## Optimal renormalization-group improvement of the perturbative series for the $e^+e^-$ annihilation cross section

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Using renormalization-group methods, we derive differential equations for the all-orders summation of logarithmic corrections to the QCD series for  $R(s) = \sigma(e^+e^- \rightarrow hadrons)/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ , as obtained from the imaginary part of the purely perturbative vector-current correlation function. We present explicit solutions for the summation of leading and up to three subsequent subleading orders of logarithms. The summations accessible from the four-loop vector correlator not only lead to a substantial reduction in sensitivity to the renormalization scale, but necessarily impose a common infrared bound on perturbative approximations to R(s), regardless of the infrared behavior of the true QCD couplant.

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For center-of-mass squared-energy *s*, QCD corrections to  $R(s) \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$  are scaled by a perturbative QCD series (*S*):

$$R(s) = 3\sum_{f} Q_{f}^{2}S\left[\frac{\alpha_{s}(\mu)}{\pi}, \log\left(\frac{\mu^{2}}{s}\right)\right].$$
 (1)

This series is extracted from the imaginary part of the modified minimal subtraction scheme ( $\overline{MS}$ ) vector-current correlation function [1,2]:

$$S[x,L] = 1 + \sum_{n=1}^{\infty} x^n \sum_{m=0}^{n-1} T_{n,m} L^m, \qquad (2)$$

with the coefficients  $T_{n,m}$  tabulated in Table I for 3–5 active flavors, as appropriate for the choice of the center-of-mass squared energy *s*. Each order of this series depends upon the  $\overline{\text{MS}}$  renormalization scale parameter  $\mu$ , both through the couplant

$$x(\mu) \equiv \alpha_s(\mu) / \pi \tag{3}$$

and through powers of the logarithm

$$L(\mu) \equiv \log(\mu^2/s). \tag{4}$$

Nevertheless, the all-orders series *S* must ultimately be independent of the renormalization scale. R(s) is a measurable physical quantity necessarily independent of  $\mu$ , the artificial scale entering QCD calculations as a by-product of the regulation of Feynman-diagrammatic infinities. Hence,

$$0 = \mu^2 \frac{dS[x(\mu), L(\mu)]}{d\mu^2} = \left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x}\right)S[x, L]. \quad (5)$$

The above renormalization-group equation (RGE) is simply a chain-rule relation in which

$$\beta(x) = \mu^2 \frac{\mathrm{d}x(\mu)}{\mathrm{d}\mu^2} = -x^2 \sum_{k=0}^{\infty} \beta_k x^k, \tag{6}$$

where known [3] MS  $\beta$ -function coefficients  $\beta_k$  are also tabulated in Table I. Thus, the RGE (5) is generally employed to provide scale dependence to the couplant *x*.

"Optimal" renormalization-group (RG) improvement is the inclusion of every term in a perturbative series of the form (2) that can be extracted by RG-methods from a perturbative computation to a given order [4]. For example, a nextto-next-to-leading (NNL) order perturbative calculation determines only the coefficients  $T_{1,0}$ ,  $T_{2,0}$ , and  $T_{2,1}$  of the series (2). However, the RGE (5) can be utilized to determine

TABLE I. Coefficients for the imaginary part of the four-looporder vector-current correlation function, as well as coefficients for the four-loop order  $\overline{\text{MS}} \beta$  function, are listed for three, four and five quark flavors.

|                  | $n_f = 3$ | $n_f = 4$  | $n_f = 5$ |
|------------------|-----------|------------|-----------|
| T <sub>1,0</sub> | 1         | 1          | 1         |
| $T_{2,0}$        | 1.63982   | 1.52453    | 1.40924   |
| $T_{2,1}$        | 9/4       | 25/12      | 23/12     |
| T <sub>3.0</sub> | -10.2839  | -11.6856   | -12.8046  |
| T <sub>3,1</sub> | 11.3792   | 9.56054    | 7.81875   |
| T <sub>3,2</sub> | 81/16     | 625/144    | 529/144   |
| $\beta_0$        | 9/4       | 25/12      | 23/12     |
| $\beta_1$        | 4         | 77/24      | 29/12     |
| $\beta_2$        | 3863/384  | 21943/3456 | 9769/3456 |
| $\beta_3$        | 47.2280   | 31.3874    | 18.8522   |

all coefficients  $T_{k+1,k}$  and  $T_{k+2,k}$  within the series (2). The contributions of this infinite set of coefficients may then be summed analytically, as described below, thereby providing an "optimal RG improvement" of the NNL expression.

The series S[x,L], as defined in Eq. (2), may be rearranged in the following form:

$$S[x,L] = 1 + \sum_{n=1}^{\infty} x^n S_n(xL),$$
(7)

where

$$S_k(u) \equiv \sum_{n=k}^{\infty} T_{n,n-k} u^{n-k}.$$
 (8)

Given knowledge of the  $k^{th}$ -order series coefficient  $T_{k,0} = S_k(0)$ , one can obtain  $S_k(x(\mu)L(\mu))$  explicitly, thereby summing over the entire set of  $k^{th}$ -order subleading logarithms contributing to the series (7). If we substitute the  $\beta$ -function series (6) into the RGE (5), we find that the aggregate coefficient of  $x^nL^{n-p}$  vanishes  $(n \ge p)$  provided the following recursion relation is upheld:

$$0 = (n-p+1)T_{n,n-p+1} - \sum_{l=0}^{p-2} (n-l-1)\beta_l T_{n-l-1,n-p}.$$
(9)

For example, if p=2, this recursion relation  $[T_{n,n-1} = \beta_0 T_{n-1,n-2}]$  relates all leading-logarithm coefficients

 $T_{n,n-1}$  within the series (2) to the known coefficient  $T_{1,0} = 1$ , thereby enabling one to sum all orders of the leading-logarithm contributions

$$xS_1(xL) = x \sum_{n=1}^{\infty} T_{n,n-1}(xL)^{n-1} = \frac{x}{1 - \beta_0 xL}$$
(10)

to the series S[x,L].

More generally, the recursion relation (9) may be utilized to obtain a succession of first-order inhomogeneous linear differential equations for the functions  $S_k(u)$  within Eq. (7). If one multiplies Eq. (9) by  $u^{n-p}$  and then sums over *n* from n=p to  $\infty$ , one finds from the definition (8) of  $S_k(u)$  that

$$0 = \frac{\mathrm{d}S_{p-1}}{\mathrm{d}u} - u \sum_{l=0}^{p-2} \beta_l \frac{\mathrm{d}S_{p-l-1}}{\mathrm{d}u} - \sum_{l=0}^{p-2} (p-l-1)\beta_l S_{p-l-1},$$
(11)

which can be trivially rearranged (k=p-1) into a set of first-order linear differential equations

$$\frac{\mathrm{d}S_{k}}{\mathrm{d}u} - \frac{k\beta_{0}}{(1-\beta_{0}u)}S_{k} = \frac{1}{(1-\beta_{0}u)}\sum_{l=1}^{k-1}\beta_{l}\left(u\frac{\mathrm{d}}{\mathrm{d}u} + k-l\right)S_{k-l},$$
(12)

with initial conditions  $S_k(0) = T_{k,0}$ .

Noting that  $S_0(u) = 1$  and that  $T_{1,0} = 1$  regardless of the number of active flavors, we find the first four solutions of Eq. (12) to be

$$S_1(xL) = \frac{1}{(1 - \beta_0 xL)},$$
(13)

$$S_2(xL) = \frac{T_{2,0} - \frac{\beta_1}{\beta_0} \log(1 - \beta_0 xL)}{(1 - \beta_0 xL)^2},$$
(14)

$$S_{3}(xL) = \left(\frac{\beta_{1}^{2}}{\beta_{0}^{2}} - \frac{\beta_{2}}{\beta_{0}}\right) / (1 - \beta_{0}xL)^{2} + \frac{T_{3,0} - \left(\frac{\beta_{1}^{2}}{\beta_{0}^{2}} - \frac{\beta_{2}}{\beta_{0}}\right) - \frac{\beta_{1}}{\beta_{0}} \left(2T_{2,0} + \frac{\beta_{1}}{\beta_{0}}\right) \log(1 - \beta_{0}xL) + \frac{\beta_{1}^{2}}{\beta_{0}^{2}} \log^{2}(1 - \beta_{0}xL)}{(1 - \beta_{0}xL)^{3}}, \quad (15)$$

$$S_{4}(xL) = -\frac{1}{2} \left[ \frac{\beta_{1}}{\beta_{0}} \left( \frac{\beta_{1}^{2}}{\beta_{0}^{2}} - 2\frac{\beta_{2}}{\beta_{0}} \right) + \frac{\beta_{3}}{\beta_{0}} \right] (1 - \beta_{0}xL)^{-2} + \left( 2T_{2,0} + \frac{\beta_{1}}{\beta_{0}} \right) \left( \frac{\beta_{1}^{2}}{\beta_{0}^{2}} - \frac{\beta_{2}}{\beta_{0}} \right) (1 - \beta_{0}xL)^{-3} \\ + 2\frac{\beta_{1}}{\beta_{0}} \left( \frac{\beta_{2}}{\beta_{0}} - \frac{\beta_{1}^{2}}{\beta_{0}^{2}} \right) (1 - \beta_{0}xL)^{-3} \log(1 - \beta_{0}xL) + \left[ T_{4,0} + \frac{\beta_{3}}{2\beta_{0}} - \frac{1}{2} \frac{\beta_{1}^{3}}{\beta_{0}^{3}} - 2T_{2,0} \left( \frac{\beta_{1}^{2}}{\beta_{0}^{2}} - \frac{\beta_{2}}{\beta_{0}} \right) \right] (1 - \beta_{0}xL)^{-4} \\ + \frac{\beta_{1}}{\beta_{0}} \left[ 2\frac{\beta_{1}^{2}}{\beta_{0}^{2}} - 3\frac{\beta_{2}}{\beta_{0}} - 2T_{2,0} \frac{\beta_{1}}{\beta_{0}} - 3T_{3,0} \right] (1 - \beta_{0}xL)^{-4} \log(1 - \beta_{0}xL) + \frac{\beta_{1}}{\beta_{0}} \left[ \frac{5\beta_{1}^{2}}{2\beta_{0}^{2}} + 3T_{2,0} \frac{\beta_{1}}{\beta_{0}} \right] (1 - \beta_{0}xL)^{-4} \\ \times \log^{2}(1 - \beta_{0}xL) - \frac{\beta_{1}^{3}}{\beta_{0}^{3}} (1 - \beta_{0}xL)^{-4} \log^{3}(1 - \beta_{0}xL).$$

$$(16)$$

To explore the near-infrared regime of perturbative QCD, we specialize to the case of three active flavors. Using Table I  $n_f=3$  values within Eqs. (13)–(16), we find that the version of the series (7) which incorporates full summation of leading and two subsequent subleading orders of logarithms is given by

$$S^{(\Sigma)}[x(\mu), L(\mu)] = 1 + xS_1(xL) + x^2S_2(xL) + x^3S_3(xL),$$
(17)

where (u = xL)

$$S_1(u) = 1 \left/ \left( 1 - \frac{9u}{4} \right),$$

$$\tag{18}$$

$$S_{2}(u) = \frac{1.63982 - \frac{16}{9}\log\left(1 - \frac{9u}{4}\right)}{\left(1 - \frac{9u}{4}\right)^{2}},$$
(19)

$$S_{3}(u) = -\frac{1.31057}{\left(1 - \frac{9}{4}u\right)^{2}} + \frac{-8.97333 - 8.99096\log\left(1 - \frac{9u}{4}\right) + 3.16049\log^{2}\left(1 - \frac{9u}{4}\right)}{\left(1 - \frac{9u}{4}\right)^{3}}.$$
(20)

Moreover, we note from Eq. (16) that

$$S_{4}(xL) = -\frac{5.35589}{\left(1 - \frac{9}{4}xL\right)^{2}} + \frac{-6.62811 + 4.65981 \log\left(1 - \frac{9}{4}xL\right)}{\left(1 - \frac{9}{4}xL\right)^{3}} + \frac{T_{4,0} + 11.9840 + 31.8738 \log\left(1 - \frac{9}{4}xL\right) + 29.5946 \log^{2}\left(1 - \frac{9}{4}xL\right) - 5.61866 \log^{3}\left(1 - \frac{9}{4}xL\right)}{\left(1 - \frac{9}{4}xL\right)^{4}}, \quad (21)$$

thereby providing for inclusion of the  $x^4S_4(xL)$  contribution to the series (7) for three active flavors. The series coefficient  $T_{4,0}$  appearing in Eq. (21) has not yet been calculated perturbatively, which is why we have not included the  $x^4S_4(xL)$ contribution to S[x,L] in Eq. (17). (An asymptotic Padé approximant estimate  $T_{4,0} \approx 1.90$  for the  $n_f = 3$  case is presented in Ref. [2].)

To examine whether the summation of leading and subsequent subleading logarithm factors decreases dependence on the unphysical renormalization-scale parameter  $\mu$ , we compare the  $\mu$  dependence of Eq. (17) for a fixed value of *s* to that of the  $n_f=3$  version of the series (2) truncated after four-loop-order (4*l*) contributions to the vector-current correlation function:

$$S^{(4l)}[x(\mu), L(\mu)] = 1 + x + (1.63982 + 9L/4)x^{2} + (-10.2839 + 11.3792L + 81L^{2}/16)x^{3}.$$
(22)

Such  $\mu$ -dependence enters Eqs. (17) and (22) both through  $L = \log(\mu^2/s)$  and through  $x = x(\mu)$ , which is assumed to

evolve via Eq. (6) (with  $n_f = 3$  choices for  $\beta_{0-3}$ ) from an initial value choice  $x(m_\tau) = \alpha_s(m_\tau)/\pi = 0.33/\pi$  [5,6]. Figure 1 displays a comparison of the  $\mu$  dependence of Eqs. (17) and (22) at fixed  $s = 1.5 \text{ GeV}^2$ . Although both expressions exhibit little variation with  $\mu$  over the 1.3 GeV  $\leq \mu$  $\leq 3$  GeV range, we see that Eq. (17) exhibits much less variation with  $\mu$  in the near-infrared regime below 1.3 GeV. These results clearly indicate that renormalization-scaleinvariance is more effectively upheld via the summations of leading and subsequent subleading orders of logarithms that occur within Eq. (17). We emphasize that Eqs. (17) and (22) both follow from "RG improvement" of the same calculational information (the coefficients  $T_{1,0}$ ,  $T_{2,0}$ , and  $T_{3,0}$ ); simply put, such RG-improvement is more effectively implemented in Eq. (17) than in Eq. (22).

The usual prescription for obtaining the purelyperturbative (non-power-law) QCD contributions to R(s) at four-loop order [6,7] is to set  $\mu = \sqrt{s}$  within Eq. (22), and then to substitute the resulting series,

$$S^{(4l)}[x(\sqrt{s}), L(\sqrt{s})] = 1 + x(\sqrt{s}) + 1.63982 x^{2}(\sqrt{s})$$
$$-10.2839 x^{3}(\sqrt{s}), \qquad (23)$$



FIG. 1. The renormalization-scale  $(\mu)$  dependence of the summation-of-logarithms  $(\Sigma)$  series (17) is compared to that of the series (22), which is truncated after four-loop (4*L*) contributions to the vector-current correlation function. For both series the physical momentum scale *s* is fixed at 1.5 GeV<sup>2</sup>. The evolution of the couplant  $x(\mu)$  for both series is referenced to the initial value  $\alpha_s(m_{\tau}) = 0.33$ .

into Eq. (1). [Note from Eq. (4) that  $L(\sqrt{s})=0$ .] This prescription follows from the presumed renormalization-scale invariance of the truncated series (22), thereby leading to an expression that depends only on the physical scale *s* for the electron-positron annihilation process. Note that all *s* dependence of Eq. (17) resides entirely in the variable *L*. Let us first fix  $\mu = m_{\tau}$  within Eq. (17) so as to incorporate the benchmark couplant value  $x(m_{\tau})=0.33/\pi$  everywhere *x* appears in Eqs. (18)–(20). With this choice, the following summation-of-logarithms series can be substituted into Eq. (1) to obtain R(s):

$$S^{(\Sigma)}[x(m_{\tau}), L(m_{\tau})] = 1 + \frac{0.33}{\pi} S_{1} \left[ \frac{0.33}{\pi} \log \left( \frac{m_{\tau}^{2}}{s} \right) \right] \\ + \left( \frac{0.33}{\pi} \right)^{2} S_{2} \left[ \frac{0.33}{\pi} \log \left( \frac{m_{\tau}^{2}}{s} \right) \right] \\ + \left( \frac{0.33}{\pi} \right)^{3} S_{3} \left[ \frac{0.33}{\pi} \log \left( \frac{m_{\tau}^{2}}{s} \right) \right].$$
(24)

In Fig. 2 we compare the *s* dependence of this series to that of Eq. (23), for which all *s* dependence resides in the evolution of  $x(\sqrt{s})$ . To make this comparison, such evolution is anchored to the initial value  $x(m_{\tau}) = 0.33/\pi$  via the differential equation (6) with  $n_f = 3$  values for  $\beta_{0-3}$  (Table I). This initial value ensures that the series (23) and (24) coincide



FIG. 2. The center-of-mass squared-energy (s) dependence of the four-loop (4L-) truncated series (23) is compared to that of the summation-of-logarithms ( $\Sigma$ ) series (24), as described in the text. In both series,  $\alpha_s(m_\tau)$  is taken to be 0.33 so that the series equilibrate at  $\sqrt{s} = m_\tau$ .

when  $\sqrt{s} = m_{\tau}$ . Figure 2 shows that both series continue to coincide over the range 750 MeV  $\leq \sqrt{s} \leq m_{\tau}$ . For values of  $\sqrt{s}$  less than 750 MeV, however, the truncated series (23) drops off quite suddenly at  $\sqrt{s} \approx 650$  MeV, a consequence of the large negative coefficient of  $x^3(\sqrt{s})$ , whereas the series (24) [as obtained from the full summation-of-logarithms series (17)] continues to probe the infrared domain of R(s)even for values of  $\sqrt{s} \approx 400$  MeV. In short, the summation of all leading and subsequent two subleading logarithms within the perturbative series (2) serves to extend the domain of the R(s) series further into the infrared. This property, as well as the reduced renormalization-scale dependence evident in Fig. 1, suggests that such summation is particularly appropriate for the near-infrared region characterizing sum-rule applications of purely perturbative QCD corrections to currentcorrelation functions.

It is also evident from Fig. 2 that the domain of the summation-of-logarithms series (24) manifests a singularity below  $\sqrt{s} = 400$  MeV, despite the fact that the *s* dependence of the series (24) is decoupled entirely from any infrared behavior of the couplant *x*, which is held constant at  $x(m_{\tau})$ . To understand this restriction on the domain of R(s), we first note that each summation (13)–(16) becomes singular when  $1 - \beta_0 x L \rightarrow 0$ . Such resummation singularities have also been observed to occur in completely different contexts, including deep inelastic structure functions [8].

The singularity property of Eqs. (13)–(16) is upheld for *all* summations  $S_k(xL)$ . The solution to the differential equation (12) is necessarily of the form



It is to be emphasized that this infrared boundary on the physical scale *s* entering term-by-term within the series (7) is *not* a manifestation of any infrared boundary [9] on the evolution of the QCD couplant  $x(\mu)$ . Even if the higher order contributions to the  $\beta$  function (6) were somehow to conspire to allow the couplant to be well-behaved in the infrared region [e.g., to have infrared-stable fixed point behavior], the restriction (26) would still apply upon making a specific choice of the renormalization-scale parameter  $\mu$  and its corresponding value of  $\alpha_s(\mu)$ . Curiously, though, this infrared restriction on *s* can be easily shown to coincide with the "infrared-slavery" Landau singularity  $\Lambda$  associated with naive evolution of the QCD couplant via a one-loop  $\beta$  function. The one-loop version of Eq. (6),

$$\mu^2 \frac{\mathrm{d}x}{\mathrm{d}\mu^2} = -\beta_0 x^2, \quad x = \frac{\alpha_s}{\pi},\tag{27}$$

is satisfied by the relation

$$\frac{\alpha_s}{\pi} = \frac{1}{\beta_0 \log(\mu^2 / \Lambda^2)},\tag{28}$$

for equally-spaced values of the renormalization scale in the range  $0.6m_{\tau} \leq \mu \leq 1.6m_{\tau}$ . The couplant  $x(\mu)$  is obtained from threeflavor four-loop evolution via Eq. (6) from the initial condition  $x(m_{\tau}) = 0.33/\pi$  using Table I values for the  $\beta$  function coefficients  $\beta_0 - \beta_3$ .

$$S_{k}(xL) = \frac{T_{k,0}}{(1 - \beta_{0}xL)^{k}} + (\text{particular solution depending on}$$
$$\{S_{k-1}, S_{k-2}, \dots, S_{1}\}).$$
(25)

Since the coefficients  $T_{k,0}$  are results of  $k^{th}$ -order Feynman diagram calculations, the  $k^{th}$ -order pole in Eq. (25) at  $1 - \beta_0 x L = 0$  is genuine and will not be canceled by particular-solution contributions that are sensitive to at most (k-1)th-order Feynman-diagrammatic coefficients  $\{T_{1,0}, T_{2,0}, \ldots, T_{k-1,0}\}$ . For a given choice of renormalization scale  $\mu$ , this singularity implies [via Eqs. (3) and (4)] that each summation  $S_k(xL)$  within the full series (7) becomes singular for a sufficiently small value of *s*:

$$1 - \beta_0 \frac{\alpha_s(\mu)}{\pi} \log \left( \frac{\mu^2}{s_{min}} \right) = 0 \quad \rightarrow s_{min} = \mu^2 \exp \left( \frac{-\pi}{\beta_0 \alpha_s(\mu)} \right).$$
(26)

For example, if the renormalization scale  $\mu$  is chosen (as in Fig. 2) to be  $m_{\tau}$ , a choice for which  $\alpha_s(m_{\tau})$  (=0.33±0.02 [6]) is phenomenologically accessible, then each term in the series (7) is seen to become progressively more singular as *s* approaches  $m_{\tau}^2 \exp[-4\pi/(9\times0.33)] = (215 \text{ MeV})^2$  from above. Furthermore, Fig. 3 shows that the low-energy behavior of the resummed expression (17) is only weakly dependent on  $\mu$  in the region  $\sqrt{s} > 600 \text{ MeV}$  for choices of  $\mu$  between  $0.6m_{\tau}$  and  $1.6m_{\tau}$ , with the appearance of singular points corresponding to 195 MeV< $s_{min} < 250 \text{ MeV}$  from (26).

which is equivalent to Eq. (26) provided  $s_{min}$  is identified with  $\Lambda^2$ . Indeed, in a one-loop world  $[x^{1l}(\mu) = 1/\beta_0 \log(\mu^2/\Lambda^2)]$  where  $\Lambda$  serves as a *universal* infrared boundary, the one-loop analogues of the summation-oflogarithms series (17) and the truncated series (23) are necessarily equivalent:

$$S_{1l}^{(\Sigma)} = 1 + x^{(1l)}(\mu) S_1[x^{(1l)}(\mu)\log(\mu^2/s)]$$

$$= 1 + \frac{1}{\beta_0 \log\left(\frac{\mu^2}{\Lambda^2}\right)} \left\{ \frac{1}{1 - \beta_0 \left[\frac{1}{\beta_0 \log\left(\frac{\mu^2}{\Lambda^2}\right)}\right] \log\left(\frac{\mu^2}{s}\right)} \right\}$$

$$= 1 + \frac{1}{\beta_0 \log\left(\frac{s}{\Lambda^2}\right)}$$

$$= 1 + x^{(1l)}(\sqrt{s})$$
(29)

We find it remarkable that  $\Lambda$ , the one-loop couplant's Landau pole, persists as an infrared boundary on the domain of each summation contributing to Eq. (7), the summation-oflogarithms formulation of the perturbative series within R(s). Consequently,  $s = \Lambda^2$  serves as an infrared boundary for any approximation to R(s) involving the truncation of the series (7), such as the expression (17) obtained via optimal RG-improvement of the four-loop vector-current correlation function's imaginary part.

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