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Geometric approach to a massive p form duality

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Massive theories of Abelian p forms are quantized in a generalized path representation that leads to a description of the phase space in terms of a pair of dual nonlocal operators analogous to the Wilson loop and the 't Hooft disorder operators. Special attention is devoted to the study of the duality between the topologically massive and self-dual models in $2+1$ dimensions. It is shown that these models share a geometric representation in which just one nonlocal operator suffices to describe the observables.

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I. INTRODUCTION

As is known, the electric-magnetic duality of free Maxwell theory may be seen as a particular case of a duality transformation relating different Abelian gauge theories of arbitrary rank in the appropriate space-time dimensions [1,2]. In D dimensions, the ranks p_1, p_2 of the generalized potentials describing the dual Abelian theories must obey the equality $p_1+p_2=D-2$. For example, in four dimensions, Maxwell theory is self-dual, while the second rank gauge theory is dual to the massles scalar field.

In Ref. [3] it was shown that the "electric-magnetic" duality of Abelian gauge theories allows us to describe their physical phase space in terms of a pair of nonlocal observables that are dual in the Kramers-Wannier sense [4]. The algebra that they obey results to be invariant under spatial diffeomorphisms. This topological algebra, the dual algebra (DA), admits a realization in terms of operators acting on functionals that depend on extended objects, inasmuch as the dual operators themselves. For instance, in the case of Maxwell theory in four space-time dimensions, the dual operators are the Wilson loop and the 't Hooft disorder operator [5]. Both operators depend on closed spatial loops, and may be realized on a loop-dependent Hilbert space (see Sec. II). The DA of the three- and four-dimensional Maxwell theory had been previously analyzed [6], due to their close relation with the Yang-Mills field. Furthermore, nonlocal operators that obey commutation relations of the DA type have been used to quantize topological excitations in interacting field theories [7].

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In this paper we discuss how the ideas of Ref. [3] can be extended to the conventional (i.e., nontopological) Abelian massive theories in arbitrary dimensions, and to the self-dual [8] and topologically massive theories [9] in $2+1$ dimensions, which are known to be dual to each other [10]. In all the cases, the program that we develop is as follows: one starts from a first-order master Lagrangian that encodes the dual theories simultaneously. We take this master Lagrangian to be of the Stückelberg form [11], in order to maintain gauge invariance even in the massive case. The master theory is then quantized within the Dirac scheme [12], and the phase space is taken into account by choosing nonlocal operators that encode all the gauge-invariant content of the original canonical operators. The algebra obeyed by these dual operators is then studied and realized onto an appropriate set of functionals.

We shall see that the DA of massive Abelian theories is also characterized by a topological quantity, namely, the intersection number between the extended objects that support the nonlocal dual operators. This contrasts with the massless case, where the DA is governed by the linking number of the closed extended objects that enter in the construction of the dual operators [3,6]. This and other differences between both cases are studied.

The case of the self-dual and topologically massive theories presents several interesting peculiarities, regarding the DA study. Perhaps the more relevant one is that instead of a pair of Wilson loop operators, as in both the massless and the conventional massive theories, only one nonlocal operator suffices to describe the gauge-invariant content of the theory. Consequently, this operator has to play both the "coordinate" and "momentum" roles. As we shall see, this feature has an interesting geometrical counterpart when the nonlocal operator is realized in a path-dependent Hilbert space. In other direction the Proca model in $2+1$ dimensions is equivalent to two noninteracting self-dual models with opposite spins.

The paper is organized as follows. In Sec. II we review the massless case, following Ref. [3], focusing mainly on the study of Maxwell theory in four dimensions. In Sec. III the DA of the Proca model in three dimensions is considered. Section IV is dedicated to the study of the self-dual and topologically massive theories. Some concluding remarks are given in the last section. In the Appendix we summarize the generalization of the study presented in Sec. III to the case of forms of arbitrary rank in arbitrary dimension.

II. MAXWELL THEORY

Let us summarize the results of Ref. [3] regarding Maxwell theory. The starting point is the first-order Lagrangian density,

$$\mathcal{L}=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu B_{\lambda\rho}-\tfrac{1}{4}(B_{\lambda\rho}+\partial_\lambda C_\rho-\partial_\rho C_\lambda)\times(B^{\lambda\rho}+\partial^\lambda C^\rho-\partial^\rho C^\lambda), \tag{1}$$

which is invariant under the simultaneous gauge transformations

$$\delta A_\mu=\partial_\mu\Lambda, \tag{2}$$

$$\delta B_{\lambda\rho}=\partial_\rho\xi_\lambda-\partial_\lambda\xi_\rho, \tag{3}$$

$$\delta C_\rho=\xi_\rho+\partial_\rho\xi. \tag{4}$$

Equation (4) shows that the field C_ρ is pure gauge. Its presence just serves to enforce gauge invariance. When this field is gauged away in Eq. (1), the equations of motion become

$$\epsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu=B^{\lambda\rho}, \tag{5}$$

$$\epsilon^{\mu\nu\lambda\rho}\partial_\mu B_{\lambda\rho}=0. \tag{6}$$

Substituting $B_{\lambda\rho}$ from Eq. (5) into the master Lagrangian (with $C_\rho=0$), one finds the standard Maxwell Lagrangian. If, instead, one solves Eq. (6) locally

$$B_{\lambda\rho}=\partial_\lambda\tilde{A}_\rho-\partial_\rho\tilde{A}_\lambda, \tag{7}$$

and substitutes the above expression into Eq. (1) (again with $C_\rho=0$) we obtain (after an integration by parts) the "dual" Lagrangian density

$$\tilde{L}=-\tfrac{1}{4}F_{\mu\nu}(\tilde{A})F^{\mu\nu}(\tilde{A}), \tag{8}$$

in correspondence with the fact that in $D=4$ Maxwell theory is self-dual.

The canonical analysis may be summarized as follows. There are three secondary first class constraints,

$$\psi=-\tfrac{1}{2}\epsilon^{ijk}\partial_i B_{jk}\approx 0, \tag{9}$$

$$\theta^i=-(\pi_C^i-\epsilon^{ijk}\partial_j A_k)\approx 0, \tag{10}$$

$$\theta=-\partial_i\pi_C^i\approx 0, \tag{11}$$

where π_c^i is the momentum canonically conjugate to C_i. We are taking $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$. These constraints are reducible $(\partial_i\theta^i-\theta=0)$ and appear associated, respectively, to A_0, B_{0i}, and C_0 as their Lagrange multipliers.

The fields A_i and $\frac{1}{2}\epsilon^{ijk}B_{jk}$ are mutually conjugate,

$$[A_k(\vec{x}),\tfrac{1}{2}\epsilon^{lij}B_{ij}(\vec{y})]=i\delta_k^l\delta^{(3)}(\vec{x}-\vec{y}), \tag{12}$$

as can be seen from the first-order BF term in the master Lagrangian (see Ref. [13]). ψ, θ^i, and θ generate the gauge transformations for A_i, B_{ij}, C_i, and π_C^i. The gauge transformations for the remaining fields are obtained imposing the gauge invariance on the extended action, taking into account the reducibility of the first class constraints.

On the physical sector, the Hamiltonian reduces to

$$H=\int d^3\vec{x}\frac{1}{2}(\mathcal{B}^i\mathcal{B}^i+\mathcal{E}^i\mathcal{E}^i), \tag{13}$$

with the magnetic and electric fields given, respectively, by

$$\mathcal{B}^k\equiv\epsilon^{ijk}\partial_j A_i, \tag{14}$$

$$\mathcal{E}^i\equiv\tfrac{1}{2}\epsilon^{ijk}[B_{jk}+F_{jk}(C)]. \tag{15}$$

The gauge-invariant combinations of the operators appearing in the above expressions indicate which are the nonlocal dual operators we are interested in. They are the Wilson loop

$$W(\gamma)=\exp\left(i\oint_\gamma dy^i A_i(\vec{y})\right), \tag{16}$$

with γ a closed spatial path, and the operator

$$\Omega(\Sigma,\Gamma)=\exp\left(i\oint_\Gamma dy^i C_i(\vec{y})\right)\exp\left(i\int_\Sigma d\Sigma_k\epsilon^{kij}B_{ij}\right), \tag{17}$$

which depends on the spatial open surface Σ whose boundary is Γ. In virtue of the constraint (9), one has

$$\Omega(\Sigma_{\mathrm{closed}})|\psi_{\mathrm{physical}}\rangle=|\psi_{\mathrm{physical}}\rangle, \tag{18}$$

i.e., Ω does not depend on the surface Σ, but only on its boundary Γ. The algebra obeyed by the dual operators (the DA) is given by

$$W(\gamma)\Omega(\Gamma)=e^{i\mathcal{L}(\gamma,\Gamma)}\Omega(\Gamma)W(\gamma), \tag{19}$$

where the quantity

$$\mathcal{L}(\gamma,\Gamma)=\frac{1}{4\pi}\oint_\gamma dx^i\oint_\Gamma dy^j\epsilon_{ijk}\frac{(\vec{x}-\vec{y})^k}{|\vec{x}-\vec{y}|^3} \tag{20}$$

measures the Gauss linking number between γ and Γ, which are closed curves in R^3, and is a topological object, since it does not depend on the metric properties of the space.

The operator $\Omega(\Gamma)$ results to be the "dual" Wilson loop, i.e., the contour integral of the dual potential $\tilde{A}$ along Γ [6].

It must be noticed, however, that these results are obtained from a formulation that does not include this potential as a Lagrangian variable, which would be redundant.

The DA (19) is satisfied if the operators are defined to act onto loop dependent functionals $\Psi(\gamma)$ as

$$W(\gamma)\Psi(\gamma_1)=\Psi(\gamma\circ\gamma_1), \tag{21}$$

$$\Omega(\Gamma)\Psi(\gamma_1)=e^{-i\mathcal{L}(\Gamma,\gamma_1)}\Psi(\gamma_1). \tag{22}$$

Here $\gamma\circ\gamma'$ denotes the Abelian group of loops product [14,15]. It is worth recalling that an Abelian loop is an equivalence class of closed curves, defined as follows. The curves γ_1 and γ_2 are equivalent if their form factors $T^i(\vec{x},\gamma_1)$ and $T^i(\vec{x},\gamma_2)$, with

$$T^i(\vec{x},\gamma)\equiv\int_\gamma dy^i\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{23}$$

are equal. With this definition it is easy to see that the usual composition of curves is lifted to a group product.

The electric and magnetic fields may be obtained from $W(\gamma)$ and $\Omega(\Gamma)$ through the expressions

$$\mathcal{B}^i(\vec{x})=-i\epsilon^{ijk}\Delta_{jk}(\vec{x})W(\gamma)|_{\gamma=0}, \tag{24}$$

$$\mathcal{E}^i(\vec{x})=-i\epsilon^{ijk}\Delta_{jk}(\vec{x})\Omega(\Gamma)|_{\Gamma=0}, \tag{25}$$

where we have made use of the loop derivative $\Delta_{ij}(x)$ of Gambini-Trias [14],

$$\delta\sigma^{ij}\Delta_{ij}(\vec{x})f(\gamma)\equiv f(\delta\gamma\circ\gamma)-f(\gamma), \tag{26}$$

that measures the change experimented by a loop dependent object $f(\gamma)$ when its argument γ is modified by attaching a small plaquette $\delta\gamma$ of area $\delta\sigma^{ij}$ at the point $\vec{x}$. In view of Eqs. (24) and (25), the Hamiltonian and the other observables of the theory may be expressed in terms of the basic operators W and Ω. Equations (19), (21), and (22) are the basic results of the geometric formulation of massless theories that we are going to extend to massive cases, with and without topological terms, in the following sections.

III. PROCA THEORY IN THREE DIMENSIONS

In order to preserve gauge invariance, we start from Lagrangian of the Proca model in the Stückelberg form,

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}m^2(A_\mu+\partial_\mu f)(A^\mu+\partial^\mu f). \tag{27}$$

It is a trivial matter to see that the equation of motion associated with the auxiliar field f is nothing but a consistence requisite for the other equation, which is the relevant one. This reflects the invariance of the Lagrangian density (27) under the gauge transformations

$$\delta A_\mu=\partial_\mu\Lambda, \tag{28}$$

$$\delta f=-\Lambda. \tag{29}$$

As in the Maxwell case, f may be eliminated by choosing $f=0$. To incorporate the dual formulation of the theory (27), we take the master Lagrangian

$$\mathcal{L}'=m\epsilon^{\mu\nu\lambda}\partial_\mu A_\nu B_\lambda+\frac{m^2}{2}(B_\mu+\partial_\mu\omega)(B^\mu+\partial^\mu\omega)+\frac{m^2}{2}(A_\mu+\partial_\mu f)(A^\mu+\partial^\mu f), \tag{30}$$

which is first order in the Proca field A_μ and the dual field B_μ. Besides f we have introduced the Stückelberg field ω, associated with B_μ, to promote gauge invariance.

It can be seen that Eq. (30) corresponds to two self-dual models [8] with opposite spins. In fact, if we do the change

$$A_\mu=\frac{1}{\sqrt{2}}(a^1_\mu+a^2_\mu),$$

$$B_\mu=\frac{1}{\sqrt{2}}(a^1_\mu-a^2_\mu),$$

$$f=\frac{1}{\sqrt{2}}(f_1+f_2),$$

$$\omega=\frac{1}{\sqrt{2}}(f_1-f_2), \tag{31}$$

we will get two decoupled self-dual Lagrangians [see Eq. (54) further] in Stückelberg form. Each of them describes a massive mode with spin $+1$ for one mode and spin -1 for the other [10]. The invariance under P and T transformations is accomplished if we exchange the fields a^1_μ and a^2_μ (and so with the fields f_1 and f_2). In this sense we see that the field B_μ behaves as a pseudovector.

The equations of motion that result after eliminating the Stückelberg fields are

$$\epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda+mA^\mu=0, \tag{32}$$

$$\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda+mB^\mu=0. \tag{33}$$

By substitution of B^λ from Eq. (33) into Eq. (30) we obtain the Proca Lagrangian (27) (with $f=0$). Doing an analogous procedure with A^μ from Eq. (32) we obtain the same Proca Lagrangian, but this time in terms of the dual field B_μ. In this sense, one says that the theory is self-dual, and the master Lagrangian $\mathcal{L}'$ is a good starting point to explore the geometrical consequences of this duality.

Let us now quantize the theory. From the "*BF*" term we directly read the commutator [13]

$$[A_i(\vec{x}),\epsilon^{kj}B_j(\vec{y})]=i\frac{1}{m}\delta_i^k\delta^{(2)}(\vec{x}-\vec{y}). \tag{34}$$

The Hamiltonian results to be

$$H=\int d^2\vec{x}\Bigg(\frac{1}{2m^2}\pi_\omega^2+\frac{1}{2m^2}\pi_f 2+\frac{m^2}{2}(B_i+\partial_i\omega)(B_i+\partial_i\omega)+\frac{m^2}{2}(A_i+\partial_i f)(A_i+\partial_i f)-A_0(\pi_f+m\epsilon^{ij}\partial_i B_j)-B_0(\pi_\omega+m\epsilon^{ij}\partial_i A_j)\Bigg), \tag{35}$$

where π_w and π_f are the momenta conjugate to ω and f, respectively. We did not consider A_0 and B_0 as canonical variables since their role as Lagrange multipiers is clear. They are associated with the first class constraints

$$\pi_f+m\epsilon^{ij}\partial_i B_j\approx 0, \tag{36}$$

$$\pi_\omega+m\epsilon^{ij}\partial_i A_j\approx 0, \tag{37}$$

that generate the time independent gauge transformations of the theory. At this point we can compare Eq. (35) with the Hamiltonian of the Proca theory obtained from the standard action. Starting from Eq. (27) with $f=0$, and following the canonical quantization procedure one obtains the Hamiltonian

$$H_{\text{Proca}}=\int d^2\vec{x}\Bigg(\frac{1}{4}F_{ij}F_{ij}+\frac{m^2}{2}A_iA_i+\frac{1}{2}\pi_i\pi_i+\frac{1}{2m^2}(\partial_i\pi^i)^2\Bigg), \tag{38}$$

after solving the second class constraint to eliminate the time component of the vector field. In Eq. (38), A_i and π^i are canonically conjugate. On the other hand, in the gauge-invariant model the Hamiltonian H may be expressed as

$$H=\int d^2\vec{x}\Bigg(\frac{1}{2}(\epsilon^{ij}\partial_i A_j)^2+\frac{1}{2}(\epsilon^{ij}\partial_i B_j)^2+\frac{m^2}{2}(B_i+\partial_i\omega)(B_i+\partial_i\omega)+\frac{m^2}{2}(A_i+\partial_i f)(A_i+\partial_i f)\Bigg). \tag{39}$$

The equivalence of the two formulations is clear after fixing $f=0$, $\omega=0$ and identifying π^i with $m\epsilon^{ij}B_j$.

Examining the first class constraints one realizes that the gauge-invariant combinations that can be formed from the canonical fields A_i, B_i, f, and ω are $A_i+\partial_i f$ and $B_i+\partial_i\omega$. Hence, it is natural to introduce the nonlocal Wilson-like operators

$$W(\gamma_x^{x'})=\exp\left(ie\int_{\gamma_x^{x'}}dx^i(A_i+\partial_i f)\right), \tag{40}$$

$$\Omega(\Gamma_y^{y'})=\exp\left(ie\int_{\Gamma_y^{y'}}dx^i(B_i+\partial_i\omega)\right), \tag{41}$$

where $\gamma_x^{x'}$ ($\Gamma_y^{y'}$) is an open curve in R^2, starting at x (y) and ending at x' (y') and e is a constant with units $L^{-1/2}$. These operators play the role of the Wilson loop and its dual [Eqs. (16) and (17)] in the four-dimensional Maxwell theory. In virtue of the constraints (36),(37) the introduction of nonlocal operators associated with π_f and π_ω would be redundant. In fact, the exponential of i times the integral of π_ω over the region of $\mathcal{R}^2$ bounded by a closed contour $\mathcal{C}$ is equivalent to $W(\mathcal{C})$. A similar argument holds for π_f and Ω.

It is simple to show that the operators (40),(41) obey

$$W(\gamma)\Omega(\Gamma)=e^{-i(e^2/m)N(\gamma,\Gamma)}\Omega(\Gamma)W(\gamma), \tag{42}$$

where

$$\begin{aligned}N(\gamma,\Gamma)&=\int_\gamma dy^i\int_\Gamma dx^j\epsilon^{ij}\delta^{(2)}(\vec{x}-\vec{y}),\\&=\int_\Gamma dx^j\epsilon^{ij}T^i(\vec{x},\gamma),\\&=\int_\gamma dx^i\epsilon^{ij}T^j(\vec{x},\Gamma),\end{aligned} \tag{43}$$

is the oriented number of intersections between the curves γ and Γ. This topological quantity obeys the relations

$$N(\gamma,\Gamma)=-N(\Gamma,\gamma), \tag{44}$$

$$N(\gamma_1,\gamma_2\gamma_3)=N(\gamma_1,\gamma_2)+N(\gamma_1,\gamma_3). \tag{45}$$

Equation (42) is the dual algebra for the $D=2+1$ Proca theory. As was pointed out in the Introduction, its topological "structure constant" involves the intersection index of the curves that enter in the definition of the nonlocal operators, instead of linking numbers, which are the objects that appear in the DA of massless theories [3].

Our next step will be to realize the DA (42) in an appropriate geometrical representation. To this end we employ the Abelian open-path representation, which has been discussed in Refs. [16,17]. The main features may be summarized as follows. One groups the piecewise continuous (and not necessarily closed) curves of R^2 in equivalence classes characterized by the equality of their form factors $T(\vec{x},\gamma)$. Then the usual composition of curves turns into a group product. It is a trivial matter to show that the Abelian group of loops is a subgroup of the Abelian group of open paths. In addition to the loop derivative Eq. (26), one can define the path derivative [16,18]

$$h^i \delta_i(\vec{x}) \Psi(\gamma) = \Psi(\delta\gamma \circ \gamma) - \Psi(\gamma), \tag{46}$$

which computes the variation of a path-dependent function when an infinitesimal open path $\delta\gamma_x^{x+h}$ going from x to $x+h$ $(h\to 0)$ is appended to γ. It is related to the loop derivative through the expression

$$\Delta_{ij}(\vec{x}) = \partial_i \delta_j(\vec{x}) - \partial_j \delta_i(\vec{x}). \tag{47}$$

The DA (42) may be realized onto open-path-dependent wave functionals in the form

$$W(\gamma)\Psi(\gamma_1) = \Psi(\gamma \circ \gamma_1), \tag{48}$$

$$\Omega(\Gamma)\Psi(\gamma_1) = e^{i(e^2/m)N(\Gamma,\gamma_1)}\psi(\gamma_1). \tag{49}$$

As in the Maxwell case, we have chosen a geometric representation in which the nonlocal operator associated with the "direct" field, i.e., the Wilson path, produces a "translation" in path space, while that associated with the dual field is diagonal. One could also interchange these roles. Since the theory is self-dual, the dual geometric representation results to be a path representation too.

With the use of the derivative (46), the basic local observables of the theory may be obtained from the nonlocal dual operators,

$$A_i + \partial_i f = -i \frac{\delta_i(\vec{x})}{e} W(\gamma)\bigg|_{\gamma=0}, \tag{50}$$

$$B_i + \partial_i \omega = -i \frac{\delta_i(\vec{x})}{e} \Omega(\Gamma)\bigg|_{\Gamma=0}. \tag{51}$$

As we show in the Appendix, the program developed in this section can also be carried out for massive p forms in arbitrary dimensions. In D dimensions, Abelian massive theories of p_1 and p_2 forms are dual for $p_1 + p_2 = D-1$.

For instance, the four-dimensional Proca theory is dual to the massive Kalb-Ramond model. On the other hand, massive p-form theories are associated with generalized Wilson surfaces $W(\Sigma_p)$, where Σ_p is an open p surface. Then the dual algebra generalizes to

$$W(\Sigma_{p_1})\Omega(\Sigma_{p_2}) = e^{-iN(\Sigma_{p_1},\Sigma_{p_2})}\Omega(\Sigma_{p_2})W(\Sigma_{p_1}), \tag{52}$$

where $N(\Sigma_{p_1},\Sigma_{p_2})$ is the intersection index of the open surfaces Σ_{p_1} and Σ_{p_2}.

IV. SELF-DUAL AND TOPOLOGICALLY MASSIVE THEORIES

A. Master Lagrangian and canonical quantization

It is well known that the topologically massive [9],

$$S_{\mathrm{TM}} = \int d^3x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{m}{4}\varepsilon^{\mu\nu\lambda}F_{\mu\nu}(A)A_\lambda \right), \tag{53}$$

and self-dual theories [8]

$$S_{AD} = \int d^3x \left(-\frac{m}{4}\varepsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda + \frac{m^2}{2}A_\mu A^\mu \right), \tag{54}$$

provide locally equivalent descriptions of spin 1 massive particles in $2+1$ dimensions [10], although they exhibit different global behaviors depending on the topological properties of the space-time where they are defined [19,20].

The local equivalence between these models may be viewed by noticing that they are dual, in the sense that both may be obtained from the master action [10]

$$S_M = \frac{m}{2}\int d^3x \left[\varepsilon^{\mu\nu\lambda}F_{\mu\nu}(A)\left(C_\lambda + \frac{1}{2}A_\lambda \right) + mC_\mu C^\mu \right], \tag{55}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We shall take $g_{\mu\nu} = \mathrm{diag}(1,-1,-1)$. The equations of motions, obtained by varying the independent fields A_μ, C_μ, are

$$\varepsilon^{\mu\nu\lambda}\partial_\nu(C_\lambda + A_\lambda) = 0, \tag{56}$$

$$\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda + mC^\mu = 0. \tag{57}$$

Using Eq. (56) to eliminate A_λ in Eq. (57) we obtain the equations of motion for the SD model. In other direction, from Eq. (57) we can eliminate C_μ in Eq. (56) to obtain the equations of of motion of the TM model. This proves the local classical equivalence.

As in the previous sections, the "C^2" term spoils gauge invariance. We remedy this fact by introducing an auxiliary Stückelberg field ω,

$$S'_M = \frac{m}{2}\int d^3x \left[\varepsilon^{\mu\nu\lambda}F_{\mu\nu}(A)\left(C_\lambda + \frac{1}{2}A_\lambda \right) + m(C_\mu + \partial_\mu\omega)(C^\mu + \partial^\mu\omega) \right]. \tag{58}$$

This action is invariant under the simultaneous gauge transformations

$$\delta A_\mu = \partial_\mu \xi, \tag{59}$$

$$\delta C_\mu = \partial_\mu \zeta, \tag{60}$$

$$\delta\omega = -\zeta. \tag{61}$$

From Eq. (61) we see that the field ω is pure gauge, as corresponds to the Stückelberg formulation.

Now we apply the canonical procedure of quantization to the master model. First, we decompose the action (58) into spatial and temporal parts,

$$S'_M = \int d^3x \left[m\varepsilon^{ij}F_{0i}\left(C_j + \frac{1}{2}A_j \right) + \frac{m}{2}\varepsilon^{ij}F_{ij}\left(C_0 + \frac{1}{2}A_0 \right) + \frac{m}{2}(C_0 + \dot{\omega})^2 - \frac{m}{2}(C_i + \partial_i\omega)(C_i + \partial_i\omega) \right]. \tag{62}$$

The canonical momenta conjugate to the dynamical variables A_i, C_i, and ω are

$$\pi_A^i = m\varepsilon^{ij}(C_j + \tfrac{1}{2}A_j), \tag{63}$$

$$\pi_C^i = 0, \tag{64}$$

$$\pi_\omega = m(C_0 + \dot{\omega}). \tag{65}$$

We consider the fields A_0 and C_0 as nondynamical. They will appear in the next step as Lagrange multipliers. Equations (63) and (64) are just primary constraints among the phase-space variables

$$\psi_i^A \equiv \pi_A^i - m\varepsilon^{ij}\left(C_j + \frac{1}{2}A_j\right) \approx 0, \tag{66}$$

$$\psi_i^C \equiv \pi_C^i \approx 0, \tag{67}$$

while Eq. (65) allows us to obtain the velocities associated with ω. Thus, the Hamiltonian on the manifold defined by the primary constraints is given by

$$H = \int d^2\vec{x}\left(\frac{1}{2m^2}\pi_\omega^2 + \frac{m^2}{2}(C_i + \partial_i\omega)^2 + A_0\theta_1 + C_0\theta_2\right), \tag{68}$$

where

$$\theta_1 \equiv -m\varepsilon^{ij}\partial_i(A_j + C_j), \quad \theta_2 \equiv -m\varepsilon^{ij}\partial_i A_j - \pi_\omega. \tag{69}$$

Following the scheme of quantization of Dirac, we extend the Hamiltonian to the whole phase space,

$$\tilde{H} = \int d^2\vec{x}\left(\frac{1}{2m^2}\pi_\omega^2 + \frac{m^2}{2}(C_i + \partial_i\omega)(C_i + \partial_i\omega) + A_0\theta_1 + C_0\theta_2 + \lambda_A^i\psi_i^A + \lambda_C^i\psi_i^C\right). \tag{70}$$

At this point we observe that the variables A_0 and C_0 are the Lagrange multipliers associated with the "secondary" constraints θ_1 and θ_2, respectively. Now, we define the Poisson brackets among the canonical variables by

$$\{A_i(\vec{x}), \pi_A^j(\vec{y})\} = \delta_i^j\delta^{(2)}(\vec{x} - \vec{y}), \tag{71}$$

$$\{C_i(\vec{x}), \pi_C^j(\vec{y})\} = \delta_i^j\delta^{(2)}(\vec{x} - \vec{y}), \tag{72}$$

$$\{\omega(\vec{x})\pi_\omega(\vec{y})'\} = \delta^{(2)}(\vec{x} - \vec{y}) \tag{73}$$

(the remaining Poisson brackets vanish) and proceed to require the preservation in time of the constraints, taking $\tilde{H}$ as the generator of time translations. This leads to determine the Lagrange multipliers associated with the primary constraints (63) and (64),

$$\lambda_A^i = \partial_i A_0 + m\varepsilon^{ij}(C_j + \partial_j\omega), \tag{74}$$

$$\lambda_C^i = \partial_i C_0 - m\varepsilon^{ij}(C_j + \partial_j\omega), \tag{75}$$

and it is seen that no further secondary constraints arise. Substituting the multipliers into the Hamiltonian yields

$$\tilde{H} = \int d^2\vec{x}\left(\frac{1}{2m^2}\pi_\omega^2 + \frac{m^2}{2}(C_i + \partial_i\omega)(C_i + \partial_i\omega) + A_0\varphi_1 + C_0\varphi_2 + \mu\varepsilon^{ij}(C_j + \partial_j\omega)\psi_i^A - \mu\varepsilon^{ij}(C_j + \partial_j\omega)\psi_i^C\right), \tag{76}$$

where

$$\varphi_1 \equiv -\partial_i\psi_i^A + \theta_1 = -\partial_i\pi_A^i - \frac{m}{2}\varepsilon^{ij}\partial_i A_j,$$

$$\varphi_2 \equiv -\partial_i\psi_i^C + \theta_2 = -\partial_i\pi_C^i - \pi_\omega - m\varepsilon^{ij}\partial_i A_j, \tag{77}$$

result to be the first class constraints of the theory.

The matrix associated with the second class constraints $\Psi_a \equiv (\psi_i^A, \psi_i^C)$ may be written as

$$C_{ab}(\vec{x},\vec{y}) \equiv \{\Psi_a(\vec{x}), \Psi_b(\vec{y})\} = -m\varepsilon^{ij}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\delta^{(2)}(\vec{x} - \vec{y}). \tag{78}$$

Its inverse is given by

$$C_{ab}^{-1}(\vec{x},\vec{y}) \equiv \{\Psi_a(\vec{x}), \Psi_b(\vec{y})\}^{-1} = -\frac{1}{m}\varepsilon^{ij}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\delta^{(2)}(\vec{x} - \vec{y}), \tag{79}$$

and allows us to define the Dirac brackets

$$\{F,G\}^* \equiv \{F,G\} - \int d^2\vec{x}\,d^2\vec{y}\{F, \Psi_a(\vec{x})\}C_{ab}^{-1}(\vec{x},\vec{y}) \times \{\Psi_b(\vec{y}), G\}. \tag{80}$$

Recalling that Dirac brackets are consistent with second class constraints, we can eliminate π_A^i and π_C^i from now on, employing the constraints Ψ_a. The brackets between the reduced phase-space variables are then

$$\{A_i(\vec{x}), A_j(\vec{y})\}^* = 0, \tag{81}$$

$$\{A_i(\vec{x}), C_j(\vec{y})\}^* = \frac{1}{m}\epsilon^{ij}\delta^{(2)}(\vec{x} - \vec{y}), \tag{82}$$

$$\{C_i(\vec{x}), C_j(\vec{y})\}^* = -\frac{1}{m}\epsilon^{ij}\delta^{(2)}(\vec{x} - \vec{y}), \tag{83}$$

$$\{\omega(\vec{x}), \pi_\omega(\vec{y})\}^* = \delta^{(2)}(\vec{x} - \vec{y}). \tag{84}$$

Once the phase space has been reduced, the first class constraints become

$$\theta_1 \approx 0, \quad \theta_2 \approx 0, \tag{85}$$

and it can be seen that the time independent gauge transformations generated by these constraints on the reduced phase-space variables are given by

$$\delta A_i(\vec{x}) = \partial_i \xi, \tag{86}$$

$$\delta C_i(\vec{x}) = \partial_i \zeta, \tag{87}$$

$$\delta \omega(\vec{x}) = -\zeta, \tag{88}$$

$$\delta \pi_\omega = 0, \tag{89}$$

as expected.

The next step in the quantization procedure is to promote the fields to operators acting on a Hilbert space $\mathcal{H}$, obeying commutation relations given by $i\{\cdot,\cdot\}^*$, and ask the physical vectors $|\psi\rangle$ to belong to the kernel of both first class constraint operators: $\theta_1|\psi\rangle = 0$ and $\theta_2|\psi\rangle = 0$. The basic observables (in the sense of Dirac), from which all relevant gauge-invariant information of the theory can be recovered, are the operators $-\epsilon^{ij}\partial_i A_j$, $C_i + \partial_i \omega$, and π_ω. It is then natural, within the spirit of the previous sections, to introduce the nonlocal operators

$$W(\mathcal{C}) \equiv \exp\left(ie \oint_{\mathcal{C}} A_i dx^i \right), \tag{90}$$

$$\Omega(\gamma_x^{x'}) \equiv \exp\left(ie \int_{\gamma_x^{x'}} (C_i + \partial_i \omega) dx^i \right). \tag{91}$$

In this expression, e is a constant with dimensions $L^{-1/2}$. The Wilson loop (W) and Wilson path (Ω) operators depend on the closed and open paths $\mathcal{C}$ and γ, respectively. It is easy to see from Eqs. (81)–(84) that these operators obey

$$W(\mathcal{C})W(\mathcal{C}') = W(\mathcal{C}')W(\mathcal{C}), \tag{92}$$

$$W(\mathcal{C})\Omega(\gamma) = e^{-i(e^2/m)N(\mathcal{C},\gamma)}\Omega(\gamma)W(\mathcal{C}), \tag{93}$$

$$\Omega(\gamma)\Omega(\Gamma) = e^{i(e^2/m)N(\gamma,\Gamma)}\Omega(\Gamma)\Omega(\gamma), \tag{94}$$

where $N(\gamma,\Gamma)$, the oriented number of intersections between γ and Γ, was defined in Eq. (43).

One could also introduce a nonlocal operator associated with π_ω: the exponential of i times the integral of π_ω over the region of $\mathcal{R}^2$ bounded by a closed contour $\mathcal{C}$. However, in virtue of the first class constraint $\theta_2 \approx 0$, this operator would be just another representation for the Wilson loop (90) when restricted to the physical space of states, so we do not gain anything with its introduction.

It should be remarked that, as in the previous cases, the local gauge-invariant operators may be obtained from the nonlocal ones. In fact, the local operator $C_i + \partial_i \omega$ can be recovered from the Wilson path by considering an infinitesimal open path $\delta\gamma$, i.e.,

$$\Omega(\delta\gamma) = 1 + ie\,\delta\gamma^i (C_i + \partial_i \omega) + O(\delta\gamma^2). \tag{95}$$

On the other hand, if $\delta\gamma$ is closed, we have

$$\Omega(\delta\gamma) = 1 + ie\,\delta\sigma^{ij}\tilde{F}_{ij} + O(\delta\sigma^2), \tag{96}$$

where $\tilde{F}_{ij} \equiv \partial_i C_j - \partial_j C_i$. In a similar way, the local gauge-invariant operator $F_{ij} = \partial_i A_j - \partial_j A_i \equiv \epsilon_{ij} B$ and the Wilson loop are related through

$$W(\delta\gamma) = 1 + ie\,\delta\sigma^{ij}\epsilon_{ij} B + O(\delta\sigma^2), \tag{97}$$

where $\delta\gamma$ is an infinitesimal loop. From the latter expansions it is straightforward to see that

$$C_i + \partial_i \omega = -\frac{i}{e}\,\delta_i(\vec{x})\Omega(\gamma)\bigg|_{\gamma=0}, \tag{98}$$

$$B = -\frac{i}{2e}\epsilon^{ij}\Delta_{ij}(\vec{x})W(C)\bigg|_{C=0}. \tag{99}$$

Finally, the evolution of the physical states is governed by the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = \int d^2\vec{x}\left(\frac{1}{2}B^2 + m^2(C_i + \partial_i\omega)^2 \right)|\psi(t)\rangle. \tag{100}$$

B. Geometrical representation and the dual algebra

The algebra of nonlocal operators (92)–(94) may be realized on the space of open-path-dependent functionals $\psi(\gamma)$ of Sec. III, if we prescribe

$$W(\mathcal{C})\psi(\gamma) \equiv \exp\left(i\frac{e^2}{m}N(\mathcal{C},\gamma) \right)\psi(\gamma), \tag{101}$$

$$\Omega(\Gamma)\psi(\gamma) \equiv \exp\left(-i\frac{e^2}{2m}N(\Gamma,\gamma) \right)\psi(\Gamma\circ\gamma). \tag{102}$$

For instance, using Eq. (102) one has

$$\begin{aligned}
&\Omega(\Gamma)\Omega(\Gamma')\psi(\gamma)\\
&\quad = \Omega(\Gamma)[e^{-i(e^2/2m)N(\Gamma',\gamma)}\psi(\Gamma'\circ\gamma)]\\
&\quad = e^{-i(e^2/2m)N(\Gamma,\gamma)}e^{-i(e^2/2m)N(\Gamma',\Gamma\circ\gamma)}\psi(\Gamma\circ\Gamma'\circ\gamma)\\
&\quad = e^{i(e^2/m)N(\Gamma,\Gamma')}\Omega(\Gamma')\Omega(\Gamma)\psi(\gamma),
\end{aligned} \tag{103}$$

in agreement with Eq. (94). In addition to realizing the nonlocal algebra, we have to consider the restrictions that the first class constraints impose onto the path dependent states. The constraint θ_2 is automatically satisfied if the nonlocal operator associated with π_ω [see comment after Eq. (94)] is realized as (essentially) the Wilson loop W. So it remains to study the constraint θ_1,

$$\epsilon^{ij}\partial_i(A_j + C_j) \approx 0, \tag{104}$$

which may be imposed onto the states as

$$
\begin{aligned}
\exp\left(ie \oint_{\mathcal{C}} (\hat{A}_i + \hat{C}_i) dx^i \right) |\psi\rangle
&= \exp\left(-\frac{1}{2} \oint_{\mathcal{C}} \oint_{\mathcal{C}} dx^i dy^j [\hat{A}_i(\vec{x}), \hat{C}_j(\vec{y})] \right) W(\mathcal{C}) \Omega(\mathcal{C}) |\psi\rangle, \\
&= W(\mathcal{C}) \Omega(\mathcal{C}) |\psi\rangle, \\
&= |\psi\rangle, \qquad (105)
\end{aligned}
$$

or, in other words,

$$W(\mathcal{C}) \approx \Omega(\overline{\mathcal{C}}), \qquad (106)$$

within the physical sector of the Hilbert space. To obtain this result we used that $N(\mathcal{C}_1, \mathcal{C}_2) = 0$ for closed paths $\mathcal{C}_1$, $\mathcal{C}_2$. Equation (106) states that instead of a pair of "Wilson" operators just one of them suffices, namely, $\Omega(\gamma)$, that simultaneously plays the role of "coordinate" and "momentum." That $N(\mathcal{C}_1, \mathcal{C}_2)$ vanishes when the curves are closed also matters to see that when Eq. (106) holds (i.e., on the physical sector), Eq. (94) already implies Eqs. (92) and (93). Therefore, it is just Eq. (94) which corresponds to the dual algebras of the previous sections [Eqs. (19) and (42)], with which it should be compared.

So far, it remains to study how Eq. (106) restricts the space of states. Combining Eq. (106) with Eqs. (101) and (102), one obtains

$$\psi(\mathcal{C} \circ \gamma) = e^{-i(e^2/2m) N(\mathcal{C}, \gamma)} \psi(\gamma). \qquad (107)$$

Taking $\mathcal{C}$ to be an infinitesimal closed path we can see that this equation is just the nonlocal version of the differential constraint recently obtained in a study of the path-space quantization of the Maxwell-Chern-Simons theory [18]

$$\left(\rho(\vec{x}, \gamma) - i \frac{m}{e^2} \epsilon^{ij} \Delta_{ij}(\vec{x}) \right) \psi(\gamma) = 0, \qquad (108)$$

where

$$\rho(\vec{x}, \gamma) \equiv -\partial_i T^i(\vec{x}, \gamma) = -\sum_s \left[\delta^2(\vec{x} - \vec{\beta}_s) - \delta^2(\vec{x} - \vec{\alpha}_s) \right] \qquad (109)$$

is a functional that depends on the boundary $(\vec{\alpha}, \vec{\beta})$ of the path γ. Since the path may comprise several pieces s, both $\vec{\alpha}$ and $\vec{\beta}$ denote the set of starting and ending points, respectively. It can be said then that $\rho(\vec{x}, \gamma)$ is the "density of extremes" of the path γ. The solution to Eq. (108) was found to be [18]

$$\psi(\gamma) = e^{i\chi(\gamma)} \Phi(\vec{\alpha}, \vec{\beta}), \qquad (110)$$

where $\Phi(\vec{\alpha}, \vec{\beta})$ is an arbitrary functional of the boundary $(\vec{\alpha}, \vec{\beta})$ and

$$
\begin{aligned}
\chi(\gamma) &= -i \frac{e^2}{4\pi m} \sum_s \int_\gamma dx^i \epsilon^{ij} \left(\frac{(x^j - \beta_s^j)}{|\vec{x} - \vec{\beta}_s|^2} - \frac{(x^j - \alpha_s^j)}{|\vec{x} - \vec{\alpha}_s|^2} \right) \\
&= -i \frac{e^2}{4\pi m} \Delta\Theta(\gamma), \qquad (111)
\end{aligned}
$$

with $\Delta\Theta(\gamma)$ being the algebraic sum of the angles subtended by the pieces of γ, measured from their ending points $\vec{\beta}$, minus that measured from their starting points $\vec{\alpha}$.

The path-dependent function $\chi(\gamma)$ is ill defined due to the ambiguous definition of the angle subtended by a path when it is measured from their own ending points. In fact, when the point from which the angle is measured coincides with one of the extremes of the path, one loses the straight line connecting that point with the extreme, which would serve as a reference to compute the desired angle. We can replace that fidutial straight line by the tangent to the path at the problematic point. For instance, if we want to compute the angle subtended by the path γ, given as a map from the interval $[0,1]$ to $R3$, measured from its starting point $\vec{y}(0) = \vec{\alpha}$, we can take the prescription

$$\Theta(\gamma, \vec{\alpha}) \equiv \lim_{a \to 0^+} \int_a^1 dt \frac{dy^i(t)}{dt} \epsilon^{ij} \frac{(y^j(t) - \alpha^j)}{|\vec{y} - \vec{\alpha}|^2}. \qquad (112)$$

It may be seen that this prescription is consistent with the fact that $\chi(\gamma)$ must be a path-dependent function, and not merely a curve-dependent one.

It is worth noticing, from Eq. (110), that the path dependence of the wave functionals is realized through the boundary points of the paths, and through the way they wind around these points. Hence, we see that in this case not only the DA shows a topological character, but also the geometrical representation of the algebra carries a topological content.

Equations (95)–(97), together with the realizations (101) and (102) for the DA, allow us to see that the gauge-invariant operators $B(\vec{x})$ and $C_i(\vec{x}) + \partial_i \omega(\vec{x})$ are realized as

$$B(\vec{x}) \to -i \frac{e^2}{m} \rho(\vec{x}, \gamma), \qquad (113)$$

$$
\begin{aligned}
C_i(\vec{x}) + \partial_i \omega(\vec{x}) &\to -i \mathcal{D}_i(\vec{x}) \\
&\equiv -i \delta_i(\vec{x}) - \frac{e^2}{2m} \epsilon^{ij} T^j(\vec{x}, \gamma), \qquad (114)
\end{aligned}
$$

whose action on gauge-invariant functionals can be seen to respect the form given in Eq. (110) [18]. The same is then true for the nonlocal operator $\Omega(\gamma)$, in view of its definition (91).

Now, let us quote the path-representation expressions for the Poincaré generators of the theory:

$$H = \frac{m^2}{e^2} \int d^2\vec{x} \left[-\Delta_{ij}(\vec{x}) \Delta_{ij}(\vec{x}) - \frac{1}{2} \mathcal{D}_i(\vec{x}) \mathcal{D}_i(\vec{x}) \right], \qquad (115)$$

$$P^i = \frac{m}{2e^2} \int d^2\vec{x} \epsilon^{jk} \Delta_{jk}(\vec{x}) i \mathcal{D}_i(\vec{x}), \qquad (116)$$

$$J=\frac{m}{e^2}\int d^2\vec{x}\,\epsilon^{ij}x^i\epsilon^{kl}\Delta_{kl}(\vec{x})\mathcal{D}_j(\vec{x}). \tag{117}$$

It can be shown that the operator P^i (J) generates rigid translations (rotations) of the path γ appearing in the argument of the wave functional $\psi(\gamma)$, as should be expected. Since $\chi(\gamma)$ is invariant under both translations and rotations, the above result does not contradict the fact that P^i and J are gauge-invariant operators. In other words, one has, for an infinitesimal translation along u^i,

$$(1+u^iP^i)\psi(\gamma)=e^{i\chi(\gamma)}(1+u^iP^i)\Phi(\vec{\alpha},\vec{\beta}). \tag{118}$$

Thus, P^i translates the boundary $(\vec{\alpha},\vec{\beta})$ of the path while maintaining the form of the wave functional given by Eq. (110), which is dictated by gauge invariance. A similar argument holds for infinitesimal rotations.

V. DISCUSSION

We have studied the duality symmetry between massive Abelian p forms, with and without topological terms, from the point of view of the geometrical representations that, in each case, generalize the loop representation of Maxwell theory. We found that in the cases without topological terms, and within the physical sector of the Hilbert space, the canonical algebra of local operators can be translated into a nonlocal algebra of a pair of gauge-invariant operators, that exhibit an interesting geometrical content, and that is characterized by a topological quantity, namely, the intersection index between the geometrical objects that constitute the argument of the gauge-invariant operators. This algebra, the dual algebra, may be realized in a basis of wave functionals depending on open paths, or p surfaces, according to the rank of the forms involved. In general, for any pair of dual theories, there is also a pair of dual geometric representations. This situation degenerates in the case of self-duality, since then the ''direct'' and the ''dual'' theories are equivalent.

Regarding the study of the TM and SD case theories in $2+1$ dimensions, we found that, as in the Proca model in $2+1$ dimensions, the topological quantity that characterizes the DA is the number of intersections of two paths. However, unlike the Proca's case, these paths are different arguments of the same operator. Another important difference is that in the TM and SD models, the open paths involved fall into equivalence classes labeled by their boundary $\partial\gamma$ and their winding properties described by $\Delta\Theta(\gamma)$. One could say that in this case the geometric representation corresponds to ''rubber bands with fixed ends'' and not to a path representation.

The results of this study could contribute to put both massless and massive Abelian gauge theories under a common scope, regarding their geometrical properties. It remains to explore whether or not these ideas find a suitable extension to the non-Abelian case. Also, it would be interesting to study how the equivalence between the Proca model and two self-dual models with opposite spins is manifested in the geometrical representation.

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APPENDIX: CONVENTIONAL MASSIVE THEORIES

In this appendix we discuss briefly how to extend the results of Sec. III to the general case of duality between massive Abelian forms $A_{\mu_1\cdots\mu_p}$ and $B_{\mu_1\cdots\mu_{D-p-1}}$, for $p=0,1,\ldots,D-1$, in D space-time dimensions. We start from the Stückelberg form of the master action, i.e.,

$$I_M^{(p,D)}=\int d^Dx\Bigg(\frac{g(-1)^{pD}}{(p+1)!(D-p-1)!}\epsilon^{\mu_1\cdots\mu_{p+1}\nu_1\cdots\nu_{D-p-1}}F_{\mu_1\cdots\mu_{p+1}}(A)B_{\nu_1\cdots\nu_{D-p-1}}$$
$$-\frac{g(-1)^p}{2(D-p-1)!}[B_{\mu_1\cdots\mu_{D-p-1}}+F_{\mu_1\cdots\mu_{D-p-1}}(C)]^2-\frac{g(-1)^p\mu^2}{2p!}[A_{\mu_1\cdots\mu_p}+F_{\mu_1\cdots\mu_p}(\omega)]^2\Bigg), \tag{A1}$$

with $F(f)\equiv df$ for any form f. Here, ω and C are auxiliary Stückelberg $p-1$ and $D-p-2$ forms, respectively. From the space-time decomposition of the master action (A1), which is given by

$$I_M^{(p,D)}=\int d^Dx\Bigg(\frac{1}{2(D-p-2)!}[B_{0i_1\cdots i_{D-p-2}}+F_{0i_1\cdots i_{D-p-2}}(C)]^2-\frac{1}{2(D-p-1)!}[B_{i_1\cdots i_{D-p-1}}+F_{i_1\cdots i_{D-p-1}}(C)]^2$$
$$+\frac{\mu^2}{2(p-1)!}[A_{0i_1\cdots i_{p-1}}+F_{0i_1\cdots i_{p-1}}(\omega)]^2-\frac{\mu^2}{2p!}[A_{i_1\cdots i_p}+F_{i_1\cdots i_p}(\omega)]^2$$
$$+\frac{g(-1)^{pD}}{p!(D-p-1)!}\epsilon^{i_1\cdots i_pj_1\cdots j_{D-p-1}}\dot{A}_{i_1\cdots i_p}B_{j_1\cdots j_{D-p-1}}$$
$$-\frac{g(-1)^{pD}}{(p-1)!(D-p-1)!}\epsilon^{i_1\cdots i_pj_1\cdots j_{D-p-1}}\partial_{i_2}A_{0i_2\cdots i_{p-1}}B_{j_1\cdots j_{D-p-1}}$$
$$-\frac{g(-1)^{pD}}{p!(D-p-2)!}(-1)^p\epsilon^{i_1\cdots i_{p+1}j_1\cdots j_{D-p-2}}\partial_{i_1}A_{i_2\cdots i_{p+1}}B_{0j_1\cdots j_{D-p-2}}\Bigg), \tag{A2}$$

we can read the fundamental Poisson bracket

$$\{A_{i_1\cdots i_p}(\vec{x}),B_{j_1\cdots j_{D-p-1}}(\vec{y})\}=g\epsilon^{i_1\cdots i_p j_1\cdots j_{D-p-1}}\delta^{(d)}(\vec{x}-\vec{y}), \tag{A3}$$

and obtain the Hamiltonian as

$$H_M^{(p,D)}=\int d^d\vec{x}\Bigg\{\frac{1}{2(D-p-2)!}(\pi_C^{i_1\cdots i_{D-p-2}})^2+\frac{1}{2(p-1)!}(\pi_\omega^{i_1\cdots i_{p-1}})^2+\frac{1}{2(D-p-1)!}[B_{i_1\cdots i_{D-p-1}}+F_{i_1\cdots i_{D-p-1}}(C)]^2$$
$$+\frac{1}{2p!}[A_{i_1\cdots i_p}+F_{i_1\cdots i_p}(\omega)]^2+B_{0i_1\cdots i_{D-p-2}}\Theta_1^{i_1\cdots i_{D-p-2}}+C_{0i_1\cdots i_{D-p-3}}\Theta_2^{i_1\cdots i_{D-p-3}}+\omega_{0i_1\cdots i_{p-2}}\Theta_3^{i_1\cdots i_{p-1}}$$
$$+A_{0i_1\cdots i_{p-1}}\Theta^{i_1\cdots i_{p-1}}\Bigg\}. \tag{A4}$$

In this equation, the quantities

$$\Theta_1^{i_1\cdots i_{D-p-2}}=-\frac{1}{(D-p-2)!}\left[\pi_C^{i_1\cdots i_{D-p-2}}+\frac{g(-1)^{p(D+1)}}{p!}\epsilon^{j_1\cdots j_{p+1}i_1\cdots i_{D-p-2}}\partial_{j_1}A_{j_2\cdots j_{p+1}}\right],$$

$$\Theta_2^{i_1\cdots i_{D-p-3}}=-\frac{1}{(D-p-3)!}\partial_i\pi_C^{ii_1\cdots i_{D-p-3}},$$

$$\Theta_3^{i_1\cdots i_{p-2}}=-\frac{1}{(p-2)!}\partial_i\pi_\omega^{ii_1\cdots i_{p-2}}, \tag{A5}$$

$$\Theta_4^{i_1\cdots i_{p-1}}=\frac{1}{(p-1)!}\left[\pi_\omega^{i_1\cdots i_{p-1}}+\frac{g(-1)^{p(D+1)}}{(D-p-1)!}\epsilon^{i_1\cdots i_{p-1}j_1\cdots j_{D-p}}\partial_{j_1}B_{j_2\cdots j_{D-p}}\right], \tag{A6}$$

are the secondary first class constraints associated with the Lagrange multipliers B_0, C_0, ω_0, and A_0, respectively.

The nonlocal and gauge-invariant "Wilson operators" of this theory are

$$W(\Sigma_p)=\exp\left(i\int_{\Sigma_p}(A+d\omega)\right), \tag{A7}$$

$$\Omega(\Sigma_{D-p-1})=\exp\left(i\int_{\Sigma_{D-p-1}}(B+dC)\right). \tag{A8}$$

They obey the dual algebra

$$W(\Sigma_p)\Omega(\Sigma_{D-p-1})=\exp[-iN(\Sigma_p,\Sigma_{D-p-1})]\Omega(\Sigma_{D-p-1})W(\Sigma_p), \tag{A9}$$

where $N(\Sigma_p,\Sigma_{D-p-1})$ is the oriented number of intersection between the hypersurfaces Σ_p and Σ_{D-p-1}. This model admits two dual geometric representations. In one of them, $W(\Sigma_p)$ appends a p surface Σ_p to the argument of the surface-dependent functional $\Psi(\Sigma_p')$ on which it acts, while $\Omega(\Sigma_{D-p-1})$ counts (i times the exponential of) how many times Σ_p' and Σ_{D-p-1} intersect each other. In the "dual" geometric representation, on the other hand, these roles are interchanged: Ω appends Σ_{D-p-1} surfaces while W counts intersection numbers. This result should be compared with the $2+1$ case discussed in Sec. III.

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