Type IIA string instanton corrections to the four-fermion correlator in the intersection of Del Pezzo surfaces

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The Becker-Becker-Strominger formula, describing the string world-sheet instanton corrections to the four-fermion correlator in the Calabi-Yau compactified type-IIA superstrings, is calculated in the special case of the Calabi-Yau threefold realized in the intersection of two Del Pezzo surfaces. We also derive the selection rules in the supersymmetric GUT of the Pati-Salam type associated with our construction.

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I. INTRODUCTION

One of the central problems in modern string and field theories is the calculation of strong-coupling effects. A calculation of the instanton corrections to various physical quantities is the important part of this problem. A study of the nonperturbative corrections in string theory due to the M-theory branes was pioneered by Becker, Becker and Strominger [1]. The simplest instantons are the so-called string world-sheet instantons whose contributions are independent of the string coupling. The string world-sheet instantons were extensively studied in the past [2] even before Ref. [1]. In the context of the type-IIA superstring compactification, the existence of the world-sheet string instantons can be related to the holomorphic curves in the internal Calabi-Yau (CY) space [2].

In the context of eleven-dimensional M theory [3], the ten-dimensional type-IIA superstring theory arises from the M-theory compactification on a circle S^1, whereas the type-IIA superstrings themselves can be understood as the double compactified (in spacetime as well as in world volume) M2-branes [4].

The M2-branes can be wrapped about the S^1 and a CY (supersymmetric) 2-cycle $\mathcal{C}_2$. They give rise to instantons in four (uncompactified) spacetime dimensions, whose effects can be computed by the standard methods of quantum field theory [5]. The low-energy effective four-dimensional field theory of the CY compactified type-IIA superstrings is given by the $N=2$ supergravity interacting with $h_{2,1}$ hypermultiplets and $h_{1,1}$ vector $N=2$ multiplets, where $h_{2,1}$ and $h_{1,1}$ are the Hodge numbers of CY [6]. The moduli space $\mathcal{M}$ of the compactified theory is given by a direct product of the hypermultiplet moduli space $\mathcal{M}_H$ and the $N=2$ vector multiplet moduli space $\mathcal{M}_V$, while the $S^1\times\mathcal{C}_2$-wrapped M2-branes correct the geometry of $\mathcal{M}_V$ only. The Bogomol'nyi-Prasad-Sommerfield (BPS) (or the supersymmetric map) condition on these wrapped M2-brane configurations just amounts to the holomorphy condition on the world-sheet instantons [2]. The same conclusion was rederived in Ref. [1] by requiring the equivalence between a global supersymmetry transformation and a kappa transformation of the Green-Schwarz superstring action,

$$\partial X^{\bar{m}}=0 \quad \text{or} \quad \bar{\partial} X^{m}=0, \tag{1.1}$$

where ∂ is the holomorphic string world-sheet exterior derivative, and X^m are the complex coordinates in CY, $m=1,2,3$.

The topological equation formally describing the string world-sheet instanton corrections to the four-point fermion (gaugino) correlator $\mathcal{F}_{IJKL}$, where $I,J,K,L=1,2,\dots,h_{1,1}$, was obtained by Becker, Becker and Strominger [1],

$$\Delta_{\mathcal{C}_2}\mathcal{F}_{IJKL}=Ne^{-\int_{\mathcal{C}_2}J-i\int_{\mathcal{C}_2}B}\int_{\mathcal{C}_2}b_I\int_{\mathcal{C}_2}b_J\int_{\mathcal{C}_2}b_K\int_{\mathcal{C}_2}b_L\,, \tag{1.2}$$

where $\mathcal{C}_2$ is the homology class of the instanton, $\{b_I\}$ is the orthonormal basis of harmonic (1,1) forms in CY, J is the Kähler (1,1) form of CY, B is the (closed) Neveu-Schwarz (1,1) form, and N is the normalization factor independent upon b_I.

Like any other (1,1) form, the form $J+iB$ can be decomposed with respect to the cohomology basis $\{b_I\}$,

$$J+iB=\sum_{I=1}^{h_{1,1}} z^I b_I\,, \tag{1.3}$$

where the complex coefficients $\{z^I\}$ are called CY moduli. Integrating Eq. (1.3) once with respect to the modulus z^I yields the famous topological formula for the world-sheet instanton corrections to the Yukawa couplings $\mathcal{F}_{IJK}$ [6]. In the case of Yukawa couplings, mirror symmetry is known to confirm the topological equation on them [7]. This fact indirectly supports the more general equation (1.2) also [1].

Like the similar equation on the Yukawa couplings, the topological Eq. (1.2) is merely a formal equation since one still has to specify how the integrals on the right-hand side of

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the equation are to be calculated. Their calculation for a generic CY space represents the important technical problem whose solution is unknown, to the best of our knowledge. There are, nevertheless, some explicit calculations of the Yukawa couplings in the literature for the special CY spaces realized as the complete intersections in a product of the projective spaces [7].

Our main purpose in this paper is to calculate Eq. (1.2) in the case of the special CY to be defined in the intersection of Del Pezzo surfaces. The Del Pezzo surface is a manifold of complex dimension 2 with a positive first Chern class [8]. We use the "old" geometrical methods first developed in Ref. [9] for computing the Yukawa couplings in a superstring model with three generations of quarks and leptons; see also Ref. [10]. The geometrical approach is based on Poincaré duality and a knowledge of the homology group basis of CY.

Our paper is organized as follows: in Sec. II we introduce into the special CY spaces realized as the complete intersections of five quadrics in a product of two projective spaces $\mathcal{P}^4\times\mathcal{P}^4$. The main body of our paper is given by Sec. III where we formulate the mathematical instruments allowing us to calculate the integrals in Eq. (1.2). Some explicit examples are given in Sec. IV. We conclude with Sec. V where some connections between our work and the recent literature are outlined.

II. QUADRICS, DEL PEZZO AND CY

Let us consider the compact CY spaces realized as the complete intersections in a product of two projective spaces, $\mathcal{P}^4\times\mathcal{P}^4$, with the configuration matrix[1]

$$\begin{pmatrix} 4 & || & 2 & 2 & 0 & 0 & 1 \\ 4 & || & 0 & 0 & 2 & 2 & 1 \end{pmatrix}. \tag{2.1}$$

To decipher this matrix, we introduce two sets of homogeneous coordinates, x and y, in each $\mathcal{P}^4$, define two Del Pezzo surfaces K_x and K_y, and a hypersurface S in $\mathcal{P}^4\times\mathcal{P}^4$ by the following constraints [12]:

$$\begin{aligned} K_x&=\left\{x\in\mathcal{P}^4:\quad P_1(x)=\sum_{i=0}^{4}x_i^2=0,\quad P_2(x)=\sum_{i=0}^{4}a_ix_i^2=0\right\},\\ K_y&=\left\{y\in\mathcal{P}^4:\quad P_3(y)=\sum_{i=0}^{4}y_i^2=0,\quad P_4(x)=\sum_{i=0}^{4}b_iy_i^2=0\right\},\\ S&=\left\{(x,y)\in\mathcal{P}^4\times\mathcal{P}^4:\quad P_5(x,y)=\sum_{i,j=0}^{4}c_{ij}x_iy_j=0\right\}. \end{aligned} \tag{2.2}$$

The sum of entries in each line of the matrix (2.1) to the right of $||$ exceeds exactly by one the dimension of the embedding space $\mathcal{P}^4$ to the left of $||$, so that

$$K_0=(K_x\times K_y)\cap S \tag{2.3}$$

appears to be Kähler and of the vanishing first Chern class, i.e., K_0 is a CY space, in agreement with the theorem of Greene, Vafa and Warner [13]. We assume that the real coefficients of the quadrics P_2, P_4, and P_5 in Eq. (2.2) are chosen to obey the transversality condition for all hypersurfaces in the definition (2.3) of K_0, i.e.,

$$dP_1\wedge dP_2\wedge dP_3\wedge dP_4\wedge dP_5\neq 0. \tag{2.4}$$

This equation guarantees the smoothness of the simply connected manifold K_0 [14].

The first column in Eq. (2.1) thus indicates that we consider a CY in the product $\mathcal{P}^4\times\mathcal{P}^4$, whereas the other columns denote bipowers of the polynomials of x and y in Eq. (2.2).

The nontrivial Hodge numbers of K_0 are given by

$$h_{2,1}(K_0)=28 \quad\text{and}\quad h_{1,1}(K_0)=12, \tag{2.5}$$

so that its Euler characteristic $\chi(K_0)$ is

$$\frac{1}{2}\chi(K_0)=h_{1,1}-h_{2,1}=-16. \tag{2.6}$$

It is not easy to construct the complete intersection CY spaces with the physically interesting values of the Euler characteristic, $\chi=\pm 6,\pm 8$. However, it easily becomes possible through the so-called 'orbifoldization' process [11]. In our case, we can introduce the quotient K of the manifold K_0 with respect to a discrete symmetry subgroup $G_F=\mathbf{Z}_2^2$ of G, which acts freely in K_0. This yields the CY space K of the Euler characteristic $\chi(K)=-8$. The G-group elements generating G_F can be chosen as follows:

$$g_1=\mathrm{diag}(1,1,1,-1,-1),\quad g_2=\mathrm{diag}(1,1,-1,1,-1), \tag{2.7}$$

so that the action of g_1 reads

[1]We use the standard notation [11].

$$g_1: \ (x_0,x_1,x_2,x_3,x_4;y_0,y_1,y_2,y_3,y_4) \rightarrow (x_0,x_1,x_2,-x_3,-x_4;y_0,y_1,y_2,-y_3,-y_4), \tag{2.8}$$

and similarly for g_2. The manifold K_0 has the hidden discrete symmetry group G isomorphic to $\mathbf{Z}_2^5$, whose action is given by

$$\begin{aligned} &\mathbf{Z}_2(A): \ A=\mathrm{diag}(-1,1,1,1,1),\\ &\mathbf{Z}_2(B): \ B=\mathrm{diag}(1,-1,1,1,1),\\ &\mathbf{Z}_2(C): \ C=\mathrm{diag}(1,1,-1,1,1),\\ &\mathbf{Z}_2(D): \ D=\mathrm{diag}(1,1,1,-1,1),\\ &\mathbf{Z}_2(S): \ S(x_i)=y_i, \quad S(y_j)=x_j. \end{aligned} \tag{2.9}$$

In the embedding space $\mathcal{P}^4\times\mathcal{P}^4$ we have

$$g_1=ABC \quad \text{and} \quad g_2=ABD. \tag{2.10}$$

The CY manifold K_0 is the simply connected covering space of the CY space K. The latter still possesses some hidden symmetries that survive after its factorization by G_F. These discrete symmetries are

$$\begin{aligned} G_H&=\frac{\mathbf{Z}_2(A)\times\mathbf{Z}_2(B)\times\mathbf{Z}_2(C)\times\mathbf{Z}_2(D)\times\mathbf{Z}_2(S)}{\mathbf{Z}_2(g_1)\times\mathbf{Z}_2(g_2)}\\ &=\mathbf{Z}_2(A)\times\mathbf{Z}_2(B)\times\mathbf{Z}_2(S). \end{aligned} \tag{2.11}$$

In the context of the type-IIA superstring compactification, the CY space K gives rise to the four-dimensional unified model with *four* generations of quarks and leptons, and an E_6 gauge group. Further breaking of E_6 by the standard mechanism of the vacuum Wilson loops yields the Pati-Salam-like unified model with a gauge group $SU(4)_c \times SU(2)_L\times SU(2)_R\times U(1)$. The Yukawa couplings in this four-generation superstring model were calculated in Ref. [12].

III. INSTANTONS IN DEL PEZZO

Equation (1.1) implies that the CY-compactified type-IIA superstring world-sheet instantons are described by the isolated holomorphic curves in CY. A single instanton corresponds to a curve of genus zero. In the case of K_0, there are 256 holomorphic or $CP(1)$ curves. A derivation of this number was given, for example, in Ref. [15] where it appeared as the leading term in the series expansion of the fundamental period as a solution to the Picard-Fuchs equation for the given CY. A geometrical derivation of the same result is given below in this section. However, first we need more information about the geometrical structure of the space K_0 defined by Eq. (2.3) and the topology of the Del Pezzo surfaces K_x and K_y.

As is well known in algebraic geometry [16], a smooth intersection of two quadrics in $\mathcal{P}^4$ is biholomorphic equivalent to the projective plane with five different blown-up points. Since the Hodge number $h_{1,1}$ of $\mathcal{P}^2$ is equal to one, after blowing up at five points $h_{1,1}$ is equal to $1+5=6$, while the other Hodge numbers remain unchanged. Next, the Del Pezzo surface K_x possesses exactly 16 complex lines $\{C_x\}$ that can be described by the relations

$$\begin{aligned} &a_{42}a_{32}a_{10}x_2-\varepsilon_1 a_{40}a_{30}a_{42}x_0-i\varepsilon_2 a_{41}a_{31}a_{20}x_1=0,\\ &a_{43}a_{32}a_{10}x_3-\varepsilon_3 a_{40}a_{31}a_{20}x_0-\varepsilon_4 a_{41}a_{30}a_{21}x_1=0,\\ &a_{43}a_{42}a_{10}x_4-\varepsilon_5 a_{41}a_{30}a_{20}x_0-i\varepsilon_6 a_{40}a_{21}a_{31}x_1=0, \end{aligned} \tag{3.1}$$

where $a_{kl}=\sqrt{a_k-a_l}$, $0\leqslant l<k\leqslant 4$, and $\varepsilon_j=\pm1$, $j=1,2,\ldots,6$. The sign coefficients ε_j and our notation for the complex lines on the Del Pezzo surfaces are collected in Table I.

TABLE I. Complex lines in the Del Pezzo surface $K_x(K_y)$ and the sign factors ε_j.

Line	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	Line	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6
E_1	+	+	−	+	+	+	F_{14}	−	−	−	+	+	+
E_2	−	+	+	+	+	−	F_{15}	+	−	−	−	+	−
E_3	+	+	+	−	−	−	F_{23}	−	+	−	−	+	−
E_4	−	−	−	+	−	−	F_{34}	−	−	+	−	−	−
E_5	+	−	−	−	−	+	F_{25}	−	−	+	−	+	+
G	+	+	−	+	−	−	F_{34}	−	−	+	−	−	−
F_{12}	−	+	+	+	−	+	F_{35}	+	−	+	+	−	+
F_{13}	+	+	+	−	+	+	F_{45}	−	+	−	−	−	+

The homology class of the Kähler form on Del Pezzo K_x can be represented by the intersection of the hyperplane S with K_x,

$$H=\{x_0=0\}\subset K_x. \tag{3.2}$$

Under the symmetry group G the 16 lines on the Del Pezzo surface K_x are naturally decomposed into *three* classes: (i) the five lines E_i, $i=1,2,3,4,5$, that (pairwise) do not intersect with each other and thus represent five linearly independent homology classes of $H_2(K_x,\mathbf{R})$; together with the hyperflat section H (dual to a Kähler form of the Del Pezzo surface K_x) they form a basis in $H_2(K_x,\mathbf{R})$, (ii) ten lines F_{ij} that have intersections only with E_i and E_j, and (iii) one line G intersecting with all E_i (see Table I also). The 256 holomorphic curves are then decomposed with respect to the $G_F=\mathbf{Z}_2\times\mathbf{Z}_2$ discrete symmetry group of order 4 into four classes that are cyclically symmetric with respect to their interchanging.

Accordingly, we get the following matrix of the intersection indices:

$$\begin{aligned} &(E_i,E_j)=-\delta_{ij}, \quad (E_i,F_{jk})=\delta_{ij}+\delta_{ik},\\ &(E_i,G)=1, \quad (E_i,H)=1, \quad (H,H)=4. \end{aligned} \tag{3.3}$$

Having obtained the holomorphic curves and the homology basis explicitly, it is not difficult to determine the action of the discrete symmetry group on the latter. The generating elements (g_1,g_2,A,B) of the group G act as follows:

$$g_1(E_1,E_2,E_3,E_4,E_5,H)=(E_3,F_{45},E_1,F_{25},F_{24},H),$$
$$g_2(E_1,E_2,E_3,E_4,E_5,H)=(E_4,F_{35},F_{25},E_1,F_{23},H),$$
$$A(E_1,E_2,E_3,E_4,E_5,H)=(F_{12},G,F_{23},F_{24},F_{25},H),$$
$$B(E_1,E_2,E_3,E_4,E_5,H)=(F_{15},F_{25},F_{35},F_{45},G,H). \tag{3.4}$$

For example, to get $g_1(E_2)=F_{45}$, we choose for definiteness $a_0=0$, $a_1=1$, $a_3=3$ and $a_4=4$ in Eq. (3.1) where the coefficients ε_j are given by Table I. We find

$$\begin{aligned} g_1(E_2)&=g_1\{x_2+\sqrt{6}x_0-i\sqrt{6}=x_3-i4x_0-3x_1\\ &=x_4-3x_0+2x_1=0\}\\ &=\{x_2+\sqrt{6}x_0-i\sqrt{6}x_1=x_3+i4x_0+3x_1\\ &=x_4+3x_0-i2x_1=0\}=F_{45}. \end{aligned} \tag{3.5}$$

The curve H is invariant under all these symmetries, whereas each line E_i goes into one of the 16 lines lying in the intersection of quadrics. The symmetry transformations act independently on each factor $\mathcal{P}^4$ in a product $\mathcal{P}^4\times\mathcal{P}^4$, so that it is enough to consider only one projective space $\mathcal{P}^4$. The action of the S symmetry of G just replaces each (1,1) form on Del Pezzo K_x by the corresponding (1,1) form on K_y. We find the following decompositions:

$$F_{ij}=\frac{1}{3}\left(\sum_{i=1}^{5}E_i+H\right)-E_i-E_j, \tag{3.6}$$

and

$$G=\frac{1}{3}\left(2H-\sum_{i=1}^{5}E_i\right). \tag{3.7}$$

For example, to prove Eq. (3.6), we begin with a decomposition

$$F_{ij}=\sum_{i=1}^{5}c_iE_i+c_6H \tag{3.8}$$

whose coefficients (c_i,c_6) are to be determined. Let us now consider the intersections of F_{ij} with H, E_i, E_j, and $M=\Sigma_{i=1}^{5}E_i$ by using the index intersection matrix (3.3). We find

$$\begin{aligned} (F_{ij},H)&=c_1+c_2+c_3+c_4+c_5+4c_6=1,\\ (F_{ij},E_i)&=-c_i+c_6=1,\\ (F_{ij},E_j)&=-c_j+c_6=1,\\ \left(F_{ij},\sum_{i=1}^{5}E_i\right)&=-(c_1+c_2+c_3+c_4+c_5)+5c_6=2. \end{aligned} \tag{3.9}$$

TABLE II. The transformation properties of the (1,1) forms in $\overline{\mathbf{27}}$ of E_6 under the discrete symmetries.

Fields	g_1	g_2	A	B	S
$(\overline{n},\overline{g},\overline{g^c})_1$	1	1	1	1	1
$(\overline{n},\overline{g},\overline{g^c})_2$	1	1	1	-1	1
$(\overline{n},\overline{g},\overline{g^c})_3$	1	1	-1	1	1
$(\overline{n},\overline{g},\overline{g^c})_4$	1	1	-1	1	-1
$(\overline{n},\overline{g},\overline{g})_5$	1	1	1	-1	-1
$(\overline{n},\overline{g},\overline{g^c})_6$	1	1	-1	1	-1
$(\overline{g},\overline{l})_1$	-1	1	-1	-1	1
$(\overline{g},\overline{l})_2$	-1	1	-1	-1	-1
$(\overline{g^c},\overline{l^c})_1$	1	-1	-1	-1	1
$(\overline{g},\overline{l})_2$	1	-1	-1	-1	-1
$\overline{H}_1$	-1	-1	-1	-1	1
$\overline{H}_2$	-1	-1	-1	-1	-1

Hence, the coefficients in Eq. (3.8) are given by

$$c_6=1/3 \quad \text{and} \quad c+i=c_j=-2/3. \tag{3.10}$$

Equation (3.7) is obtained similarly.

In the grand unification theories of the Pati-Salam type, based on the gauge group E_6 that is supposed to be broken by Wilson lines as

$$E_6\to SU(4)_c\times SU(2)_L\times SU(2)_R\times U(1), \tag{3.11}$$

the representation $\overline{\mathbf{27}}$ of E_6 is decomposed as follows:

$$\begin{aligned} \overline{\mathbf{27}}=&[(q,l)=(4c,2L,1R)]+[q^c,l^c=(\overline{4c},1L,\overline{2}R)]\\ &+[H=(1c,2L,2R)]+[g,g^c=(6c,1L,1R)]\\ &+[n=(1c,1L,1R)], \end{aligned} \tag{3.12}$$

where $(q_{R,L},l_{R,L})$ stand for the quark-lepton families, H are the new leptons, g are the new quarks and n is the singlet.

As was demonstrated in Ref. [14], the particle spectrum corresponding to the (2,1) forms in K is given by

$$h_{2,1}:\quad 10(n,g,g^c)+6(f,H), \tag{3.13}$$

where f stands for (q,l,q^c,l^c). The antiparticles corresponding to the (1,1) forms in K are given by

$$h_{1,1}:\quad 6(\overline{n},\overline{g},\overline{g^c})+2(\overline{f},H). \tag{3.14}$$

The transformation properties of the fields, in accordance with the decomposition (3.12), are collected in Table II.

Equations (3.4) also allow us to identify the special combinations of the basic (1,1) homology elements that are invariant under the CY symmetry group G_H of Eq. (2.11). They are

$$F_i^{x,y}=H^{x,y}+6E_i^{x,y}-2\sum_{i=1}^{5}E_i^{x,y},$$
$$H^{\pm}=H^x\pm H^y,\quad F_i^{\pm}=F_i^x\pm F_i^y, \tag{3.15}$$

where $i=1,2,3,4,5$.

In the context of the CY superstring compactification, the invariant elements of the (1,1) cohomology basis correspond to the physical matter fields transforming in $\overline{\mathbf{27}}$ of E_6 [6,11]. A direct calculation yields the following set of twelve invariant combinations in the given homology basis of $H_2(K^x,\mathbf{R})$ dual to the cohomology group $H^{1,1}$ [14]:

$$(\overline{n},\overline{g},\overline{g^c})_1=H^x+H^y,$$
$$(\overline{n},\overline{g},\overline{g^c})_2=F_2^+,\quad(\overline{n},\overline{g},\overline{g^c})_3=F_5^+,$$
$$(\overline{n},\overline{g},\overline{g^c})_4=H^x-H^y\equiv H^-,$$
$$(\overline{n},\overline{g},\overline{g})_5=F_2^-,\quad(\overline{n},\overline{g},\overline{g^c})_6=F_5^-, \tag{3.16}$$
$$(\overline{g},\overline{l})_1=F_3^+,\quad(\overline{g},\overline{l})_2=F_3^-,$$
$$(\overline{g^c},\overline{l^c})_1=F_4^+,\quad(\overline{g^c},\overline{l^c})_2=F_4^-,$$
$$\overline{H}_1=F_1^+,\quad\overline{H}_2=F_1^-,$$

where the quark-lepton families (q,l) and extra leptons (H) are merely considered here as the formal notation. There are no two different combinations of the cycles that would have the same transformation properties under the discrete symmetries. We verified this statement by a straightforward calculation (see Table II). This means that our identification of cycles is unique.

The instantons in the Del Pezzo intersection have the form $C_x\times C_y$ that yields $16\times16=256$, as it should. The intersection of these 256 surfaces with the hyperplane S in Eq. (2.3) yields 128 complex curves of genus zero on one of the Del Pezzo surfaces $\times$ point on the other Del Pezzo surface. Accordingly, there are two ways of choosing on which Del Pezzo surface we take the line to lie on, while there are four ways of choosing a point on the other Del Pezzo surface. This yields in total $2\times4\times16=128$ different instantons of the type *line*$\times$*point*, and, in addition, 128 different instantons of the type *line*$\times$*line*. Unlike Ref. [9], where a similar problem was solved in the case of the *cubic* Del Pezzo intersection in $\mathcal{P}^3\times\mathcal{P}^3$, we have a more degenerate (and more symmetric) situation.

Each holomorphic curve corresponding to an instanton is thus given by an intersection of $C_x\times C_y$ with the hyperplane S in accordance with Eq. (2.3), where C_x are 16 lines in the Del Pezzo surface K_x and similarly for K_y,

$$\mathcal{L}=(C_x\times C_y)\cap S. \tag{3.17}$$

There are four classes amongst the 256 instantons that are cyclically connected in our case. To calculate Eq. (1.2) we have to choose a representative $\mathcal{L}$ from each class. The holomorphic curve $\mathcal{L}$ is the image of the string world sheet in the CY space under the instanton map.

Now, on the one hand, the integral of any closed form ω of the maximal degree over $K_0=(K_x\times K_y)\cap S$ can be represented by the value of the cohomology class of ω on the cycle K_0. On the other hand, the cycle intersection in the homologies is dual to the exterior multiplication in the cohomologies. Hence, we have

$$\int_{K_0}\omega=(w\wedge H)[K_x\times K_y], \tag{3.18}$$

where we have introduced the class of cohomologies w of ω, and the image H of the cohomology class of the Kähler form in $\mathcal{P}^{24}$ dual to the hyperplane S. The hyperplane S represents the hypersurface S after Segre embedding [16] restricted on $K_x\times K_y$. The Segre embedding in this case means the embedding of a product $\mathcal{P}^4\times\mathcal{P}^4$ into the projective space $\mathcal{P}^{24}$ by the coordinate identification $w_{ij}=x_iy_j$ where w_{ij} are the homogeneous coordinates of $\mathcal{P}^{24}$.

It is worth mentioning that only one component $H^x(H^y)$ remains in what follows from $H=H^x+H^y$. Hence, our problem reduces to a calculation of the intersection indices from the homology group $H_2(K_x,\mathbf{Z})$ only. In general, the Poincaré duality establishes the isomorphism between the closed (DeRham) homologies and compact-dual cohomology classes, as well as the isomorphism between compact (DeRham) homologies and cohomologies [17]. In the *compact* CY case we consider, there is no difference between the closed and compact classes.

Taken together, this allows us to replace the integral of the (1,1) form b_I along the curve $\mathcal{L}$ in Eq. (1.2) by the intersection index of this curve $\mathcal{L}$ with the cycle F_I that is Poincaré dual to b_I,

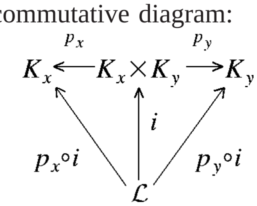
$$\int_{\mathcal{L}}b_I=(F_I\cdot\mathcal{L}). \tag{3.19}$$

As a result, the instanton correction to the four-fermion correlator in Eq. (1.2) is proportional to a product

$$\int_{\mathcal{L}}b_I\int_{\mathcal{L}}b_J\int_{\mathcal{L}}b_K\int_{\mathcal{L}}b_L=(F_I\cdot\mathcal{L})(F_J\cdot\mathcal{L})(F_K\cdot\mathcal{L})(F_L\cdot\mathcal{L}), \tag{3.20}$$

where the brackets with dots stand for the intersection indices of the corresponding cycles, which are to be determined from the matrix (3.3) of our basic curves (E_i^x,H^x,E_i^y,H^y)—see Sec. IV for some explicit examples.

We label our 256 curves by the corresponding lines in one of the following forms: *line*$\times$*line*, *point*$\times$*point*, and *line* $\times$*line* in the cover K_0. The validity of the cycle intersection matrix for both Del Pezzo surfaces K_x and K_y is justified by the fundamental commutative diagram:



In a more explicit notation we just get

$$\int_{\mathcal{L}} b_I^x \equiv \int_{\mathcal{L}} p_x^* b_I^x|_{\mathcal{L}} = \int_{\mathcal{L}} p^* i^* b_I^x = \int_{\mathcal{L}} (p_x \circ i)^* b_I^x = \int_{(p_x \circ i)_* \mathcal{L}} b_I^x x$$
$$\equiv \int_{\mathcal{L}_x} b_I^x = (F_x, \mathcal{L}_x), \tag{3.21}$$

where the projection p_x lifts the form b_I on K_x to $K_x \times K_y$ with its simultaneous restriction on the curve $\mathcal{L}_x$. The notation (i) in Eq. (3.21) stands for the embedding of $\mathcal{L}$ into $K_x \times K_y$.

IV. EXAMPLES

Equations (3.19) and (3.20) reduce a calculation of Eq. (1.2) to a calculation of the homology intersection indices. In turn, this can be easily done when instantons are labeled by lines in the intersection of quadrics. As was demonstrated in Sec. III, there are the 128 holomorphic curves of the type *point*×*line* and *line*×*point*, and the 128 curves of the type *line*×*line*. The intersection of these curves with our homology basis is given by Eq. (3.3), in agreement with Eq. (3.18). The 16 lines on Del Pezzo are divided into three classes: one G, five E_i^x that intersect with G, and ten F_{ij}^x (see Table I). Hence, each type of lines receives further classification according to its class. It is worth mentioning that the instanton contributions of the type *point*×*line* and *line*×*point* do not always coincide.

As our first (simple) example, let us consider the correlator given by

$$\int H^+ \int H^+ \int H^+ \int H^+ = (H^+ \cdot \mathcal{L})^4 \tag{4.1}$$

over $\mathcal{L} = G^x \times G^y$. Let us recall (Sec. III) that $H^+ = H^x + H^y$, while

$$\int_{G^x \times \{point\}} H^+ = \int_{\{point\} \times G^y} H^+ = 1 \tag{4.2}$$

and

$$\int_{G^x \times \{point\}} H^- = 1, \quad \int_{\{point\} \times G^y} H^- = -1,$$
$$\int_{G^x \times G^y} H^- = 0. \tag{4.3}$$

Therefore, we obtain

$$(H^+ \cdot \mathcal{L})^4 = [(H^x + H^y) \cdot (G^x + G^y)]^4 = (H^x \cdot G^x + H^y \cdot G^y)^4$$
$$= (1+1)^4 = 2^4. \tag{4.4}$$

Similarly, we find

$$\int_{G^x \times G^y} H^- \int_{G^x \times G^y} H^- \int_{G^x \times G^y} H^- \int_{G^x \times G^y} H^-$$
$$= (H^- \cdot [G^x \times G^y])^4 = [(H^x - H^y) \cdot (G^x \times G^y)]^4$$
$$= (H^x \cdot G^x - H^y \cdot G^y)^4 = (1-1)^4 = 0. \tag{4.5}$$

In fact, any correlator containing the factor $\int H^-$ vanishes. Thus we see that some fermionic correlators are equal to zero despite the fact that the discrete symmetries allow nonvanishing values for them.

As another (less trivial) example, let us consider the factor

$$\int_{F_{ij} \times \{point\}} H^+ = \left([H^x + H^y] \cdot \left[\frac{1}{3} \left(\sum E_i + H \right) - E_i - E_j \right]^x \right)$$
$$= \frac{5}{3} + \frac{4}{3} - 2 = 1. \tag{4.6}$$

Similarly, we find

$$\int_{\{point\} \times F_{ij}^y} H^+ = \int_{F_{ij}^x \times \{point\}} H^+ = 1, \quad \int_{F_{ij}^x \times F_{ij}^y} H^+ = 2, \tag{4.7}$$

and

$$\int_{F_{ij}^x \times \{point\}} H^- = \int_{\{point\} \times F_{ij}^y} H^- = -1, \quad \int_{F_{ij}^x \times F_{ij}^y} H^- = 0. \tag{4.8}$$

A more complicated example is given by

$$\int_{G^x \times \{point\}} F_5^+ \equiv (F_5^+, G^x \times \{point\}) = [(F_5^x + F_5^y), (G^x \times \{point\})] = (F_5^x \cdot G^x) = \left(H^x + 6E_5^x - 2\sum_i E_i^x \right) \cdot G^x$$
$$= \left(H^x + 6E_5^x - 2\sum_i E_i^x \right) \cdot \frac{1}{3} \left(2H^x - \sum_i E_i^x \right)$$
$$= \frac{2}{3} H^x \cdot H^x + 4H^x \cdot E_5^x - \frac{4}{3} \sum_i E_i^x \cdot H^x - \frac{1}{3} H^x \cdot \sum_i E_i^x - 2E_5^x \cdot \sum E_i^x + \frac{2}{3} \sum_i E_i^x \cdot \sum_j E_j^x$$
$$= \frac{8}{3} + 4 - \frac{20}{3} - \frac{5}{3} + 2 - \frac{10}{3} = -3. \tag{4.9}$$

Similarly, we find

$$\int_{\{point\}\times G^y} F_5^+ = -3, \tag{4.10}$$

and, hence,

$$\left(\int_{G^x\times G^y} F_5^+\right)^4 = (-3-3)^4 = 6^4. \tag{4.11}$$

We find, in addition,

$$\int_{G^x\times\{point\}} F_i^- = \int_{G^x\times\{point\}} F_i^+ = -3, \tag{4.12}$$

and

$$\int_{\{point\}\times G^y} F_i^- = -\int_{\{point\}\times G^y} F_i^+ = +3. \tag{4.13}$$

The similar contributions are given by

$$\int_{G^x\times G^y} F_i^- = \int_{F_{ij}^x\times F_{ij}^y} F_i^- = 0, \tag{4.14}$$

$$\int_{\{point\}\times F_{ij}^y} F_i^- = -\int_{F_{ij}^x\times\{point\}} F_i^- = -7+6\delta_{ij}, \tag{4.15}$$

and

$$\int_{F_{ij}^x\times F_{ij}^y} F_i^+ = 14-12\delta_{ij}. \tag{4.16}$$

The rest of the integrals is given by

$$\begin{gathered}
\int_{F_{ij}^x\times G^y} F_i^+ = \int_{G^x\times F_{ij}^y} F_i^+ = 4-6\delta_{ij}, \\
\int_{\{point\}\times E_i^y} F_i^- = -\int_{\{point\}\times E_i^y} F_i^+ = -\int_{E_i^x\times\{point\}} F_i^- = -\int_{E_i^x\times\{point\}} F_i^+ = 3, \\
\int_{E_i^x\times E_i^y} F_i^+ = 9, \quad \int_{E_i^x\times E_i^y} F_i^- = 0, \quad \int_{E_j^x\times E_i^y} F_i^- = -6+6\delta_{ij}, \\
\int_{E_j^x\times\{point\}} F_i^+ = -\int_{\{point\}\times E_j^y} F_i^- = 3-6\delta_{ij}, \quad \int_{E_j^x\times E_i^y} F_i^- = 6-6\delta_{ij}, \\
\int_{E_i^x\times\{point\}} H^+ = \int_{\{point\}\times E_i^y} H^+ = \int_{E_i^x\times\{point\}} H^- = -\int_{\{point\}\times E_j^y} H^- = 1, \\
\int_{E_i^x\times E_j^y} H^+ = \int_{E_i^x\times G^y} H^+ = \int_{G^x\times E_i^y} H^+ = 2, \\
\int_{G^x\times F_{ij}^y} H^+ = \int_{F_{ij}^x\times E_i^y} H^+ = 2, \\
\int_{G^x\times F_{ij}^y} H^- = \int_{E_i^x\times F_{ij}^y} H^- = \int_{F_{ij}^x\times E_i^y} H^- = 0.
\end{gathered} \tag{4.17}$$

The fermionic correlators in Eq. (1.2) are given by various products of four factors calculated above.

V. CONCLUSION

It is surprising, from the mathematical viewpoint, that the fermionic correlators (1.2) are entirely determined by topology so that they can be explicitly calculated. Their physical significance is yet to be understood. At the very least, however, all the fermionic correlators (1.2) vanish in the (classical) tree approximation by index theorems [5], so that the instanton corrections obtained are actually the leading contributions to these correlators. This leads to the highly nontrivial selection rules for the physical processes described by the fermionic correlators in the CY compactified type-II strings. For example, some correlators exactly vanish (Sec. IV) even though the discrete symmetries of CY allow nonvanishing values for them.

Though our geometrical approach is similar to the one used earlier for other string models in Refs. [2,9], there are also some conceptual differences. We find it simpler to consider instantons in the simply connected covering space (CY) manifold K_0 of K, instead of the CY space K. Each instanton in K has four representatives in K_0, which are all equivalent as regards the G-invariant real quantities.

We merely discussed the *one*-instanton corrections to the four-fermion correlators. One may expect the existence of the *multi*-instanton corrections from the maps of higher degree (more than one). Unfortunately, the status of multi-instantons in the context of type-IIA superstrings is not quite clear [1] (see, however, Ref. [18]).

One might also think that the intersection of Del Pezzo surfaces is the very special case of the type-IIA string or M Theory compactification. In fact, as was recently noticed in Ref. [19], there is a non-trivial duality between toroidal compactifications of M theory and Del Pezzo surfaces. According to Ref. [19], a group of the classical symmetries of Del Pezzo (i.e., the global diffeomorphisms preserving the canonical class of Del Pezzo) corresponds to the U-duality symmetries of the toroidally compactified M theory. The M theory (BPS) branes are mapped under this ''mysterious duality'' to rational curves on Del Pezzo, so that the electric-magnetic duality of M theory receives a nice geometrical description in terms of the Del Pezzo surfaces [19]. In particular, the bound states of the (1/2)-BPS branes in M theory can be related to the intersections of spheres in Del Pezzo [19]. Further developments of this new duality require counting intersections of the holomorphic curves of higher genus in CY [20].

In the context of Horava-Witten theory [21], similar calculations of instanton corrections are needed when one considers a torus-fibered CY threefold $\mathcal{Z}$ over the Del Pezzo base, with the nontrivial first homotopy group $\pi_1(\mathcal{Z})=\mathbf{Z}_2$. When a gauge vacuum on the hidden brane is trivial, the threefold $\mathcal{Z}$ admits three families of the semistable holomorphic vector bundles associated with an $N=1$ supersymmetric gauge theory having three (chiral) quark-lepton familes and the GUT group $SU(5)$ in the observable brane [20]. Both five-branes in this Horawa-Witten type construction are wrapped about holomorphic curves in $\mathcal{Z}$ whose homology classes are exactly calculable.

Our investigation is also relevant for studying the supersymmetric Pati-Salam-type models from intersecting D-branes (see, e.g., Ref. [22]), and the nonperturbative flipped $SU(5)$ vacua in heterotic M theory [23,24].

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