Orbifolds, Penrose limits, and supersymmetry enhancement

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We consider supersymmetric pp-wave limits for different $\mathcal{N}=1$ orbifold geometries of the five-sphere $\mathbf{S}^5$ and the five-dimensional Einstein manifold $\mathbf{T}^{1,1}$. As there are several interesting ways to take the Penrose limits, the pp-wave geometry can be either maximally supersymmetric $\mathcal{N}=4$ or half-maximally supersymmetric $\mathcal{N}=2$. We discuss in detail the cases $\mathrm{AdS}_5\times\mathbf{S}^5/\mathbf{Z}_3$, $\mathrm{AdS}_5\times\mathbf{S}^5/(\mathbf{Z}_m\times\mathbf{Z}_n)$, and $\mathrm{AdS}_5\times\mathbf{T}^{1,1}/(\mathbf{Z}_m\times\mathbf{Z}_n)$ and we identify the gauge invariant operators that correspond to stringy excitations for the different limits.

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I. INTRODUCTION

The duality between open strings and closed strings has been explored extensively over recent years. One important example is the AdS conformal field theory (CFT) conjecture between the $\mathcal{N}=4$ field theory and type IIB strings on $\mathbf{AdS}_5\times\mathbf{S}^5$ [1–3]. The conjecture has been generalized to orbifolds of $\mathbf{S}^5$ [4,5] and to conifolds [6]. The supergravity limit of the string has been mainly considered so far because of the difficulties in quantizing strings in the presence of Ramond-Ramond (RR) fluxes. On the other hand, another maximally supersymmetric background, the pp wave, has been discussed recently in [7], and string theory on the pp wave is an exactly solvable model, where one can identify all the string oscillators [8].

The pp-wave solution appear, as a Penrose limit of the $\mathbf{AdS}_5\times\mathrm{S}^5$ solution [7,9] so it can be used to obtain information about the AdS/CFT correspondence. The authors of [9] have extended the AdS/CFT conjecture to the case of strings moving on a pp-wave background where the corresponding field theory operators are the ones with high R charge, and in this case the field theory describes not only the supergravity but also the full closed string theory.

The idea of [9] has been extended in many directions [10–17]. The direction we are pursuing in this work was initiated in a series of papers [11–17] and involves geometries more complicated than the $\mathbf{S}^5$. Especially interesting are the cases of orbifolds of $\mathbf{S}^5$ or conifolds where one can take two kinds of pp-wave limit: one which preserves the supersymmetry and the other one which enlarges the supersymmetry. As discussed in [7], if we take the Penrose limit on directions orthogonal to the orbifolding direction, then we expect to get the same amount of supersymmetry, but a Penrose limit along the orbifolding direction will get an increase of supersymmetry. One example of the second type was described in [11–13] for the case of D3-branes at a conifold singularity, where the Penrose limit gives a maximally supersymmetric solution. In this case we expect a supersymmetry enhancement in field theory, from $\mathcal{N}=1$ to $\mathcal{N}=4$, and the relevant $\mathcal{N}=1$ multiplets which give rise to an $\mathcal{N}=4$ multiplet have been identified. In [15,16] a similar discussion was developed for the supersymmetry enhancement from $\mathcal{N}=2$ to $\mathcal{N}=4$ in the case of $\mathbf{S}^5/\mathbf{Z}_k$.

In the present work we study the supersymmetry enhancement in the Penrose limit for several examples of orbifolds. As only the infinitesimal neighborhood of the null geodesic is probed in the pp-wave limit, the orbifold action disappears unless it is considered locally around the null geodesic. In other words, the orbifolding action also changes in the pp-wave limit. Thus, in general, it is not possible to build duals to string oscillators in the Penrose limit from gauge invariant operators of the original orbifold theory. We have found that, in the Penrose limit, one needs to consider operators from the covering space of the original space. We also comment on anomalous dimensions and correlation functions for the orbifold theories and on the interpretation as a limit of a discrete light-cone quantization (DLCQ) theory with the light-cone momentum p^+ fixed.

In Sec. II we will describe examples of $\mathcal{N}=1$ orbifolds of $\mathbf{S}^5$. The first model is $\mathbf{S}^5/\mathbf{Z}_3$ whose Penrose limit was outlined in [12], for which we describe the string/field theory matching. As a second example we consider different boostings for the $\mathbf{S}^5/(\mathbf{Z}_k\times\mathbf{Z}_l)$ orbifold which can give an enlargement of supersymmetry from $\mathcal{N}=1$ to $\mathcal{N}=2$ or $\mathcal{N}=4$. In Sec. III we consider the Penrose limits of $\mathbf{T}^{1,1}/(\mathbf{Z}_k\times\mathbf{Z}_l)$ along the fixed circles of the quotienting action.

II. $\mathcal{N}=1$ ORBIFOLDS OF $\mathbf{S}^5$

A. Review of the $\mathrm{AdS}_5\times\mathbf{S}^5$ result

We start with a brief review of the result of [9], pointing out the features that we expect to get from the orbifold discussion.

Consider $\mathrm{AdS}_5\times\mathbf{S}^5$ where the anti–de Sitter space AdS_5 is represented as a universal covering of a hyperboloid of radius R in the flat space $\mathbf{R}^{2,4}$ and a sphere $\mathbf{S}^5$ of radius R in the flat space $\mathbf{R}^{0,6}$. One may regard the AdS_5 ($\mathbf{S}^5$) as a foliation of a timelike direction and a three-sphere Ω_3 (a circle parametrized by ψ and a three-sphere Ω_3'). Then the induced metric on $\mathrm{AdS}_5\times\mathbf{S}^5$ becomes

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$$ds^2 = R^2[-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\psi^2\cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2]. \quad (1)$$

One now considers the pp limit by boosting along the ψ direction around $\rho = 0$. The metric in this limit can be obtained by taking $R \to \infty$ after introducing coordinates

$$x^+ = \frac{1}{2}(t+\psi), \quad x^- = \frac{R^2}{2}(t-\psi) \quad (2)$$

and rescaling $\rho = r/R, \theta = y/R$ as follows:

$$ds^2 = -4dx^+dx^- - (\mathbf{r}\cdot\mathbf{r} + \mathbf{y}\cdot\mathbf{y})dx^{+2} + d\mathbf{y}^2 + d\mathbf{r}^2 \quad (3)$$

where $\mathbf{y}$ and $\mathbf{r}$ parametrize points on $\mathbf{R}^4$. Only the components of the RR five-form F with a plus index survive in this limit.

The energy is given by $E = i\partial_t$ and the angular momentum in the direction ψ is $J = -i\partial_\psi$ and the latter is seen as a generator that rotates a two-plane inside the original $\mathbf{R}^6$.

In terms of the dual $\mathcal{N}=4$ theory, the energy E is related to the conformal weight Δ and the angular momentum to the R charge. As discussed in [9], the relation between the oscillations of the string in the pp-wave geometry (3) and the field theory quantities is

$$(\Delta - J)_n = \sqrt{1 + \frac{4\pi g N n^2}{J^2}} \quad (4)$$

where N stands for the rank of the gauge theory and g is the string coupling constant. The vacuum has $\Delta - J = 0$.

In the $\mathcal{N}=4$ field theory, the interpretation of the string vacuum and of the string oscillators is made in terms of the gauge invariant operators. Consider the $\mathcal{N}=4$ multiplet in terms of a triplet of $\mathcal{N}=1$ multiplets, denoted by Z, Y^1, Y^2, the dimension of each field being 1. The complex field Z is on the directions whose rotation generator is J, so the value of J for the field Z is 1; therefore for the field Z we have $\Delta - J = 0$. The other fields Y^1, Y^2 (and their complex conjugates $\overline{Y^1}, \overline{Y^2}$) have $J = 0$ and $\Delta - J = 1$.

We can proceed to compare the stringy results with the field theory results. The string vacuum is given by $\mathrm{Tr}[Z^J]$ and the stringy oscillators are given by inserting $Y^1, Y^2, \overline{Y^1}, \overline{Y^2}$, i.e., the operators

$$\mathrm{Tr}[Z^{J-1}]Y^i, \quad \mathrm{Tr}[Z^{J-1}]\overline{Y^i}, \quad i = 1,2. \quad (5)$$

We can also have gauge invariant operators $\mathrm{Tr}[Z^{J-1}]\overline{Z}, \mathrm{Tr}[Z^{J-2}]Y^i\overline{Y^j}$, etc., but in [9] arguments have been given that such operators will get infinite mass.

B. String oscillators in the pp limit of the $\mathrm{AdS}_5 \times \mathrm{S}^5/\mathbf{Z}_3$

The geometry $\mathrm{AdS}_5 \times \mathbf{S}^5/\mathbf{Z}_3$ is obtained as a near horizon geometry of N D3-branes placed at a $\mathbf{C}^3/\mathbf{Z}_3$ orbifold. The generator g of $\mathbf{Z}_3$ acts on $\mathbf{C}^3$ by

$$g\cdot(z_1, z_2, z_3) \to (\omega z_1, \omega z_2, \omega z_3), \quad \omega^3 = 1. \quad (6)$$

We consider the boosting along the direction of the orbifolding which was studied in [12]. We need to consider a metric for the threefold covering of $\mathbf{S}^5$. As in [12], it is convenient to consider $\mathbf{S}^5$ as a Hopf fibration over $\mathbf{CP}^2$. The metric can be written as

$$ds^2 = (3d\psi + A)^2 + d^2_{\mathbf{CP}^2} \quad (7)$$

where $dA/2$ gives the Kähler class of $\mathbf{CP}^2$. As ψ ranges from 0 to 2π, we get a threefold of $\mathbf{S}^5$. More generally, we may take an orbifold theory on $\mathbf{C}^3/\mathbf{Z}_m$ where the generator g of $\mathbf{Z}_m$ acts on $\mathbf{C}^3$ by

$$g\cdot(z_1, z_2, z_3) \to (\omega^{a_1}, z_1, \omega^{a_2} z_2, \omega^{a_3} z_3), \quad \omega^m = 1,$$
$$a_1 + a_2 + a_3 = 0(\mathrm{mod\ m}), \quad a_i > 0. \quad (8)$$

Then the $\mathbf{S}^5$ is a Hopf fibration over a weighted projective space $\mathbf{CP}(a_1, a_2, a_3)$. As long as the null geodesic does not lie over the singular locus of the weighted projective space $\mathbf{CP}(a_1, a_2, a_3)$, there will no change in the argument.

We now choose the null coordinates as

$$x^+ = \frac{1}{2}\left(t + \frac{1}{3}\psi\right),$$
$$x^- = \frac{R^2}{2}\left(t - \frac{1}{3}\psi\right). \quad (9)$$

In the limit $R \to \infty$ and after rescaling the transversal direction $\mathbf{CP}^2$, we obtain the maximally supersymmetric pp-wave metric (3) as in [12]. The light-cone momenta can be written in terms of the conformal weight Δ and the angular momentum $J = -i\partial_\psi$:

$$2p^- = i\partial_{x^+} = i(\partial_t + 3\partial_\psi) = \Delta - 3J,$$
$$2R^2 p^+ = i\partial_{x^-} = i(\partial_t - 3\partial_\psi) = \Delta + 3J. \quad (10)$$

Before we describe the duality of string and field theory in the Penrose limit, we recall the results of [4,5] concerning the field theory on D3-branes at $\mathbf{C}^3/\mathbf{Z}_3$ singularities. By starting with $3N$ D3-branes in the covering space of $\mathbf{C}^3/\mathbf{Z}_3$ orbifold, the $\mathrm{SU}(3N)$ gauge group is broken to $\mathrm{SU}(N)^3$ by orbifold action on the Chan-Paton factors and there are three fields in the bifundamental representation for each pair of gauge groups, denoted by X_i, Y_i, Z_i, $i = 1,2,3$ (they come as $3\ N \times N$ blocks inside each of the $3N \times 3N$ matrices X, Y, Z describing the transversal motion of the D-branes). The surviving Kaluza-Klein (KK) modes are of the form [18]

$$\mathrm{Tr}(X_i^{m_1} Y_{i+1}^{m_2} Z_{i+2}^{m_3}), \quad m_1 + m_2 + m_3 = 0 \ (\mathrm{mod}\ 3),$$
$$i = 1,2,3 \ \ (\mathrm{mod}\ 3). \quad (11)$$

The quiver gauge theories have a quantum $\mathbf{Z}_3$ symmetry and the surviving KK modes have to be invariant under it. In the Penrose limit, the effect of the $\mathbf{Z}_3$ action on the transversal direction to the boosting direction disappears as the string probes an infinitesimally small neighborhood of the boosting circle parametrized by ψ. In the quantum vacua, the $\mathbf{Z}_3$ action remains along the boosting direction as we see in Eq. (10). In the orbifold theory $\mathbf{S}^5/\mathbf{Z}_3$, the global symmetry $SO(6)\approx SU(4)$ is broken up into $U(1)\times\mathbf{Z}_3$. Before the limit, the Hopf fibration is nontrivial, so even if the $\mathbf{Z}_3$ acts only along the Hopf fiber, this does not imply the breaking of global SO(6) isometry. In the pp limit, the fibration becomes trivial and it breaks the global symmetry SO(6) to $SO(4)\times SO(2)$, with SO(2) being in the boosting direction and SO(4) in the transverse directions.

To describe the string/field theory duality, we denote by Z the boosted direction and by X,Y the transverse directions where the orbifold does not act so X,Y do not enter in a gauge invariant form.[1] The action of the $\mathbf{Z}_3$ orbifold is only on the Hopf fiber parametrized by Z. We identify the scalar field along the Hopf fiber as $Z=Z_1Z_2Z_3$ where Z_i are the above fields in the bifundamental representation of $SU(N)_i \times SU(N)_{i+1}$, $i=1,2,3$. The field Z is in the adjoint representation of $SU(N)$ and has angular momentum in the U(1) direction equal to 3. The fields X,Y are also in the adjoint representation of the same $SU(N)$ and together with Z they form an $\mathcal{N}=4$ multiplet.

The vacuum of the string in the presence of $\mathbf{Z}_3$ is

$$\frac{1}{\sqrt{3J}N^{3J/2}}\,\mathrm{Tr}[Z^J]. \tag{12}$$

The first excited states are obtained by insertions of $X,Y,\overline{X},\overline{Y}$ for the string in the pp-wave background, these states being obtained by acting with a single oscillator on the ground states. Because there are eight bosonic zero-mode oscillators, we expect to find eight bosonic states with $\Delta-3J=1$. They are

$$\mathrm{Tr}[Z^JX],\quad \mathrm{Tr}[Z^J\overline{X}]\quad \text{or}\quad \mathrm{Tr}[Z^JY],\quad \mathrm{Tr}[Z^JY] \tag{13}$$

and the ones with the covariant derivative

$$\mathrm{Tr}[Z^JD_\mu Z]. \tag{14}$$

The nonsupergravity modes are obtained by acting with creation operators which imply the introduction of a position dependent phase, in addition to the above insertions [9].

Because we discuss the $\mathbf{Z}_3$ orbifold, we do not have a DLCQ limit as in [15,16], which holds only for $\mathbf{Z}_n$ with large n. Therefore, if we make the identification of the radius of the x^- direction as in [15,16],

$$\frac{\pi R^2}{n}=2\pi R_-, \tag{15}$$

where R^2 is approximately N (the rank of the gauge group), we see that when n is small the radius R_- of the x^- direction is infinite, so we are not allowed to use a discrete light cone quantization. There is no winding mode discussion for the $\mathbf{Z}_3$ orbifold and the insertions corresponding to the nonsupergravity modes are identical to the ones of [9].

An interesting case of supersymmetry enhancement was treated in [15,16] for the $\mathcal{N}=2$ orbifolds $\mathbf{S}^5/Z_n$. By boosting along the nonfixed directions of the orbifold, one gets a maximal $\mathcal{N}=4$ theory. One interesting related development would be to consider the supersymmetry enhancement when D3-branes probe backgrounds of D7/O7 planes [19,20]. The Penrose limit in the fixed direction(orthogonal to O7) was considered in [17] but the discussion of Penrose limits in the nonfixed directions still remains to be discussed. One step further in this direction would be to consider the Penrose limit for the case when D3 branes probe geometries with orthogonal D7 branes as in [20–22].

C. $\mathbf{Z}_m\times\mathbf{Z}_n$ orbifolds of $\mathbf{S}^5$

In this subsection we consider the geometry $\mathrm{AdS}_5\times\mathbf{S}^5/(\mathbf{Z}_m\times\mathbf{Z}_n)$ which is the near-horizon limit of the D3-branes placed at the tip of $\mathbf{C}^3/(\mathbf{Z}_m\times\mathbf{Z}_n)$.[2] The coordinates of $\mathbf{C}^3$ are z_1,z_2,z_3 and the generators g_m,g_n of $\mathbf{Z}_m,\mathbf{Z}_n$ act on (z_1,z_2,z_3) as

$$g_m:\quad (z_1,z_2,z_3)\to(e^{2\pi i/m}z_1,e^{-2\pi i/m}z_2,z_3), \tag{16}$$

$$g_n:\quad (z_1,z_2,z_3)\to(e^{2\pi i/n}z_1,z_2,e^{-2\pi i/n}z_3). \tag{17}$$

The singular points in the quotient are points left invariant under elements of the discrete group. The complex curve $z_1=z_2=0$, parametrized by z_3, is invariant under the $\mathbf{Z}_m$ and becomes a curve of $\mathbf{A}_{m-1}$ singularities, the complex curve $z_1=z_3=0$, parametrized by z_2, is invariant under the $\mathbf{Z}_n$ and becomes a curve of $\mathbf{A}_{n-1}$ singularities, and the complex curve $z_2=z_3=0$, parametrized by z_1, is invariant under the $\mathbf{Z}_r$, $r=\gcd(m,n)$, and becomes a curve of $\mathbf{A}_{r-1}$ singularities.

The field theory on D3-branes at $\mathbf{C}^3/(\mathbf{Z}_m\times\mathbf{Z}_n)$ singularity is $\mathcal{N}=1$ theory with gauge group $\Pi_{i=1}^m\Pi_{j=1}^n SU(N)_{(i,j)}$ and chiral bifundamentals [23,24]. The gauge invariant operators are

$$\mathrm{Tr}\,H_{(i,j)(i+1,j)}D_{(i+1,j)(i,j-1)}V_{(i,j-1)(i,j)} \tag{18}$$

where $H_{(i,j)(i+1,j)}$ are in the bifundamental representation of $SU(N)_{(i,j)}\times SU(N)_{(i+1,j)}$, $V_{(i,j)(i,j+1)}$ are in the bifundamental representation of $SU(N)_{(i,j)}\times SU(N)_{(i,j+1)}$, and $D_{(i+1,j+1)(i,j)}$ are in the bifundamental representation of $SU(N)_{(i+1,j+1)}\times SU(N)_{(i,j)}$. If D3-branes move to the points of $\mathbf{A}_{m-1}$ ($\mathbf{A}_{n-1}$ or $\mathbf{A}_{r-1}$) singularities described above, the field theory on the D3-branes becomes $\mathcal{N}=2$ with

[1] This set of X,Y,Z is different from the original complex coordinates of $\mathbf{C}^3$ in Eq. (11). But by a change of complex structures we may identify them as complex coordinates of the infinitesimal neighborhood of the boosting circle.

[2] This model was also discussed in [15].

gauge group $SU(N)^m$ [$SU(N)^n$ or $SU(N)^r$]. Hence there are flat directions in the $\mathcal{N}=1$ theory which connect it to an $\mathcal{N}=2$ theory.

In the $\mathbf{S}^5/(\mathbf{Z}_m\times\mathbf{Z}_n)$ geometry, there are many interesting directions along which we can consider the boosting, and the amount of the supersymmetry enhancement will depend on both the direction and the locality of the trajectories. We now classify the different possibilities.

Case 1. Boosting in the direction of the $\mathbf{Z}_m$ orbifolding (the same discussion holds for the direction of the $\mathbf{Z}_n$ or $\mathbf{Z}_r$ orbifolding).

We understand that the direction of $\mathbf{Z}_m$ orbifolding is a $U(1)$ direction in which $\mathbf{Z}_m$ is embedded. For this purpose, it is convenient to consider $\mathbf{S}^5$ as a foliation of the $\mathbf{S}^3$ in $\mathbf{C}^2$ with coordinates z_1,z_2 and the $\mathbf{S}^1$ in $\mathbf{C}^1$ with coordinates z_3. Furthermore, we consider $\mathbf{S}^3$ as a Hopf fibration over $\mathbf{CP}^1$ after changing the complex structure z_2 to $\overline{z_2}$ and the $\mathbf{Z}_3$ will locally act along the Hopf fiber. From this geometric description of $\mathbf{S}^5$, we obtain the metric for the $AdS_5\times\mathbf{S}^5$ as

$$dS^2_{\rm AdS}=R^2(-\cosh^2\rho dt^2+d\rho^2+\sinh^2\rho d\Omega_3^2),$$

$$\begin{aligned}ds^2_{\mathbf{S}^5}=R^2\Bigg(&d\theta^2+\sin^2\theta d^2_{\mathbf{S}^1}+\cos^2\theta\\&\times\Bigg\{\left[d\tau+\frac{1}{2}(\cos\chi-1)d\phi\right]^2\\&+\frac{1}{4}(d\chi^2+\sin^2\chi d\phi^2)\Bigg\}\Bigg).\end{aligned} \tag{19}$$

where τ is the coordinate for the fiber direction and $d\chi^2+\sin^2\chi d\phi^2$ is the metric for the base $\mathbf{CP}^1$ in the Hopf fibration of $\mathbf{S}^3$.

We need to consider an m-fold cover of $\mathbf{S}^5$. As the string probes only an infinitesimal neighborhood of the boosting direction, the action on the directions transverse to the Hopf fiber is not seen. For simplicity we take an m-fold covering of the $\mathbf{S}^5$ where the $\mathbf{S}^3$ part of the metric changes to

$$\frac{1}{m^2}\left(d\tau+\frac{m}{2}(\cos\chi-1)d\phi\right)^2+\frac{1}{4}(d\chi^2+\sin^2\chi d\phi^2). \tag{20}$$

We now choose the null coordinates as

$$x^+=\frac{1}{2}\left(t+\frac{\tau}{m}\right),\quad x^-=\frac{R^2}{2}\left(t-\frac{\tau}{m}\right), \tag{21}$$

and consider a scaling limit $R\to\infty$ around $\theta=\chi=0$ with

$$\rho=\frac{r}{R},\quad \theta=\frac{u}{R},\quad \chi=\frac{v}{R}. \tag{22}$$

In this limit, the metric becomes

$$\begin{aligned}ds^2=&-(R^2+r^2)dt^2+dr^2+r^2d^2\Omega_3+du^2+u^2d^2_{\mathbf{S}^1}\\&+R^2\left(1-\frac{u^2}{R^2}\right)\left(\frac{d\tau^2}{m^2}-\frac{v^2}{2mR^2}d\phi d\tau+\frac{v^4}{16R^4}d\phi^2\right.\\&\left.+\frac{dv^2+v^2d\phi^2}{4R^2}\right)\\=&-4dx^+dx^-+dr^2+r^2d\Omega_3^2-r^2dx^{+2}+du^2+u^2d^2_{\mathbf{S}^1}\\&-\frac{v^2dx^+d\phi}{2}-u^2dx^{+2}+\frac{dv^2+v^2d\phi^2}{4}\\=&-4dx^+dx^--\left(r^2+u^2+\frac{v^2}{4}\right)dx^{+2}+dr^2+r^2d\Omega_3^2\\&+du^2+u^2d^2_{\mathbf{S}^1}+\frac{dv^2+v^2d\phi'^2}{4}\end{aligned} \tag{23}$$

where $\phi'=\phi-x^+$.

After replacing v by $2v$, we go to the rectangular coordinate system and rewrite the metric as

$$ds^2=-4dx^+dx^--(\mathbf{r}^2+\mathbf{u}^2+\mathbf{v}^2)dx^{+2}+d\mathbf{r}^2+d\mathbf{u}^2+d\mathbf{v}^2. \tag{24}$$

The pp wave has a natural decomposition of the $\mathbf{R}^8$ transverse space into $\mathbf{R}^4\times\mathbf{R}^2\times\mathbf{R}^2$ where $\mathbf{R}^4$ is parametrized by $\mathbf{r}$ and the $\mathbf{R}^2\times\mathbf{R}^2$ by $\mathbf{u}$ and $\mathbf{v}$, respectively. The covariantly constant flux of the RR field is on $(x^+,\mathbf{r})$ and $(x^+,\mathbf{u},\mathbf{v})$. In this geometry, the light cone momenta are

$$2p^-=i\partial_{x^+}=i(\partial_t+m\partial_\tau)=\Delta-mJ,$$

$$2R^2p^+=i\partial_{x^-}=i(\partial_t-m\partial_\tau)=\Delta+mJ. \tag{25}$$

The effective angular momentum in the boosting direction is mJ and this is the quantity which should be large in the Penrose limit of the AdS/CFT correspondence. Therefore we have two options, the first one being to consider a discrete orbifold group $\mathbf{Z}_m$ with a very large m and finite J and the second a discrete orbifold group $\mathbf{Z}_m$ with finite m and with a large value for J [15,16].

In the field theory, the supersymmetry is enlarged from $\mathcal{N}=1$ to $\mathcal{N}=4$ and the corresponding global symmetries are $SO(2)$ in the boosted direction and $SO(4)$ in the direction transverse to the boosting. To identify the gauge invariant operators, we need to use the fact that we boost along the direction of the $\mathbf{Z}_m$ orbifolding and the rest of the space is invariant. The direction of the boosting is denoted by Z and, as in the previous subsection, we denote the coordinates transverse to the boosting by X and Y. In terms of the fields of the $\mathcal{N}=1$ theory, Z should be in a gauge invariant form and is written as a product $Z=\Pi_{i=1}^m Z_i$ where Z_i are either $H_{(i,j)(i+1,j)}$ for fixed j, $D_{(i+1,j+1)(i,j)}$ for fixed j, or

$V_{(i,j)(i,j+1)}$ for fixed i. The above fields are in the bifundamental representation of $\mathrm{SU}(N)_i \times \mathrm{SU}(N)_{i+1}$; the field Z transforms in the adjoint representation of the group $\mathrm{SU}(N)$. Together with the scalar fields denoting the transverse direction, X and Y, they form an $\mathcal{N}=4$ multiplet.

The coupling of the $\mathrm{SU}(N)$ gauge theory is of order $g_{YM}^2 = g_s m$ and the effective 't Hooft parameter is $g_{YM}^2 N/J^2 m^2$ which is finite, being of the order of $g_s Nm/R^4$, which is finite. Therefore we can treat the $\mathrm{SU}(N)$ gauge theory perturbatively.

We can now proceed to describe the gauge invariant operators corresponding to the stringy ground state and excitations. The gauge invariant operators $\mathrm{Tr}(Z^J)$ have angular momentum mJ in the boosted direction due to the action of $\mathbf{Z}_m$, and this corresponds to the vacuum of the string theory. To describe the excitations, we need to consider the two cases discussed above, i.e., when m is either small or large.

For the case of small m and large J, the first level eight bosonic zero-mode oscillators are

$$\mathrm{Tr}(Z^J X), \quad \mathrm{Tr}\ (Z^J Y), \quad \mathrm{Tr}\ (Z^J \bar{X}), \quad \mathrm{Tr}(Z^J \bar{Y}) \tag{26}$$

together with $\mathrm{Tr}(Z^J D_\mu Z)$. In this case x^- is not compact as it was for the $\mathbf{S}^5/\mathbf{Z}_3$ case discussed in the previous section. The insertions of $X,Y,\bar{X},\bar{Y}$ should be made as $\mathrm{Tr}(Z^l X Z^{J-l})$, etc. The nonsupergravity oscillations are obtained by introducing extra phases in the above operators.

More interesting is the case when m is very large and the light cone is compact with radius $\pi R^2/m$, the light-cone momentum being quantized as $2p^+ = m/R_-$. The string theory has a matrix string description which mimics that of the flat space as pointed out in [15,25]. In [15] string propagation in the DLCQ pp wave was considered and the states were labeled by two quantum numbers, the first being the DLCQ momentum k and the second being the winding number m in the x^- direction.

The vacuum corresponds to $\mathrm{Tr}(Z^J)$ which has $2p^+ = m/R_-$ and zero winding number. As J is finite, we can consider $J=1$. The insertions of the fields $X,Y,\bar{X},\bar{Y}$ should now be made into the trace of the string of Z_i fields. To do this, we also need to consider the splitting of the matrices X,Y into $mN\times N$ blocks, each one being inserted in m different positions and then a summation over position is required to ensure gauge invariance. In terms of the original $\mathcal{N}=1$ theory, if we choose Z_i to be the fields $H_{(i,j)(i+1,j)}$ for fixed j, then the fields X and Y are built of $mN\times N$ blocks which can be either $D_{(i+1,j+1)(i,j)}$ for fixed j (we denote these by X_i) or $V_{(i,j)(i,j+1)}$ for fixed i (we denote these by Y_i. By choosing the Z_i transform in the bifundamental representation of $\mathrm{SU}(N)_i \times \mathrm{SU}(N)_{i+1}$, the result is that X_i transform in the bifundamental of $\mathrm{SU}(N)_{i+1}\times \mathrm{SU}(N)_i$ and Y^i are in the adjoint representation of $\mathrm{SU}(N)_i$. Therefore the fields X_i should be inserted between Z_i and Z_i and the fields Y_i should be inserted between Z_{i-1} and Z_i. The first oscillators with zero winding number will then be

$$\sum_{i=1}^m \mathrm{Tr}(Z_1 Z_2 \cdots Z_i X_i Z_i \cdots Z_m),$$

$$\sum_{i=1}^m \mathrm{Tr}(Z_1 Z_2 \cdots Z_{i-1} Y_i Z_i \cdots Z_m), \tag{27}$$

and

$$\sum_{i=1}^m \mathrm{Tr}(Z_1 Z_2 \cdots Z_{i-1} \bar{X}_i Z_{i+1} \cdots Z_m),$$

$$\sum_{i=1}^m \mathrm{Tr}(Z_1 Z_2 \cdots Z_{i-1} \bar{Y}_i Z_i \cdots Z_m), \tag{28}$$

where the summation over i ensures the gauge invariance. The states that have winding numbers are built with an additional factor $e^{2\pi i/m}$ in the above formulas.

In this form, the stringy operators have an expansion which is similar to the Kaluza-Klein expansion of a generic field of five-dimensional theory reduced on a circle used in [26,27] to conjecture the deconstruction of a five-dimensional theory for large m quiver theories in four dimensions. Our $\mathbf{S}^5/(\mathbf{Z}_m\times\mathbf{Z}_n)$ model should actually be related to a (1,1) theory in six dimensions [27], but we expect to get a five-dimensional theory as long as we boost along the orbifolding directions. The two directions needed to deconstruct a six-dimensional theory are obtained in different boostings, one discussed in this subsection and the other discussed in the next subsection.

The conclusion is that a fast moving particle in the τ direction reduces the gauge group to $\mathrm{SU}(N)$ and enhances the supersymmetry from $\mathcal{N}=1$ to $\mathcal{N}=4$.

Case 2. Boosting in the direction of the fixed locus of the $\mathbf{Z}_m$ orbifolding (the same discussion holds for the $\mathbf{Z}_n$ or $\mathbf{Z}_r$ orbifolding).

We take the same form of the metric as in Eq. (19), we parametrize the angle of $\mathbf{S}^1$ by ψ, the phase of z_3, and we boost along the ψ direction. Since $\mathbf{Z}_n$ acts on z_3, we take an n-fold covering of $\mathbf{S}^5$, replacing ψ by ψ/n in the metric.

The metric for the spherical part is

$$R^2\left\{ d\theta^2 + \frac{1}{n^2}\sin^2\theta d\psi^2 + \cos^2\theta\left[\left(d\tau + \frac{1}{2}(\cos\chi - 1)d\phi\right)^2 + \frac{1}{4}(d\chi^2 + \sin^2\chi d\phi^2)\right]\right\}. \tag{29}$$

We introduce the null coordinates

$$x^+ = \frac{1}{2}\left(t + \frac{\psi}{n}\right), \quad x^- = \frac{R^2}{2}\left(t - \frac{\psi}{n}\right), \tag{30}$$

and consider a scaling limit $R\to\infty$ around $\theta = \pi/2$ with

$$\rho = \frac{r}{R}, \quad \theta - \frac{\pi}{2} = \frac{u}{R}. \tag{31}$$

The computation is essentially the same as in [14]. The transversal $\mathbf{S}^3$ part of the metric $\{[d\tau+\frac{1}{2}(\cos\chi-1)d\phi]^2+\frac{1}{4}(d\chi^2+\sin^2\chi d\phi^2)\}$ is left intact in this limit and hence the $\mathbf{Z}_m$ action remains. The result is a metric that is similar to a maximally supersymmetric one, the difference being the $\mathbf{Z}_n$ action (seen from the range of ψ), which tells us that the supersymmetry preserved is $\mathcal{N}=2$ and not $\mathcal{N}=4$.

We now denote the scalar field parametrizing the boosted direction by $z_3=Z$ and the scalar fields parametrizing the transverse directions by $z_1=X, z_2=Y$. The $\mathbf{Z}_m$ discrete group acts now on X,Y,Z as

$$X\to e^{2\pi i/m}X,\quad Y\to e^{-2\pi i/m}Y,\quad Z\to Z, \tag{32}$$

and there is also an action of the $\mathbf{Z}_n$ discrete group on the boosting direction:

$$Z\to e^{-2\pi i/n}Z. \tag{33}$$

Because of the last action, the field Z should enter at the power n, and this is obtained if we consider that Z is a product of the $\mathcal{N}=1$ fields $V_{(ij)(ij+1)}$ for fixed i. We introduce the notation

$$Z^n=V_{(i,j)(i,j+1)}V_{(i,j+1)(i,j+2)}\cdots V_{(i,j+n-1)(i,j+n)} \tag{34}$$

where $j=j+n \pmod n$. The field Z^n is in the adjoint representation of $\mathrm{SU}(N)_{i,j}$ for fixed i,j. For future use, we also introduce the notation $Z_j=V_{(i,j)(i,j+1)}$.

In this case the field theory after the boosting becomes $\mathcal{N}=2\Pi_{i=1}^m\,\mathrm{SU}(N)_i$, the gauge coupling constants of the gauge groups are of order $g_{YM}^2=g_s n$, and the effective 't Hooft parameters are g_{YM}^2N/J^2n^2, which are finite, being of the order of g_sNm/R^4.

Because of the $\mathbf{Z}_m$ projection, the field Z is actually promoted to an $mN\times mN$ matrix, with $mN\times N$ blocks, each block being in the adjoint representation of $\mathrm{SU}(N)_{i,j}$. Together with the corresponding vectors of $\mathrm{SU}(N)_{i,j}$, they form $\mathcal{N}=2$ multiplets. The effective angular momentum in the boosting direction for $\mathrm{Tr}\,Z^J$ being nJ, we again have two choices, one when n is small and the other when n is big.

Consider first the case when n is small. The vacuum of the string theory corresponds to the $\mathbf{Z}_m$ invariant operators

$$\frac{1}{\sqrt{mJ}}\ \mathrm{Tr}[S^qZ^{nJ}] \tag{35}$$

where $S=(1,e^{2\pi i/m},\dots,e^{2\pi i(m-1)/m})$ denotes the qth twisted sector. The oscillations of the string belong to the untwisted modes, which are of the type

$$\mathrm{Tr}[S^qZ^{nJ}D_\mu(Z)] \tag{36}$$

and

$$\mathrm{Tr}[S^qZ^{nJ}\chi] \tag{37}$$

where D_μ is the covariant derivative and χ is the supersymmetric partner of the scalar Z. The scalar fields X and Y are now $mN\times mN$ matrices with m $N\times N$ extra diagonal blocks denoted by X_i and Y_i, each one transforming in the bifundamental representation of the group $\mathrm{SU}(N)_{ij}\times\mathrm{SU}(N)_{i+1j}$. For the twisted sectors we need to consider states built with oscillators with fractional modes. These are obtained by multiplying with X and Y, which are acted upon by the $\mathbf{Z}_m$ group, together with a position independent phase factor $e^{(2\pi i/J)n(q)}$ when inserting X,Y and $e^{(2\pi i/J)n(-q)}$ for insertions of $\overline{X},\overline{Y}$.

The discussion changes when m is very large. In this case we have a compact light cone with radius $\pi R^2/n$ and the light-cone momentum is quantized as $2p^+=n/R_-$. The vacuum and the oscillations of the string belonging to the untwisted modes are the same as before, but we have a change in the definition of the oscillations of the twisted sectors. The insertions of X and Y should now be made between Z_{j-1} and Z_j. To do this, we have to consider all the blocks X_i,Y_i, $i=1,\dots,m$, as $nN\times nN$ matrices and split each one of them into n diagonal $N\times N$ blocks denoted by X_{ij},Y_{ij}, $i=1,\dots,n$, for fixed i. The insertions will then be

$$\sum_{j=1}^n \mathrm{Tr}(Z_1\cdots Z_{j-1}X_{ij}Z_j\cdots Z_n) \tag{38}$$

or

$$\sum_{j=1}^n \mathrm{Tr}(Z_1\cdots Z_{j-1}Y_{ij}Z_j\cdots Z_n) \tag{39}$$

where j denotes the insertion and i denotes the twisted sector. The winding modes are obtained by using the same formulas with an extra $e^{1\pi i/n}$ factor.

The conclusion is that a fast moving particle moving in the ψ direction reduces the gauge group to $\Pi_{i=1}^m\,\mathrm{SU}(N)_i$ and enhances the supersymmetry from $\mathcal{N}=1$ to $\mathcal{N}=2$.

Case 3. Boosting in a general direction which is neither *Case 1* nor *Case 2*.

In this case both discrete groups $\mathbf{Z}_m$ and $\mathbf{Z}_n$ are in the direction of the boosting and the string probes only a small strip along this direction; therefore there is no orbifold action on the scalar fields and the result is a maximally supersymmetric Penrose limit. Because we do not have any orbifold projection on the three scalar fields Z,X,Y, the situation is similar to moving the D3-brane from the tip of the $\mathbf{Z}_m\times\mathbf{Z}_n$ orbifold into the bulk, when the supersymmetry is changed from $\mathcal{N}=1$ to $\mathcal{N}=4$. The string/field theory duality then reduces to that of Sec. II A. There is no change in the angular momentum in the boosted directions due to the orbifolding.

We have identified several boosting directions which imply an enlargement of supersymmetry. Three directions are along the $\mathbf{Z}_m$, $\mathbf{Z}_n$, or $\mathbf{Z}_r$ orbifolding, which give maximally supersymmetric pp limits, three directions are along the fixed loci of $\mathbf{Z}_m$, $\mathbf{Z}_n$, or $\mathbf{Z}_r$ orbifolds, which give $\mathcal{N}=2$ supersymmetry, and an infinite number of boosting directions are along a general direction, which would give $\mathcal{N}=4$.

The discussion is different for large m,n as compared to the case of small m,n. For the first case we get a compact light cone and this can be used to describe the pp wave as the limit of a DLCQ theory with fixed p^+. In terms of the choice of boosting, we get a specific circle so we get a two-dimensional torus when both m and n are large. These two directions are the ones used in [27] to describe the deconstruction of the six-dimensional (1,1) theories.

D. Correlation functions and supersymmetry enhancement

In [28], a detailed analysis was made for the anomalous dimensions and three-point functions for the chiral and almost chiral operators introduced in [9] (see also [29] for a similar discussion). In particular, the authors of [28] identified the parameter $g_2 = J^2/N$ as the genus counting parameter in the free Yang-Mills theory, so the correlation function of $\mathrm{Tr}\,\bar{Z}^J$ and $\mathrm{Tr}\,ZJ$ has a contribution $JN^J g_2^{2h}$ from the genus h Feynman diagram.

We want to see what happens for the pp-wave limits of orbifold theories. To do this, we start from the observation that the correlation functions for the orbifold theories coincide with those of $\mathcal{N}=4$ theory, modulo the rescaling of the gauge coupling constant, as observed in [30] with string theory methods and [31] using field theory methods. If we consider the case $\mathbf{S}^5/\mathbf{Z}_n$, in the $\mathcal{N}=2$ theory we have a factor of $1/n$ in front of the correlation functions. After the Penrose limit in the nonfixed direction, we go from the orbifold theory to the covering space and therefore the factor $1/n$ disappears. The correlation functions for the orbifold theories are then expected to have a similar expansion in the genus as in the $\mathbf{S}^5$ case. It would be interesting to show this in detail, by analogous computations to those of [30,31].

III. THE $\mathcal{N}=1$ ORBIFOLDS OF $\mathbf{T}^{1,1}$

The case of D3-branes at the conifold or at orbifolds of the conifold has been discussed extensively in the literature [6,32–35]. The conifold is a three-dimensional hypersurface singularity in $\mathbf{C}^4$ defined by

$$z_1 z_2 - z_3 z_4 = 0 \tag{40}$$

which is a metric cone over the five-dimensional Einstein manifold $\mathbf{T}^{1,1} = \mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)$. The conifold can be realized as a holomorphic quotient of $\mathbf{C}^4$ by the $\mathbf{C}^*$ action given by [6]

$$(A_1, A_2, B_1, B_2) \to (\lambda A_1, \lambda A_2, \lambda^{-1} B_1, \lambda^{-1} B_2). \tag{41}$$

The map

$$z_1 = A_1 B_1, \quad z_2 = A_2 B_2, \quad z_3 = A_1 B_2, \quad z_4 = A_2 B_1 \tag{42}$$

provides an isomorphism between these two representations of the conifold. The horizon $\mathbf{T}^{11}$ can be identified with the $|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = 1$ quotient by a U(1) action induced by Eq. (41). Following [32], we can parametrize A_i, B_i in terms of Euler angles of $\mathrm{SU}(2)\times\mathrm{SU}(2)$:

$$A_1 = \cos\frac{\theta_1}{2}\exp\frac{i}{2}(\psi_1 + \phi_1), \quad A_2 = \sin\frac{\theta_1}{2}\exp\frac{i}{2}(\psi_1 - \phi_1),$$
$$B_1 = \cos\frac{\theta_2}{2}\exp\frac{i}{2}(\psi_2 + \phi_2), \quad B_2 = \sin\frac{\theta_2}{2}\exp\frac{i}{2}(\psi_2 - \phi_2), \tag{43}$$

and the U(1) is diagonally embedded in $\mathrm{SU}(2)\times\mathrm{SU}(2)$. After taking a further quotient by the remaining U(1) factor of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, we obtain a product of two projective spaces $\mathbf{CP}^1_1 \times \mathbf{CP}^1_2$, and may identify the parameters θ_i, ϕ_i with the spherical coordinates of $\mathbf{CP}^1_i$ for $i = 1,2$. Now $\mathbf{T}^{1,1}$ is a U(1) fibration over $\mathbf{CP}^1_1 \times \mathbf{CP}^1_2$ and the U(1) fiber can be parametrized by $\psi := \psi_1 + \psi_2$. The Einstein metric on $\mathbf{T}^{1,1}$ of radius R is

$$ds^2_{\mathbf{T}^{1,1}} = R^2\left[\frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)\right]. \tag{44}$$

Consider an orbifold theory of the conifold where the discrete group $\mathbf{Z}_m \times \mathbf{Z}_n$ acts on A_i, B_j by

$$(A_1, A_2, B_1, B_2) \to (e^{-2\pi i/m} A_1, A_2, e^{2\pi i/m} B_1, B_2) \tag{45}$$

and

$$(A_1, A_2, B_1, B_2) \to (e^{-2\pi i/n} A_1, A_2, B_1, e^{2\pi i/n} B_2). \tag{46}$$

The action (45) descends to the horizon $\mathbf{T}^{1,1}$ and yields two fixed circles $|A_2|^2 = |B_2|^2 = 1, A_1 = B_1 = 0$ [mod U(1)] and $|A_1|^2 = |B_1|^2 = 1, A_2 = B_2 = 0$ [mod U(1)] [35]. Similarly, the action (46) yields two fixed circles $|A_2|^2 = |B_1|^2 = 1, A_1 = B_2 = 0$ [mod U(1)] and $|A_1|^2 = |B_2|^2 = 1, A_2 = B_1 = 0$ [mod U(1)].The horizon $\mathbf{T}^{11}/(\mathbf{Z}_m \times \mathbf{Z}_n)$ is singular along these circles, having an $\mathbf{A}_{m-1}$ singularity along the first two circles and an $\mathbf{A}_{n-1}$ singularity along the last two circles. The discrete group $\mathbf{Z}_m \times \mathbf{Z}_n$ breaks the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ part of the isometry group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ of $\mathbf{T}^{1,1}$ and the U(1) part remains as the global R symmetry.

In terms of Euler angles of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, the discrete group $\mathbf{Z}_m \times \mathbf{Z}_n$ action is given by

$$(\psi_1, \phi_1, \psi_2, \phi_2) \to (\psi_1 - 2\pi i/m, \phi_1 - 2\pi i/m, \psi_2 + 2\pi i/m, \phi_2 + 2\pi i/m),$$
$$(\psi_1, \phi_1, \psi_2, \phi_2) \to (\psi_1 - 2\pi i/n, \phi_1 - 2\pi i/n, \psi_2 + 2\pi i/n, \phi_2 - 2\pi i/n). \tag{47}$$

What we see from the above equations is that the coordinate of the U(1) fiber ($\psi = \psi_1 + \psi_2$) is left invariant under the action of $\mathbf{Z}_m \times \mathbf{Z}_n$, as it should be in order to preserve the $\mathcal{N}=1$ supersymmetry.

Now we study the Penrose limits of $\mathrm{AdS}_5 \times \mathbf{T}^{1,1}/\mathbf{Z}_m \times \mathbf{Z}_n$. The metric for $\mathrm{AdS}_5 \times \mathbf{T}^{1,1}$ is

$$ds^2=R^2\left[-\cosh^2\rho dt^2+d\rho^2+\sinh^2\rho d\Omega_3^2\right.$$
$$+\frac{1}{9}(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2)^2$$
$$\left.+\frac{1}{6}(d\theta_1^2+\sin^2\theta_1 d\phi_1^2+d\theta_2^2+\sin^2\theta_2 d\phi_2^2)\right]. \quad (48)$$

The Penrose limit for the conifold was studied in [11–13]. As in the previous section, there are many directions of boosting. We want to study the boosting along the fixed locus of the discrete group action. Consider first the boosting along the circle $|A_1|^2=|B_1|^2=1, A_2=B_2=0$ [mod U(1)] which is a fixed locus of the $\mathbf{Z}_m$ action. In terms of the parameters used in Eq. (44), this is located at $\theta_1=\theta_2=0$ and can be parametrized by $\psi+\phi_1+\phi_2$. Because of the action of $\mathbf{Z}_n$, we are actually dealing with an n-covering of $\mathbf{T}^{1,1}$ whose metric looks the same as before:

$$\frac{1}{9}(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2)^2$$
$$+\frac{1}{6}(d\theta_1^2+\sin^2\theta_1 d\phi_1^2+d\theta_2^2+\sin^2\theta_2 d\phi_2^2) \quad (49)$$

but the ranges of ϕ_i and ψ are $1/n$ of the initial ones.

We introduce the null coordinates

$$x^+=\frac{1}{2}\left(t+\frac{1}{3}(\psi+\phi_1+\phi_2)\right),$$
$$x^-=\frac{R^2}{2}\left(t-\frac{1}{3}(\psi+\phi_1+\phi_2)\right), \quad (50)$$

and consider a scaling limit $R\to\infty$ around $\theta_1=\theta_2=0$ with

$$\rho=\frac{r}{R}, \quad \theta_i=\frac{\sqrt{6}}{R}\xi_i, \quad i=1,2, \quad (51)$$

and in the limit $R\to\infty$, the metric becomes

$$ds^2=-4dx^+dx^- - r^2dx^{+2}+dr^2+r^2d\Omega_3^2$$
$$+\sum_{i=1,2}(d\xi_i^2+\xi_i^2 d\phi_i^2-2\xi_i^2 d\phi_i dx^+)$$
$$=-4dx^+dx^-+d\mathbf{r}^2-(\mathbf{r}\cdot\mathbf{r}+\mathbf{w}\cdot\overline{\mathbf{w}})dx^{+2}+d\mathbf{w}d\overline{\mathbf{w}} \quad (52)$$

where $\mathbf{w}=(\xi_1 e^{i(\phi_1-x^+)},\xi_2 e^{i(\phi_2-x^+)})$.

In Eq. (52) and in the definition of $\mathbf{w}$, we understand that the range of ϕ_i is $(0,2\pi/n)$. The change in the range of ϕ_i from $(0,2\pi)$ to $(0,2\pi/n)$ is due to a remnant $\mathbf{Z}_n$ orbifolding and the presence of this orbifolding is telling us that the theory is not maximally supersymmetric but $\mathcal{N}=2$ supersymmetric.

The same discussion can be extended to the other fixed circles $A_1=B_1=0$, $A_1=B_2=0$, and $A_2=B_1=0$, where the boosting is again in the direction $\psi+\phi_1+\phi_2$, but around $\theta_1=\theta_2=\pi$, $\theta_1=\pi,\theta_2=0$, and $\theta_1=0,\theta_2=\pi$, respectively. The Penrose limit will be identical with Eq. (52) after redefining $\mathbf{w}$ appropriately. The transverse space $\mathbf{R}^8$ decomposes into a product of R^4 which is in the r^i direction and $\mathbf{C}^2$ whose coordinates are given by w_1 and w_2. We now investigate the effect of the orbifolding on the geometry of the pp limit. Note that if we project the conifold (40) in $\mathbf{C}^4$ to $\mathbf{C}^3$ by $(z_1,z_2,z_3,z_4)\to(z_1,z_3,z_4)$, we can identify the boosting direction of $\mathbf{T}^{1,1}$ with the angular direction of z_1, which is parametrized by $(1/2)(\psi+\phi_1+\phi_2)$ as in Eqs. (42) and (43), and the transversal space $\mathbf{C}^2$ can be parametrized by z_3 and z_4. At the pp limit, $\mathbf{Z}_n$ acts on the boosting direction as

$$z_1\to e^{-2\pi i/n}z_1 \quad (53)$$

and on the transversal direction trivially, and, on the other hand, $\mathbf{Z}_m$ acts on the transversal direction as

$$(z_3,z_4)\to(e^{-2\pi i/m}z_3,e^{2\pi i/m}z_4) \quad (54)$$

and acts trivially on the boosting direction z_1 which is along the circle of boosting. In terms of the coordinate of the boosting direction, there is a $\mathbf{Z}_n$ action on $\tilde{\psi}=\psi+\phi_1+\phi_2$ as

$$\tilde{\psi}\to\tilde{\psi}-\frac{\pi i}{n}. \quad (55)$$

On the string theory side, our conclusion is that, by boosting along the fixed angular directions with respect to the $\mathbf{Z}_m$ or $\mathbf{Z}_n$ actions, we get half-maximally supersymmetric solutions which should correspond to $\mathcal{N}=2$ supersymmetric gauge theories.

We now go to the gauge theories and identify the gauge invariant operators which are dual to the string modes in the above pp-wave geometry. The transverse space is $\mathbf{S}^5/\mathbf{Z}_n$ or $\mathbf{S}^5/\mathbf{Z}_m$, so the field theory is $\mathcal{N}=2$, $\Pi_{i=1}^n \mathrm{SU}(N)_i$ or $\Pi_{i=1}^m \mathrm{SU}(N)_i$. Before the boosting the field theory is $\mathcal{N}=1$:

$$\prod_{i=1}^m\prod_{j=1}^n \mathrm{SU}(N)_{ij}\times\prod_{i=1}^m\prod_{j=1}^n \mathrm{SU}(N)'_{ij}, \quad (56)$$

and there are bifundamental fields $(A_1)_{i,j;i,j}$ in $\mathrm{SU}(N)_{i,j}\times\mathrm{SU}(N)'_{i,j}$, $(A_2)_{i+1,j+1;i,j}$ in $\mathrm{SU}(N)_{i+1,j+1}\times\mathrm{SU}(N)'_{i,j}$, $(B_1)_{i,j;i,j+1}$ in $\mathrm{SU}(N)'_{i,j}\times\mathrm{SU}(N)_{i,j+1}$, and $(B_2)_{i,j;i+1,j}$ in $\mathrm{SU}(N)'_{i,j}\times\mathrm{SU}(N)'_{i+1,j}$. The products of A_i,B_j, which enter in the definitions of z_1,z_3,z_4 are $(A_1B_1)_{i,j;i,j+1}$ in $\mathrm{SU}(N)_{i,j}\times\mathrm{SU}(N)_{i,j+1}$, $(A_1B_2)_{i,j;i+1,j}$ in $\mathrm{SU}(N)_{i,j}\times\mathrm{SU}(N)_{i+1,j}$, and $(A_2B_1)_{i+1,j+1;i,j+1}$ in $\mathrm{SU}(N)_{i+1,j+1}\times\mathrm{SU}(N)_{i,j+1}$.

We want to see the change in field theory after the boosting. There are four possible particular cases of particular gauge groups which correspond to D3-branes at four-dimensional $\mathbf{A}_{m-1}$ or $\mathbf{A}_{n-1}$ singularities, and the field theory becomes $\mathcal{N}=2$, $\Pi_{i=1}^m \mathrm{SU}(N)_{ij}$ or $\Pi_{i=1}^m \mathrm{SU}(N)'_{ij}$ for fixed j and $\Pi_{j=1}^n \mathrm{SU}(N)_{ij}$ or $\Pi_{j=1}^n \mathrm{SU}(N)'_{ij}$ for fixed i. The chiral primaries are constructed from sums of gauge invariant products of chiral superfields, modulo F- and D-flatness conditions [34]. They are products of the form $A_{i_1}B_{j_1}A_{i_2}B_{j_2}\cdots A_{i_{mn}}B_{j_{mn}}$, symmetrized in A_i and B_j. For fixed i, a particular example of a chiral primary involving only A_1 and B_1 fields is

$$\begin{aligned}\mathrm{Tr}[&(A_1)_{i,j;i,j}(B_1)_{i,j;i,j+1}(A_1)_{i,j+1;i,j+1}\\ &\times(B_1)_{i,j+1;i,j+2}\cdots(A_1)_{i,j+n-1;i,j+n-1}\\ &\times(B_1)_{i,j+n-1;ij+n}]\end{aligned} \tag{57}$$

where $j+n=j \pmod n$ so the trace is taken over the adjoint representation of $\mathrm{SU}(N)_{i,j}$. The R charges of the fields A_i, B_i are not changed by the quotienting so the R charge of the gauge invariant operator (57) is n.

We now relate the field theory R charge to the other U(1) charges that appear in the field theory and geometry. In the geometry we have two rotation charges for the $\mathrm{U}(1)\times\mathrm{U}(1)$ isometry group which are denoted by J_1 and J_2, and they are related to the Cartan generators of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ global symmetry of the dual superconformal field theory by [11,12]

$$J_a=-i\frac{\partial}{\partial\phi_a}\Big|_{x^\pm}=-i\frac{\partial}{\partial\phi_a}\Big|_{t,\psi}+i\frac{\partial}{\partial\psi}\Big|_{t,\phi_i}=Q_a-\frac{1}{2}R$$
$$a=1,2. \tag{58}$$

Because of the $\mathbf{Z}_n$ action on the fixed circle of the quotiented conifold, the above relation becomes

$$nJ_a=nQ_a-\frac{R}{2}. \tag{59}$$

We use the convention that A_1 has $Q_1=\frac{1}{2}$ and B_1 has $Q_2=\frac{1}{2}$. In [11–13], the vacuum of the string theory has been identified with the state $J_1=J_2=0$ and the first oscillations of the strings with $J_1=\pm1, J_2=0$ and $J_1=0, J_2=\pm1$.

Consider now boosting along the z_1 direction. We want to identify the gauge invariant operators which correspond to the string theory ground state and first oscillation modes. In the case of the conifold, the ground state $J_1=J_2=0$ was identified with the gauge theory operators [11–13]

$$\mathrm{Tr}(A_1B_1)^J, \tag{60}$$

the first oscillations $J_1=-1, J_2=0$ or $J_1=0, J_2=-1$ were identified with multiplication by

$$A_1B_2 \quad \text{or} \quad A_2B_1, \tag{61}$$

and the first oscillations $J_1=1, J_2=0$ or $J_1=0, J_2=1$ were identified with multiplication by

$$A_1\bar{A}_2 \quad \text{or} \quad \bar{B}_2B_1, \tag{62}$$

where A_i, B_i are all $N\times N$ matrices. $A_1\bar{A}_2$ or $\bar{B}_2B_1$ came from the semiconserved currents of the SU(2) groups and were introduced in [36]. When there is a quotient action on the SU(2) groups, $A_1\bar{A}_2$ or $\bar{B}_2B_1$ are not invariant so they do not appear in the spectrum. Because the supersymmetry in the Penrose limit is $\mathcal{N}=2$, we do not need the semiconserved currents to build $\mathcal{N}=2$ multiplets and we only need A_1B_2 and A_2B_1 in order to build the field theory duals to the twisted sectors of the string theory.

For the quotiented conifold, the matrix A_1B_1 is promoted to an $mN\times mN$ matrix, which splits into $mN\times N$ diagonal matrices in the adjoint representation $\mathrm{SU}(N)_{i,j}$, $i=1,\ldots,m$, for fixed j. The matrices A_1B_2 and A_2B_1 become $mN\times mN$ matrices which also split into m extradiagonal $N\times N$ matrices and each block corresponds to fields transforming in the bifundamental representation of $\mathrm{SU}(N)_{i,j}\times\mathrm{SU}(N)_{i+1,j}$. The boosted direction is acted upon by the discrete group $\mathbf{Z}_n$ so the invariant quantity is a product as in Eq. (57) with n copies of A_1 and n copies of B_1, of the form

$$\begin{aligned}&(A_1)_{i,j;i,j}(B_1)_{i,j;i,j+1}\cdots(A_1)_{i,j+n-1;i,j+n-1}\\ &\quad\times(B_1)_{i,j+n-1;i,j+n},\end{aligned} \tag{63}$$

which is indeed in the adjoint representation of $\mathrm{SU}(N)_{i,j}$. Denoting this by $(A_1B_1)^n$, we see that Eq. (59) implies that it has $J_1=J_2=0$ and it is the ground state of the string. The vector field for all $\mathrm{SU}(N)_{i,j}$ with fixed j, together with the field $[(A_1B_1)^n]_i$, form an $\mathcal{N}=2$ multiplet. The ground state is given by m mutually orthogonal $\mathbf{Z}_m$ invariant single trace operators

$$\mathrm{Tr}[S^q(A_1B_1)^{nJ}] \tag{64}$$

where S is defined as $S=(1,e^{2\pi i/m},\ldots,e^{2\pi i(m-1)/m})$ denotes the qth twisted sector.

The first level untwisted sectors are built with derivatives and descendants of $(A_1B_1)^n$ and are of the form

$$\mathrm{Tr}[S^q(A_1B_1)^{nJ}D_\mu(A_1B_1)^n] \tag{65}$$

and

$$\mathrm{Tr}[S^q(A_1B_1)^{nJ}\chi] \tag{66}$$

where D_μ is the covariant derivative and χ is the supersymmetric partner of the scalar $(A_1B_1)^n$.

The first level twisted sectors are written with insertions of A_1B_2 and A_2B_1, which are acted upon by $\mathbf{Z}_m$ but are invariant under $\mathbf{Z}_n$. They have zero angular momentum in the boosted direction so they are used to build first level string oscillations. The discussion is similar to that in [14].

As the effective angular momentum of the string states is nJ, we can again choose n to be either small or large. For the case of large n, the insertions of A_1B_2 and A_2B_1 should be made between different $(A_1)_{i,j;i,j}(B_1)_{i,j;i,j+1}$. The Penrose limits of the quotiented conifold will then be the limit of a DLCQ theory with constant p^+.

IV. CONCLUSIONS

In this paper we studied the Penrose limits of different $\mathcal{N}=1$ orbifold geometries of $\mathbf{S}^5$ and $\mathbf{T}^{11}$ which lead to supersymmetric *pp*-wave backgrounds with enlarged supersymmetry. We considered the gauge invariant chiral operators in the different Penrose limits and we identified the string oscillations in terms of the gauge invariant operators. We discussed the different choices for the rank of the quotient groups.

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