# Approximations to the QED fermion Green's function in a constant external field

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An exact representation of the causal QED fermion Green's function, in an arbitrary external electromagnetic field, derived in Fried, Gabellini, and McKellar, and which naturally allows for nonperturbative approximations, is here used to calculate nonperturbative approximations to the Green's function in the simple case of a constant external field. Schwinger's famous exact result is obtained as the limit as the order of the approximation approaches infinity.

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## I. INTRODUCTION

An exact representation for the causal QED fermion Green's function, $G_c(x,y|A)$, in an arbitrary external field, was derived by Fried et al. [1] in such a way as to obtain an exact representation which naturally allows for nonperturbative approximations. The rather intimidating exact representation is

$$\begin{aligned} G_c(x,y|A) = i\int_0^\infty ds e^{-ism^2}\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \\ \times \int \frac{d^4z d^4P}{(2\pi)^4} e^{iP\cdot(z-y)+i(s/4)P^2} \prod_{N=1}^{\infty}{}' \frac{(-i)^2}{(2\pi)^4} \\ \times \int d^4P_N d^4Q_N e^{(i/2)(P_N^2+Q_N^2)} e^{-i\int_0^s ds'[p-\Omega(s')]^2} \\ \times e^{-g\int_0^s ds'(\partial/\partial z_\mu)A_\mu(\zeta(s')-2\int_0^{s'}\Omega)} \\ \times \{m - i\gamma\cdot[p-\Omega(s)]\} \\ \times (e^{g\int_0^s ds'\sigma\cdot F(\zeta(s')-2\int_0^{s'}\Omega)})_+, \end{aligned} \tag{1.1}$$

where $\Omega(s')$ is the solution to the "map,"

$$\Omega(s') = gA\left(\zeta(s') - 2\int_0^{s'}\Omega\right), \tag{1.2}$$

and $\zeta(s')$ is given by

$$\begin{aligned} \zeta(s') = z + s'(2p+P) - \frac{2\sqrt{s}}{\pi}\sum_{N=1}^{\infty}{}' \frac{1}{N}\left[P_N \cos\left(\frac{N\pi s'}{s}\right)\right. \\ \left. + Q_N \sin\left(\frac{N\pi s'}{s}\right)\right]. \end{aligned} \tag{1.3}$$

$\Pi_{N=1}^{\infty\prime}$ and $\Sigma_{N=1}^{\infty\prime}$ represent the product and sum, respectively, over all odd natural numbers. It is easy to see that we may approximate the Green's function nonperturbatively by retaining a finite number of integrations, in particular, for the $n$th approximation $G_c^{(n)}(x,y|A)$, where $n=0,1,2,\ldots$, we retain $2n+3$ integrations. This means that for $n\neq 0$, the odd number $N$ takes the values $N=1,3,\ldots,2n-1$, while for the zeroth approximation all $N$ dependence is neglected. The exact result is of course recovered in the limit $n\to\infty$, that is, $G_c(x,y|A)=\lim_{n\to\infty}G_c^{(n)}(x,y|A)$.

It is well known that there is an exact expression, first obtained by Schwinger [2], for the fermion Green's function in the simple case of a constant, but otherwise arbitrary, external field, which (in an arbitrary gauge) reads

$$\begin{aligned} G_c(x,y|A) = \Phi(x,y|A)\frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{s^2} e^{-ism^2} \\ \times e^{gs\sigma\cdot F}\left(\det\frac{\sinh gFs}{gFs}\right)^{-1/2} \\ \times e^{(i/4)(x-y)gF\coth gFs(x-y)} \\ \times \left[m - \frac{i}{2}\gamma\cdot(gF\coth gFs + gF)(x-y)\right], \end{aligned} \tag{1.4}$$

where the holonomy factor $\Phi(x,y|A)$ is

$$\Phi(x,y|A) = e^{ig\int_y^x d\xi_\mu [A_\mu(\xi)+(1/2)F(\xi-y)]}, \tag{1.5}$$

and carries the complete gauge dependence of the Green's function. It is convenient to employ matrix notation, in which we regard the field strength tensor as a constant, antisymmetric $4\times 4$ matrix $F$.

It is obvious that Schwinger's result must somehow be contained as a special case of the exact representation (1.1)–(1.3). Furthermore, the Fradkin representation [3,4], from which the above representation was derived, almost trivially yields Schwinger's result. This leads us to expect that the latter may be extracted, analytically, from the above representation, and here we demonstrate that this is indeed the case.

We proceed by evaluating that $n$th nonperturbative approximation to the Green's function in a constant field as

 

given by Eqs. (1.1)–(1.3) Schwinger's result is then recovered in the limit as the order of this approximation approaches infinity. That we may carry out this program relies on two things: for the case of constant $F$ the ordered exponential of Eq. (1.1) becomes an ordinary exponential, and the $2n+3$ integrations we must perform are all Gaussian. Indeed a large part of the evaluation of the $n$th approximation is an extension of our matrix notation to account for this latter fact. $2n+3$ Gaussian integrals may be expressed as one Gaussian integral over a $(2n+3)$-dimensional space. The integration is then, with the appropriate notation, essentially trivial. In Sec. II we make the change of notation, perform the integration, and express the $n$th approximation in terms of the two resulting, order $2n+3$, determinants. In Sec. III we evaluate the determinants and take the limit $n\to\infty$ to recover Schwinger's result.

We emphasize that the result when stopped at a finite $n$ (except $n=0$) is essentially nonperturbative, in that it is *not* a polynomial in $g$. The result of Eqs. (1.1)–(1.3) is thus, as $n$ is increased, a systematically improving, but nonperturbative, method for calculating the Green's function in an external field.

## II. THE INTEGRATION

We must first choose a gauge to work in. It shall be convenient to work exclusively in the Schwinger-Fock ($SF$) gauge

$$A_\mu^{SF}(z)=-\frac{1}{2}F_{\mu\nu}(z-y)_\nu, \tag{2.1}$$

the initial motivation for which is that we may forget about the holonomy factor, which reduces to 1. An immediate conseqence is that $(\partial/\partial z_\mu)A_\mu^{SF}(z)=0$, due to the antisymmetry of $F$; the factor in Eq. (1.1) containing this term in the exponent also reduces to 1. With the above simplifications, the $n$th approximation to the fermion Green's function in a constant field, in the $SF$ gauge, as given by Eq. (1.1), is thus

$$G_c^{(n)}(x,y|A^{SF})=\int_0^\infty ds e^{-ism^2}e^{gs\sigma\cdot F}\frac{i}{(2\pi)^8}\frac{(-i)^{2n}}{(2\pi)^{4n}}\int dpdzdP\prod_{N=1}^{2n-1}{}' dP_N dQ_N$$
$$\times e^{ip\cdot(x-y)+iP\cdot(z-y)+i(s/4)P^2+\Sigma'_{N=1}^{2n-1}(i/2)(P_N^2+Q_N^2)}e^{-i\int_0^s ds'[p-\Omega^{(n)}(s')]^2}\{m-i\gamma\cdot[p-\Omega^{(n)}(s)]\}, \tag{2.2}$$

where $\Omega^{(n)}(s')$ is the solution to Eqs. (1.2), (1.3), but with the sum in Eq. (1.3) terminating at $N=2n-1$, and in the spirit of using matrix notation we have used $dp$ instead of $d^4p$ and so on. In the $SF$ gauge, Eqs. (1.2), (1.3) yield a simple integral equation for $\Omega^{(n)}(s')$:

$$\Omega^{(n)}(s')-gF\int_0^{s'}\Omega^{(n)}=-\frac{1}{2}gF\Bigg\{z-y+s'(2p+P)-\frac{2\sqrt{s}}{\pi}\sum_{N=1}^{2n-1}{}'\frac{1}{N}\Bigg[P_N\cos\left(\frac{N\pi s'}{s}\right)+Q_N\sin\left(\frac{N\pi s'}{s}\right)\Bigg]\Bigg\}. \tag{2.3}$$

The equivalent differential equation plus boundary condition may be solved with only elementary integrals. It is the combination $p-\Omega^{(n)}(s')$ which appears in Eq. (2.2), we find

$$p-\Omega^{(n)}(s')=e^{gFs'}p+\frac{1}{2}gFe^{gFs'}(z-y)+\frac{1}{2}(e^{gFs'}-1)P-\frac{1}{\sqrt{s}}\sum_{N=1}^{2n-1}{}'\frac{1}{1+\Lambda_N^2}\Bigg[\sin\left(\frac{N\pi s'}{s}\right)+\Lambda_N\cos\left(\frac{N\pi s'}{s}\right)+\frac{1}{\Lambda_N}e^{gFs'}\Bigg]P_N$$
$$-\frac{1}{\sqrt{s}}\sum_{N=1}^{2n-1}{}'\frac{1}{1+\Lambda_N^2}\Bigg[-\cos\left(\frac{N\pi s'}{s}\right)+\Lambda_N\sin\left(\frac{N\pi s'}{s}\right)+e^{gFs'}\Bigg]Q_N, \tag{2.4}$$

where $\Lambda_N=N\pi/gFs$. Our choice of gauge allows us to make the change of variable $z-y\to z$, after which $p-\Omega^{(n)}(s')$ is independent of $y$, so that the only $x$ and $y$ dependence in the exponent of Eq. (2.2) appears in the term $ip\cdot(x-y)$.

If we imagine substituting Eq. (2.4) into Eq. (2.2), we recognize that all of the terms in the exponent of the integrand, save the term $ip\cdot(x-y)$, are able to be expressed in the form $(i/2)X_i^TA_{ij}^{(n)}X_j$, where the $X_i$ are the $2n+3$ 4-vector variables we must integrate over, the $A_{ij}^{(n)}$ are some matrix functions of $F$, and the $i/2$ is a convenient normalization factor. This suggests that we extend our matrix notation, and write this part of the exponent of Eqs. (2.2) as $(i/2)X^TA^{(n)}X$, where $X$ is a $(2n+3)\times 1$ column vector of 4-vector variables, and $A^{(n)}$ is a $(2n+3)\times(2n+3)$ symmetric matrix with matrix elements $A_{ij}^{(n)}$. Now $dpdzdP\Pi_{N=1}^{2n-1}{}'dP_NdQ_N=d^{2n+3}X$. Let us define the column vector $X$ such that

$$X^T=[p\ \ z\ \ P\ \ P_1\ \ Q_1\ldots P_N\ \ Q_N\ \ \ldots\ \ P_{2n-1}\ \ Q_{2n-1}]. \tag{2.5}$$

The term $ip\cdot(x-y)$ is included by introducing

$$B^{(n)T}=[(x-y)0\ \ 0\ \ \ldots\ \ 0], \tag{2.6}$$

whence $e^{ip\cdot(x-y)}=e^{iB^{(n)T}X}$. A construction for the matrix $A^{(n)}$ is obtained by defining

$$p-\Omega^{(n)}(s')=C^{(n)T}(s')X, \tag{2.7}$$

so that $e^{-i\int_0^s ds'[p-\Omega^{(n)}(s')]^2}=e^{-iX^T\int_0^s ds' C^{(n)}(s')C^{(n)T}(s')X}$. The elements of the row vector $C^{(n)T}(s')$ are read straight from Eq. (2.4). The transpose of the row vector is

$$C^{(n)}(s')=\begin{bmatrix} e^{-gFs'} \\ -\frac{1}{2}gFe^{-gFs'} \\ \frac{1}{2}(e^{-gFs'}-1) \\ \frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_1^2}\left[-\sin\left(\frac{\pi s'}{s}\right)+\Lambda_1\cos\left(\frac{\pi s'}{s}\right)+\frac{1}{\Lambda_1}e^{-gFs'}\right. \\ \left.\frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_1^2}\left[\cos\left(\frac{\pi s'}{s}\right)+\Lambda_1\sin\left(\frac{\pi s'}{s}\right)-e^{-gFs'}\right]\right. \\ \vdots \\ \frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_N^2}\left[-\sin\left(\frac{N\pi s'}{s}\right)+\Lambda_N\cos\left(\frac{N\pi s'}{s}\right)+\frac{1}{\Lambda_N}e^{-gFs'}\right] \\ \left.\frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_N^2}\left[\cos\left(\frac{N\pi s'}{s}\right)+\Lambda_N\sin\left(\frac{N\pi s'}{s}\right)-e^{-gFs'}\right]\right. \\ \vdots \end{bmatrix}, \tag{2.8}$$

where we have used $F^T=-F$ ($\Lambda_N^T=-\Lambda_N$), and omitted the last two $N=2n-1$ elements for brevity. We also write $e^{iP\cdot z+i(s/4)P^2+\Sigma_{N=1}^{2n-1\prime}(i/2)(P_N^2+Q_N^2)}=e^{(i/2)X^TD^{(n)}X}$, which defines the matrix $D^{(n)}$:

$$D^{(n)}=\begin{bmatrix} 0&0&0&0&0&\cdots&0&0\\ 0&0&1&0&0&\cdots&0&0\\ 0&1&\frac{s}{2}&0&0&\cdots&0&0\\ 0&0&0&1&0&\cdots&0&0\\ 0&0&0&0&1&\cdots&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ 0&0&0&0&0&\cdots&1&0\\ 0&0&0&0&0&\cdots&0&1 \end{bmatrix}. \tag{2.9}$$

Our change of notation complete, Eqs. (2.2) and (2.4) become

$$G_c^{(n)}(x,y|A^{SF})=\int_0^\infty ds e^{-ism^2}e^{gs\sigma\cdot F}\frac{i}{(2\pi)^8}\frac{(-i)^{2n}}{(2\pi)^{4n}} \times\int d^{2n+3}X\, e^{(i/2)X^TA^{(n)}X+iB^{(n)T}X} \times[m-i\gamma\cdot C^{(n)T}(s)X], \tag{2.10}$$

where

$$A^{(n)}=-2\int_0^s ds' C^{(n)}(s')C^{(n)T}(s')+D^{(n)}. \tag{2.11}$$

All that we have done is to reexpress, in the usual way, the product of $2n+3$ Gaussian integrals as one $(2n+3)$-dimensional Gaussian integral. After a change of variable $X\to X-A^{(n)-1}B^{(n)}$, the integration is trivial, we obtain

$$G_c^{(n)}(x,y|A^{SF})=\frac{1}{4\pi^2}\int_0^\infty ds e^{-ism^2}e^{gs\sigma\cdot F}(\mathrm{Det}A^{(n)})^{-1/2} \times e^{-(i/2)B^{(n)T}A^{(n)-1}B^{(n)}} \times[m+i\gamma\cdot C^{(n)T}(s)A^{(n)-1}B^{(n)}], \tag{2.12}$$

where $\mathrm{Det}A^{(n)}$ is the determinant of $A^{(n)}$.

The matrix $A^{(n)}$ is a $4(2n+3)\times4(2n+3)$ matrix which is naturally partitioned into the $(2n+3)\times(2n+3)$ $4\times4$ matrices $A_{ij}^{(n)}=-2\int_0^s ds' C_i^{(n)}(s')C_j^{(n)T}(s')+D_{ij}^{(n)}$. The $A_{ij}^{(n)}$, as (matrix) functions of $F$ only, commute, and are thus referred to as the "elements" of $A^{(n)}$. Since $A^{(n)}$ is symmetric, we have the relations $A_{ii}^{(n)}=A_{ii}^{(n)T}$ and $A_{ji}^{(n)}=A_{ij}^{(n)T}$. If we form a determinant using the elements $A_{ij}^{(n)}$, the result will

be a $4\times4$ matrix, also some function of $F$, which we denote $\det A^{(n)}$. It is easy to convince oneself that the determinant of $A^{(n)}$, which may be partitioned in this way, is the determinant of the matrix we have called $\det A^{(n)}$, that is, $\mathrm{Det}\,A^{(n)} = \det(\det A^{(n)})$.

Recalling that $B_i^{(n)} = (x-y)\,\delta_{i1}$, and with the notation discussed above, we may write

$$C^{(n)T}(s)A^{(n)-1}B^{(n)} = \frac{\det A_1^{(n)}}{\det A^{(n)}}(x-y), \tag{2.13}$$

where $A_1^{(n)}$ is the matrix obtained by replacing the first row of $A^{(n)}$ by $C^{(n)T}(s)$. Similarly,

$$B^{(n)T}A^{(n)-1}B^{(n)} = (x-y)\frac{\det \tilde{A}^{(n)}}{\det A^{(n)}}(x-y), \tag{2.14}$$

where $\det \tilde{A}^{(n)}$ is the (1,1) cofactor of $A^{(n)}$, that is, $\tilde{A}^{(n)}$ is the matrix obtained by deleting the first row and the first column of $A^{(n)}$. In fact it will be shown in the next section that $\det \tilde{A}^{(n)}/\det A^{(n)}$ is the part of $\det A_1^{(n)}/\det A^{(n)}$ symmetric with respect to the interchange of space-time indices, for all $n$, and denoted by a superscript $S$. The $n$th approximation to the Green's function, in terms of the two determinants $\det A^{(n)}$ and $\det A_1^{(n)}$, is then

$$\begin{aligned} G_c^{(n)}(x,y|A^{SF}) = \frac{1}{4\pi^2}\int_0^\infty ds e^{-ism^2} e^{gs\sigma\cdot F}(\mathrm{Det}\,A^{(n)})^{-1/2} \\ \times e^{-(i/2)(x-y)(\det A_1^{(n)}/\det A^{(n)})^S(x-y)} \\ \times\left[m + i\gamma\cdot\frac{\det A_1^{(n)}}{\det A^{(n)}}(x-y)\right]. \end{aligned} \tag{2.15}$$

Comparing Eq. (2.15) with Schwinger's result (1.4), and letting $\det A = \lim_{n\to\infty}\det A^{(n)}$ and $\det A_1 = \lim_{n\to\infty}\det A_1^{(n)}$, it is necessary that

$$\det A = 2s\frac{\sinh gFs}{gFs}, \tag{2.16}$$

so that $(\mathrm{Det}\,A)^{-1/2} = (1/4s^2)(\det(\sinh gFs/gFs))^{-1/2}$, and

$$\det A_1 = -e^{gFs}, \tag{2.17}$$

so that the quotient $\det A_1/\det A = -\frac{1}{2}(gF\coth gFs + gF)$. Note that the symmetric part of $\det A_1/\det A$ is $(\det A_1/\det A)^S = -\frac{1}{2}gF\coth gFs$, we mentioned above that this relation holds for all values of $n$. Thus $\det A^{(n)}$ and $\det A_1^{(n)}$ are yet to be determined, $n$th, nonperturbative approximations to Eqs. (2.16) and (2.17) respectively. The quotient $\det A_1^{(n)}/\det A^{(n)}$ provides a nonperturbative approximation to $-\frac{1}{2}(gF\coth gFs + gF)$, the symmetric part of the former a nonperturbative approximation to the symmetric part of the latter. In the next section we calculate exact expressions for these non-perturbative approximations, and show that we can obtain Eqs. (2.16) and (2.17), and thus Schwinger's result, in the limit $n\to\infty$.

## III. THE DETERMINANTS

An obvious but important fact is that we only need to find the determinants of the matrices $A^{(n)}$ and $A_1^{(n)}$, not the matrices themselves. This means that we can simplify $C^{(n)}(s')$, its transpose, and $D^{(n)}$, with any row and column operations which do not alter the determinant, before using Eq. (2.17) to find $A^{(n)}$. We now find the reduced form of $A^{(n)}$, from this it will be easy to obtain the reduced form of $A_1^{(n)}$ and of $\tilde{A}^{(n)}$. We perform the following sets of row operations on the column vector $C^{(n)}(s')$: use the first element to remove those terms proportional to $e^{-gFs'}$ from all other elements, noting that the second element requires the row operation $\mathrm{row}_2 \to \mathrm{row}_2 + \frac{1}{2}gF\,\mathrm{row}_1$; then $\mathrm{row}_{N+3}\to\mathrm{row}_{N+3} - \Lambda_N\mathrm{row}_{N+4}$; then $\mathrm{row}_{N+4}\to\mathrm{row}_{N+4} + ([\Lambda_N - (1/\Lambda_N)]/(1+\Lambda_N^2))\mathrm{row}_{N+3}$; the last two sets for all rows $N = 1,3,\ldots,2n-1$. $C^{(n)}(s')$ becomes

$$C^{(n)}(s') \sim \begin{bmatrix} e^{-gFs'} \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{\sqrt{s}}\sin\left(\frac{\pi s'}{s}\right) \\ \frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_1^2}\left[\cos\left(\frac{\pi s'}{s}\right) + \frac{1}{\Lambda_1}\sin\left(\frac{\pi s'}{s}\right)\right] \\ \vdots \\ -\frac{1}{\sqrt{s}}\sin\left(\frac{N\pi s'}{s}\right) \\ \frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_N^2}\left[\cos\left(\frac{N\pi s'}{s}\right) + \frac{1}{\Lambda_N}\sin\left(\frac{N\pi s'}{s}\right)\right] \\ \vdots \end{bmatrix}. \tag{3.1}$$

The row operations are performed on $C^{(n)}(s')$ and $D^{(n)}$. To keep things symmetric we perform the transposed operations on $C^{(n)T}(s')$ and $D^{(n)}$. The integrals required by Eq. (2.17) are elementary and we easily obtain the reduced form of the matrix $A^{(n)}$:

$$A^{(n)}\sim\begin{bmatrix} -2s & 0 & -\frac{1}{gF}(e^{-gFs}-1) & x_1^T & 0 & \cdots & x_N^T & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \frac{1}{gF}(e^{gFs}-1) & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ x_1 & 0 & 0 & -\Lambda_1^2 & -\frac{3\Lambda_1}{1+\Lambda_1^2} & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{3\Lambda_1}{1+\Lambda_1^2} & \frac{5-\Lambda_1^2}{(1+\Lambda_1^2)^2} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ x_N & 0 & 0 & 0 & 0 & \cdots & -\Lambda_N^2 & -\frac{3\Lambda_N}{1+\Lambda_N^2} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{3\Lambda_N}{1+\Lambda_N^2} & \frac{5-\Lambda_N^2}{(1+\Lambda_N^2)^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.2}$$

where $x_N=(2/\sqrt{s})[\Lambda_N/1+\Lambda_N^2](1/gF)(1+e^{gFs})$ and we have further used the (2,3) and (3,2) elements of the reduced $A^{(n)}$ to eliminate those elements to the right of the former and below the latter. After trivially expanding the determinant along the second row and down the second column we are left with the determinant of a $(2n+1)\times(2n+1)$ bordered matrix. The first set of row and column operations and the orthogonality of the sine and cosine functions have ensured the matrix is block diagonal, the other sets of operations that every second element of the border is zero. Expanding along the first row and down the first column we obtain the following expression for the $n$th nonperturbative approximation to Eq. (2.16):

$$\det A^{(n)}=2s\left(\prod_{N=1}^{2n-1}{}' \alpha_N\right)\left[1+\frac{8}{\lambda^2}\cosh^2\left(\frac{\lambda}{2}\right)\left(\sum_{N=1}^{2n-1}{}' \beta_N\right)\right], \tag{3.3}$$

where

$$\alpha_N=\frac{1+\frac{4\lambda^2}{N^2\pi^2}}{\left(1+\frac{\lambda^2}{N^2\pi^2}\right)^2},\quad \beta_N=\frac{5-\frac{N^2\pi^2}{\lambda^2}}{\left(4+\frac{N^2\pi^2}{\lambda^2}\right)\left(1+\frac{N^2\pi^2}{\lambda^2}\right)^2}, \tag{3.4}$$

and we have written the approximation in terms of $\lambda=gFs$, $(\Lambda_N=N\pi/\lambda)$. That Eq. (3.3) is an approximation to Eq. (2.16) can be seen with the help of the relations [5]

$$\cosh x=\prod_{N=1}^{\infty}{}'\left(1+\frac{4x^2}{N^2\pi^2}\right),\quad \frac{x}{2}\tanh x=\sum_{N=1}^{\infty}{}'\frac{1}{1+\frac{N^2\pi^2}{4x^2}}, \tag{3.5}$$

whence

$$\prod_{N=1}^{\infty}{}'\alpha_N=\frac{\cosh\lambda}{\cosh^2\left(\frac{\lambda}{2}\right)},\quad \sum_{N=1}^{\infty}{}'\beta_N=\frac{\lambda}{8}\tanh\lambda-\frac{\lambda^2}{8}\operatorname{sech}^2\left(\frac{\lambda}{2}\right). \tag{3.6}$$

The nature of the approximation is now evident. The function $\sinh\lambda/\lambda$ is rewritten as $[\cosh\lambda/\cosh^2(\lambda/2)]\{1+(8/\lambda^2)\cosh^2(\lambda/2)[(\lambda/8)\tanh\lambda-(\lambda^2/8)\operatorname{sech}^2(\lambda/2)]\}$, the foremost factor is expressed exactly as the infinite product of the $\alpha_N$s, and the expression in the square brackets as the infinite sum of the $\beta_N$s. The $n$th approximation is then defined by including the first $n$ terms in the product and in the sum.

Note that $\det A^{(n)}$ is an even function of $\lambda$ (of $F$), and hence symmetric, for all $n$. This is desirable since the exact Eq. (2.16) is symmetric. The zeroth approximation, in which Eq. (2.16) is approximated by $\det A^{(0)}=2s$, is the only perturbative result, of order $(gFs)^0$. Every approximation order greater than zero contains all (natural number) powers of $gFs$.

The matrix (3.2) shall be our starting point for finding $\det A_1^{(n)}$. Before replacing the first row of Eq. (3.2) with the column-reduced form of $C^{(n)T}(s)$ [the transpose of Eq. (3.1) with $s'=s$], we must reinstate the second row via $\text{row}_2$

$\rightarrow \mathrm{row}_2 - \frac{1}{2} gF \mathrm{row}_1$. This procedure is valid since the second element of $C^{(n)}(s')$ in Eq. (2.8) is proportional to the first element, and we could have used the second element to eliminate the $e^{-gFs'}$ terms instead of the first element. After thus reinstating the second row and replacing the first row with the column-reduced $C^{(n)T}(s)$, we obtain the reduced form of the (neither symmetric nor antisymmetric) matrix $A_1^{(n)}$:

$$\begin{bmatrix}
e^{gFs} & 0 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_1^2} & \cdots & 0 & -\frac{1}{\sqrt{s}}\frac{1}{1+\Lambda_N^2} & \cdots \\
gFs & 0 & \frac{1}{2}(e^{-gFs}+1) & -\frac{1}{2}gFx_1^T & 0 & \cdots & -\frac{1}{2}gFx_N^T & 0 & \cdots \\
\frac{1}{gF}(e^{gFs}-1) & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
x_1 & 0 & 0 & -\Lambda_1^2 & -\frac{3\Lambda_1}{1+\Lambda_1^2} & \cdots & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{3\Lambda_1}{1+\Lambda_1^2} & \frac{5-\Lambda_1^2}{(1+\Lambda_1^2)^2} & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
x_N & 0 & 0 & 0 & 0 & \cdots & -\Lambda_N^2 & -\frac{3\Lambda_N}{1+\Lambda_N^2} & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \frac{3\Lambda_N}{1+\Lambda_N^2} & \frac{5-\Lambda_N^2}{(1+\Lambda_N^2)^2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots
\end{bmatrix}. \tag{3.7}$$

We may now expand the determinant to obtain the following $n$th approximation to Eq. (2.17):

$$\det A_1^{(n)} = -\frac{1}{2}\left(\prod_{N=1}^{2n-1}{}' \alpha_N\right)\left[1+\lambda+e^{\lambda}+\frac{16}{\lambda}\cosh^2\left(\frac{\lambda}{2}\right)\times\left(\sum_{N=1}^{2n-1}{}' \gamma_N\right)\right], \tag{3.8}$$

where

$$\gamma_N = \frac{1-\frac{2N^2\pi^2}{\lambda^2}}{\left(4+\frac{N^2\pi^2}{\lambda^2}\right)\left(1+\frac{N^2\pi^2}{\lambda^2}\right)^2}. \tag{3.9}$$

Using the second relation of Eq. (3.5), we find

$$\sum_{N=1}^{\infty}{}' \gamma_N = \frac{\lambda}{8}\tanh\lambda - \frac{\lambda}{8}\tanh\left(\frac{\lambda}{2}\right) - \frac{\lambda^2}{16}\mathrm{sech}^2\left(\frac{\lambda}{2}\right). \tag{3.10}$$

The function $2e^{\lambda}$ is thus rewritten exactly as

$$\frac{\cosh\lambda}{\cosh^2\left(\frac{\lambda}{2}\right)}\left\{1+\lambda+e^{\lambda}+\frac{16}{\lambda}\cosh^2\left(\frac{\lambda}{2}\right)\times\left[\frac{\lambda}{8}\tanh\lambda-\frac{\lambda}{8}\tanh\left(\frac{\lambda}{2}\right)-\frac{\lambda^2}{16}\mathrm{sech}^2\left(\frac{\lambda}{2}\right)\right]\right\},$$

the first factor again expressed as the infinite product of the $\alpha_N$s, the expression in the square brackets as the infinite sum of the $\gamma_N$s, and approximated by taking the first $n$ terms in the product and in the sum. Note that the zeroth approximation $\det A_1^{(0)} = -\frac{1}{2}(1+gFs+e^{gFs})$ to Eq. (2.17) is nonperturbative.

The $n$th nonperturbative approximation to $-\frac{1}{2}(gF\coth gFs + gF)$ is the ratio of Eqs. (3.8) and (3.3) (note that the product over the $\alpha_N$s cancel) is the object that appears in the $n$th approximation to the Green's function (2.15). It is not obvious from the reduced form of the matrix $A_1^{(n)}$ that the symmetric part of the determinant of the same is the determinant of the matrix $\tilde{A}^{(n)}$. That is no problem, we find $\det\tilde{A}^{(n)}$ in much the same manner as we found $\det A_1^{(n)}$: take the reduced form of $A_1^{(n)}$, reinstate the second column with the operation $\mathrm{col}_2 \rightarrow \mathrm{col}_2 + \frac{1}{2}gF\mathrm{col}_1$, and discard the first row and column to obtain the symmetric, reduced form of $\tilde{A}^{(n)}$:

$$\tilde{A}^{(n)}\sim\begin{bmatrix} 0 & \frac{1}{2}(e^{-gFs}+1) & -\frac{1}{2}gFx_1^T & 0 & \cdots & -\frac{1}{2}gFx_N^T & 0 & \cdots \\ \frac{1}{2}(e^{gFs}+1) & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \frac{1}{2}gFx_1 & 0 & -\Lambda_1^2 & -\frac{3\Lambda_1}{1+\Lambda_1^2} & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \frac{3\Lambda_1}{1+\Lambda_1^2} & \frac{5-\Lambda_1^2}{(1+\Lambda_1^2)^2} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ \frac{1}{2}gFx_N & 0 & 0 & 0 & \cdots & -\Lambda_N^2 & -\frac{3\Lambda_N}{1+\Lambda_N^2} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{3\Lambda_N}{1+\Lambda_N^2} & \frac{5-\Lambda_N^2}{(1+\Lambda_N^2)^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.11}$$

Expanding the determinant yields

$$\det\tilde{A}^{(n)}=-\cosh^2\left(\frac{\lambda}{2}\right)\left(\prod_{N=1}^{2n-1}{}'\ \alpha_N\right). \tag{3.12}$$

Upon comparison of Eqs. (3.12) and (3.8), noting that $\alpha_N$, $\beta_N$, and $\gamma_N$ are all even functions of $\lambda$ $(F)$, and using $\frac{1}{2}(1+\cosh\lambda)=\cosh^2(\lambda/2)$, our earlier statement that $\det\tilde{A}^{(n)}$ is the symmetric part of $\det A_1^{(n)}$ is apparent.

Finally then, Eq. (3.8), the symmetric part thereof, Eq. (3.12), and Eq. (3.3) substituted into Eq. (2.15) give the $n$th approximation to the fermion Green's function in a constant external field, in the Schwinger-Fock gauge, as defined through the exact representation (1.1)–(1.3). The approximation easily leads to Schwinger's exact result (1.4) in the limit $n\rightarrow\infty$, and we repeat and emphasize that only the zeroth approximation corresponds to a perturbative result.

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