Domain wall fermion and *CP* symmetry breaking

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We examine the CP properties of chiral gauge theory defined by a formulation of the domain wall fermion, where the light field variables q and $\bar{q}$ together with Pauli-Villars fields Q and $\bar{Q}$ are utilized. It is shown that this domain wall representation in the infinite flavor limit $N=\infty$ is valid only in the topologically trivial sector, and that the conflict among lattice chiral symmetry, strict locality and CP symmetry still persists for finite lattice spacing a. The CP transformation generally sends one representation of lattice chiral gauge theory into another representation of lattice chiral gauge theory, resulting in the inevitable change of propagators. A modified form of lattice CP transformation motivated by the domain wall fermion, which keeps the chiral action in terms of the Ginsparg-Wilson fermion invariant, is analyzed in detail; this provides an alternative way to understand the breaking of CP symmetry at least in the topologically trivial sector. We note that the conflict with CP symmetry could be regarded as a topological obstruction. We also discuss the issues related to the definition of Majorana fermions in connection with the supersymmetric Wess-Zumino model on the lattice.

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I. INTRODUCTION

It has been recently shown that CP symmetry in chiral gauge theory [1–3] and also the Majorana reduction in the presence of chiral symmetric Yukawa couplings [4] have a certain conflict with lattice chiral symmetry, doubler-free, and locality conditions in the framework of Ginsparg-Wilson operators [5–19]. There exists a closely related formulation of lattice fermions which is called the domain wall fermion [20–26]. In one representation of the domain wall fermion in the infinite flavor limit, the domain wall fermion becomes identical to the overlap fermion [27–29] and thus to the Ginsparg-Wilson fermion. In such a case, the conflict with CP symmetry in chiral theory naturally persists if one uses the conventional representation of Ginsparg-Wilson fermions. There are, however, other representations of the domain wall fermion when discussing chiral symmetry [21,22,24], and in those representations (and also in the conventional overlap fermion [27–29]), the conflict with CP symmetry is less obvious. It is therefore desirable to examine in detail how the conflict observed in the framework of Ginsparg-Wilson fermions persists in the domain wall fermion.

We analyze this issue in a formulation of the domain wall fermion where the light field variables q and $\bar{q}$ together with Pauli-Villars fields Q and $\bar{Q}$ are utilized [22–24]. To make this analysis as definite as possible, we concentrate on the infinite flavor $N=\infty$ limit of the domain wall fermion, where chiral symmetry is well defined. It is shown that this representation of the domain wall fermion is valid only in the topologically trivial sector and that the conflict with CP symmetry persists. We also analyze in detail a modified form of lattice CP transformation motivated by the domain wall fermion, which keeps the chiral action in the Ginsparg-Wilson fermion invariant, and show that CP symmetry is still violated. In the analysis of CP symmetry, it turns out that topological considerations play an essential role and, in fact, the conflict with CP symmetry could be regarded as a topological obstruction.

In connection with the definition of Majorana fermions and its application to supersymmetry, we note a possibility of replacing the Pauli-Villars fields in the domain wall formulation by the auxiliary field in the Wess-Zumino model. In fact this formulation agrees with a past suggestion [4,30] of the Wess-Zumino action in terms of the Ginsparg-Wilson operators.

In this paper we take as a basis of our analysis a Hermitian lattice operator defined by

$$H=a\gamma_5 D=H^\dagger=aD^\dagger\gamma_5, \tag{1.1}$$

where D stands for the lattice Dirac operator and a is the lattice spacing. The Ginsparg-Wilson operator is then defined by the algebraic relation

$$\gamma_5 H+H\gamma_5=2H^2 \tag{1.2}$$

and its solution agrees with the overlap operator [7] (and its variants).

Although the above simplest form of the Ginsparg-Wilson relation is relevant to our analysis of the domain wall fermion, the generality of the conflict with CP (or C) symmetry is best understood if one considers a more general algebraic relation [2]

$$\gamma_5 H+H\gamma_5=2H^2 f(H^2), \tag{1.3}$$

where $f(H^2)$ is assumed to be a regular function of H^2 and $f(H^2)^\dagger=f(H^2)$: $f(x)$ is assumed to be monotonous and nondecreasing for $x\geqslant 0$. The explicit construction of the operator D is known for $f(H^2)=H^{2k}$ with nonnegative integers k

[18,19], and $k=0$ gives rise to the conventional Ginsparg-Wilson relation [17]. In our analysis of CP symmetry, the operator defined by

$$\Gamma_5=\gamma_5-Hf(H^2) \tag{1.4}$$

or $\gamma_5\Gamma_5$ plays a central role. This operator satisfies the relation

$$\Gamma_5 H+H\Gamma_5=0 \tag{1.5}$$

and $\gamma_5\Gamma_5$ vanishes for some momentum variables inside the basic Brillouin zone.

This vanishing of $\gamma_5\Gamma_5$ is shown on the general ground of locality and species doubler-free conditions of H. We here briefly illustrate the basic reasoning, since it is closely related to the basic issue of the domain wall fermion: One can confirm the relation

$$\gamma_5 H^2=(\gamma_5 H+H\gamma_5)H-H(\gamma_5 H+H\gamma_5)+H^2\gamma_5=H^2\gamma_5 \tag{1.6}$$

which implies $H^2=\gamma_5 H^2\gamma_5$ and thus $DH^2=H^2D$. The above defining relation (1.3) is also written as

$$\gamma_5 H+H\hat{\gamma}_5=0, \quad \gamma_5 D+D\hat{\gamma}_5=0, \tag{1.7}$$

and $\hat{\gamma}_5^2=1$, where

$$\hat{\gamma}_5=\gamma_5-2Hf(H^2). \tag{1.8}$$

We note that

$$D\gamma_5\Gamma_5-\gamma_5\Gamma_5 D=0 \tag{1.9}$$

and also the relation

$$\begin{aligned}\gamma_5\Gamma_5\hat{\gamma}_5&=\gamma_5 2\Gamma_5^2-\gamma_5\Gamma_5\gamma_5=\gamma_5(\gamma_5\Gamma_5+\Gamma_5\gamma_5)-\gamma_5\Gamma_5\gamma_5\\&=\gamma_5(\gamma_5\Gamma_5).\end{aligned} \tag{1.10}$$

We now examine the action defined by

$$S=\int \bar{\psi}D\psi\equiv\sum_{x,y} a^4\bar{\psi}(x)D(x,y)\psi(y) \tag{1.11}$$

which is invariant under the lattice chiral transformation

$$\delta\psi=i\epsilon\hat{\gamma}_5\psi, \quad \delta\bar{\psi}=\bar{\psi}i\epsilon\gamma_5. \tag{1.12}$$

If one considers the field redefinition

$$q=\gamma_5\Gamma_5\psi, \quad \bar{q}=\bar{\psi} \tag{1.13}$$

the above action is written as

$$S=\int \bar{q}D\frac{1}{\gamma_5\Gamma_5}q \tag{1.14}$$

which is invariant under the naive chiral transformation

$$\begin{aligned}\delta q&=\gamma_5\Gamma_5\delta\psi=\gamma_5\Gamma_5 i\epsilon\hat{\gamma}_5\psi=i\epsilon\gamma_5 q,\\ \delta\bar{q}&=\bar{q}i\epsilon\gamma_5.\end{aligned} \tag{1.15}$$

This chiral symmetry implies the relation

$$\left\{\gamma_5, D\frac{1}{\gamma_5\Gamma_5}\right\}=0. \tag{1.16}$$

On the basis of the standard argument of the no-go theorem, $D/(\gamma_5\Gamma_5)$ and thus $1/(\gamma_5\Gamma_5)$ have singularities inside the Brillouin zone for local and species doubler-free $H=a\gamma_5 D$. In fact it is shown that [18]

$$\Gamma_5=0 \tag{1.17}$$

just on top of the would-be species doublers for $f(H^2)=H^{2k}$ with non-negative integers k in the case of free fermions and also for the topological modes in the presence of instantons (see also the Appendix). The field q, which plays a central role in the domain-wall fermion [21–24], is thus ill defined for these configurations.

It is shown that the domain wall variables q and $\bar{q}$ in the infinite flavor limit satisfy the normal charge conjugation properties as well as the continuum chiral symmetry, though they are defined in terms of the nonlocal action. Moreover, one can rewrite all the correlation functions for q and $\bar{q}$ in terms of the local variables by using $q=\gamma_5\Gamma_5\psi$ and $\bar{q}=\bar{\psi}$ [24]. One might thus naively expect that we do not encounter any difficulty associated with CP and charge conjugation properties. The purpose of this paper is to clarify this and related issues.

II. DOMAIN WALL FERMIONS AND CP TRANSFORMATION

A. Chiral properties

The domain wall fermion is defined by a set of coupled fermion fields [20,21]

$$\begin{aligned}a_5\mathcal{L}_{\mathrm{DW}}=&\bar{\psi}_1[(\gamma_5 a_5 H_{\mathrm{W}}+1)\psi_1-P_-\psi_2+\mu P_+\psi_N]\\&+\sum_{i=2}^{N-1}\bar{\psi}_i[(\gamma_5 a_5 H_{\mathrm{W}}+1)\psi_i-P_-\psi_{i+1}-P_+\psi_{i-1}]\\&+\bar{\psi}_N[(\gamma_5 a_5 H_{\mathrm{W}}+1)\psi_N+\mu P_-\psi_1-P_+\psi_{N-1}],\end{aligned} \tag{2.1}$$

where N is chosen to be a positive even integer, and

$$H_{\mathrm{W}}\equiv\gamma_5\left(D_{\mathrm{W}}-\frac{m_0}{a}\right) \tag{2.2}$$

with the Wilson fermion operator D_{W} (with the Wilson parameter $r=1$) and $0<m_0<2$; a_5 is the lattice spacing in the fifth (or "flavor") direction. Note that $H_{\mathrm{W}}^\dagger=H_{\mathrm{W}}$. We use the conventional chiral projection operators

$$P_{\pm}=\frac{1\pm\gamma_5}{2}. \tag{2.3}$$

The parameter μ is chosen to be $\mu=0$ for the domain wall variables and $\mu=1$ for the Pauli-Villars variables to subtract heavy fermion degrees of freedom. After performing the path integral over all the fermion variables one obtains [23,25]

$$\det[\gamma_5(1-a_5H_WP_-)]^N\det[(P_- -\mu P_+) - T^{-N}(P_+-\mu P_-)], \tag{2.4}$$

where the transfer operator is given by

$$T=\frac{1}{1+a_5H_WP_+}(1-a_5H_WP_-)=\frac{1+\mathcal{H}_W}{1-\mathcal{H}_W} \tag{2.5}$$

with

$$\mathcal{H}_W=\frac{-1}{2+a_5H_W\gamma_5}a_5H_W=a_5H_W\frac{-1}{2+\gamma_5a_5H_W}. \tag{2.6}$$

Note that both of H_W and $\mathcal{H}_W$ are Hermitian. If one subtracts the contributions of heavy fermions (by setting $\mu=1$) from the above determinant with $\mu=0$, one obtains the "truncated" overlap or Ginsparg-Wilson operator D_N

$$\begin{aligned}\det aD_N &\equiv \det(P_- - T^{-N}P_+)/\det[(P_- - P_+) - T^{-N}(P_+-P_-)]\\ &=\det\left[\frac{1}{2}\left(1+\gamma_5\frac{1-T^N}{1+T^N}\right)\right],\end{aligned} \tag{2.7}$$

where a is the lattice spacing in four-dimensional Euclidean space, and the effective Lagrangian for the physical fermion

$$\mathcal{L}=\bar{\psi}D_N\psi. \tag{2.8}$$

Note that D_N is well-defined for $N=$even and $a_5/a\ll 1$, since $\|a_5H_W\|\leqslant a_5/a$ and T is a well-defined Hermitian operator.

On the other hand, if one defines the light fermion degrees of freedom by [21]

$$q\equiv\frac{a}{a_5}(P_-\psi_1+P_+\psi_N), \quad \bar{q}\equiv\bar{\psi}_1P_+ + \bar{\psi}_NP_- \tag{2.9}$$

and integrates over all the remaining degrees of freedom in Eq. (2.1), one obtains after subtracting the heavy fermion contributions by the Pauli-Villars bosonic spinors Q and $\bar{Q}$ [24],

$$\begin{aligned}\mathcal{L}_{DW}&=\bar{q}\frac{a_5}{a}D_N^{\rm eff}q+\bar{Q}(1+a_5D_N^{\rm eff})Q\\ &=\bar{q}\frac{D_N}{1-aD_N}q+\bar{Q}\frac{1}{1-aD_N}Q.\end{aligned} \tag{2.10}$$

If one performs the path integral over all the variables, one obtains the same result $\det D_N$ as above. We denote this Lagrangian (and its $N=\infty$ limit) by $\mathcal{L}_{DW}$ hereafter.

If one takes the infinite flavor limit $N\to\infty$, one obtains (in the limit $a_5\to 0$)

$$D_N\to D=\frac{1}{2a}\left(1+\gamma_5\frac{H_W}{\sqrt{H_W^2}}\right) \tag{2.11}$$

which gives the Neuberger's overlap operator [7] satisfying the simplest Ginsparg-Wilson relation $\gamma_5D+D\gamma_5=2aD\gamma_5D$. In this limit we can write the domain wall fermion as

$$\mathcal{L}_{DW}=\bar{q}D\frac{1}{\gamma_5\Gamma_5}q+\bar{Q}\frac{1}{\gamma_5\Gamma_5}Q \tag{2.12}$$

which is valid for a general class of Ginsparg-Wilson operators. In our analysis of *CP* and related problems, we utilize this $N=\infty$ expression.

We here note that

$$1-aD_N=\frac{1}{2}\left(1+\gamma_5\frac{T^N-1}{T^N+1}\right)\neq 0 \tag{2.13}$$

since

$$\left\|\frac{T^N-1}{T^N+1}\right\|<1 \tag{2.14}$$

for finite even N and sufficiently small a_5/a. Consequently, $D_N/(1-aD_N)$ is a well-defined and local operator (see Ref. [39] for the locality of D_N), and

$$\left\{\gamma_5,\frac{D_N}{1-aD_N}\right\}\neq 0 \tag{2.15}$$

since D_N with finite N does not satisfy the Ginsparg-Wilson relation. On the other hand, the $N=\infty$ expression satisfies

$$\left\{\gamma_5,\frac{D}{1-aD}\right\}=0 \tag{2.16}$$

and thus the operator $1/(1-aD)$ becomes singular. It is interesting that good locality (analytic property) and good chiral symmetry for the operator $D_N/(1-aD_N)$ are traded in the limit $N=\infty$.

The locality of $D_N/(1-aD_N)$ is understood intuitively, since the defining Lagrangian of the domain wall fermion for finite N couples N fields by the operator H_W, which causes correlation over the finite distances $\sim Na$ in four-dimensional Euclidean space. In the limit $N\to\infty$, the operator $D/(1-aD)$ could thus become non-local. By subtracting the contributions from far apart fields with the parameter $\mu=1$ in the defining Lagrangian, the Pauli-Villars fields Q and $\bar{Q}$ could restore the locality: In fact, the singular factor $1/(1-aD)$ is canceled by the Pauli-Villars fields.

The explicit expression of the Lagrangian for chiral gauge theory is not specified by the domain wall prescription,[1] since precise chiral symmetry is not defined for finite N. It is, however, natural to analyze chiral theory based on the above correspondence to the Ginsparg-Wilson operator

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(\int \bar{\psi}D\psi\right) = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}Q\mathcal{D}\bar{Q}\exp \times\left(\int \bar{q}D\frac{1}{\gamma_5\Gamma_5}q+\int \bar{Q}\frac{1}{\gamma_5\Gamma_5}Q\right). \tag{2.17}$$

We first note

$$D=P_+D\hat{P}_-+P_-D\hat{P}_+ \tag{2.18}$$

with

$$\hat{P}_\pm=\frac{1\pm\hat{\gamma}_5}{2} \tag{2.19}$$

and (by using the relations such as $P_\pm\hat{P}_\pm=P_\pm\gamma_5\Gamma_5$ and $\gamma_5\Gamma_5\hat{P}_\pm=P_\pm\gamma_5\Gamma_5$)

$$\begin{aligned}\hat{P}_-&=\hat{P}_-\hat{P}_-\\&=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}\gamma_5\Gamma_5\hat{P}_-\\&=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-\gamma_5\Gamma_5\hat{P}_-+\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_+\gamma_5\Gamma_5\hat{P}_-\\&=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-\hat{P}_-\hat{P}_-\\&=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-P_-\gamma_5\Gamma_5.\end{aligned} \tag{2.20}$$

We then have the chiral Lagrangian

$$\mathcal{L}_L=\int \bar{\psi}P_+D\hat{P}_-\psi =\int \bar{q}P_+D\frac{1}{\gamma_5\Gamma_5}P_-q \tag{2.21}$$

with

$$q(x)=\gamma_5\Gamma_5\psi(x),\qquad \bar{q}(x)=\bar{\psi}(x) \tag{2.22}$$

and

[1]See, however, Ref. [26].

$$\psi_L\equiv\hat{P}_-\psi=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-(P_-q)=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-q_L,\qquad \bar{\psi}_L\equiv\bar{\psi}P_+=\bar{q}P_+=\bar{q}_L. \tag{2.23}$$

The path integral is then given by taking the Jacobian associated with the above change of variables into account,

$$\int \mathcal{D}\psi_L\mathcal{D}\bar{\psi}_L\exp\left(\int \bar{\psi}P_+D\hat{P}_-\psi\right) =\int \mathcal{D}q_L\mathcal{D}\bar{q}_L\mathcal{D}Q_L\mathcal{D}\bar{Q}_R\exp\left(\int \bar{q}P_+D\frac{1}{\gamma_5\Gamma_5}P_-q +\int \bar{Q}\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-Q\right), \tag{2.24}$$

where we defined the bosonic Pauli-Villars spinors

$$Q_L(x)=P_-Q(x),\qquad \bar{Q}_R(x)=\bar{Q}(x)\hat{P}_-. \tag{2.25}$$

This is consistent if one recalls

$$D\frac{1}{\gamma_5\Gamma_5}=P_+D\frac{1}{\gamma_5\Gamma_5}P_-+P_-D\frac{1}{\gamma_5\Gamma_5}P_+,\qquad \frac{1}{\gamma_5\Gamma_5}=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-+\hat{P}_+\frac{1}{\gamma_5\Gamma_5}P_+. \tag{2.26}$$

The chiral transformation laws of various fields are defined by

$$\psi\to e^{i\alpha\hat{\gamma}_5}\psi,\quad \bar{\psi}\to\bar{\psi}e^{i\alpha\gamma_5},\qquad q\to e^{i\alpha\gamma_5}q,\quad \bar{q}\to\bar{q}e^{i\alpha\gamma_5},\qquad Q\to e^{i\alpha\gamma_5}Q,\quad \bar{Q}\to\bar{Q}e^{-i\alpha\hat{\gamma}_5} \tag{2.27}$$

and, similarly, the fermion number transformation by

$$\psi\to e^{-i\alpha}\psi,\quad \bar{\psi}\to\bar{\psi}e^{i\alpha},\qquad q\to e^{-i\alpha}q,\quad \bar{q}\to\bar{q}e^{i\alpha},\qquad Q\to e^{-i\alpha}Q,\quad \bar{Q}\to\bar{Q}e^{i\alpha}. \tag{2.28}$$

These transformation rules are fixed if one formally gauges those degrees of freedom.

Based on this formulation of chiral gauge theory, we make the following observations. (i) One may take the Ginsparg-Wilson variables ψ and $\bar{\psi}$, which are defined by a local Lagrangian, as the primary variables. One may thus add the source terms to both hand-sides of the path integral

$$\mathcal{L}_{\rm source} = \bar{\eta}_R \psi_L + \bar{\psi}_L \eta_R$$
$$= \bar{\eta}_R \hat{P}_- \psi + \bar{\psi} P_+ \eta_R$$
$$= \bar{\eta}_R \hat{P}_- \frac{1}{\gamma_5 \Gamma_5} P_-(P_- q) + \bar{q} P_+ \eta_R . \quad (2.29)$$

To avoid the singularities appearing in various expressions for domain wall variables, one needs to work in the functional space *without* the modes

$$\Gamma_5 \varphi_n = 0 \quad (2.30)$$

in the domain wall representation. We here recall that the index for the Ginsparg-Wilson operator is given by [8,9,31–33]

$$\mathrm{Tr}\, \Gamma_5 = n_+ - n_- = N_- - N_+ , \quad (2.31)$$

where $n_\pm$ stand for the modes $H\varphi_n = a\gamma_5 D\varphi_n = 0$ with $\gamma_5 \varphi_n = \pm \varphi_n$, respectively, and $N_\pm$ stand for the modes $\Gamma_5 \varphi_n = [\gamma_5 - Hf(H^2)]\varphi_n = 0$ with $\gamma_5 \varphi_n = \pm \varphi_n$, respectively (see the Appendix). The constraint $N_+ = N_- = 0$ thus implies that we work in the *topologically trivial sector* with $\mathrm{Tr}\, \Gamma_5 = 0$. This constraint is consistent with the above fermion number transformation: For the Ginsparg-Wilson variables, we obtain the Jacobian factor[2]

$$\ln J_\psi = i\alpha \mathrm{Tr}\, \hat{P}_- - i\alpha\, \mathrm{Tr}\, P_+ = -i\alpha\, \mathrm{Tr}\, \Gamma_5 , \quad (2.32)$$

whereas for the domain wall variables q and $\bar{q}$, we obtain

$$\ln J_q = i\alpha\, \mathrm{Tr}\, P_- - i\alpha\, \mathrm{Tr}\, P_+ = -i\alpha\, \mathrm{Tr}\, \gamma_5 . \quad (2.33)$$

We thus have $\mathrm{Tr}\, \Gamma_5 = \mathrm{Tr}\, \gamma_5 = 0$.

(ii) One may take the domain wall variables as the primary variables and add the source terms

$$\mathcal{L}_{\rm source} = \bar{\eta}_R q_L + \bar{q}_L \eta_R$$
$$= \bar{\eta}_R P_- q + \bar{q} P_+ \eta_R$$
$$= \bar{\eta}_R P_- \gamma_5 \Gamma_5 \psi + \bar{\psi} P_+ \eta_R$$
$$= \bar{\eta}_R P_- \psi_L + \bar{\psi}_L \eta_R , \quad (2.34)$$

where we used $P_- \gamma_5 \Gamma_5 = P_- \hat{P}_-$. One might attempt to interpret the chiral domain wall representation with these source terms in the following way: In any fermion loop diagram such as in the determinant factor, we combine the variables q, $\bar{q}$ and Q, $\bar{Q}$ together and obtain the determinant without the $1/(\gamma_5 \Gamma_5)$ factor. For the external fermion lines connected to the source terms, we use the variables q_L and $\bar{q}_L$ by replacing those variables later by $P_- \psi_L$ and $\bar{\psi}_L$. By this way, we do not encounter any singularity even in topologically nontrivial sectors. This is perhaps the simplest view based on the domain wall fermion. This view, however, has a fatal difficulty in topological properties, namely, one cannot generate the fermion number anomaly by a transformation of variables q_L and $\bar{q}_L$ alone

$$\ln J_q = i\alpha\, \mathrm{Tr}\, P_- - i\alpha\, \mathrm{Tr}\, P_+ = -i\alpha\, \mathrm{Tr}\, \gamma_5 = 0 \quad (2.35)$$

if one works in the complete functional space in topologically nontrivial sectors. We thus have to exclude the modes $\Gamma_5 \varphi_n = 0$ by hand, for example, but we may still work in all the topological sectors on the basis of the domain wall variables q_L and $\bar{q}_L$. In this case, the topological properties are maintained since one can confirm [33]

$$\mathrm{Tr}'\, \gamma_5 = n_+ - n_- , \quad (2.36)$$

where Tr' is taken in the functional space with the modes $\Gamma_5 \varphi_n = 0$ excluded. This index relation has the same form as in continuum theory [34]. This exclusion of the modes $\Gamma_5 \varphi_n = 0$ is consistent with the replacement $q \to \gamma_5 \Gamma_5 \psi$ since the factor $\gamma_5 \Gamma_5$ projects out those modes [10,35]. This operation is, however, apparently nonlocal.[3]

Based on these considerations we conclude that the domain wall fermion representation in the limit $N = \infty$, where chiral symmetry is well defined, is valid as a local field theory (in the above interpretation) only in the topologically trivial sector with $\mathrm{Tr}\, \Gamma_5 = 0$. The primary variables, which describe the full physical contents expressed by various correlation functions in topologically trivial as well as nontrivial sectors, are thus given by the Ginsparg-Wilson fermions ψ_L and $\bar{\psi}_L$, and hereafter we analyze the domain wall representation in the topologically trivial sector with the source terms (2.29) added. Note that source terms specify the correlation functions, and the component $P_+ \psi_L$ is missing in Eq. (2.34); to maintain the consistency of internal and external fermion lines, which is related to unitarity, we need to use the source (2.29).

In passing, another interesting representation, which is equivalent to the domain wall fermion, is given by

$$\int \mathcal{D}\psi_L \mathcal{D}\bar{\psi}_L \exp\left(\int \bar{\psi} P_+ D \hat{P}_- \psi \right)$$
$$= \int \mathcal{D}q_L \mathcal{D}\bar{q}_L \mathcal{D}S_L \mathcal{D}\bar{S}_R \exp\left(\int \bar{q} P_+ D \frac{1}{\gamma_5 \Gamma_5} P_- q \right.$$
$$\left. + \int \bar{S} P_- \gamma_5 \Gamma_5 \hat{P}_- S \right), \quad (2.37)$$

where we defined the *fermionic* auxiliary fields

[2]The path integral for chiral non-Abelian gauge theory has not been completely understood yet. But the chiral U(1) anomaly and associated index are insensitive to the details of the path integral measure.

[3]The exclusion of the modes $\Gamma_5 \varphi_n(x) = 0$ in all the topological sectors is apparently a nonlocal operation in spacetime, though it is a local operation in the "mode space," since the functional value of $\varphi_n(x)$ is fixed over the entire space once its value at one point is fixed. The exclusion of the modes $\Gamma_5 \varphi_n(x) = 0$ by hand corresponds to the exclusion of would-be species doublers by hand.

$$S_L(x)=\hat{P}_- S(x),$$

$$\overline{S_R}(x)=\overline{S(x)}P_- \tag{2.38}$$

by noting

$$P_+ D\hat{P}_- = \left(P_+ D\frac{1}{\gamma_5\Gamma_5}P_-\right)(P_-\gamma_5\Gamma_5\hat{P}_-) \tag{2.39}$$

and

$$\gamma_5\Gamma_5 = P_-\gamma_5\Gamma_5\hat{P}_- + P_+\gamma_5\Gamma_5\hat{P}_+ . \tag{2.40}$$

The vectorlike theory is then defined by

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left(\int \overline{\psi}D\psi\right)$$
$$=\int \mathcal{D}q\mathcal{D}\overline{q}\mathcal{D}S\mathcal{D}\overline{S}\exp$$
$$\times\left(\int \overline{q}D\frac{1}{\gamma_5\Gamma_5}q+\int \overline{S}\gamma_5\Gamma_5 S\right). \tag{2.41}$$

This representation is applicable only to the topologically trivial sector, but it turns out to be convenient when we discuss Majorana fermions in the domain wall representation later.

B. CP symmetry in chiral gauge theory

We recall the charge conjugation properties of various operators. We employ the convention of the charge conjugation matrix C

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T, \tag{2.42}$$

$$C\gamma_5 C^{-1} = \gamma_5^T, \tag{2.43}$$

$$C^\dagger C = 1, \quad C^T = -C. \tag{2.44}$$

We then have[4]

$$WD(U^{CP})W^{-1} = D(U)^T,$$
$$W\gamma_5\Gamma_5(U^{CP})W^{-1} = [\gamma_5\Gamma_5(U)]^T,$$
$$WH(U^{CP})W^{-1} = -[\gamma_5 H(U)\gamma_5]^T,$$
$$WH^2(U^{CP})W^{-1} = [H^2(U)]^T,$$
$$W\Gamma_5(U^{CP})W^{-1} = -[\gamma_5\Gamma_5(U)\gamma_5]^T,$$
$$W(\Gamma_5/\Gamma)(U^{CP})W^{-1} = -[(\gamma_5\Gamma_5\gamma_5/\Gamma)(U)]^T \tag{2.45}$$

where

$$\Gamma=\sqrt{\Gamma_5^2}=\sqrt{(\gamma_5\Gamma_5\gamma_5)^2}=\sqrt{1-H^2 f^2(H^2)}. \tag{2.46}$$

Here we imposed the relation $WD(U^{CP})W^{-1}=D(U)^T$ or $[CD(U)]^T=-CD(U^C)$ which is consistent with the defining Ginsparg-Wilson relation.

We also have the properties

$$W\hat{\gamma}_5(U^{CP})W^{-1} = -[\gamma_5\hat{\gamma}_5(U)\gamma_5]^T$$

$$W\frac{1}{\gamma_5\Gamma_5(U^{CP})}W^{-1} = \left[\frac{1}{\gamma_5\Gamma_5(U)}\right]^T. \tag{2.47}$$

We now examine the CP symmetry in chiral gauge theory

$$\mathcal{L}_L = \overline{\psi}_L D\psi_L , \tag{2.48}$$

where we defined the (general) projection operators

$$D=\overline{P}_L D P_L + \overline{P}_R D P_R ,$$

$$\psi_{L,R}=P_{L,R}\psi, \quad \overline{\psi}_{L,R}=\overline{\psi P}_{L,R} . \tag{2.49}$$

Under the standard CP transformation[5]

$$\overline{\psi}\to\psi^T W,$$

$$\psi\to -W^{-1}\overline{\psi}^T \tag{2.50}$$

the chiral action is invariant only if

$$WP_L W^{-1} = \overline{P}_L^T, \quad W\overline{P}_L W^{-1} = P_L^T. \tag{2.51}$$

It was shown elsewhere that the unique solution for this condition in the framework of the Ginsparg-Wilson operators is given by [2]

$$P_{L,R}=\frac{1}{2}(1\mp\Gamma_5/\Gamma),$$

$$\overline{P}_{L,R}=\frac{1}{2}(1\pm\gamma_5\Gamma_5\gamma_5/\Gamma), \tag{2.52}$$

but these projection operators suffer from singularities in $1/\Gamma$. Namely, it is impossible to maintain the manifest invariance of the local and chiral Lagrangian under the CP transformation [1,2].[6]

If one stays in the well-defined local Lagrangian

$$\int \mathcal{L}_L = \int \overline{\psi}P_+ D\hat{P}_-\psi \tag{2.53}$$

it is not invariant under the standard CP transformation as

$$WP_\pm W^{-1}=P_\mp^T \neq \hat{P}_\mp^T(U),$$

[4]We define the CP operation by $W=C\gamma_0=\gamma_2$ with Hermitian γ_2 and the CP transformed gauge field by U^{CP}, and then $WD(U^{CP})W^{-1}=D(U)^T$. If the parity is realized in the standard way, we have $CD(U^C)C^{-1}=D(U)^T$.

[5]The vectorlike theory is invariant under this CP transformation.

[6]This, however, shows that one can maintain manifest CP invariance, if one ignores the singularities associated with $\gamma_5\Gamma_5=0$.

$$W\hat{P}_{\pm}(U^{CP})W^{-1}=\frac{1\mp[\gamma_5\hat{\gamma}_5(U)\gamma_5]^T}{2}$$
$$=[\gamma_5\hat{P}_{\mp}(U)\gamma_5]^T\neq P_{\mp}^T,$$
$$[WP_+D(U^{CP})\hat{P}_-(U^{CP})W^{-1}]^T=\gamma_5\hat{P}_+(U)\gamma_5D(U)P_-$$
$$=P_+D(U)\hat{P}_-(U)-D(U)[\gamma_5-\Gamma_5(U)]\neq P_+D\hat{P}_-. \tag{2.54}$$

Since one can show that

$$(\gamma_5\hat{P}_\pm\gamma_5)(\gamma_5\hat{P}_\pm\gamma_5)=(\gamma_5\hat{P}_\pm\gamma_5),$$
$$D=(\gamma_5\hat{P}_+\gamma_5)DP_-+(\gamma_5\hat{P}_-\gamma_5)DP_+, \tag{2.55}$$

the CP transformation actually maps one specific representation of chiral gauge theory to another representation of chiral gauge theory

$$\int \mathcal{L}=\int \bar{\psi}P_+D(U)\hat{P}_-(U)\psi$$
$$\rightarrow\int \mathcal{L}=\int \bar{\psi}\gamma_5\hat{P}_+(U)\gamma_5D(U)P_-\psi \tag{2.56}$$

based on the *same* vectorlike theory defined by the lattice operator D.

It may be appropriate to recall here the essence of our previous analysis [3]. The functional space in our problem is naturally spanned by the eigenfunctions of the basic Hermitian operator $H=a\gamma_5D$

$$H\varphi_n=\lambda_n\varphi_n. \tag{2.57}$$

However, this eigenvalue equation is gauge covariant as are all the quantities in the gauge invariant lattice regularization. To accommodate the gauge noncovariant quantities such as a consistent form of anomaly, one defines the path integral in a specific topological sector specified by M by

$$Z_M(U)=\exp[i\vartheta_M(\langle w_n|v_m\rangle;\langle\bar{w}_n|\bar{v}_m\rangle)]$$
$$\times\int\prod_{n,l}da_nd\bar{a}_l\exp\left[\int\mathcal{L}(\bar{\psi},\psi,U)\right], \tag{2.58}$$

where we expanded fermionic variables as

$$\hat{P}_-\psi=\sum_n a_nw_n,$$
$$\bar{\psi}P_+=\sum_n \bar{a}_n\bar{w}_n. \tag{2.59}$$

The basis vectors $\{w_n\}$ and $\{\bar{w}_n\}$, which satisfy

$$\hat{P}_-w_n=w_n,\quad \bar{w}_nP_+=\bar{w}_n \tag{2.60}$$

are suitable linear combinations of $\{\varphi_n\}$ and $\{\varphi_n^\dagger\}$, respectively. The "measure factor" ϑ_M, which is the Jacobian for the transformation from *ideal* bases $\{v_n\}$ and $\{\bar{v}_n\}$ to the bases specified by H and thus crucially depends on the ideal bases,[7] is not specified at this stage and it is later determined by imposing several physical conditions.

When one considers the change of fermionic variables which formally corresponds to gauge transformation $\psi\rightarrow\psi'$ and $\bar{\psi}\rightarrow\bar{\psi}'$, the expansion coefficients with the fixed basis vectors are transformed as $\{a_n\}\rightarrow\{a_n'\}$ and $\{\bar{a}_n\}\rightarrow\{\bar{a}_n'\}$. Since the naming of path integral variables does not matter, one obtains the identity

$$\exp[i\vartheta_M(\langle w_n|v_m\rangle;\langle\bar{w}_n|\bar{v}_m\rangle)]\int\prod_{n,l}da_nd\bar{a}_l\exp\left[\int\mathcal{L}(\bar{\psi},\psi,U)\right]$$
$$=\exp[i\vartheta_M(\langle w_n|v_m\rangle;\langle\bar{w}_n|\bar{v}_m\rangle)]\int\prod_{n,l}da_n'd\bar{a}_l'\exp\left[\int\mathcal{L}(\bar{\psi}',\psi',U)\right]. \tag{2.61}$$

In this form of identity, the Jacobian of path integral measure gives a lattice version of covariant anomaly and the variation of the action gives the divergence of covariant current. The gauge covariant fermion number anomaly is naturally derived in this way.

If one performs the simultaneous gauge transformation of the link variables U in the above path integral $Z_M(U)$, the action becomes invariant but one needs to take into account the variation of the measure factor $\delta\vartheta_M(\langle w_n|v_m\rangle;\langle\bar{w}_n|\bar{v}_m\rangle)$ induced by the gauge transformation of U. This variation $\delta\vartheta_M$ converts the covariant anomaly to a lattice form of consistent anomaly, which is one of the requirements on the measure factor. In the anomaly free theory, $\delta\vartheta_M$ should completely cancel the non-vanishing Jacobian arising from lattice artifacts. The current associated to $\delta\vartheta_M$ should be local and satisfy several other requirements: The existence proof of such a measure factor ϑ_M amounts to a definition of lattice chiral gauge theory [11–16].

[7]The measure factor is thus chosen to be a constant for the ideal bases.

A characteristic property of the Ginsparg-Wilson algebra is that among the eigenfunctions of $H\varphi_n=\lambda_n\varphi_n$ the eigenstates corresponding to zero modes and also those eigenstates corresponding to the largest values of $|\lambda_n|$ are chosen to be the simultaneous eigenstates of γ_5, and that those states corresponding to the largest values of $|\lambda_n|$ are annihilated by $\Gamma_5\varphi_n=0$. By noting

$$\hat{P}_\pm=\frac{1\pm\hat{\gamma}_5}{2}=P_\mp\pm\Gamma_5, \tag{2.62}$$

$\hat{P}_\pm$ is replaced by $P_\mp$ when acting on the modes annihilated by Γ_5. We also have the chirality sum rule arising from $\mathrm{Tr}\,\gamma_5=0$, $n_++N_+=n_-+N_-$, where $n_\pm$ and $N_\pm$ stand for the numbers of zero modes and largest eigenmodes with chirality $\pm$, respectively (see the Appendix).

In terms of the eigenfunctions of the basic operator H, one can describe the change of the action under the standard CP transformation as follows. One starts with

$$\int \mathcal{L}=\int \bar{\psi}P_+D(U)\hat{P}_-(U)\psi \tag{2.63}$$

which is characterized by

$$\bar{\psi}=\begin{pmatrix} n_+ \\ N_+ \end{pmatrix}, \quad \psi=\begin{pmatrix} n_- \\ N_+ \end{pmatrix}, \quad n_+-n_-=\mathrm{Tr}\,\Gamma_5(U). \tag{2.64}$$

Here we write only the number of simultaneous chiral eigenstates explicitly, since the same number of eigenstates belonging to other eigenvalues are included in ψ and $\bar{\psi}$. The CP conjugate theory is defined by

$$\int \mathcal{L}^{CP}=\int \bar{\psi}^{CP}P_+D(U^{CP})\hat{P}_-(U^{CP})\psi^{CP} \tag{2.65}$$

which is characterized by

$$\bar{\psi}^{CP}=\begin{pmatrix} n_+^{CP} \\ N_+^{CP} \end{pmatrix}, \quad \psi^{CP}=\begin{pmatrix} n_-^{CP} \\ N_+^{CP} \end{pmatrix},$$
$$n_+^{CP}-n_-^{CP}=\mathrm{Tr}\,\Gamma_5(U^{CP})=-\mathrm{Tr}\,\Gamma_5(U). \tag{2.66}$$

A regular renaming of fermionic variables in $\mathcal{L}^{CP}$

$$\bar{\psi}^{CP}=\psi'W, \quad \psi^{CP}=-W^{-1}\bar{\psi}' \tag{2.67}$$

gives rise to

$$\int \mathcal{L}^{CP}=\int \bar{\psi}'\gamma_5\hat{P}_+(U)\gamma_5D(U)P_-\psi' \tag{2.68}$$

which is characterized by

$$\bar{\psi}'=\begin{pmatrix} n_+'=n_-^{CP} \\ N_-'=N_+^{CP} \end{pmatrix}, \quad \psi'=\begin{pmatrix} n_-'=n_+^{CP} \\ N_-'=N_+^{CP} \end{pmatrix},$$
$$n_+'-n_-'=\mathrm{Tr}\,\Gamma_5(U). \tag{2.69}$$

This analysis suggests that we may define the CP transformed theory by means of the chiral theory defined by projection operators P_- and $\gamma_5\hat{P}_+\gamma_5$. It is shown that this is in fact consistent including the measure factor [3].

When one compares the original theory to the CP transformed theory, the topological index is identical for these two theories. Although the number of heaviest modes is different $N_-'=N_-\neq N_+$ in general, one may expect that these two theories when summed over all the topological sectors give rise to an identical result. After all, it should not matter how one chooses a specific chiral projection of the original vectorlike theory specified by D, as long as it is not singular. (The continuum limit is expected to be identical, if it is well defined.) This expectation is in fact born out by a detailed analysis, and the difference $N_-'\neq N_+$ is taken care of by suitably choosing the weight factors for different topological sectors when summing those sectors [3]. The different actions however give rise to different propagators (for finite a)

$$\langle\psi_L(x)\overline{\psi_L}(y)\rangle=\hat{P}_-\frac{1}{D}P_+\to P_-\frac{1}{D}\gamma_5\hat{P}_+\gamma_5\neq\hat{P}_-\frac{1}{D}P_+ \tag{2.70}$$

which manifest CP breaking in this formulation. From this view point, if one chooses projection operators for which the chiral theories before and after CP transformation coincide, one inevitably encounters a singularity in the topologically nontrivial sector because of $N_-'\neq N_+$. The conflict with CP symmetry could thus be regarded as a topological obstruction.

The CP noninvariance in the action level persists in the domain wall representation

$$\int \mathcal{L}_L=\int \bar{q}P_+D\frac{1}{\gamma_5\Gamma_5}P_-q+\int \bar{Q}\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-Q. \tag{2.71}$$

A natural definition of CP transformation is

$$\bar{q}\to q^TW,$$
$$q\to -W^{-1}\bar{q}^T,$$
$$\bar{Q}\to Q^TW,$$
$$Q\to -W^{-1}\bar{Q}^T. \tag{2.72}$$

This transformation leaves the vectorlike theory invariant up to the overall signature of the second term in Eq. (2.17), which is immaterial.[8] We note that one cannot keep Q and $\bar{Q}$ invariant under CP since the gauge field is transformed under CP by

[8]One may define a transformation law $Q\to W^{-1}\bar{Q}^T$, for example, to keep the action invariant. In this case, however, the CP transformation applied twice gives rise to $Q\to -Q$ and $\bar{Q}\to -\bar{Q}$.

$$W\frac{1}{\gamma_5\Gamma_5(U^{CP})}W^{-1}=\left[\frac{1}{\gamma_5\Gamma_5(U)}\right]^T. \tag{2.73}$$

The part containing the field q and $\bar{q}$ in the above chiral Lagrangian is invariant under the CP transformation, but the part containing Q and $\bar{Q}$ is not invariant under the CP transformation

$$\begin{aligned}\left(W\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-W^{-1}\right)^T&=P_+\frac{1}{\gamma_5\Gamma_5}\gamma_5\hat{P}_+\gamma_5\\&=\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-+\gamma_5\neq\hat{P}_-\frac{1}{\gamma_5\Gamma_5}P_-.\end{aligned} \tag{2.74}$$

The conflict with CP symmetry persists as far as the invariance of the action is concerned. We recall that Q and $\bar{Q}$ are essential to maintain the full physical contents described by the variables ψ and $\bar{\psi}$. To analyze the effects of CP violation in the sector of Q and $\bar{Q}$ precisely, we discuss a modified CP transformation for ψ_L and $\bar{\psi}_L$ in the next section, which includes the effects of both of $q,\bar{q}$ and $Q,\bar{Q}$.

As we noted above, this CP transformation is regarded as a change of representation specified by $P_\pm$ and $\hat{P}_\pm$ to another representation specified by $\gamma_5\hat{P}_\pm\gamma_5$ and $P_\pm$. To be specific, we have

$$\begin{aligned}\gamma_5\hat{P}_+\gamma_5&=(\gamma_5\Gamma_5)P_+P_+\left(\frac{1}{\gamma_5\Gamma_5}\right)\gamma_5\hat{P}_+\gamma_5,\\ \bar{\psi}\gamma_5\hat{P}_+\gamma_5&=\bar{\psi}(\gamma_5\Gamma_5)P_+P_+\left(\frac{1}{\gamma_5\Gamma_5}\right)\gamma_5\hat{P}_+\gamma_5\\&=\bar{q}P_+P_+\left(\frac{1}{\gamma_5\Gamma_5}\right)\gamma_5\hat{P}_+\gamma_5,\\ \mathcal{L}&=\bar{\psi}\gamma_5\hat{P}_+\gamma_5DP_-\psi=\bar{q}P_+\left(\frac{1}{\gamma_5\Gamma_5}\right)DP_-q,\end{aligned} \tag{2.75}$$

where we defined

$$\bar{q}=\bar{\psi}\gamma_5\Gamma_5,\quad q=\psi. \tag{2.76}$$

We thus have

$$\begin{aligned}&\int\mathcal{D}\psi_L\mathcal{D}\bar{\psi}_L\exp\left(\int\bar{\psi}\gamma_5\hat{P}_+\gamma_5DP_-\psi\right)\\&=\int\mathcal{D}q_L\mathcal{D}\bar{q}_L\mathcal{D}Q_R\mathcal{D}\bar{Q}_L\exp\left[\int\bar{q}P_+\left(\frac{1}{\gamma_5\Gamma_5}\right)DP_-q\right.\\&\qquad\left.+\int\bar{Q}P_+\frac{1}{\gamma_5\Gamma_5}\gamma_5\hat{P}_+\gamma_5Q\right],\end{aligned} \tag{2.77}$$

where the action for q and $\bar{q}$ formally retains the form before the CP transformation, though the definitions of q and $\bar{q}$ in terms of ψ and $\bar{\psi}$ are not the same as before. To the extent one can define the left-hand side consistently, one can define the right-hand side consistently except for topological properties. The source terms and the resulting propagator, which need to be defined in terms of the local variables ψ and $\bar{\psi}$, however, change under the CP transformation as in Eq. (2.70). We reiterate that the variables q and $\bar{q}$ cannot describe the essential properties such as the fermion number nonconservation and chirality selection rules (in vectorlike theory), which are described by the local variables ψ and $\bar{\psi}$.

The same conclusion is obtained, namely, CP noninvariance of the action, for the ‘‘fermionic’’ representation of the domain wall fermion

$$\mathcal{L}_L=\int\bar{q}P_+D\frac{1}{\gamma_5\Gamma_5}P_-q+\int\bar{S}P_-\gamma_5\Gamma_5\hat{P}_-S \tag{2.78}$$

if one assigns the natural CP transformation law

$$\begin{aligned}\bar{S}&\to S^TW,\\ S&\to -W^{-1}\bar{S}^T\end{aligned} \tag{2.79}$$

which keeps the vectorlike theory invariant. We then have

$$\begin{aligned}(WP_-\gamma_5\Gamma_5\hat{P}_-W^{-1})^T&=\gamma_5\hat{P}_+\gamma_5\gamma_5\Gamma_5P_+\\&=P_-\gamma_5\Gamma_5\hat{P}_-+\gamma_5\Gamma_5^2\neq P_-\gamma_5\Gamma_5\hat{P}_-\end{aligned} \tag{2.80}$$

which is again interpreted as a change of representation of lattice chiral theory based on the same vectorlike theory.

III. MODIFIED LATTICE CP FOR GINSPARG-WILSON OPERATORS

The part of the Lagrangian for chiral domain wall fermions in Eq. (2.71), which includes the light variables q and $\bar{q}$, is invariant under CP transformation. This property together with $q=\gamma_5\Gamma_5\psi$ and $\bar{q}=\bar{\psi}$ suggest a modified lattice CP transformation (which is fixed by first going to q and then coming back to ψ after CP operation)

$$\begin{aligned}\psi_L&=\hat{P}_-\psi\to\psi_L^{CP}=-W^{-1}\left[\bar{\psi}_L\frac{1}{\gamma_5\Gamma_5(U)}\right]^T,\\ \bar{\psi}_L&=\bar{\psi}P_+\to\bar{\psi}_L^{CP}=[\gamma_5\Gamma_5(U)\psi_L]^TW\end{aligned} \tag{3.1}$$

for the chiral theory defined in terms of the Ginsparg-Wilson fermion

$$\mathcal{L}_L=\bar{\psi}_LD\psi_L=\bar{\psi}\frac{1+\gamma_5}{2}D\frac{1-\hat{\gamma}_5}{2}\psi. \tag{3.2}$$

One can confirm that the chiral Lagrangian is invariant under the above modified lattice CP transformation. If one establishes that the Jacobian for the above modified CP transformation gives unity, all the effects of CP violation (or abnormal C) effects appear in the propagator, which is derived by considering source terms (2.29) for ψ_L and $\overline{\psi}_L$,

$$\int \mathcal{L}^{CP}_{\text{source}}=\int [\overline{\eta}^{CP}_R\psi^{CP}_L+\overline{\psi}^{CP}_L\eta^{CP}_R]$$
$$=\int\left[\overline{\eta}_R\gamma_5\Gamma_5(U)\psi_L+\overline{\psi}_L\frac{1}{\gamma_5\Gamma_5(U)}\eta_R\right], \tag{3.3}$$

and the propagator becomes after CP transformation

$$(\gamma_5\Gamma_5)\hat{P}_-\frac{1}{D}P_+\frac{1}{(\gamma_5\Gamma_5)}=P_-\frac{1}{D}\gamma_5\hat{P}_+\gamma_5\neq\hat{P}_-\frac{1}{D}P_+ \tag{3.4}$$

in pure chiral gauge theory, to be consistent with our previous result [3]. We here assumed the natural CP transformation for the source functions $\overline{\eta}_R\rightarrow\eta^T_R W$ and $\eta_R\rightarrow -W^{-1}\overline{\eta}^T_R$.

This analysis turned out to be rather limited in its scope and it is applicable only to the topologically trivial sector, as it is directly related to the domain wall representation. It is, however, nice to examine the modified lattice CP transformation, since, after all, the invention of a lattice version of chiral transformation was the starting point of the analysis of lattice chiral gauge theory. Also, this analysis illustrates an alternative picture about what is going on in the analysis of CP symmetry, together with general topological complications associated with the transformation which keeps action invariant.

In passing, we note that the vectorlike theory defined by the Ginsparg-Wilson fermion is invariant under the modified CP transformation

$$\psi\rightarrow-\frac{1}{\gamma_5\Gamma_5}W^{-1}\overline{\psi}^T=-W^{-1}\left(\overline{\psi}\frac{1}{\gamma_5\Gamma_5}\right)^T,$$
$$\overline{\psi}\rightarrow\psi^T(\gamma_5\Gamma_5)^T W=(\gamma_5\Gamma_5\psi)^T W \tag{3.5}$$

and the Jacobian for this transformation is unity up to the possible singularity associated with $1/(\gamma_5\Gamma_5)$.

We now present a precise analysis of the Jacobian factor associated with the above modified CP transformation in chiral gauge theory, and show that we arrive at precisely the same conclusion, at least in the topologically trivial sector, as in our previous analysis based on the more conventional CP transformation [3]. The analysis in this section also provides some of the mathematical details briefly sketched in the previous section.

We start with the definition of expectation values in the fermion sector of the chiral gauge theory

$$\langle\mathcal{O}\rangle=\int\mathcal{D}\psi_L\mathcal{D}\overline{\psi}_L\mathcal{O}\exp\left(\int\overline{\psi}_L D\psi_L\right) \tag{3.6}$$

and

$$\mathcal{D}\psi_L\mathcal{D}\overline{\psi}_L=\prod_j dc_j\prod_k d\overline{c}_k. \tag{3.7}$$

In this expression, c_j and $\overline{c}_k$ are the expansion coefficients of fermion fields.

$$\psi_L(x)=\sum_j v_j(x)c_j,\quad c_j=(v_j,\psi_L)\equiv a^4\sum_x v_j^\dagger(x)\psi_L(x) \tag{3.8}$$

and

$$\overline{\psi}_L(x)=\sum_k \overline{c}_k\overline{v}_k(x),\quad \overline{c}_k=(\overline{\psi}^\dagger_L,\overline{v}^\dagger_k)\equiv a^4\sum_x\overline{\psi}_L(x)\overline{v}^\dagger_k(x). \tag{3.9}$$

Basic requirements for the (ideal) basis vectors are

$$\hat{P}_-v_j=v_j,\quad (v_j,v_k)=\delta_{jk} \tag{3.10}$$

and

$$\overline{v}_kP_+=\overline{v}_k,\quad (\overline{v}^\dagger_j,\overline{v}^\dagger_k)=\delta_{jk} \tag{3.11}$$

so that

$$\hat{P}_-\psi_L=\psi_L,\quad \overline{\psi}_LP_+=\overline{\psi}_L. \tag{3.12}$$

Let us consider how $\langle\mathcal{O}\rangle$ changes under the CP transformation of the gauge field $U\rightarrow U^{CP}$. The above framework gives

$$\langle\mathcal{O}\rangle(U^{CP})=\int\mathcal{D}\psi^{CP}_L\mathcal{D}\overline{\psi}^{CP}_L\mathcal{O}^{CP}$$
$$\times\exp\left[\int\overline{\psi}^{CP}_LD(U^{CP})\psi^{CP}_L\right] \tag{3.13}$$

and

$$\mathcal{D}\psi^{CP}_L\mathcal{D}\overline{\psi}^{CP}_L=\prod_j dc^{CP}_j\prod_k d\overline{c}^{CP}_k. \tag{3.14}$$

Here the expansion coefficients are defined by

$$\psi^{CP}_L(x)=\sum_j v^{CP}_j(x)c^{CP}_j,\quad c^{CP}_j=(v^{CP}_j,\psi^{CP}_L) \tag{3.15}$$

and

$$\overline{\psi}^{CP}_L(x)=\sum_k\overline{c}^{CP}_k\overline{v}^{CP}_k(x),\quad \overline{c}^{CP}_k=(\overline{\psi}^{CP\dagger}_L,\overline{v}^{CP\dagger}_k). \tag{3.16}$$

The (ideal) basis vectors satisfy

$$\hat{P}_-(U^{CP})v^{CP}_j=v^{CP}_j,\quad (v^{CP}_j,v^{CP}_k)=\delta_{jk} \tag{3.17}$$

and

$$\overline{v_k^{CP}}P_+=\overline{v_k^{CP}}, \quad (\overline{v_j^{CP\dagger}},\overline{v_k^{CP\dagger}})=\delta_{jk}. \tag{3.18}$$

In what follows, we take basis vectors as

$$\overline{v_k^{CP}}=\overline{v_k} \tag{3.19}$$

because both satisfy the *same* chirality constraint that is independent of gauge fields.

We thus examine the following modified substitution rule:

$$\psi_L^{CP}=-W^{-1}\left[\overline{\psi_L}\frac{1}{\gamma_5\Gamma_5(U)}\right]^T=-\sum_k W^{-1}\left[\overline{v_k}\frac{1}{\gamma_5\Gamma_5(U)}\right]^T\overline{c_k} \tag{3.20}$$

and

$$\overline{\psi_L^{CP}}=[\gamma_5\Gamma_5(U)\psi_L]^TW=\sum_j [\gamma_5\Gamma_5(U)v_j]^TWc_j. \tag{3.21}$$

This substitution is in fact consistent with the chirality constraint $\hat{P}_-(U^{CP})\psi_L^{CP}=\psi_L^{CP}$ and $\overline{\psi_L^{CP}}P_+=\overline{\psi_L^{CP}}$. Moreover, the action takes the form identical to the original one under this substitution, as we already noted. The appearance of the singular factor $1/(\gamma_5\Gamma_5)$ is consistent with our "no-go theorem" [2]. The question related to the existence of the inverse $1/(\gamma_5\Gamma_5)$ is discussed later. These observations show that we should consider a change of integration variables from $(c_j^{CP},\overline{c_k^{CP}})$ to $(c_j,\overline{c_k})$. These two sets are connected by

$$c_j^{CP}=-\sum_k a^4\sum_x v_j^{CP\dagger}(x)W^{-1}\left[\overline{v_k}(x)\frac{1}{\gamma_5\Gamma_5}\right]^T\overline{c_k} \tag{3.22}$$

and

$$\overline{c_k^{CP}}=\sum_j a^4\sum_x [\gamma_5\Gamma_5v_j(x)]^TW\overline{v_k^\dagger}(x)c_j. \tag{3.23}$$

This transformation is, however, regular only if $\mathrm{Tr}\,\Gamma_5=n_+-n_-=0$, because

$$\begin{aligned}\text{No. of } c_j^{CP}-\text{No. of }\overline{c_k}&=\mathrm{Tr}\,\hat{P}_-(U^{CP})-\mathrm{Tr}\,P_+\\&=\mathrm{Tr}\,\hat{P}_+(U)-\mathrm{Tr}\,P_+=\mathrm{Tr}\,\Gamma_5\end{aligned} \tag{3.24}$$

and

$$\text{No. of } \overline{c_k^{CP}}-\text{No. of } c_j=\mathrm{Tr}\,\hat{P}_+-\mathrm{Tr}\,\hat{P}_-=\mathrm{Tr}\,\Gamma_5. \tag{3.25}$$

So we assume $\mathrm{Tr}\,\Gamma_5=n_+-n_-=0$ in what follows; this is also necessary (though not sufficient) for the existence of the inverse of $\gamma_5\Gamma_5$.

By defining

$$\prod_j dc_j^{CP}\prod_k d\overline{c_k^{CP}}=J^{-1}\prod_j dc_j\prod_k d\overline{c_k} \tag{3.26}$$

we have

$$\begin{aligned}J&=\det\left\{-a^4\sum_x v_j^{CP\dagger}(x)W^{-1}\left[\overline{v_k}(x)\frac{1}{\gamma_5\Gamma_5}\right]^T\right\}\det\left\{a^4\sum_x [\gamma_5\Gamma_5v_j(x)]^TW\overline{v_k^\dagger}(x)\right\}\\
&=\det\left\{a^4\sum_x [\gamma_5\Gamma_5v_j(x)]^TW\overline{v_k^\dagger}(x)\right\}\det\left[-a^4\sum_x \overline{v_k}(x)\frac{1}{\gamma_5\Gamma_5}(W^{-1})^Tv_j^{CP*}(x)\right]\\
&=\det\left\{-a^4\sum_x [\gamma_5\Gamma_5v_j(x)]^TWP_+\frac{1}{\gamma_5\Gamma_5}(W^{-1})^Tv_k^{CP*}(x)\right\}\\
&=\det\left\{-a^4\sum_x v_j^{CP\dagger}(x)W^{-1}\left[\frac{1}{\gamma_5\Gamma_5(U)}\right]^TP_+^TW^T\gamma_5\Gamma_5(U)v_k(x)\right\}\\
&=\det\left[-a^4\sum_x v_j^{CP\dagger}(x)\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\Gamma_5(U)v_k(x)\right],\end{aligned} \tag{3.27}$$

where we have used $\Sigma_k\overline{v_k^\dagger}(x)\overline{v_k}(y)=P_+\delta_{x,y}$ in deriving the third line, and $W^T=W$ and $P_-\gamma_5\Gamma_5=\gamma_5\Gamma_5\hat{P}_-$ in the fourth line.

Clearly, whether the Jacobian J is unity or not depends on the relation between v_k and v_j^{CP} which may be quite arbitrary (because these refer to different gauge fields U and U^{CP}, respectively). To investigate a minimal condition on v_k and v_j^{CP} such that $J=1$, we consider an infinitesimal variation of the gauge field specified by

$$\delta_\eta U(x,\mu)=a\,\eta_\mu(x)U(x,\mu). \tag{3.28}$$

Under this variation, the Jacobian $J=\det M$ changes as $\delta_\eta\ln J=\mathrm{tr}\,\delta_\eta MM^{-1}$, where

$$\delta_\eta M_{jk}=-a^4\sum_x \delta_\eta v_j^{CP\dagger}(x)\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\Gamma_5(U)v_k(x)$$
$$+a^4\sum_x v_j^{CP\dagger}(x)\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\delta_\eta\Gamma_5(U^{CP})$$
$$\times\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\Gamma_5(U)v_k(x)$$
$$-a^4\sum_x v_j^{CP\dagger}(x)\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\delta_\eta\Gamma_5(U)v_k(x)$$
$$-a^4\sum_x v_j^{CP\dagger}(x)\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\Gamma_5(U)\delta_\eta v_k(x) \tag{3.29}$$

and

$$M_{kj}^{-1}=\left(v_k,\frac{1}{\gamma_5\Gamma_5(U)}\gamma_5\Gamma_5(U^{CP})v_j^{CP}\right). \tag{3.30}$$

Using $\Sigma_j v_j(x)v_j^\dagger(y)=\hat{P}_-(U)(x,y)$, $\Sigma_j v_j^{CP}(x)v_j^{CP\dagger}(y)=\hat{P}_-(U^{CP})(x,y)$ and

$$\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\Gamma_5(U)\hat{P}_-(U)$$
$$=\hat{P}_-(U^{CP})\frac{1}{\gamma_5\Gamma_5(U^{CP})}\gamma_5\Gamma_5(U) \tag{3.31}$$

together with Eq. (2.45), we arrive at

$$\delta_\eta \ln J=-i\mathcal{L}_\eta+i\mathcal{L}_\eta^{CP}-\delta_\eta \mathrm{Tr}\,\Gamma_5(U), \tag{3.32}$$

where $\mathcal{L}_\eta$ and $\mathcal{L}_\eta^{CP}$ are so-called measure terms [11–16].

$$\mathcal{L}_\eta=i\sum_j (v_j,\delta_\eta v_j),\quad \mathcal{L}_\eta^{CP}=i\sum_j (v_j^{CP},\delta_\eta v_j^{CP}) \tag{3.33}$$

which specify how the fermion path integral measure changes according to a change of gauge fields.

Recalling that $\mathrm{Tr}\,\Gamma_5$ is an integer which cannot change under an infinitesimal variation of the gauge field (or simply that we have set $\mathrm{Tr}\,\Gamma_5=0$), we see that the necessary condition for $J=1$ is $\mathcal{L}_\eta^{CP}=\mathcal{L}_\eta$. Namely, for the CP invariance, the (ideal) basis vectors have to be chosen such that $\mathcal{L}_\eta^{CP}=\mathcal{L}_\eta$. Conversely, if $\mathcal{L}_\eta^{CP}=\mathcal{L}_\eta$, we see that the Jacobian J is a constant which can depend only on the topological properties of each sector. In the vacuum sector, in which the vacuum $U_0=1$ is contained, we can determine this constant and obtain $J=1$ because $U_0^{CP}=1=U_0$. So, for the vacuum sector, $\mathcal{L}_\eta^{CP}=\mathcal{L}_\eta$ implies $J=1$.

In our previous work, we have shown that the conditions on the ideal measure factor (which appear in the reconstruction theorem of chiral gauge theory [11,13]) are consistent with the choice $\mathcal{L}_\eta^{CP}=\mathcal{L}_\eta$ [3]. The unit Jacobian condition (in the vacuum sector) is thus equivalent to the existence of the ideal measure factor in this sense.

In fact, the CP invariance in the sense that we can ignore the Jacobian associated with the above modified CP transformation is shown more generally, when there is no modes such that $\gamma_5\Gamma_5\Psi(x)=0$, namely, $N_+=N_-=0$. In this case, one can show that the Jacobian is a pure phase, $J=e^{i\theta}$. With the CP invariant choice of the fermion measure terms $\mathcal{L}_\eta^{CP}=\mathcal{L}_\eta$, the phase θ is a constant depending only on the topological sector, as we have shown above. Such a constant breaking of CP, however, may be reabsorbed into the basis vectors v_j and v_j^{CP} (this operation does not change the measure terms), or equivalently may be absorbed into the phase factor ϑ_M for each topological sector. This apparent CP breaking is thus harmless. This is completely consistent with our result in the previous work [3] where the CP invariance of path integral (in the topologically trivial sector) except for propagators is shown. We present the proof of the above statement below.

Proof of $|J|^2=1$: Our Jacobian factor J is expressed as

$$J=\det\left[-a^4\sum_x v_j^{CP\dagger}(x)\frac{1}{\gamma_5\Gamma_5(U^{CP})}t_k(x)\right]$$
$$\times\det\left[a^4\sum_x t_j^\dagger(x)\gamma_5\Gamma_5(U)v_k(x)\right], \tag{3.34}$$

where $\{t_j(x)\}$ is any orthonormal complete set of vectors such that $P_-t_j=t_j$. First, one can easily see that $|J|^2$ is invariant under a unitary transformation of bases v_j^{CP} and v_j. We may therefore choose any bases (as long as they are consistent with the chirality constraints) in evaluating $|J|^2$. A convenient choice is the "auxiliary basis" defined by H:

$$v_j(x)=w_j(x)=\hat{P}_-u_j(x),\quad (w_j,w_k)=\delta_{jk} \tag{3.35}$$

or, more explicitly,

$$w_j=\varphi_0^- \tag{3.36}$$

which satisfies $H^2w_j=0$, and

$$w_j=\frac{1}{\sqrt{2[1-\lambda_j f(\lambda_j^2)]}}\{\sqrt{1-\lambda_j^2 f^2(\lambda_j^2)}\varphi_j-[1-\lambda_j f(\lambda_j^2)]\tilde{\varphi}_j\} \tag{3.37}$$

which satisfies $H^2w_j=\lambda_j^2 w_j$. (We use basically half of the eigenstates of H.) See the Appendix for notational conventions. Similarly we set

$$v_j^{CP}(x)=w_j^{CP}(x)=\hat{P}_-(U^{CP})u_j^{CP}(x),\quad (w_j^{CP},w_k^{CP})=\delta_{jk}, \tag{3.38}$$

where $u_j^{CP}(x)$ is the eigenfunction of $H^2(U^{CP})$, $H^2(U^{CP})u_j^{CP}(x)=\lambda_j^{CP2}u_j^{CP}(x)$. We also use

$$t_j(x)=P_-u_j(x), \quad (t_j,t_k)=\delta_{jk}, \tag{3.39}$$

namely,

$$t_j=\varphi_0^- \tag{3.40}$$

and

$$t_j=\frac{1}{\sqrt{2[1-\lambda_j f(\lambda_j^2)]}}\{\sqrt{1-\lambda_j^2 f^2(\lambda_j^2)}\varphi_j-[1+\lambda_j f(\lambda_j^2)]\tilde{\varphi}_j\}. \tag{3.41}$$

When there are no modes such that $\gamma_5\Gamma_5\Psi(x)=0$, the above vectors span complete sets in the space restricted by the chirality constraints. Then, using properties of Γ_5, it is straightforward to see that

$$a^4\sum_x t_j^\dagger(x)\gamma_5\Gamma_5(U)v_k(x)=\sqrt{1-\lambda_j^2 f^2(\lambda_j^2)}\delta_{jk} \tag{3.42}$$

and

$$a^4\sum_x v_j^{CP\dagger}(x)\frac{1}{\gamma_5\Gamma_5(U^{CP})}t_k(x)=\frac{1}{\sqrt{1-\lambda_j^{CP2}f^2(\lambda_j^{CP2})}}\delta_{jk}. \tag{3.43}$$

Therefore, we have

$$|J|^2=\frac{\prod_j [1-\lambda_j^2 f^2(\lambda_j^2)]}{\prod_k [1-\lambda_k^{CP2}f^2(\lambda_k^{CP2})]}. \tag{3.44}$$

This combination is, however, unity because for $\lambda_j\neq 0$ (and for $\lambda_j\neq\Lambda$, which is our assumption), the eigenvalues are degenerate as $\lambda_j^2=\lambda_j^{CP2}$, as one can confirm by using the relations in the Appendix and Eq. (2.45).

IV. CP (OR C) TRANSFORMATION AND YUKAWA COUPLINGS

The CP symmetry is of course broken in the presence of the Higgs coupling in chiral gauge theory. For example,[9]

$$\begin{aligned}\mathcal{L}&=\bar{\psi}_L D(U_1)\psi_L+\bar{\psi}_R D(U_2)\psi_R+2g(\bar{\psi}_L\phi\psi_R+\bar{\psi}_R\phi^\dagger\psi_L)\\ &=\bar{\psi}P_+D(U_1)\hat{P}_-(U_1)\psi+\bar{\psi}P_-D(U_2)\hat{P}_+(U_2)\psi\\ &\quad+2g[\bar{\psi}P_+\phi\hat{P}_+(U_2)\psi+\bar{\psi}P_-\phi^\dagger\hat{P}_-(U_1)\psi],\end{aligned} \tag{4.1}$$

[9]We assume that the left-handed fermion $\psi_L(x)$ belongs to the representation R_L of the gauge group and the right-handed fermion $\psi_R(x)$ belongs to R_R [the Higgs field $\phi(x)$ transforms as $R_L\otimes(R_R)^*$]. The gauge couplings in the Dirac operators $D(U_1)$ and $D(U_2)$, and correspondingly in $\hat{P}_-(U_1)$ and $\hat{P}_+(U_2)$, are thus defined with respect to the representations R_L and R_R, respectively.

where CP is broken not only in the kinetic term but also in the Higgs couplings. Under the CP transformation

$$\begin{aligned}&U_1\to U_1^{CP}, \quad U_2\to U_2^{CP}\\ &\bar{\psi}\to\psi^T W, \quad \psi\to -W^{-1}\bar{\psi}^T,\\ &\phi\to\phi^*\end{aligned} \tag{4.2}$$

this Lagrangian is transformed to

$$\begin{aligned}\mathcal{L}^{CP}&=\bar{\psi}\gamma_5\hat{P}_+(U_1)\gamma_5 D(U_1)P_-\psi\\ &\quad+\bar{\psi}\gamma_5\hat{P}_-(U_2)\gamma_5 D(U_2)P_+\psi\\ &\quad+2g[\bar{\psi}\gamma_5\hat{P}_+(U_1)\gamma_5\phi P_+\psi\\ &\quad+\bar{\psi}\gamma_5\hat{P}_-(U_2)\gamma_5\phi^\dagger P_-\psi].\end{aligned} \tag{4.3}$$

This is again interpreted as a change of representation of chiral projection operators, from $P_\pm$ and $\hat{P}_\pm$ to $\gamma_5\hat{P}_\pm\gamma_5$ and $P_\pm$, constructed from a vectorlike Ginsparg-Wilson theory if one introduces two sets of fermion fields $\psi^{(1)}$ and $\psi^{(2)}$ in Eq. (4.1)

$$\begin{aligned}&\psi_L=\hat{P}_-(U_1)\psi^{(1)}, \quad \bar{\psi}_L=\bar{\psi}^{(1)}P_+,\\ &\psi_R=\hat{P}_+(U_2)\psi^{(2)}, \quad \bar{\psi}_R=\bar{\psi}^{(2)}P_-.\end{aligned} \tag{4.4}$$

In a perturbative treatment of the Higgs coupling, the analysis of CP symmetry becomes identical to that of the pure chiral gauge theory, as was shown elsewhere [3]. For a nonperturbative treatment of the Higgs coupling but in the topologically trivial sector, one can use the modified CP transformation motivated by the domain wall fermion

$$\begin{aligned}&\psi_L\to\psi_L^{CP}=-W^{-1}\left[\bar{\psi}_L\frac{1}{\gamma_5\Gamma_5(U_1)}\right]^T,\\ &\bar{\psi}_L\to\bar{\psi}_L^{CP}=[\gamma_5\Gamma_5(U_1)\psi_L]^T W,\\ &\psi_R\to\psi_R^{CP}=-W^{-1}\left[\bar{\psi}_R\frac{1}{\gamma_5\Gamma_5(U_2)}\right]^T,\\ &\bar{\psi}_R\to\bar{\psi}_R^{CP}=[\gamma_5\Gamma_5(U_2)\psi_R]^T W\end{aligned} \tag{4.5}$$

which keeps the action (4.1) invariant. The invariance of the Higgs coupling is confirmed by noting, for example,

$$\begin{aligned}&\bar{\psi}_L^{CP}P_+\phi^{CP}\hat{P}_+(U_2^{CP})\psi_R^{CP}\\ &\quad=-[\gamma_5\Gamma_5(U_1)\psi_L]^T WP_+\phi^*\\ &\qquad\times P_+\gamma_5\Gamma_5(U_2^{CP})W^{-1}\left[\bar{\psi}_R\frac{1}{\gamma_5\Gamma_5(U_2)}\right]^T\\ &\quad=-[\gamma_5\Gamma_5(U_1)\psi_L]^T WP_+\phi^*P_+W^{-1}\bar{\psi}_R^T\\ &\quad=\bar{\psi}_R P_-\phi^\dagger P_-\gamma_5\Gamma_5(U_1)\psi_L\end{aligned}$$

$$=\overline{\psi}_R P_- \phi^\dagger \hat{P}_-(U_1)\psi_L, \quad (4.6)$$

where we used $P_+\hat{P}_+(U_2^{CP})=P_+P_+\gamma_5\Gamma_5(U_2^{CP})$ and $P_-P_-\gamma_5\Gamma_5(U_1)=P_-\hat{P}_-(U_1)$. We can thus repeat the analysis of the previous section and confirm that the path integral is invariant under the modified CP transformation except for the propagators which are determined by the source terms. The essence of CP analysis in the domain wall representation is included in this analysis.

It would be interesting if one can generally establish the CP invariance except for the propagators

$$\langle\psi_L(x)\overline{\psi}_L(y)\rangle=\hat{P}_-(U_1)\frac{1}{D(U_1)-2g\phi\frac{1}{D(U_2)}2g\phi^\dagger}P_+,$$

$$\langle\psi_L(x)\overline{\psi}_R(y)\rangle=-\hat{P}_-(U_1)\frac{1}{D(U_1)-2g\phi\frac{1}{D(U_2)}2g\phi^\dagger}2g\times\phi\frac{1}{D(U_2)}P_-,$$

$$\langle\psi_R(x)\overline{\psi}_R(y)\rangle=\hat{P}_+(U_2)\frac{1}{D(U_2)-2g\phi^\dagger\frac{1}{D(U_1)}2g\phi}P_-,$$

$$\langle\psi_R(x)\overline{\psi}_L(y)\rangle=-\hat{P}_+(U_2)\frac{1}{D(U_2)-2g\phi^\dagger\frac{1}{D(U_1)}2g\phi}\times2g\phi^\dagger\frac{1}{D(U_1)}P_+, \quad (4.7)$$

which depend on the specific choice of chiral projection operators $P_\pm$ and $\hat{P}_\pm$ as in Eq. (4.7) (or $\gamma_5\hat{P}_\pm\gamma_5$ and $P_\pm$ after CP transformation), after summing over the topological sectors but without using the explicit diagonal representation of the action (which was used in our previous paper [3]).

It is shown that CP is broken even in the vectorlike theory in the presence of chiral symmetric Yukawa couplings. For example, one may consider a theory with Abelian flavor symmetry (by using $P_\pm\hat{P}_\pm=P_\pm\gamma_5\Gamma_5$)

$$\begin{aligned}\mathcal{L}&=\overline{\psi}_R D\psi_R+\overline{\psi}_L D\psi_L-m(\overline{\psi}_R\psi_L+\overline{\psi}_L\psi_R)\\&\quad+2g(\overline{\psi}_L\phi\psi_R+\overline{\psi}_R\phi^\dagger\psi_L)\\&=\overline{\psi}D\psi-m\overline{\psi}\gamma_5\Gamma_5\psi+2g\overline{\psi}(P_+\phi\hat{P}_++P_-\phi^\dagger\hat{P}_-)\psi\\&=\overline{\psi}D\psi-m\overline{\psi}\gamma_5\Gamma_5\psi+2g\overline{\psi}(P_+\phi P_++P_-\phi^\dagger P_-)\gamma_5\Gamma_5\psi.\end{aligned} \quad (4.8)$$

The Yukawa coupling in this Lagrangian is not invariant under CP transformation[10]

$$\begin{aligned}&\overline{\psi}\to\psi^TW,\quad \psi\to-W^{-1}\overline{\psi}^T,\\&WD(U^{CP})W^{-1}=D(U)^T,\\&W\gamma_5\Gamma_5(U^{CP})W^{-1}=[\gamma_5\Gamma_5(U)]^T,\\&W\phi W^{-1}=\phi^*.\end{aligned} \quad (4.9)$$

This noninvariance arises from

$$[\gamma_5,\gamma_5\Gamma_5]\neq0,\quad[\phi(x),\gamma_5\Gamma_5]\neq0. \quad (4.10)$$

For a real constant ϕ, these conditions are cleared, and the Yukawa coupling is reduced to the mass term. The above CP noninvariance is of course interpreted as a change from one representation of lattice chiral projectors to another, just as we discussed in the case of pure chiral gauge theory.

One can rewrite the above Lagrangian in terms of the domain wall fermion as

$$\begin{aligned}\mathcal{L}&=\overline{q}D\frac{1}{\gamma_5\Gamma_5}q-m\overline{q}q+2g\overline{q}(P_+\phi P_++P_-\phi^\dagger P_-)q\\&\quad+\overline{Q}\frac{1}{\gamma_5\Gamma_5}Q\end{aligned} \quad (4.11)$$

which is *invariant* under CP transformation

$$\begin{aligned}&\overline{q}\to q^TW,\quad q\to-W^{-1}\overline{q}^T,\\&\overline{Q}\to Q^TW,\quad Q\to-W^{-1}\overline{Q}^T,\\&WD(U^{CP})W^{-1}=D(U)^T,\\&W\gamma_5\Gamma_5(U^{CP})W^{-1}=[\gamma_5\Gamma_5(U)]^T,\\&W\phi W^{-1}=\phi^*\end{aligned} \quad (4.12)$$

if one notes that the overall signature of the last term in Eq. (4.11) is immaterial.

To see the breaking of CP symmetry in this context of the domain wall representation, we introduce the source terms for the fermion fields which specifies general correlation functions. For the local Ginsparg-Wilson variables, we have

$$\int\mathcal{L}_{\rm source}=\int(\overline{\psi}\eta+\overline{\eta}\psi) \quad (4.13)$$

which is invariant under CP transformation

$$\begin{aligned}&\overline{\psi}\to\psi^TW,\quad \psi\to-W^{-1}\overline{\psi}^T,\\&\overline{\eta}\to\eta_w^TW,\quad \eta\to-W^{-1}\overline{\eta}_w^T.\end{aligned} \quad (4.14)$$

[10]Under parity we have $\phi\to\phi^*$, and thus under CP we have $\phi\to\phi^*$.

The source terms are translated in the language of the domain wall fermion as

$$\int \mathcal{L}_{\rm source}=\int\left(\bar{q}\eta+\bar{\eta}\frac{1}{\gamma_5\Gamma_5}q\right) \tag{4.15}$$

which is transformed under CP symmetry

$$\bar{q}\to q^TW,\quad q\to -W^{-1}\bar{q}^T \tag{4.16}$$

to

$$\int\left(\bar{q}\frac{1}{\gamma_5\Gamma_5}\eta_w+\bar{\eta}_w q\right). \tag{4.17}$$

To recover the original source terms,[11] we need to perform the redefinition of field variables

$$q\to\frac{1}{\gamma_5\Gamma_5}q,\quad \bar{q}\to\bar{q}\gamma_5\Gamma_5 \tag{4.18}$$

but the Yukawa coupling is not invariant under this redefinition because of $[\gamma_5,\gamma_5\Gamma_5]\neq 0$ and $[\phi(x),\gamma_5\Gamma_5]\neq 0$. The propagator is thus modified under CP as

$$\frac{1}{D/(\gamma_5\Gamma_5)-m+2g(P_+\phi P_++P_-\phi^\dagger P_-)}\times\frac{1}{\gamma_5\Gamma_5}$$
$$\neq\frac{1}{\gamma_5\Gamma_5}\times\frac{1}{D/(\gamma_5\Gamma_5)-m+2g(P_+\phi P_++P_-\phi^\dagger P_-)}. \tag{4.19}$$

We arrive at the same conclusion by using the fermionic representation of the domain wall fermion with a chiral symmetric Yukawa coupling

$$\mathcal{L}=\bar{q}D\frac{1}{\gamma_5\Gamma_5}q-m\bar{q}q+2g\bar{q}(P_+\phi P_++P_-\phi^\dagger P_-)q$$
$$+\bar{S}\gamma_5\Gamma_5 S \tag{4.20}$$

if one assigns CP transformation

$$\bar{S}\to S^TW,\quad S\to -W^{-1}\bar{S}^T. \tag{4.21}$$

The action is invariant under CP transformation, but to keep the source terms invariant one needs to perform a field redefinition which is not compatible with the Yukawa coupling.

V. MAJORANA FERMION

The above complication of CP symmetry (or equivalently charge conjugation symmetry since the parity is normal in the above model) for the vectorlike theory with the chiral invariant Yukawa coupling gives rise to a difficulty in defining Majorana fermions in a Euclidean sense [2,4]. Following the standard procedure, we replace the field variables [36–38]

$$\psi=(\chi+i\eta)/\sqrt{2},$$
$$\bar{\psi}=(\chi^TC-i\eta^TC)/\sqrt{2} \tag{5.1}$$

in the Lagrangian written in the Ginsparg-Wilson fermions. We naively expect[12]

$$\mathcal{L}=\frac{1}{2}\chi^TCD\chi-\frac{1}{2}m\chi^TC\gamma_5\Gamma_5\chi+g\chi^TC(P_+\phi\hat{P}_+$$
$$+P_-\phi^\dagger\hat{P}_-)\chi+\frac{1}{2}\eta^TCD\eta-\frac{1}{2}m\eta^TC\gamma_5\Gamma_5\eta$$
$$+g\eta^TC(P_+\phi\hat{P}_++P_-\phi^\dagger\hat{P}_-)\eta. \tag{5.2}$$

One would then define the Majorana fermion χ (or η) and the resulting Pfaffian. But this actually fails since the cross terms between χ and η do not quite vanish due to the complications in the charge conjugation.

If one uses the domain wall fermion with "fermionic" variables, one may make the replacement[13]

$$q=(\chi+i\eta)/\sqrt{2},\bar{q}=(\chi^TC-i\eta^TC)/\sqrt{2},$$
$$S=(\lambda+i\rho)/\sqrt{2},$$
$$\bar{S}=(\lambda^TC-i\rho^TC)/\sqrt{2}. \tag{5.3}$$

One can then define the Majorana fermions χ or η (and λ or ρ) by

$$\mathcal{L}=\frac{1}{2}\chi^TCD\frac{1}{\gamma_5\Gamma_5}\chi-\frac{1}{2}m\chi^TC\chi+g\chi^TC(P_+\phi P_+$$
$$+P_-\phi^\dagger P_-)\chi+\frac{1}{2}\eta^TCD\frac{1}{\gamma_5\Gamma_5}\eta-\frac{1}{2}m\eta^TC\eta$$
$$+g\eta^TC(P_+\phi P_++P_-\phi^\dagger P_-)\eta+\frac{1}{2}\lambda^TC\gamma_5\Gamma_5\lambda$$
$$+\frac{1}{2}\rho^TC\gamma_5\Gamma_5\rho, \tag{5.4}$$

namely, one may define a Majorana fermion by

[11]This complication does not appear to be resolved by an argument of the use of equations of motion for external field lines in the nonperturbative treatment of the Yukawa coupling. See Ref. [3].

[12]If $(CO)^T=-CO$ or equivalently $COC^{-1}=O^T$ for a general operator O, the cross term vanishes $\eta^TCO\chi-\chi^TCO\eta=0$ by using the anticommuting property of χ and η. In the presence of background gauge field, we assume that the representation of gauge symmetry is real.

[13]The Majorana reduction of the bosonic fermion Q in the conventional domain wall fermion is nontrivial, since $\int\lambda^TC(1/\gamma_5\Gamma_5)\lambda=0$ for a bosonic spinor.

$$\mathcal{L}_M=\frac{1}{2}\chi^T CD\frac{1}{\gamma_5\Gamma_5}\chi-\frac{1}{2}m\chi^T C\chi+g\chi^T C(P_+\phi P_+ +P_-\phi^\dagger P_-)\chi+\frac{1}{2}\lambda^T C\gamma_5\Gamma_5\lambda. \tag{5.5}$$

This theory is, however, nonlocal due to the singularities in $1/(\gamma_5\Gamma_5)$.

In the level of path integral, one may modify the above Lagrangian by writing

$$mpos;l\int\mathcal{D}\chi\mathcal{D}\lambda\,\exp\left(\int\mathcal{L}_M\right)$$
$$=\int\mathcal{D}\chi\,\exp\left\{\int\left[\frac{1}{2}\chi^T CD\chi-\frac{1}{2}m\chi^T C\gamma_5\Gamma_5\chi+g\chi^T C\sqrt{\gamma_5\Gamma_5}(P_+\phi P_+ +P_-\phi^\dagger P_-)\sqrt{\gamma_5\Gamma_5}\chi\right]\right\}, \tag{5.6}$$

where we made a formal rescaling

$$\chi\to\sqrt{\gamma_5\Gamma_5}\chi,\quad \sqrt{\gamma_5\Gamma_5}\lambda\to\lambda. \tag{5.7}$$

This rescaling formally removes the singular factor $1/(\gamma_5\Gamma_5)$ and makes the auxiliary fermion λ decouple. This final path integral is, however, not what we expect for the Ginsparg-Wilson fermion because of $[\gamma_5,\gamma_5\Gamma_5]\neq 0$ and $[\phi(x),\gamma_5\Gamma_5]\neq 0$, which caused the failure of the charge conjugation symmetry. As for the Pfaffian and the determinant factor without the external fermion lines, one may adopt the above definition of the Majorana fermion, which is consistent up to a possible non-locality arising from $\sqrt{\gamma_5\Gamma_5}$.

A difficulty in defining the Majorana fermion is clearly seen when one considers the source terms for the Ginsparg-Wilson fermion, as we did in the analysis of CP symmetry

$$\int(\bar{J}\psi+\bar{\psi}J)=\int(\chi^T CJ_1+\eta^T CJ_2) \tag{5.8}$$

where the Majorana sources are defined by

$$J=(J_1+iJ_2)/\sqrt{2},\quad \bar{J}=(J_1^T C-iJ_2^T C)/\sqrt{2}. \tag{5.9}$$

The derivatives with respect to the source J_1 give rise to correlation functions of the would-be Majorana fermion χ, which we failed to define for the Ginsparg-Wilson fermion.

The corresponding source terms for the domain wall fermion are given by

$$\int\left(\bar{J}\frac{1}{\gamma_5\Gamma_5}q+\bar{q}J\right)$$
$$=\int\frac{1}{2}\left\{\left[\left(1+\frac{1}{\gamma_5\Gamma_5}\right)\chi\right]^T C-i\left[\left(1-\frac{1}{\gamma_5\Gamma_5}\right)\eta\right]^T C\right\}J_1$$
$$+\int\frac{1}{2}\left\{\left[\left(1+\frac{1}{\gamma_5\Gamma_5}\right)\eta\right]^T C+i\left[\left(1-\frac{1}{\gamma_5\Gamma_5}\right)\chi\right]^T C\right\}J_2, \tag{5.10}$$

where we used the variables supposed to describe Majorana fermions in the domain wall representation (5.4). This expression of source terms shows that neither of the Majorana fermions χ and η, defined by the domain wall fermion correspond to the Majorana fermion generated by the source J_1, for example. In addition, the correlation functions generated by differentiating with respect to J_1 contain the species-doubler poles in $1/(\gamma_5\Gamma_5)$. This shows that we cannot define the Majorana fermion consistently for physical processes in the presence of the chiral symmetric Yukawa coupling. The conflict among chiral symmetry, strict locality and Majorana condition persists. The condition for the presence of Majorana fermions is in a sense more demanding than the CP invariance. The Majorana fermion requires a Lagrangian self-symmetric under charge conjugation, while CP symmetry requires the invariance of the path integral after summing over all topological sectors.

In a supersymmetric Wess-Zumino model on the lattice, one needs to define the constraint-free Majorana fermion.[14] A past attempt to define the Wess-Zumino model is given by [4,30]

$$\mathcal{L}_{\rm WZ}=\frac{1}{2}\chi^T C\frac{1}{\gamma_5\Gamma_5}D\chi-\phi^\dagger D^\dagger D\phi+F^\dagger\frac{1}{\Gamma_5^2}F+\frac{1}{2}m\chi^T C\chi+m[F\phi+(F\phi)^\dagger]+g\chi^T C(P_+\phi P_+ +P_-\phi^\dagger P_-)\chi+g[F\phi^2+(F\phi^2)^\dagger], \tag{5.11}$$

where ϕ stands for the complex scalar field and F for the auxiliary field. The operator D is the (free) Ginsparg-Wilson operator, and when $D^\dagger D$ appears in the bosonic sector we adopt the convention to discard the (unit) Dirac matrix. The Majorana fermion χ and its Yukawa couplings are the same as those we find for the above domain wall representation. However, a crucial difference is that the Pauli-Villars field S is now replaced by the "physical" field F. For this reason, we regard the field χ in $\mathcal{L}_{\rm WZ}$ as a primary definition of the Majorana fermion, though it is defined by a nonlocal Lagrangian. The singular factor $1/(\gamma_5\Gamma_5)$ for the Majorana fermion is canceled by the same factor coming from the auxil-

[14]If one uses the Weyl fermion defined by the Ginsparg-Wilson operator, the constant spinor parameter appearing in supersymmetry transformation is constrained by projection operators. This leads to complications, in particular, in the presence of the background gauge field.

iary field $F^{\dagger}[1/(\Gamma_5^2)]F$. In fact one can confirm that the free part of $\mathcal{L}_{\mathrm{WZ}}$ is invariant under a lattice version of supersymmetry [30]

$$\delta\chi=-\Gamma_5\frac{1}{a}H(A-i\gamma_5B)\epsilon-(F-i\gamma_5G)\epsilon,$$

$$\delta A=\epsilon^TC\chi=\chi^TC\epsilon,$$

$$\delta B=-i\epsilon^TC\gamma_5\chi=-i\chi^TC\gamma_5\epsilon,$$

$$\delta F=\epsilon^TC\Gamma_5\frac{1}{a}H\chi,$$

$$\delta G=i\epsilon^TC\Gamma_5\frac{1}{a}H\gamma_5\chi \tag{5.12}$$

with a constant Majorana-type Grassmann parameter ϵ. Here we defined

$$\phi\rightarrow\frac{1}{\sqrt{2}}(A+iB),\quad F\rightarrow\frac{1}{\sqrt{2}}(F-iG), \tag{5.13}$$

and $H=a\gamma_5D$. This construction of $\mathcal{L}_{\mathrm{WZ}}$ is not completely satisfactory, but it may be amusing to see that a certain aspect of the domain wall fermion may play an essential role in the construction of Majorana fermions.

VI. DISCUSSION

We have examined the CP properties of a domain wall fermion where light field variables q and $\bar{q}$ and the Pauli-Villars fields Q and $\bar{Q}$ are used. It was first shown that the variables q_L and $\bar{q}_L$ cannot describe the topological properties, and the full physical contents are only described by the local Ginsparg-Wilson variables ψ_L and $\bar{\psi}_L$. The domain wall variables q and $\bar{q}$ in the infinite flavor limit, which themselves exhibit nice CP and charge conjugation properties, cannot help to resolve the difficulty associated with CP symmetry in chiral gauge theory [1] and the failure of the Majorana condition in the presence of chiral symmetric Yukawa couplings [4].

The conflict among the good chiral property, strict locality, and CP (or charge conjugation) symmetry thus persists. The CP transformation sends one representation of lattice chiral gauge theory into another representation of lattice chiral gauge theory, which are constructed from the same vectorlike theory defined by the Ginsparg-Wilson operator D. The violation of CP symmetry in the Lagrangian level is partly resolved by summing over various topological sectors [3], and the CP noninvariance is manifested by the change of propagators. In the presence of Higgs couplings, the complications with CP symmetry become more involved since the chiral projection operators are determined by the Ginsparg-Wilson operator which depends only on the gauge field whereas the nonperturbative fermion propagator contains Higgs couplings as well. As for a definition of Majorana fermions in the presence of chiral symmetric Yukawa couplings, an action which is symmetric under the charge conjugation is required. The Ginsparg-Wilson fermions cannot be used in this context. As a tentative (and not complete) resolution of this conflict, we mentioned a use of the domain wall-like representation for the supersymmetric Wess-Zumino model where the auxiliary field F plays a role of the Pauli-Villars fields.

We have analyzed only the infinite flavor limit $N\rightarrow\infty$ in the domain wall fermion, where chiral symmetry is well defined. It will be interesting to examine if the above conflict is already seen for the finite N domain wall fermion where the operator $D_N/(1-aD_N)$ is local (see Ref. [39] for the locality of D_N), though precise chiral symmetry is not defined.[15]

Our analysis of various complications is based on the singular behavior of

$$\frac{1}{\gamma_5\Gamma_5}=\frac{1}{1-aD(\gamma_5aD)^{2k}} \tag{6.1}$$

in the context of general Ginsparg-Wilson operators. This factor contains poles at the positions of the would-be species doublers which have a mass $1/a$ in the case of free fermions, and topological poles in the presence of instantons. This mass value approaches ∞ in the limit $a\rightarrow0$, and those particles are naively expected to decouple from the Hilbert space in the same limit. The singularity at $1/a$ causes nonlocality in a strict sense and thus cannot be consistent in all respects [40], but one might hope that the singularity may not be so serious in a suitable limit $a\rightarrow0$ in some practical applications. This issue may deserve further analyses and, in any case, would lead to a better understanding of the domain wall fermion.

Note added. We have emphasized that the free fermion operator $1/[1-aD(\gamma_5aD)^{2k}]$ in Eq. (6.1) contains poles at the positions of would-be species doublers. The operator $1/[1-aD(\gamma_5aD)^{2k}]$ could contain poles even in the presence of topologically *trivial* gauge fields. See, for example, Ref. [41]. If the functional measure of topologically trivial gauge fields which give rise to the possible poles is substantial, the domain wall representation not only for chiral theory but also for vectorlike theory would be significantly influenced.

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[15]One may, for example, argue that the domain wall variables q_L and $\bar{q}_L$, which become nonlocal and cannot describe topological properties in the limit $N=\infty$, are not the suitable variables to describe physical correlation functions even for finite N, to the extent that the finite N theory is intended to be an approximation to the theory with $N=\infty$.

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APPENDIX: REPRESENTATION OF THE GINSPARG-WILSON ALGEBRA

We here summarize the representation of the general Ginsparg-Wilson relation [2,3] $H\gamma_5+\gamma_5 H=2H^2 f(H^2)$. Let us consider the eigenvalue problem

$$H\varphi_n(x)=\lambda_n\varphi_n(x),\quad (\varphi_n,\varphi_m)=\delta_{nm}. \tag{A1}$$

We first note $H\Gamma_5\varphi_n(x)=-\Gamma_5 H\varphi_n(x)=-\lambda_n\Gamma_5\varphi_n(x)$ and

$$(\Gamma_5\varphi_n,\Gamma_5\varphi_m)=[1-\lambda_n^2 f^2(\lambda_n^2)]\delta_{nm}. \tag{A2}$$

These relations show that eigenfunctions with $\lambda_n\neq 0$ and $\lambda_n f(\lambda_n^2)\neq\pm 1$ come in pairs as λ_n and $-\lambda_n$ [when $\lambda_n=0$, $\varphi_0(x)$ and $\Gamma_5\varphi_0(x)$ are not necessarily linearly independent].

We can thus classify eigenfunctions as follows.

(i) $\lambda_n=0$ $[H\varphi_0(x)=0]$. For this one may impose the chirality on $\varphi_0(x)$ as

$$\gamma_5\varphi_0^{\pm}(x)=\Gamma_5\varphi_0^{\pm}(x)=\pm\varphi_0^{\pm}(x). \tag{A3}$$

We denote the number of $\varphi_0^{+}(x)$ $(\varphi_0^{-}(x))$ as n_+ (n_-).

(ii) $\lambda_n\neq 0$ and $\lambda_n f(\lambda_n^2)\neq\pm 1$. As shown above,

$$H\varphi_n(x)=\lambda_n\varphi_n(x),\quad H\tilde{\varphi}_n(x)=-\lambda_n\tilde{\varphi}_n(x), \tag{A4}$$

where

$$\tilde{\varphi}_n(x)=\frac{1}{\sqrt{1-\lambda_n^2 f^2(\lambda_n^2)}}\Gamma_5\varphi_n(x). \tag{A5}$$

We have

$$\Gamma_5\varphi_n(x)=\sqrt{1-\lambda_n^2 f^2(\lambda_n^2)}\,\tilde{\varphi}_n(x),$$
$$\Gamma_5\tilde{\varphi}_n(x)=\sqrt{1-\lambda_n^2 f^2(\lambda_n^2)}\,\varphi_n(x), \tag{A6}$$

and

$$\gamma_5\varphi_n(x)=\sqrt{1-\lambda_n^2 f^2(\lambda_n^2)}\,\tilde{\varphi}_n(x)+\lambda_n f(\lambda_n^2)\varphi_n(x),$$
$$\gamma_5\tilde{\varphi}_n(x)=\sqrt{1-\lambda_n^2 f^2(\lambda_n^2)}\,\varphi_n(x)-\lambda_n f(\lambda_n^2)\tilde{\varphi}_n(x). \tag{A7}$$

(iii) $\lambda_n f(\lambda_n^2)=\pm 1$ or

$$H\Psi_{\pm}(x)=\pm\Lambda\Psi_{\pm}(x),\quad \Lambda f(\Lambda^2)=1. \tag{A8}$$

In this case we see

$$\Gamma_5\Psi_{\pm}(x)=0 \tag{A9}$$

and

$$\gamma_5\Psi_{\pm}(x)=\pm\Lambda f(\Lambda^2)\Psi_{\pm}(x)=\pm\Psi_{\pm}(x). \tag{A10}$$

We denote the number of $\Psi_+(x)$ $[\Psi_-(x)]$ as N_+ (N_-). From the relation Tr $\gamma_5=0$ valid on the lattice, one can derive the chirality sum rule [32,33]

$$n_+-n_-+N_+-N_-=0. \tag{A11}$$

The explicit form of the operator H is known for $f(H^2)=H^{2k}$ with non-negative integers k [18].

[1] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) **106**, 159 (2002), and references therein.
[2] K. Fujikawa, M. Ishibashi, and H. Suzuki, Phys. Lett. B **538**, 197 (2002).
[3] K. Fujiwawa, M. Ishibashi, and H. Suzuki, J. High Energy Phys. **04**, 046 (2002).
[4] K. Fujikawa and M. Ishibashi, Nucl. Phys. **B622**, 115 (2002); Phys. Lett. B **528**, 295 (2002).
[5] P.H. Ginsparg and K.G. Wilson, Phys. Rev. D **25**, 2649 (1982).
[6] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) **63**, 53 (1998); *ibid.* **B525**, 401 (1998).
[7] H. Neuberger, Phys. Lett. B **417**, 141 (1998); **427**, 353 (1998).
[8] P. Hasenfratz, V. Laliena, and F. Niedermayer, Phys. Lett. B **427**, 125 (1998).
[9] M. Lüscher, Phys. Lett. B **428**, 342 (1998).
[10] R. Narayanan, Phys. Rev. D **58**, 097501 (1998).
[11] M. Lüscher, Nucl. Phys. **B549**, 295 (1999).
[12] H. Suzuki, Prog. Theor. Phys. **101**, 1147 (1999).
[13] M. Lüscher, Nucl. Phys. **B568**, 162 (2000).
[14] H. Suzuki, Nucl. Phys. **B585**, 471 (2000); H. Igarashi, K. Okuyama, and H. Suzuki, hep-lat/0012018.
[15] Y. Kikukawa and Y. Nakayama, Nucl. Phys. **B597**, 519 (2001).
[16] M. Lüscher, J. High Energy Phys. **06**, 028 (2000).
[17] F. Niedermayer, Nucl. Phys. B (Proc. Suppl.) **73**, 105 (1999); M. Golterman, *ibid.* **94**, 189 (2001); H. Neuberger, Annu. Rev. Nucl. Part. Sci. **51**, 23 (2001); M. Lüscher, lectures given at the International School of Subnuclear physics, Erice 2000, hep-th/0102028; Y. Kikukawa, Nucl. Phys. B (Proc. Suppl.) **106**, 71 (2002).
[18] K. Fujikawa, Nucl. Phys. **B589**, 487 (2000).
[19] K. Fujikawa and M. Ishibashi, Nucl. Phys. **B587**, 419 (2000); **B605**, 365 (2001); Phys. Rev. D **65**, 114504 (2002); T.W. Chiu, Nucl. Phys. **B588**, 400 (2000); Phys. Lett. B **498**, 111 (2001); Nucl. Phys. B (Proc. Suppl.) **94**, 733 (2001).
[20] D.B. Kaplan, Phys. Lett. B **288**, 342 (1992).
[21] Y. Shamir, Nucl. Phys. **B406**, 90 (1993); V. Furman and Y. Shamir, *ibid.* **B439**, 54 (1995).
[22] P. Vranas, Phys. Rev. D **57**, 1415 (1998).
[23] H. Neuberger, Phys. Rev. D **57**, 5417 (1998).
[24] Y. Kikukawa and T. Noguchi, hep-lat/9902022.
[25] A. Boriçi, Nucl. Phys. B (Proc. Suppl.) **83**, 771 (2000).
[26] Y. Kikukawa, Phys. Rev. D **65**, 074504 (2002), and references therein.

[27] R. Narayanan and H. Neuberger, Phys. Lett. B **302**, 62 (1993); Nucl. Phys. **B412**, 574 (1994); Phys. Rev. Lett. **71**, 3251 (1993); Nucl. Phys. **B443**, 305 (1995).
[28] S. Randjbar-Daemi and J. Strathdee, Phys. Lett. B **348**, 543 (1995); Nucl. Phys. **B443**, 386 (1995); **B466**, 335 (1996); **B402**, 134 (1997).
[29] H. Neuberger, Phys. Rev. D **59**, 085006 (1999).
[30] K. Fujikawa, Nucl. Phys. **B636**, 80 (2002).
[31] Y. Kikukawa and A. Yamada, Phys. Lett. B **448**, 265 (1999); D.H. Adams, Ann. Phys. (N.Y.) **296**, 131 (2002); H. Suzuki, Prog. Theor. Phys. **102**, 141 (1999); K. Fujikawa, Nucl. Phys. **B546**, 480 (1999); T. Fujiwara, K. Nagao, and H. Suzuki, J. High Energy Phys. **09**, 025 (2002).
[32] T.W. Chiu, Phys. Rev. D **58**, 074511 (1998).
[33] K. Fujikawa, Phys. Rev. D **60**, 074505 (1999).
[34] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); Phys. Rev. D **21**, 2848 (1980); **22**, 1499(E) (1980).
[35] S. Chandrasekharan, Phys. Rev. D **60**, 074503 (1999).
[36] H. Nicolai, Nucl. Phys. **B140**, 294 (1978); **B156**, 157 (1979).
[37] P. van Nieuwenhuizen and A. Waldron, Phys. Lett. B **389**, 29 (1996).
[38] H. Suzuki, J. High Energy Phys. **10**, 039 (2000).
[39] Y. Kikukawa, Nucl. Phys. **B584**, 511 (2000).
[40] T.W. Chiu, C.W. Wang, and S.V. Zenkin, Phys. Lett. B **438**, 321 (1998).
[41] D. Adams, Nucl. Phys. **B589**, 633 (2000).