Bosonization and duality in arbitrary dimensions: New results

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A generic massive Thirring model in three space-time dimensions exhibits a correspondence with a topologically massive bosonized gauge action associated with a self-duality constraint, and we write down a general expression for this relationship. We also generalize this structure to d dimensions, by adopting the so-called doublet approach, recently introduced. In particular, a nonconventional formulation of the bosonization technique in higher dimensions (in the spirit of $d=3$) is proposed and, as an application, we show how fermionic (Thirring-like) representations for bosonic topologically massive models in four dimensions may be built-up.

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I. INTRODUCTION

This paper has a twofold purpose: to establish both bosonic first-order (gauge noninvariant) and fermionic Thirring-like formulations for very general topologically massive theories [1–3] in arbitrary dimensions. We show these correspondences by extending the techniques typically used for duality and bosonization in three-dimensional models via the doublet formalism [4,5], which appear insensitive to the space-time dimensionality.

Duality has a fundamental importance in our understanding of various nonperturbative aspects of point-particle and string theories.

Some years ago [6], Deser and Jackiw developed the concept of the parent action approach [7] and showed the duality between the so-called self-dual (SD) theory [8] in three dimensions and the topologically massive gauge theory, referred to as Maxwell-Chern-Simons (MCS) theory. Furthermore, it was shown [9] that the SD model is connected, via the so-called bosonization technique, to the Thirring model:

$$S^{(ferm)}(\psi,\overline{\psi})\equiv\int d^3x\left(\overline{\psi}(i\partial\!\!\!/-m)\psi-\frac{g^2}{2}j^{\mu}j_{\mu}\right),$$

$$j^{\mu}\equiv\overline{\psi}\gamma^{\mu}\psi. \tag{1}$$

Bosonization is the mapping of a quantum field theory for interacting fermions onto an equivalent theory for interacting bosons [10].

Recently, Tripathy and Khare [11] considered a modification of this model by replacing the Maxwell term by $\sqrt{1-F^2}$, the Born-Infeld Lagrangian [12]. Bosonization and dual-correspondences of its topologically massive version, the Born-Infeld-Chern-Simons theory [13,14], have recently been studied motivated by the fact that these theories naturally appear in the context of Dp-branes [15] whose dynamics is described by Born-Infeld-Chern-Simmons-actions in $d=(p+1)$ dimensions. In particular, the D2-brane is described by the 3D-Born-Infeld-Chern-Simmons model. This result is one of the main motivations for the study carried out in our paper, whose purpose is precisely the extension of the above result to a general dimension d. Despite the notion of self-duality in arbitrary dimensions introduced in Ref. [4], which has proven to be a crucial hint in order to establish dual correspondences [5], this extension is not set in a straightforward way. This becomes clearer mainly in Sec. IV, where nonconventional fermionic currents must be introduced in order to describe topologically massive models as purely fermionic theories. Indeed, we succeed in setting up fermionic representations for the topologically massive Cremmer-Scherk-Kalb-Ramond model in four space-time dimensions and also for more general gauge models, for instance involving a Born-Infeld theory topologically coupled to a Kalb-Ramond field (this theory shall be referred to as Born-Infeld-Kalb-Ramond). Some interesting technical particularities also appear when the bosonization procedure, initially thought for $d=3$ [9], is reproduced for $d=4$.

The main goal of this paper is thus to focus all these issues in a more general context.

We shall construct this generalized framework by investigating two principal types of extension for this structure: to consider arbitrary (d) dimensions and more general nonlinearities (arbitrary functions of the squared field strength).

This work is organized as explained below. In Sec. II, we briefly review the bosonization of the Thirring model in three dimensions into a SD model and the SD-MCS duality. In Sec. III, we generalize this to arbitrary nonlinearities in the Maxwell term: we show that this is always equivalent to a SD model in a generalized sense and find a formula to relate the theories of this correspondence. Afterwards, we use a direct procedure to bosonize a generic Thirring model with an arbitrary current-current coupling and connect it to the nonlinearity of its bosonic representations.

Generalization of this structure to higher dimensions is the matter of Sec. IV. We shall show in this section (for the particular case $d=4$, but indicating the way for generalizing to higher dimensions elsewhere), that bosonization may be implemented *in the same way* as in 3D, via the recently introduced doublet formalism [4,5], resulting in an alternative formulation of the bosonization technique in four dimen-

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sions [16]. Such as in the 3D case, fermionic models bosonize to topologically massive ones; in particular, we concentrate our discussion in specially interesting topologically massive gauge theories in four dimensions: Born-Infeld-Kalb-Ramond and Cremmer-Scherk-Kalb-Ramond [1–3].

Finally, in Sec. V, we draw our general conclusions and emphasize the aspects that concern generalization to arbitrary dimensions.

II. A SHORT INTRODUCTORY REVIEW

Let us briefly review how the low-energy sector of a theory of massive, electrically charged, self-interacting fermions (the massive Thirring model) in (2+1)-dimensions may be bosonized into a gauge theory, the Maxwell-Chern-Simons gauge theory [6,9].

A. SD-MCS duality

In $2+1$ dimensions, one currently defines the Hodge-Duality operation by

$$^{\star}a_{\mu}=\frac{1}{m}\ \epsilon_{\mu\nu\lambda}\partial^{\nu}a^{\lambda}, \tag{2}$$

where m is a parameter that renders the * operation dimensionless.

We refer to *self(anti-self)-duality* whenever the relations $^{\star}a=+a,\ -a$ are, respectively, satisfied. Throughout this paper, we shall introduce a parameter $\chi=\pm 1$ to express this self/anti-self-duality.

The so-called self-dual model (Townsend, Pilch, and van Nieuwenhuizen [8]) is described by the following action:

$$S_{\mathrm{SD}}(a)=\int d^3x\left(\frac{\chi}{2m}\epsilon_{\mu\nu\lambda}a^{\mu}\partial^{\nu}a^{\lambda}-\frac{1}{2}a_{\mu}a^{\mu}\right). \tag{3}$$

The equation of motion is the self-duality relation:

$$a_{\mu}=\frac{\chi}{m}\epsilon_{\mu\nu\lambda}\partial^{\nu}a^{\lambda}. \tag{4}$$

This model is claimed to be chiral, and the chirality results defined precisely by this self-duality.

On the other hand, the gauge-invariant combination of a Chern-Simons and a Maxwell term,

$$S_{\mathrm{MCS}}[A]=\int d^3x\left(\frac{1}{4m^2}F^{\mu\nu}F_{\mu\nu}-\frac{\chi}{2m}\epsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda}\right), \tag{5}$$

is the topologically massive theory, which is known to be equivalent [6] to the self-dual model (3). $F_{\mu\nu}$ is the usual Maxwell field strength,

$$F_{\mu\nu}[A]\equiv\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}=2\partial_{[\mu}A_{\nu]}\,. \tag{6}$$

This equivalence may be verified with the parent action approach [7]. We write down the general parent action proposed by Deser and Jackiw in [6], which proves this equivalence:

$$\mathcal{S}_{\mathrm{Parent}}[A,a]=\chi\mathcal{S}_{\mathrm{CS}}[A]-\int d^3x[\epsilon^{\mu\nu\lambda}F_{\nu\lambda}[A]a_{\mu}+ma_{\mu}a^{\mu}], \tag{7}$$

where

$$\mathcal{S}_{\mathrm{CS}}[A]\equiv\int d^3x\,\epsilon^{\mu\nu\lambda}(A_{\mu}\partial_{\nu}A_{\lambda}), \tag{8}$$

is the Chern-Simons action [17].[1]

B. Bosonization and Thirring-MCS correspondence

On the other hand, the (Euclidean) fermionic partition function for the three-dimensional massive Thirring reads as below:

$$Z^{(ferm)}=\int\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-\int(\overline{\psi}(\partial\!\!\!/+m)\psi-(g^2/2)j^{\mu}j_{\mu})d^3x}, \tag{11}$$

with the coupling constant g^2 having dimensions of inverse mass and m is the fermion mass.

It is well-known that this model can be bosonized to the self-dual model [9],

$$Z^{(ferm)}\approx Z^{\mathrm{SD}}, \tag{12}$$

in the low-energy limit.

Thus thanks to the equivalence between Eqs. (3) and (5), one can establish the following bosonization identity:

$$Z^{(ferm)}\approx Z^{\mathrm{MCS}}. \tag{13}$$

This equation, together with Eq. (12), both connected by SD-MCS duality (7), constitutes the kernel of this work. Our main purpose is to actually study the generalizations of this structure along two *independent* lines: for Thirring-like models with an arbitrary current-current coupling: correspondence rule with self-dual and nonlinear topologically massive theories; for arbitrary dimensions: fermionic Thirring-like models in general dimensions correspond to topologically massive theories, such as in 3D.[2] Clearly, both generalizations are suitable to be connected to one another.

[1] In fact, varying this action with respect to f, and eliminating this in the action from the equation of motion, one gets $S_{\mathrm{SD}}(a^{\mu})$. Varying $\mathcal{S}_{\mathrm{Parent}}$ with respect to a, we obtain

$$a^{\mu}=-\frac{1}{2m}\epsilon^{\mu\nu\lambda}\ F_{\nu\lambda}[A]; \tag{9}$$

plugging this back into Eq. (7) and using

$$\epsilon^{\mu\nu\alpha}\epsilon_{\mu\nu\lambda}=2\ \delta^{\alpha}_{\lambda}, \tag{10}$$

we recover the MCS action, Eq. (5).

[2] In general dimensions, the Abelian gauge field generalizes to a pair of field forms.

III. DUALITY BETWEEN NONLINEAR SELF-DUAL AND TOPOLOGICALLY MASSIVE MODELS IN THREE DIMENSIONS

In this section, we shall generalize the correspondence SD-MCS to account for arbitrary nonlinearities. We will show here that the TM model with nonlinearity described by a function $U(F^2)$,[3]

$$S_{U(F^2)}[A]=\int d^3x\ (U(F^{\mu\nu}F_{\mu\nu})-\chi\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda), \tag{14}$$

corresponds to the also general nonlinear self-dual model, with nonlinearity given by a potential $V(a^2)$:

$$\mathcal{S}_{V(a^2)}[a]=\int d^3x V(a_\mu a^\mu)-\chi\mathcal{S}_{\rm CS}[a], \tag{15}$$

which is the nonlinear version of the self-dual action introduced in [8]. We shall refer to this theory as nonlinear self-dual model.

It is useful to briefly clarify why the property of self-duality can be attributed to this model. The equations of motion derived from Eq. (42) are given by

$$a_\mu=\frac{\chi}{2\ V'}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda, \tag{16}$$

where the prime denotes a derivative with respect to the argument. This nonlinear SD model possesses a well-defined self-dual property in the same manner as its linear counterpart. This can be seen as follows. Define a field, ${}^\star a_\mu$, dual to a_μ as

$${}^\star a_\mu\equiv\frac{1}{2\ V'}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda, \tag{17}$$

and repeat this dual operation to find that, as a consequence of the equations of motion (16),

$${}^\star({}^\star a_\mu)=a_\mu. \tag{18}$$

Dual correspondences for this type of nonlinear systems have recently been studied in the particular case of Born-Infeld [14] and also in other specific cases in Ref. [18] [for instance, a power law $U(z)=z^r,\ r\epsilon Q$]; which use a method recently proposed [19] based on the traditional idea of a local lifting of a global symmetry that may be realized by iterative embedding Noether counterterms.

These approaches treat the nonlinearities by introducing auxiliary fields. In this section, we are going to confirm the previous results by adopting the parent action approach and generalize them further *without* introducing auxiliary fields; of course, this enforces the evidence in favor of this so-called gauging Noether method [19] as a useful dualization procedure.

[3]When U is linear the theory is commonly referred to as MCS.

To derive our results, we consider the following nonlinear generalization of the Deser-Jackiw parent action [6]:

$$\mathcal{S}_{\rm Parent}[A,a]=\chi\mathcal{S}_{\rm CS}[A]-\int d^3x\ [\epsilon^{\mu\nu\lambda}F_{\nu\lambda}[A]a_\mu + V(a_\mu a^\mu)]. \tag{19}$$

Varying it with respect to A,

$$\epsilon_{\mu\nu\lambda}\partial^\nu[A^\lambda-a^\lambda]=0, \tag{20}$$

we write its solution as

$$A^\lambda=a^\lambda+\Delta^\lambda, \tag{21}$$

where $\Delta^\lambda=\partial^\lambda\Delta$ is pure gauge. Putting this back into Eq. (19), we recover $S_{V(a^2)}[a]$, Eq. (15).

Now, strictly following the standard program of the master action approach [7], we must vary the parent action with respect to A, and use the resulting equation to solve A in terms of the other field, a. Finally, one shall eliminate A from the parent action.

Varying $\mathcal{S}_{\rm Parent}$ with respect to a, we obtain

$$-2V'(a^2)a^\mu=\epsilon^{\mu\nu\lambda}F_{\nu\lambda}[A], \tag{22}$$

from which it follows that

$$-2a^2V'(a^2)=a_\mu\epsilon^{\mu\nu\lambda}F_{\nu\lambda}[A] \tag{23}$$

and

$$\epsilon^{\mu\nu\lambda}F_{\nu\lambda}[A]\epsilon_{\mu\rho\alpha}F_{\rho\alpha}[A]=2F^2=4a^2[V'(a^2)]^2. \tag{24}$$

Formally, one can solve this for a^2 in terms of $F^2[A]$, and put the result back into Eq. (19) to express this action in terms of the field A, which results to be a TM theory. Defining a function W trough its inverse (whenever it exists),

$$W^{-1}(v)\equiv 2v[V'(v)]^2,\quad v\epsilon\mathbb{R}, \tag{25}$$

and substituting in the parent action by Eq. (23), we recover the generalized nonlinear topologically massive theory; the gauge invariant combination of a Chern-Simons with a nonlinear Maxwell term

$$S_{\rm MCS}[A]=\int d^3x(U(F^{\mu\nu}F_{\mu\nu})-\chi\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda), \tag{26}$$

where the functional U is related to V (that characterizes the nonlinearity of the self-dual model) by the formula:

$$U(q)=-2W(q)V'[W(q)]+V[W(q)], \tag{27}$$

with $q\epsilon\mathbb{R}^+$.

At the end of the next section, we shall mention some more relevant examples of solutions to this equation.

Bosonization of Thirring models with arbitrary (current-current) coupling in $d=3$

In this section, we are going to find bosonization identities for the most general Thirring (fermionic) model, i.e., with an arbitrary dependence on the current-current coupling; this remarkably corresponds to a version of the MCS with *the same* dependence on the square of the field strength [20]. We use a direct procedure such as in the traditional case [Eq. (11)].

The particular case of Born-Infeld-Chern-Simons has already been studied in [11,14]; clearly, these results are contained in the scheme presented here.

In fact, consider a generalization of the Thirring model to have a term depending arbitrarily on j^μ. By relativistic invariance, the only possibility is the generalized nonlinear model:

$$Z_{T(j^2)}^{(ferm)}=\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-\int(8\pi\ \bar{\psi}(\partial\!\!\!/+m)\psi-T(j^\mu j_\mu/2))d^3x}, \quad (28)$$

where the function T is analytic and real-valued.

Next, we eliminate the nonlinear interaction by introducing a vector field, a^μ, and using the identity:

$$e^{\int d^3x T(j^\mu j_\mu/2)}=\int \mathcal{D}a_\mu e^{-\int d^3x\ \mathrm{tr}[V(a^\mu a_\mu)+j^\mu a_\mu]}, \quad (29)$$

where V is related to T. We shall find this relation varying the exponent of the right-hand side with respect to a to obtain

$$-2V'(a^\nu a_\nu)a^\mu=j^\mu, \quad (30)$$

from which there follow the relations

$$-2a^2V'(a^2)=a_\mu j^\mu \quad (31)$$

and

$$j^\mu j_\mu=4a^2V'(a^2). \quad (32)$$

In principle, one can solve for a (or a^2) from Eq. (32) in terms of j^2, and put the result back into Eq. (29) to express this action in terms of the current j, and recover the nonlinear Thirring model. Let us define again the function W through its inverse, assuming it to be

$$W^{-1}(v)=2v[V'(v)]^2, \quad (33)$$

therefore $W(q)=v$. Plugging these equations back into Eq. (29), we recover the generalized nonlinear Thirring Model, Eq. (28), where T is given by

$$T(q)=-2W(q)V'[W(q)]+V[W(q)], \quad (34)$$

Notice that, by virtue of Eq. (33), Eq. (34) coincides with Eq. (27); then, one obtains

$$T(q)=U(q), \quad q\epsilon\mathbb{R}^+, \quad (35)$$

in agreement with the formal correspondence rule [20], $j\to{}^*F$.[4] This shall provide us with a general correspondence bosonization identity for very general Thirring-like models and topologically massive gauge theories.

So, thanks to these results, one can represent the Thirring model as

$$\mathcal{Z}_{T(j^2/2)}=\int \mathcal{D}a_\mu \det(i\partial\!\!\!/+m+a\!\!\!/)e^{-\int d^3x V(a^\mu a_\mu)}. \quad (36)$$

Now, we proceed in the same way as in the typical case $[T(j^2/2),\ \text{linear}]$ to evaluate the determinant. The determinant of the Dirac operator is an unbounded operator and requires regularization.

For $d=3$, the actual computation of this determinant will give parity-breaking and parity-preserving terms that are computed in powers of the inverse mass,

$$\ln\det(i\partial\!\!\!/+m+a\!\!\!/)=\frac{\chi}{16\pi}\mathcal{S}_{\mathrm{CS}}[a]+I_{PC}[a]+O(\partial^2/m^2). \quad (37)$$

Here, $\mathcal{S}_{\mathrm{CS}}$ is given by

$$\mathcal{S}_{\mathrm{CS}}[a]=\int d^3x i\epsilon^{\mu\nu\lambda}(F_{\mu\nu}a_\lambda), \quad (38)$$

it is the Abelian Chern-Simons action, and the parity-preserving contributions, to first-order, lead to the Maxwell action

$$I_{PC}[a]=-\frac{1}{24\pi m}\mathrm{tr}\int d^3xF^{\mu\nu}F_{\mu\nu}. \quad (39)$$

In the low-energy regime, only the Chern-Simons action survives yielding a closed expression for the determinant:

$$\ln\det(i\partial\!\!\!/+m+a\!\!\!/)=\frac{\chi}{16\pi}\mathcal{S}_{\mathrm{CS}}[a]+o(m^{-1}). \quad (40)$$

Using this result, we may write

$$\lim_{m\to\infty}\mathcal{Z}_{T(j^2)/2}^{(ferm)}=\int \mathcal{D}a_\mu \exp(-\mathcal{S}_{V(a^2)}[a]), \quad (41)$$

where $\mathcal{S}_{V(a^2)}$ is the nonlinear version of the self-dual action introduced in [8],

$$\mathcal{S}_{V(a^2)}[a]=\int d^3xV(a_\mu a^\mu)-\chi\mathcal{S}_{\mathrm{CS}}[a]. \quad (42)$$

Therefore, to the leading order in $1/m$, we have established the identification with the nonlinear self-dual theory:

$$\mathcal{Z}_{T(j^2)/2}^{(ferm)}\approx\mathcal{Z}_{V(a^2)}. \quad (43)$$

Finally, recalling that the model with dynamics defined by the nonlinear self-dual action ($\mathcal{S}_{V(a^2)}$) is equivalent to a non-

[4]$()^*$ is the usual Hodge's operation.

linear Maxwell-Chern-Simons theory ($\mathcal{S}_{U(F[a]^2)}$), we use the relation (35) to establish the bosonization identity of the nonlinear massive Thirring model with the topologically massive theory along with the remarkable identification of the potentials, as

$$\mathcal{Z}^{(ferm)}_{U(j^2/2)} \approx \mathcal{Z}_{U(F^2)}. \tag{44}$$

In some cases, it is relatively simple to solve Eq. (33) [or, by virtue of Eq. (35), Eq. (27)]. Let us illustrate this by mentioning some relevant examples: Taking a Thirring model with current-current interaction described by a function $T(j^2) \propto (j^\mu j_\mu)^k$, then, it is equivalent to a self-dual model with nonlinearity described by another power law: $V(a^2) \propto (a^\mu a_\mu)^{k/(2k-1)}$, and by virtue of Eq. (35), the corresponding model has a Maxwell term substituted by $U(F^2) \propto (F^{\mu\nu}F_{\mu\nu})^k$. A simple inspection shows that this result agrees with the one obtained in [18], which enforces the validity of the method proposed there.

The Born-Infeld-Chern-Simons example sets a special case since, as it can be directly verified from Eq. (27), the functional forms of the three models, *coincide* [11,18,19]; i.e., $T(q) = U(q) \propto V(q) \propto \sqrt{1-(\text{const}\times q^2)}$, for all $q \epsilon \mathbb{R}$.

IV. GENERAL DIMENSIONS: BORN-INFELD-KALB-RAMOND AND CREMMER-SCHERK-KALB-RAMOND GAUGE THEORIES

In this section, by considering doublets of field-forms, we show how the structure described above may also be established in d dimensions.

For general dimensions, it is possible to define self- (and anti-self-) duality for pairs (doublets) of form-fields with different ranks [5], close in spirit to the self-duality in $(2+1)$ dimensions due to Townsend, Pilch, and van Nieuwenhuizen [8]. Remarkably, as it has been shown in Ref. [5], the actions which describe this doublet-self-duality result to be *dual-equivalent* to topologically massive theories in d dimensions, which involve *BF* terms (topological coupling between different Abelian gauge forms [1,3]); in the same way as the SD-MCS duality [in $(2+1)$].

In this work, this parallel shall be enforced and generalized with novel consequences on bosonization in high dimensions; besides, new dualities between theories shall be established.

Let us consider a d-dimensional space-time with signature s:[5] we consider the tensor doublet,

$$\mathcal{F} := (f_{\mu_1\cdots\mu_p}, g_{\mu_1\cdots\mu_{d-p-1}}), \tag{45}$$

where f is a $p(<d)$-form [a totally antisymmetric tensor type $(0;p)$], and g is a $(d-p-1)$-form. $\mathcal{F}$ is an element of the space $\Delta_p \equiv \Lambda_p \times \Lambda_{d-[p+1]}$.[6]

There is a well-defined notion of self- (and anti-self-) duality for the objects in this space based on the standard Hodges operation, $()^*$[7] in a fashion extremely similar to the $(2+1)$-dimensional case described above. Consider the action with topological coupling:

$$S_{\text{DSD}}[\mathcal{F}] \equiv \int dx^d \bigg[\frac{-1}{m} g_{\mu_1\cdots\mu_{d-p-1}} \epsilon^{\mu_1\cdots\mu_d} \partial_{\mu_{d-p}} \times f_{\mu_{d-p+1}\cdots\mu_d} + \rho(\mathcal{F}) \bigg], \tag{47}$$

where $\rho(\mathcal{F})$ collects the explicit mass terms as

$$\rho(\mathcal{F}) \equiv \frac{1}{2}([p+1]!\, g_{\mu_1\cdots\mu_{d-p-1}} g^{\mu_1\cdots\mu_{d-p-1}} + (-1)^s [d-p]!\, f_{\mu_1\cdots\mu_p} f^{\mu_1\cdots\mu_p}). \tag{48}$$

For a more concise notation, in terms of forms, consider the following definitions: $d(f,g) \equiv (df, dg)$, and

$$*(df,\ dg) \equiv [*dg, (-1)^{p+1}\ S_{p+1}\ *df], \tag{49}$$

where S_q is a number defined by the double dualization operation, for a q-form A: $*_{(*A)=S_q A}$, this depends on the signature (s) and dimension of the space-time in the form $S_q = (-1)^{s+q[d-q]}$.

Notice that $*$ applied to doublets is defined such that its components are interchanged with a supplementary change of sign for the second component.

In so doing, the equations of motion derived from the action (47) read as

$$\mathcal{F} = \frac{1}{m} {}^* d\mathcal{F}, \tag{50}$$

where m is a mass parameter introduced for dimensional reasons. It may trivially be verified that these equations require that $\mathcal{F}$ satisfies a Proca equation with mass m.

Notice that Eq. (50) looks like Eq. (4). In that sense, we state that S_{DSD} describes *doublet*-self-duality.

In the previous section, we considered nonlinear generalizations of SD models; in the same sense, we may replace ρ by $V(\rho)$ in the action (47) and obtain nonlinear generalizations of the model. Below, we are going to prove that these theories are dual equivalent to also nonlinear topologically massive ones. The form of this correspondence shall result the same as to the 3D case [Eq. (27)], which constitutes an additional motivation to interpretate Eq. (47) as a self-dual system.

[5]That is, this is the number of minuses occurring in the metric.

[6]A more detailed discussion on this construction and its motivations may be found in Refs. [4,5]; however, the notion presented here is sufficient to make this paper self-contained.

[7]For a generic q-form, A, the Hodge dual is defined by

$$({}^*A)^{\mu_{q+1}\cdots\mu_d} = \frac{1}{q!} \epsilon^{\mu_1\cdots\mu_d} A_{\mu_1\cdots\mu_q}. \tag{46}$$

Consider the doublet of gauge fields $\mathcal{A} \equiv (a_{\mu_1\cdots\mu_p}, b_{\mu_1\cdots\mu_{d-p-1}})$ in addition to $\mathcal{F} = (f_{\mu_1\cdots\mu_p}, g_{\mu_1\cdots\mu_{d-p-1}})$; we now propose the following parent action:

$$\mathcal{S}_P[\mathcal{A},\mathcal{F}] = \mathcal{S}_{BF}[\mathcal{A}] - \int dx^d\, \epsilon^{\mu_1\cdots\mu_d}[b_{\mu_1\cdots\mu_{d-p-1}}\partial_{\mu_{d-p}} \times f_{\mu_{d-p+1}\cdots\mu_d} + g_{\mu_1\cdots\mu_{d-p-1}}\partial_{\mu_{d-p}} a_{\mu_{d-p+1}\cdots\mu_d}] + V[\rho(\mathcal{F})], \tag{51}$$

where

$$\mathcal{S}_{BF}[\mathcal{A}] \equiv \int dx^d [b_{\mu_1\cdots\mu_{d-p-1}}\epsilon^{\mu_1\cdots\mu_d}\partial_{\mu_{d-p}} a_{\mu_{d-p+1}\cdots\mu_d}] \tag{52}$$

is the *BF* action.

Varying $\mathcal{S}_P$ with respect to $\mathcal{F}$, we obtain

$$\mathcal{F} = -\frac{1}{V'(\rho)} * d\mathcal{A}, \tag{53}$$

which looks like nonlinear self-duality, Eq. (16). Plugging this relation back into Eq. (51), we recover the generalized (nonlinear) topologically massive action:

$$\mathcal{S}_{TM}[\mathcal{A}] = \mathcal{S}_{BF}[\mathcal{A}] - \int d^d x\, U(\theta), \tag{54}$$

where θ encodes the Maxwell-type terms:

$$\theta \equiv \frac{1}{2}((-1)^s[d-p-1]!(\partial_{[\mu} a_{\mu_1\cdots\mu_p]})^2 + [p+1]!(\partial_{[\mu} b_{\mu_1\cdots\mu_{d-p-1}]})^2). \tag{55}$$

Thus the same algebraic manipulations in the 3D case lead to relating U and V again in terms of V by Eq. (27).

We shall observe that this is invariant under the gauge transformations; $\mathcal{A} \to \mathcal{A} + d\mathcal{D}$, where $d\mathcal{D}$ is a *pure gauge doublet*, i.e., it is a pair of exact differentials of $(p-1, d-p-2)$-forms.

Now, we vary $\mathcal{S}_P$ with respect to $\mathcal{A}$ and obtain:

$$*d(\mathcal{A}-\mathcal{F}) = 0, \tag{56}$$

or in components,

$$*d(a-f)=0,$$
$$*d(b-g)=0. \tag{57}$$

This implies that the differences $a-f$ and $b-g$ may locally be written as exact forms; therefore it is possible to express the solution to these equations as

$$\mathcal{A} = \mathcal{F} + d\mathcal{D}. \tag{58}$$

Putting this back into the action (51), we recover the generalized SD theory up to topological terms:

$$S_{\rm DSD}[\mathcal{F}] \equiv \int dx^d \Big[-\frac{1}{m} g_{\mu_1\cdots\mu_{d-p-1}}\epsilon^{\mu_1\cdots\mu_d}\partial_{\mu_{d-p}} \times f_{\mu_{d-p+1}\cdots\mu_d} + V[\rho(\mathcal{F})]\Big]. \tag{59}$$

Whenever V (or U) is linear, we get the so-called Cremmer-Scherk-Kalb-Ramond-type models, and the present result reproduces the dual correspondence obtained by Harikumar *et al.* in the recent work of Ref. [21] for $d = 3+1$, recently generalized, in [5], to arbitrary dimensions and all possible tensorial ranks.

A. More general nonlinearities

It is not a general fact that $V = V[\rho(\mathcal{F})]$. Besides the requirement of Lorentz invariance, one may also require that the two gauge forms which compose the doublet do not interact with one another, apart from the interaction due to the *BF* term.

Consider $\mathcal{F} \equiv (f_1, f_2)$ and $\mathcal{A} \equiv (a_1, a_2)$, both in Δ_p, and the nonlinearity given by

$$V = V_1[N_2\ (f_1)^2] + (-1)^s V_2[N_1\ (f_2)^2], \tag{60}$$

where $N_i \equiv [p_i+1]!/2$, $i=1, 2$ and p_i denotes the rank of f_i $(p_1+p_2+1=d)$.[8]

Then, the variation of Eq. (51) with respect to $\mathcal{F}$ yields

$$(V_1'[N_2(f_1)^2]f_1; V_2'[N_1(f_2)^2]f_2) = -*d\mathcal{A}. \tag{61}$$

Thus, by repeating the previous calculations, we can readily check the duality between

$$\mathcal{S}_{V_1,V_2}[\mathcal{F}] = \mathcal{S}_{BF}[\mathcal{F}] + \int d^d x (V_1[N_2\ (f_1)^2] + V_2[N_1(f_2)^2]) \tag{62}$$

and

$$\mathcal{S}_{U_1,U_2}[\mathcal{A}] = \mathcal{S}_{BF}[\mathcal{A}] - \int d^d x (U_1[p_2!(da_1)^2] + U_2[p_2!(da_2)^2]). \tag{63}$$

Thus we come to the known relation

$$U_i(q) = -2W_i(q)V_i'[W_i(q)] + V_i[W_i(q)], \quad q \epsilon R^+, \tag{64}$$

where the functions W_i are again defined by

$$W_i^{-1}(v) \equiv 2v[V_i'(v)]^2 \quad v \epsilon R^+. \tag{65}$$

In $d=3+1$, an interesting duality can be established by applying this result to a topologically massive combination of a Born-Infeld with a (rank two) Kalb-Ramond field.

[8] $(f_i)^2$ denotes $f_{\mu_1\cdots\mu_{p_i}} f_{\mu_1\cdots\mu_{p_i}}$.

The Born-Infeld-Kalb-Ramond (BIKR) theory[9]

$$S_{\mathrm{BIKR}}(\mathcal{A})=\int d^4x\Bigg(\beta^2\sqrt{\left[1-\frac{2}{\beta^2}\partial_{[\rho}A_{\mu]}\partial^{[\rho}A^{\mu]}\right]}-\partial_{[\rho}B_{\mu\nu]}\partial^{[\rho}B^{\mu\nu]}+mB_{\mu\nu}\epsilon^{\rho\mu\nu\sigma}\partial_\rho A_\sigma\Bigg), \tag{66}$$

for the doublet of gauge fields $\mathcal{A}=(A_\mu,B_{\mu\nu})$,[10] is dual-equivalent to the first-order model:

$$S_{\mathrm{DSD}}[\tilde{\mathcal{A}}\equiv(\tilde{A}_\sigma,\tilde{B}_{\mu\nu})]=\int d^4x\Bigg(-\beta^2\sqrt{\left[1+\frac{1}{\beta^2}\tilde{A}_\sigma\tilde{A}^\sigma\right]}+\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu}+\frac{1}{m}\tilde{A}_\sigma\epsilon^{\sigma\rho\mu\nu}\partial_{[\rho}\tilde{B}_{\mu\nu]}\Bigg), \tag{67}$$

which is a gauge noninvariant theory, also associated to a nonlinear doublet-self-duality constraint.

B. Bosonization in (3+1)-*d*

Here, we present a novel approach to bosonization in $d=3+1$, valid for length scales long compared with the Compton wavelength of the fermion.

In a four-dimensional massive fermionic model with $U(1)$ charge; just like in the 3D case, one defines a current: $j^\mu\equiv\bar{\psi}\gamma^\mu\psi$, where ψ are N_f four-component Dirac spinors.[11]

However, one can also define a rank-two current, $j^{\mu\nu}\equiv\bar{\psi}\gamma_5[\gamma^\mu,\gamma^\nu]\psi$; let us now define the doublet-current:

$$\mathcal{J}=(j^\mu,j^{\mu\nu}). \tag{68}$$

The appearance of the γ_5 matrix in $j^{\mu\nu}$ follows from requiring that $j^{\mu\nu}$ as well as j^μ are both odd under charge conjugation: $\bar{\psi}\gamma_5[\gamma^\mu,\gamma^\nu]\psi=-\bar{\psi}^C\gamma_5[\gamma^\mu,\gamma^\nu]\psi^C$. Notice that $\mathcal{J}$ is a well-formed doublet ($\mathcal{J}\,\epsilon\,\Delta_p$).

Now, we can write a nonconventional (Euclidean)[12] massive Thirring model in a similar fashion to the 3D case:

$$\mathcal{Z}^{(ferm)}\equiv\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \times e^{-\int(\bar{\psi}(\partial\!\!\!/+m)\psi-(g^2/2N_fm)[2\ j_{\mu\nu}j^{\mu\nu}-j_\mu j^\mu])d^4x}, \tag{69}$$

where m is the fermion mass and g a coupling constant of the model, such that g^2 have dimensions of inverse mass.

[9]Here, Born-Infeld means that the free action is proportional to $\sqrt{1-\mathrm{const}\times F_{\mu\nu}F^{\mu\nu}}$.

[10]β is a parameter introduced for dimensional reasons.

[11]In this calculation, N_f will be simply considered as a parameter.

[12]In an Euclidean space-time, $j^{\mu\nu}$ is purely imaginary; thus, in order to render it real, we may redefine this bilinear by multiplying it by an i.

We are going to show that this bosonizes into the CSKR model, a gauge, topologically massive theory.

Such as in the 3D case, we get the identity,

$$e^{-(g^2/2\ N_fm)\int d^4x[2\ j_{\mu\nu}j^{\mu\nu}-j_\mu j^\mu]}=\int \mathcal{D}\mathcal{A}\ e^{\int d^4x\ \mathrm{tr}(1/2[1/2b_{\mu\nu}b^{\mu\nu}-a_\mu a^\mu]+(g/\sqrt{mN_f})[b_{\mu\nu}j^{\mu\nu}-a_\mu j^\mu])}, \tag{70}$$

which introduces the doublet of bosonic fields $\mathcal{A}\equiv(a_\mu,b_{\mu\nu})$.

Defining the doublet-slash by

$$\mathcal{A}\!\!\!/\equiv\gamma^\mu a_\mu+\gamma_5[\gamma^\mu,\gamma^\nu]b_{\mu\nu}, \tag{71}$$

the partition function reduces to

$$\mathcal{Z}^{(ferm)}=\int \mathcal{D}\mathcal{A}\det(i\partial\!\!\!/+m+\mathcal{A}\!\!\!/)e^{1/2\int d^4x[1/2b_{\mu\nu}b^{\mu\nu}-a_\mu a^\mu]}. \tag{72}$$

Next, we must evaluate this determinant.

A straightforward perturbative expansion yields

$$S_{eff}[\mathcal{A},m]=N_f\mathrm{tr}[\ln(\partial\!\!\!/+m)]+N_f\ \frac{g}{\sqrt{N_fm}}\mathrm{tr}\left(\frac{1}{\partial\!\!\!/+m}\mathcal{A}\!\!\!/\right)+\frac{N_f}{2}\ \left(\frac{g^2}{N_fm}\right)\mathrm{tr}\left(\frac{1}{\partial\!\!\!/+m}\mathcal{A}\!\!\!/\frac{1}{\partial\!\!\!/+m}\mathcal{A}\!\!\!/\right)+\cdots. \tag{73}$$

The first term is just the free $(A=0)$ case, which is subtracted, while the second term is simply two tadpoles accommodated in the doublet. Thus we draw our attention to the quadratic term (in the bosonic fields $\mathcal{A}$) in the effective action. In momentum space, this reads

$$S_{eff}^{quad}[A,m]=\frac{g^2}{2m}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\ \frac{d^4k}{(2\pi)^4}\times\left[\mathcal{A}\!\!\!/(-p)\frac{ip\!\!\!/+ik\!\!\!/-m}{(p+k)^2+m^2}\ \mathcal{A}\!\!\!/(p)\frac{ik\!\!\!/-m}{k^2+m^2}\right]. \tag{74}$$

Terms of the form $\mathcal{A}\!\!\!/(-p)k\!\!\!/\mathcal{A}\!\!\!/(p)k\!\!\!/$ and $\mathcal{A}\!\!\!/(-p)p\!\!\!/\mathcal{A}\!\!\!/(p)k\!\!\!/$ in the numerator of the integrand will contribute at most to second order in p_μ.[13] Since we are seeking for the low energy limit, like in the 3D case, we can approximate this by

[13]They also cancel the terms tr $[m^2\mathcal{A}\!\!\!/(-p)\mathcal{A}\!\!\!/(p)]$ that appear in the numerator.

$$S^{quad}_{eff}[A,m]\approx i\frac{g^2}{2m}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\times\left[\not{A}(-p)\frac{(\not{p}+\not{k})\not{A}(p)-\not{A}(p)\not{k}}{[(p+k)^2+m^2][k^2+m^2]}\right], \tag{75}$$

using also the fact that the trace of an odd number of gamma-matrices is zero. We obtain only a topological contribution:

$$S^{quad}_{eff}[A,m]\approx\frac{g^2}{2}\int\frac{d^4p}{(2\pi)^4}[a_\mu(-p)\Gamma^{\mu\nu\alpha}(p)b_{\nu\alpha}(p)], \tag{76}$$

where, by virtue of the special property of the gamma matrices (here, Euclidean) in $(3+1)$-d,

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma_5[\gamma^\rho,\gamma^\alpha])=-8\epsilon^{\mu\nu\rho\alpha}, \tag{77}$$

the kernel takes the form:

$$\Gamma^{\mu\nu\alpha}(p,m)=\epsilon^{\mu\rho\nu\alpha}p_\rho\Pi(p^2,m), \tag{78}$$

where $\Pi(p^2,m)$ is the contribution corresponding to the one-fermion-loop self-energy diagram. For the sake of computing the loop integral and factoring out the divergent part, we go over to $d=4-\epsilon$-dimensions, following the procedure of dimensional regularization (see Ref. [22]):

$$\Pi(p^2,m)=(\mu)^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{1}{[(p+k)^2+m^2][k^2+m^2]}=\frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma-\ln\frac{p^2}{\mu^2}-I\left(\frac{p^2}{m^2}\right)\right]+o(\epsilon), \tag{79}$$

μ is a parameter and the finite part reads as below:

$$I\left(\frac{p^2}{m^2}\right)=a\ln a-(a-1)\ln(a-1)+b\ln|b|+(1-b)\ln(1-b)-2, \tag{80}$$

where

$$a=\frac{1}{2}\left[1\pm\sqrt{1+4\frac{m^2}{p^2}}\right],$$
$$b=\frac{1}{2}\left[1\pm\sqrt{1-4\frac{m^2}{p^2}}\right]. \tag{81}$$

In the long wavelength $(p\to 0)$ and large mass $(m\to\infty)$ limit, $a\to\infty$, $b\to-\infty$; thus it is easily verifiable that $I\to -2$. Therefore we find the finite part of the kernel:

$$\Gamma^{\mu\nu\alpha}(p,m)\sim\frac{2}{(4\pi)^2}\epsilon^{\mu\rho\nu\alpha}p_\rho. \tag{82}$$

Inserting the leading term into the quadratic effective action (76) and going back to configuration space (Lorentzian), we find an induced *BF* term

$$S_{eff}=-8\frac{g^2}{(4\pi)^2}\int d^4x\,\epsilon^{\mu\nu\rho\alpha}a_\mu\partial_\nu b_{\rho\alpha}=-8\frac{g^2}{(4\pi)^2}\mathcal{S}_{BF}(\mathcal{A}). \tag{83}$$

Putting this result back into Eq. (36), we obtain

$$\mathcal{Z}^{(ferm)}\approx\int\mathcal{D}\mathcal{A}\;e^{\mathcal{S}_{BF}(\mathcal{A})+1/2\int d^4x[1/2b_{\mu\nu}b^{\mu\nu}-a_\mu a^\mu]}, \tag{84}$$

which, via the correspondence proven before, is equivalent to the gauge invariant Cremer-Sherk-Kalb-Ramond model which describes a massive spin-one (bosonic) particle. The boson mass is given by the inverse of the factor in front of $\mathcal{S}_{BF}$ in Eq. (83), $m^{-1}_{\mathrm{boson}}\sim g^2/2\pi^2$.

Notice also that, if one rescales the doublet current as $(j^\mu,j^{\mu\nu})\to(sj^\mu,tj^{\mu\nu})$, the single effect of this is that the boson mass results are rescaled as $m_{\mathrm{boson}}\to m_{\mathrm{boson}}/(st)$.

Finally, a *fermionic representation* for the CSKR model is given by the partition function:

$$\mathcal{Z}^{(ferm)}=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\times e^{-\int(\bar{\psi}(\not{\partial}+m)\psi-(g^2/2N_fm)[2j_{\mu\nu}j^{\mu\nu}+j_\mu j^\mu])d^3x}\approx\mathcal{Z}_{\mathrm{CSKR}}. \tag{85}$$

Now, by repeating the calculations of the previous sections, one may study nonlinear generalizations of the fermionic model (69). In fact, substituting $j_{\mu\nu}j^{\mu\nu}+j_\mu j^\mu\to U_1(j_{\mu\nu}j^{\mu\nu})+U_2(j_\mu j^\mu)$ in the expression (69), one can bosonize this into a nonlinear SD theory given by Eq. (62),[14] whose nonlinearities are related to $U_{1/2}$ by the expressions (64). And once more, for composing this with the duality proven in Sec. IV B, this corresponds to a topologically massive gauge theory (so as in the Thirring-MCS correspondence) given by the action (63).

In particular, we can write down the fermionic counterpart of the Born-Infeld-Kalb-Ramond gauge theory. This may be cast as

$$\mathcal{Z}_{BI-KR}\approx\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\times e^{-\int(\bar{\psi}(\not{\partial}+m)\psi-(g^2/2N_fm)[2j_{\mu\nu}j^{\mu\nu}+\beta^2\sqrt{1-j_\mu j^\mu/\beta^2}])d^3x}. \tag{86}$$

[14]For simplicity, we are discussing the case $d=4$ and doublets in Δ_1.

Let us conclude this section by mentioning that the operator correspondence underlying this structure reads as

$$\mathcal{J} \to {}^{*}d\mathcal{A}. \tag{87}$$

V. FINAL REMARKS

We have presented here a new approach to study the bosonization of a model of interacting fermions in terms of topologically massive models, similar to what happens in $d=3$. In general, this involves two gauge fields with different tensorial ranks (*BF*-type theories). We have actually discussed this point for $d=4$, but we showed the road to reproduce this construction in higher dimensions (one simply should build up the currents as elements in some Δ_p). These results have been emphasized for theories which appear to be very important in field theory and/or dynamics of D*p*-branes [Chern-Simons-Kalb-Ramond (CSKR) and BIKR theories].

A comment is in order that regards the two-form current, $j^{\mu\nu}$, appearing in the Thirring model. It may look somewhat artificial, since it is not necessarily conserved. Nevertheless, we try here to show that it is actually a natural piece of the formalism, since it is related to topologically massive gauge invariant models: it is crucial for the attainment of a bosonic topologically massive theory in the large fermionic mass limit. Bosonization in the case of nonconserved fermionic currents has already been contemplated by other authors [23].

We conclude this paper by stressing a motivation for the proposed generalization of the self-duality to $d>3$, via doublets [4,5]. It appears to be appropriate to highlight such a point, since, despite the use of the doublet procedure proposed here to bosonize a 3D-Thirring model, one recovers the well-known results in 3D, i.e., the doublet disappears and reduces to a single dynamic self-dual field. In fact, for a Thirring model in 3D, with a $U(1)$ interaction, we can only construct a current doublet in Δ_1, $\mathcal{J}=(j^\mu, j^\mu)$, where $j^\mu \equiv \overline{\psi}\gamma^\mu \psi$. After introducing, as usual, a bosonic doublet $\mathcal{A}=(a_\mu, b_\mu)$, the partition function may be cast as

$$\mathcal{Z}_{(ferm)} \equiv \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\, \mathcal{D}a\mathcal{D}b \times e^{-\int(\overline{\psi}(\not{\partial}+m)\psi-(g^2/2)j^\mu\, [a_\mu+b_\mu]-(a^2+b^2)/2)d^4x}. \tag{88}$$

By changing coordinates to $c^{\pm}_\mu \equiv (a_\mu \pm b_\mu)/2$, the field c^+_μ appears decoupled from c^-_μ (the latter without dynamics), whose action, induced by the fermionic model, is precisely given by a self-dual model [Eq. (3)], as expected. This fact seems to be an additional motivation to think of the (current) doublets as more general objects.

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