5\pi m}{16b}\left((\boldsymbol{a}\times\bar{\boldsymbol{e}})-\frac{\{(\boldsymbol{a}\times\boldsymbol{b})\cdot\bar{\boldsymbol{e}}\}\boldsymbol{b}}{b^2}\right), \tag{41}$$

so that the deflection angle in Eq. (38) is rewritten as

$$\boldsymbol{\alpha}=\frac{4m\bar{\boldsymbol{b}}}{\bar{b}^2}+\frac{15\pi m^2\bar{\boldsymbol{b}}}{4\bar{b}^3}+O(a^2,m^3). \tag{42}$$

However, $\boldsymbol{\theta}-\boldsymbol{\beta}$ is not invariant under this *local* transformation. As a consequence, the lens equation is not invariant, so that we can distinguish the rotational effect on *multiple* images of a *point* source, such as changes in relative positions of images. Furthermore, we can recognize it for an *extended* source (e.g. spherical stars, binary stars and luminous disks) and even for a point source if it *moves* for instance on a straight line or a Keplerian orbit.

In order to illustrate the rotational effects on the relative separation between images, let us consider the lens equation in the unit of the Einstein ring radius angle as

$$\boldsymbol{\theta}_S=\boldsymbol{\theta}_I-\frac{\boldsymbol{\theta}_I}{\theta_I^2}-\lambda\frac{\boldsymbol{\theta}_I}{\theta_I^3}+\lambda\left(\frac{\boldsymbol{s}\times\bar{\boldsymbol{e}}}{\theta_I^2}-\frac{2\{(\boldsymbol{s}\times\bar{\boldsymbol{e}})\cdot\boldsymbol{\theta}_I\}\boldsymbol{\theta}_I}{\theta_I^4}\right)+\frac{4}{3}\lambda^2\left[\frac{\boldsymbol{s}\times\bar{\boldsymbol{e}}}{\theta_I^3}-\frac{3\{(\boldsymbol{s}\times\bar{\boldsymbol{e}})\cdot\boldsymbol{\theta}_I\}\boldsymbol{\theta}_I}{\theta_I^5}\right]+O(s^2,\lambda^3), \tag{43}$$

where we defined [3]

$$\theta_E=\sqrt{\frac{4mD_{LS}}{D_L D_S}}, \tag{44}$$

$$\boldsymbol{\theta}_S=\frac{\boldsymbol{\beta}}{\theta_E}, \tag{45}$$

$$\boldsymbol{\theta}_I=\frac{\boldsymbol{\theta}}{\theta_E}, \tag{46}$$

$$\lambda=\frac{15\pi D_S\theta_E}{64D_{LS}}, \tag{47}$$

$$\boldsymbol{s}=\frac{16}{15\pi}\frac{\boldsymbol{a}-(\boldsymbol{a}\cdot\bar{\boldsymbol{e}})\bar{\boldsymbol{e}}}{m}, \tag{48}$$

$$s=|\boldsymbol{s}|. \tag{49}$$

Here, it should be noted that the rotational effect comes from $\boldsymbol{s}$ which is proportional to the projection of the spin vector onto the lens plane. For a nearby stellar mass black hole and a supermassive one in our galactic center, the dimensionless parameter λ becomes respectively

$$\lambda \sim 10^{-7}\left(\frac{m}{10M_{\odot}}\right)^{1/2}\left(\frac{100\ \mathrm{pc}}{D_S}\right)^{1/2}, \tag{50}$$

$$\lambda \sim 10^{-5}\left(\frac{m}{10^{6}M_{\odot}}\right)^{1/2}\left(\frac{8\ \mathrm{kpc}}{D_S}\right)^{1/2}, \tag{51}$$

where we assumed $D_L \sim D_{LS}$.

For simplicity, we solve perturbatively Eq. (43) for sources on the equatorial plane, namely $\boldsymbol{\beta}\cdot\boldsymbol{a}=0$. The solutions which are on the equatorial plane take a form as

$$\theta_{\pm}=\phi_{\pm}+\lambda\chi_{\pm}+\lambda^{2}\psi_{\pm}+O(\lambda^{3}), \tag{52}$$

where we defined

$$\phi_{\pm}=\frac{1}{2}(\theta_S \pm \sqrt{4+\theta_S^2}), \tag{53}$$

$$\chi_{\pm}=\frac{(1\pm s)}{\phi_{\pm}\sqrt{4+\theta_S^2}}, \tag{54}$$

$$\psi_{\pm}=\left[\frac{1}{2}\theta_S \mp \frac{6+6\theta_S^2+\theta_S^4}{2(4+\theta_S^2)^{3/2}} + \frac{s}{3}\left(\frac{14+6\theta_S^2+\theta_S^4}{(4+\theta_S^2)^{3/2}} \mp \theta_S\right)\right]. \tag{55}$$

Hence, the angular separation between the double images, which is one of the important observables, becomes

$$\Delta\theta \equiv \theta_{+}-\theta_{-}=\sqrt{4+\theta_S^2}+\lambda\left(1-s\frac{\theta_S}{\sqrt{4+\theta_S^2}}\right) -\lambda^{2}\left(\frac{6+6\theta_S^2+\theta_S^4}{(4+\theta_S^2)^{3/2}}+\frac{2}{3}s\theta_S\right)+O(s^2,\lambda^3). \tag{56}$$

The term of $O(\lambda s)$ can be absorbed into the leading term as $\sqrt{4+(\theta_S-\lambda s)^2}$, which corresponds to the global translation given by Eq. (39). The correction due to the terms at $O(m^2 a)$ is of the order of $\lambda^2 s$, which becomes $O(10^{-10})$ for the supermassive black hole in our galactic center even if s is of the order of unity. It might be interesting to study some models in detail. For instance, (1) how do light curves change due to a Kerr lens?, (2) what changes occur in image positions and motions when a source is a binary star particularly a binary pulsar?, and (3) what do images look like when a source is an accretion disk?

Our result is in marked contrast to rotational effects on the polarization: The difference in the polarization angle between *double* images from a fixed point source appears at exactly the same order $O(m^2 a)$ [13,14]. In practice, however, these nonlinear gravitational lensing effects are too small to detect by near-future observations [15–18].

ACKNOWLEDGMENTS

This work was supported in part by the Japanese Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, No. 13740137 (H.A.) and by the Sumitomo Foundation (H.A. and M.K.).

[1] R. Epstein and I.I. Shapiro, Phys. Rev. D **22**, 2947 (1980).
[2] C. Baraldo, A. Hosoya, and T.T. Nakamura, Phys. Rev. D **59**, 083001 (1999).
[3] H. Asada and M. Kasai, Prog. Theor. Phys. **104**, 95 (2000).
[4] J.J. Drake *et al.*, Astrophys. J. **572**, 996 (2002).
[5] P. Slane, D.J. Helfand, and S.S. Murray, Astrophys. J. Lett. **571**, L45 (2002).
[6] F.M. Walter and J. M. Lattimer, Astrophys. J. Lett. **576**, L145 (2002).
[7] C.T. Cunningham and J.M. Bardeen, Astrophys. J. Lett. **173**, L137 (1972).
[8] C.T. Cunningham and J.M. Bardeen, Astrophys. J. **183**, 237 (1973).
[9] J.M. Bardeen, W.H. Press, and S.A. Teukolsky, Astrophys. J. **178**, 347 (1972).
[10] B. Carter, Phys. Rev. **174**, 1559 (1968).
[11] P. Schneider, J. Ehlers, and E.E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992).
[12] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).
[13] H. Ishihara, M. Takahashi, and A. Tomimatsu, Phys. Rev. D **38**, 472 (1988).
[14] M. Nouri-Zonoz, Phys. Rev. D **60**, 024013 (1999).
[15] Overwhelmingly Large Telescope (OWL), http://www.eso.org/projects/owl/
[16] Space Interferometry Mission (SIM), http://sim.jpl.nasa.gov/
[17] Global Astrometric Interferometer for Astrophysics (GAIA), http://astro.estec.esa.nl/GAIA/
[18] Japan Astrometry Satellite Mission for INfrared Exploration (JASMINE), http://www.jasmine-galaxy.org/