Non-Abelian fluid dynamics in Lagrangian formulation

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(Received 16 October 2002; published 27 January 2003)

Non-Abelian extensions of fluid dynamics, which can have applications to the quark-gluon plasma, are given. These theories are presented in a symplectic or Lagrangian formulation and involve a fluid generalization of the Kirillov-Kostant form well known in Lie group theory. In our simplest model the fluid flows with velocity $\mathbf{v}$ and, in the presence of non-Abelian chromoelectric or magnetic $\mathbf{E}^a/\mathbf{B}^a$ fields, the fluid feels a Lorentz force of the form $Q_a\mathbf{E}^a+(\mathbf{v}/c)\times Q_a\mathbf{B}^a$, where Q_a is a space-time local non-Abelian charge satisfying a fluid Wong equation $[(D_t+\mathbf{v}\cdot\mathbf{D})Q]_a=0$ with gauge covariant derivatives.

DOI: 10.1103/PhysRevD.67.025013 PACS number(s): 03.50.Kk, 11.10.Ef, 47.10.+g, 47.75.+f

I. INTRODUCTION

When different species of particles compose a fluid, it may be that the collective variables describing the fluid (density ρ, velocity $\mathbf{v}$ in an Eulerian formulation), and the dynamical equations that govern them, can reflect the compositional variety. In this paper we develop models for fluids where the variety of constituents arises from an internal symmetry group. We envision applying our theory to high densities of non-Abelian quarks, with or without additional interaction to a dynamical gauge field.

High energy collisions of heavy nuclei can produce a plasma state of quarks and gluons. This new state of matter has recently been of great interest both theoretically and in experiments at the Relativistic Heavy Ion Collider (RHIC) facility and at CERN. Most of the theoretical work in this area has been based on perturbative quantum chromodynamics at high temperatures with hard thermal loop resummations and other improvements [1]. This can be a good description at high temperatures and for plasma states that are not too far from equilibrium. An alternative approach, which may be more suitable for nondilute plasmas or for situations far from equilibrium, would be to use a fluid mechanical description.

It is well known that the equations of fluid mechanics can be derived from particle dynamics by taking suitable averages of Boltzmann-type equations. Specifically for the quark-gluon plasma, some work along these lines was done many years ago using single particle kinetic equations [2]. These kinetic equations form a hierarchy, the so-called Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, involving higher and higher correlated N-particle distribution functions. To be able to solve them, one needs to truncate the hierarchy, very often at just the single particle distribution function. Therefore, the feasibility of solving these equations limits the kinetic approach to dilute systems near equilibrium, where the truncation can be justified. We might therefore expect that the regime of validity of equations derived within kinetic theory is likewise limited. However, fluid dynamical equations can also be derived from very general principles, showing that they have a much wider regime of validity, and, indeed in practice, we apply them over such a wider range. This is the ''universality'' of fluid dynamics.

In the context of a non-Abelian plasma, in analogy with the ordinary fluid mechanics, we may therefore ask for an *a priori* derivation of a non-Abelian fluid mechanics, which incorporates the non-Abelian degrees of freedom, coupling to a non-Abelian gauge field, etc. This theory may be valid for dense, nonperturbative and nondilute systems. Further, a canonical or symplectic formulation (at least in the conservative limit) is important for quantization. At the same time, the analysis based on the kinetic equations remains useful to us as a guide for arriving at a Lagrangian, canonical description.

Our goal is to provide plausible equations that generalize to the non-Abelian situation the continuity

$$\partial_t\rho(t,\mathbf{r})+\nabla\cdot(\rho(t,\mathbf{r})\mathbf{v}(t,\mathbf{r}))=0 \tag{1}$$

and Euler equations

$$\partial_t\mathbf{v}(t,\mathbf{r})+\mathbf{v}\cdot\nabla\mathbf{v}(t,\mathbf{r})=\textit{force} \tag{2}$$

which govern conventional Eulerian perfect fluids. In Eq. (2) *force* denotes forces acting on a unit volume of the fluid. The above four equations or their relativistic versions can be equivalently presented as the four conservation laws for the energy and momentum densities (either nonrelativistic or relativistic).

In the non-Abelian generalizations one must first fix the group transformation law for the collective variables and the covariance properties of the equations, which now may number more than four, and cannot be comprised solely in energy-momentum conservation. An obvious additional equation is the (covariant) conservation equation for the non-

Abelian current, but the relation of this current to the collective variables needs to be established.

Our constructions begin by postulating collective variable Lagrangians that are invariant against space-time translations and internal symmetry transformations. Thus energy-momentum conservation and current conservation are assured by Noether's theorem, while the Euler-Lagrange equations describe the temporal evolution of the collective variables. In Sec. II we review the Lagrangian as well as other aspects of nonrelativistic and relativistic Abelian fluids. This is done to motivate and contrast our non-Abelian constructions, which we develop in Secs. III and IV. Derivations are postponed to the Appendixes; in the text only results are presented and discussed.

II. REVIEW OF ABELIAN FLUIDS

Equations (1) and (2) can be found as the Euler-Lagrange equations for the Lagrange density [3]

$$\mathcal{L}=-j^{\mu}a_{\mu}+\frac{1}{2}\rho\mathbf{v}^2-V \tag{3}$$

where our coordinates are $x^{\mu}=(ct,\mathbf{r})$. The various quantities are defined as

$$j^{\mu}=(c\rho,\rho\mathbf{v}) \tag{4}$$

$$a_{\mu}\equiv\partial_{\mu}\theta+\alpha\partial_{\mu}\beta \tag{5}$$

and V is a ρ-dependent potential that gives rise to the *force*. (In the usual theory, V depends only on ρ, but we allow dependence on derivatives of ρ as well.) The canonical 1-form is determined by the first contribution to $\mathcal{L}$ while the dynamics is encoded in the $\frac{1}{2}\rho\mathbf{v}^2-V$ term. Specifically, variation of $\mathbf{v}$ shows that $\mathbf{v}$ is given by the Clebsch parametrization [4]

$$\mathbf{v}=\nabla\theta+\alpha\nabla\beta. \tag{6}$$

A variation of θ results in Eq. (1), while a variation of the Clebsch potentials (α,β) produces subsidiary equations

$$\begin{aligned} j^{\mu}\partial_{\mu}\alpha&=\rho(\partial_t\alpha+\mathbf{v}\cdot\nabla\alpha)=0\\ j^{\mu}\partial_{\mu}\beta&=\rho(\partial_t\beta+\mathbf{v}\cdot\nabla\beta)=0 \end{aligned} \tag{7}$$

which are needed in the subsequent derivation of the Euler equation. Finally, the "Bernoulli" equation emerges by varying ρ:

$$\partial_t\theta+\alpha\partial_t\beta+\frac{\mathbf{v}^2}{2}=-\frac{\delta}{\delta\rho}\int dr\,V. \tag{8}$$

$[(\delta/\delta\rho)\int dr\,V$ is just the Euler-Lagrange derivative of $V.]$ The Euler equation (2) then follows by taking the gradient of Eq. (8) and using Eq. (7). The force on the right of Eq. (2) is now seen to be

$$\mathit{force}=-\nabla\frac{\delta}{\delta\rho}\int dr\,V. \tag{9}$$

The energy $\mathcal{E}$ and momentum $\boldsymbol{\mathcal{P}}$ densities carried by the fields are

$$\mathcal{E}=\frac{1}{2}\rho\mathbf{v}^2+V \tag{10}$$

$$\boldsymbol{\mathcal{P}}=\rho\mathbf{v}. \tag{11}$$

These satisfy continuity equations as a consequence of (or equivalent to) Eqs. (1) and (2).

Interaction with electromagnetic fields is accommodated by including the Maxwell Lagrange density and adding the electromagnetic vector potential to a_{μ}, whereupon j^{μ} becomes the electromagnetic current. This gives rise to the Lorentz force in the Euler equations, thereby producing magnetohydrodynamics. Note that in this model the mass density moves with the same velocity as the charge density as is seen by inspecting $\boldsymbol{\mathcal{P}}$ and $\mathbf{j}$.

An interesting realization of the above equations is provided by Madelung's "hydrodynamical" rewriting of the Schrödinger equation [5]

$$i\hbar\,\partial_t\psi(t,\mathbf{r})=-\frac{\hbar^2}{2m}\nabla^2\psi(t,\mathbf{r}) \tag{12}$$

where the wave function is presented as

$$\psi(t,\mathbf{r})=\sqrt{\rho(t,\mathbf{r})}\,e^{im\theta(t,\mathbf{r})/\hbar}. \tag{13}$$

The imaginary part of Eq. (12) reproduces the continuity equation (1), when $\nabla\theta$ is identified as the velocity $\mathbf{v}$, with vanishing vorticity $\nabla\times\mathbf{v}=0$. Also the quantum current $(\hbar/m)\mathrm{Im}\,\psi^*\nabla\psi$ becomes $\rho\mathbf{v}$. The real part of Eq. (12) gives the Bernoulli equation with α and β set to zero and with a "quantum" force derived from

$$V=\frac{\hbar^2}{2m^2}(\nabla\sqrt{\rho})^2=\frac{\hbar^2}{8m^2}\frac{(\nabla\rho)^2}{\rho}. \tag{14}$$

The Euler equation (2) follows by taking the gradient of the Bernoulli equation.

The Lagrange density for a relativistic fluid is chosen as [3]

$$\mathcal{L}=-j^{\mu}a_{\mu}-f(n),\quad n\equiv\sqrt{j^{\mu}j_{\mu}/c^2}. \tag{15}$$

The canonical 1-form is as in the nonrelativistic case (3), (4), (5) and now j^{μ} is written in the Eckart form [6]

$$\begin{aligned} &j^{\mu}=nu^{\mu},\quad u^{\mu}u_{\mu}=c^2,\quad u^{\mu}=(c,\mathbf{v})/\sqrt{1-v^2/c^2},\\ &n=\rho\sqrt{1-v^2/c^2}. \end{aligned} \tag{16}$$

$f(n)$ encodes the dynamics; for the free theory $f(n)=nc^2$. Variations of θ, α and β still produce Eqs. (1) and (7) while a variation of j^{μ} evaluates a_{μ}

$$a_{\mu}=-\frac{u_{\mu}}{c^2}f'(n) \tag{17}$$

where the prime denotes differentiation with respect to the argument of the function. We call (17) the relativistic Bernoulli equation, which leads to the Euler equation with the following steps. The curl of Eq. (17) reads

$$\partial_\mu a_\nu - \partial_\nu a_\mu = \partial_\nu\left(\frac{u_\mu}{c^2} f'(n)\right) - \partial_\mu\left(\frac{u_\nu}{c^2} f'(n)\right). \tag{18}$$

Since according to Eq. (6) the left side is equal to $\partial_\mu\alpha\partial_\nu\beta - \partial_\nu\alpha\partial_\mu\beta$, it vanishes by Eq. (7) when projected on u^μ. Thus there remains the relativistic Euler equation

$$\frac{u^\mu}{c^2}\partial_\mu(u_\nu f'(n)) - \frac{u^\mu}{c^2}\partial_\nu(u_\mu f'(n)) = \frac{u^\mu}{c^2}\partial_\mu(u_\nu f'(n)) - \partial_\nu f'(n) = 0. \tag{19}$$

It is easy to show that in the nonrelativistic limit, with

$$n \approx \rho - \frac{1}{2c^2}\rho \mathbf{v}^2 \tag{20}$$

$$u^\mu \approx (c, \mathbf{v}) \tag{21}$$

and

$$f = nc^2 + V(n), \tag{22}$$

the Lagrange density (15) passes to Eq. (3) (apart from an irrelevant term $-\rho c^2$) and the spatial component of Eq. (19) reproduces the Euler equation (2).

The energy-momentum tensor for Eq. (15) reads, after a_μ is eliminated with Eq. (17) [7]

$$T^{\mu\nu} = -g^{\mu\nu}[n f'(n) - f(n)] + \frac{u^\mu u^\nu}{c^2} n f'(n). \tag{23}$$

Its divergence $\partial_\mu T^{\mu\nu}$ can be expressed as

$$\partial_\mu T^{\mu\nu} = \partial_\mu(n u^\mu)\frac{u^\nu f'(n)}{c^2} + n\left[\frac{u^\mu}{c^2}\partial_\mu(u^\nu f'(n)) - \partial^\nu f'(n)\right]. \tag{24}$$

The first term on the right-hand side vanishes by the virtue of the continuity equation (1) and the second term vanishes by the relativistic Euler equation (19), or vice versa: since the two terms in Eq. (24) are linearly independent (the first is parallel to u^ν and the second is orthogonal to it), conservation of $T^{\mu\nu}$ implies Eqs. (1) and (19). Relativistic magnetohydrodynamics is achieved, as in the nonrelativistic case, by adding the electrodynamical potential to a^μ.

Finally we record one fact, which we shall use below, about the alternative, Lagrange, formulation of fluid mechanics. Here one describes the fluid with co-moving coordinates $\mathbf{X}(t,\mathbf{x})$ where $\mathbf{x}$ is a (continuously) varying particle label. It may be chosen to coincide with $\mathbf{X}$ at $t=0$. The relation between Euler and Lagrange variables is the following. For the density we have

$$\rho(t,\mathbf{r}) = \int dx\ \delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r}) \tag{25}$$

which is normalized to unity at $t=0$. (The integral and the δ function follow the dimensionality of space.) Evidently, in the course of the x integral, the δ function evaluates $\mathbf{x}$ at an expression $\boldsymbol{\chi}(t,\mathbf{x})$ such that $\mathbf{X}(t,\boldsymbol{\chi}(t,\mathbf{r})) = \mathbf{r}$, i.e., $\boldsymbol{\chi}$ is the inverse of $\mathbf{X}$. There is also a Jacobian. Thus

$$\frac{1}{\rho} = \det \frac{\partial X^i}{\partial x^j}\bigg|_{\mathbf{x}=\boldsymbol{\chi}}. \tag{26}$$

The Euler velocity is given by

$$\mathbf{v} = \partial_t \mathbf{X}|_{\mathbf{x}=\boldsymbol{\chi}} \tag{27}$$

or

$$\rho(t,\mathbf{r})\mathbf{v}(t,\mathbf{r}) = \int dx\ \partial_t \mathbf{X}(t,\mathbf{x})\,\delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r}). \tag{28}$$

A simple calculation shows that the density ρ and current $\mathbf{j} = \rho\mathbf{v}$ defined by these equations satisfy the continuity equation (1). [The Euler equation follows by differentiating Eq. (28) with respect to time, positing a force that determines $\partial^2\mathbf{X}/\partial t^2$, and performing the corresponding $\mathbf{x}$ integral; we shall not need this here.]

III. NON-ABELIAN MODELS BASED ON A PARTICLE SUBSTRATUM

A. Non-Abelian current

Before presenting a specific model, we give a general analysis of the non-Abelian current $J_a^\mu = (c\rho_a, \mathbf{J}_a)$. The conventional formula for the current of a single, non-Abelian point particle, moving in a 4-dimensional space-time $\{t,\mathbf{r}\}$, along a space-time path $X^\mu(\tau)$ (τ parametrizes the path) is

$$J_a^\mu(t,\mathbf{r}) = c\int d\tau\, \mathcal{Q}_a(\tau)\frac{dX^\mu(\tau)}{d\tau}\,\delta(X^0(\tau) - ct)\,\delta(\mathbf{X}(\tau) - \mathbf{r}). \tag{29}$$

This is covariantly conserved,

$$\partial_\mu J_a^\mu + f_{abc}A_\mu^b J_c^\mu \equiv (D_\mu J^\mu)_a = 0, \tag{30}$$

provided that $\mathcal{Q}_a$ satisfies the Wong equation [8]

$$\frac{d\mathcal{Q}_a(\tau)}{d\tau} + f_{abc}\frac{d}{d\tau}X^\mu(\tau)A_\mu^b(X(\tau))\mathcal{Q}_c(\tau) = 0, \tag{31}$$

For several particles $\mathcal{Q}_a$ and X^μ are indexed by a discrete particle label n, which is summed in the definition of the current J_a^μ. In a continuum limit $n \to \mathbf{x}$ we have

$$J_a^\mu(t,\mathbf{r}) = c\int dx\, d\tau\, \mathcal{Q}_a(\tau,\mathbf{x})\frac{\partial X^\mu(\tau,\mathbf{x})}{\partial\tau} \times \delta(X^0(\tau,\mathbf{x})-ct)\,\delta(\mathbf{X}(\tau,\mathbf{x})-\mathbf{r}) \tag{32}$$

$$\frac{\partial}{\partial\tau}\mathcal{Q}_a(\tau,\mathbf{x}) + f_{abc}\frac{\partial}{\partial\tau}X^\mu(\tau,\mathbf{x})A_\mu^b(X(\tau,\mathbf{x}))\mathcal{Q}_c(\tau,\mathbf{x}) = 0. \tag{33}$$

[Notice that, since we replace a sum over n with an integral over $\mathbf{x}$, $\mathcal{Q}_a(\tau,\mathbf{x})$ is now a charge density.] The parametrization may be fixed at $X^0(\tau,\mathbf{x}) = c\tau$ so that the equations read

$$\rho_a(t,\mathbf{r}) = \int dx\ \mathcal{Q}_a(t,\mathbf{x})\,\delta(\mathbf{X}(t,\mathbf{x})-\mathbf{r})$$
$$\mathbf{J}_a(t,\mathbf{r}) = \int dx\ \mathcal{Q}_a(t,\mathbf{x})\,\partial_t\mathbf{X}(t,\mathbf{x})\,\delta(\mathbf{X}(t,\mathbf{x})-\mathbf{r}) \tag{34}$$

$$\partial_t\mathcal{Q}_a(t,\mathbf{x}) + f_{abc}[cA_0^b(t,\mathbf{X}(t,\mathbf{x})) + \partial_t\mathbf{X}(t,\mathbf{x})\cdot\mathbf{A}^b(t,\mathbf{X}(t,\mathbf{x}))]\mathcal{Q}_c(t,\mathbf{x}) = 0. \tag{35}$$

Observe that just as in the Abelian case discussed above, the $\mathbf{x}$ integration evaluates $\mathbf{x}$ at $\boldsymbol{\chi}(t,\mathbf{r})$, the inverse of $\mathbf{X}$, and the Jacobian factor $(\det\partial X^i/\partial x^j)^{-1}$ is just the Abelian charge density ρ [see Eq. (26)]. Thus

$$\rho_a(t,\mathbf{r}) = Q_a(t,\mathbf{r})\rho(t,\mathbf{r})$$
$$\mathbf{J}_a(t,\mathbf{r}) = Q_a(t,\mathbf{r})\rho(t,\mathbf{r})\mathbf{v}(t,\mathbf{r}) \tag{36}$$

or

$$J_a^\mu(t,\mathbf{r}) = Q_a(t,\mathbf{r})j^\mu(t,\mathbf{r}), \tag{37}$$

where

$$Q_a(t,\mathbf{r}) = \mathcal{Q}_a(t,\mathbf{x})|_{\mathbf{x}=\boldsymbol{\chi}} \tag{38}$$

$$\rho(t,\mathbf{r})Q_a(t,\mathbf{r}) = \int dx\,\mathcal{Q}_a(t,\mathbf{x})\,\delta(\mathbf{X}(t,\mathbf{x})-\mathbf{r}). \tag{39}$$

Moreover, differentiating Eq. (39) with respect to time and using Eqs. (1) and (35) results in an equation for $\partial Q^a/\partial t$,

$$\partial_t Q_a(t,\mathbf{r}) + \mathbf{v}(t,\mathbf{r})\cdot\nabla Q_a(t,\mathbf{r}) = -f_{abc}(cA_0^b(t,\mathbf{r}) + \mathbf{v}(t,\mathbf{r})\cdot\mathbf{A}^b(t,\mathbf{r}))Q_c(t,\mathbf{r}) \tag{40}$$

which can also be written as

$$j^\mu(D_\mu Q)_a = 0. \tag{41}$$

This is analogous to the Abelian equations (7). Equations (40) and (41) can be understood from the fact that the covariantly conserved current (30) factorizes according to Eq. (37) into a group variable Q_a and a conserved Abelian current j^μ. Consistency of Eqs. (1), (30) and (37) then enforces Eq. (41).

We recognize that formulas (32) and (34) are the non-Abelian version of the Lagrange variable–Euler variable correspondence [see Eqs. (25) and (28)]. Also, Eqs. (35), (40) and (41) are the field generalizations of the particle Wong equation (31), with Eq. (35) being presented in the Lagrange formalism and Eqs. (40), (41) in the Euler formalism. The decomposition of the non-Abelian current in Eq. (37) is the non-Abelian version of the Eckart decomposition (16) [2]. Indeed, Eq. (37) may be further factored as in Eq. (16)

$$J_a^\mu(t,\mathbf{r}) = Q_a(t,\mathbf{r})n(t,\mathbf{r})u^\mu(t,\mathbf{r}). \tag{42}$$

In the remainder of Sec. III, we are guided in our construction of a dynamical model for non-Abelian fluid mechanics and "color" hydrodynamics by the above properties of the non-Abelian current, which follow from the very general arguments, based on a particle picture for the substratum of a fluid. In Sec. IV we present a different model, based on a field-theoretic fluid substratum.

B. A model for non-Abelian color hydrodynamics

The model that we present is based on a group with group elements g, and anti-Hermitian Lie algebra elements with basis T^a,

$$[T^a,T^b] = f^{abc}T^c \tag{43}$$

$$\mathrm{tr}(T^aT^b) = -\frac{1}{2}\delta^{ab}. \tag{44}$$

The Lagrange density for an Eulerian fluid built on such a group is taken to be the following generalization of the Abelian expression (15):

$$\mathcal{L} = j^\mu 2\ \mathrm{tr}[Kg^{-1}D_\mu g] - f(n) + \mathcal{L}_{gauge}. \tag{45}$$

Here j^μ is an Abelian vector field (current) which can also be decomposed as in Eq. (16)

$$j^\mu = (c\rho,\rho\mathbf{v}) = nu^\mu, \quad u^\mu u_\mu = c^2. \tag{46}$$

The covariant derivative

$$D_\mu g = \partial_\mu g + A_\mu g \tag{47}$$

involves a dynamical non-Abelian gauge potential $A_\mu = A_\mu^a T_a$ whose dynamics is provided by $\mathcal{L}_{gauge}$. K is a fixed, constant element of the Lie algebra. The first term in $\mathcal{L}$ contains the canonical 1-form for our theory and determines the canonical brackets, while $f(n)$ describes the fluid dynamics. The theory is invariant under gauge transformations with group element U

$$g \to U^{-1}g$$
$$A_\mu \to U^{-1}(A_\mu + \partial_\mu)U. \tag{48}$$

According to Eq. (45), the current J_a^μ to which A_μ^a couples is of the Eckart form (42)

$$J_a^\mu = Q_a j^\mu \tag{49}$$

with

$$Q \equiv Q_a T^a = g K g^{-1}. \tag{50}$$

For consistency of the gauge field dynamics $J^\mu \equiv J_a^\mu T^a$ must be covariantly conserved. To satisfy the general arguments of Sec. III A, j^μ must be divergenceless and then the Wong equation (41) will follow. In Appendix A we show that both conservation laws are a consequence of invariance of the action with respect to variations of the group element g; arbitrary variations of g lead to covariant conservation of J^μ in Eq. (30) while the particular variation $\delta g = gK\lambda$ ensures that j^μ is conserved as in Eq. (1). Therefore the Wong equation (41) is also a consequence.

Thus our model reproduces all the equations satisfied by the current that were established in Sec. III A from general principles. Indeed the canonical (first) term of the Lagrangian in Eq. (45) is like a Kirillov-Kostant 1-form, which has been previously used to give a Lagrangian for the point particle Wong's equation (31) [9]. Moreover, as we show in Appendix B, the canonical brackets implied by the canonical 1-form ensure that the charge density algebra is represented canonically

$$\{\rho_a(t,\mathbf{r}),\rho_b(t,\mathbf{r}')\} = f_{abc}\rho_c(t,\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}'). \tag{51}$$

It remains to derive the Euler equation. This is accomplished by varying j^μ; stationary variation requires

$$2\ \mathrm{tr}\,[Q(D_\mu g)g^{-1}] = \frac{u_\mu}{c^2} f'(n) \tag{52}$$

which we call the non-Abelian Bernoulli equation. The Euler equation then follows, as in the Abelian case, by taking the curl

$$\partial_\mu\{2\ \mathrm{tr}[Q(D_\nu g)g^{-1}]\} - \partial_\nu\{2\ \mathrm{tr}[Q(D_\mu g)g^{-1}]\} = \partial_\mu\left(\frac{u_\nu}{c^2} f'(n)\right) - \partial_\nu\left(\frac{u_\mu}{c^2} f'(n)\right). \tag{53}$$

In Appendix C we show that manipulating the left-hand side allows rewriting Eq. (53) as

$$2\ \mathrm{tr}\,[(D_\mu Q)(D_\nu g)g^{-1}] + 2\ \mathrm{tr}\,[QF_{\mu\nu}] = \partial_\mu\left(\frac{u_\nu}{c^2} f'(n)\right) - \partial_\nu\left(\frac{u_\mu}{c^2} f'(n)\right). \tag{54}$$

Finally, contracting with $j^\mu = nu^\mu$ and using Eq. (41) produces the relativistic, non-Abelian Euler equation

$$\frac{nu^\mu}{c^2}\partial_\mu(u_\nu f'(n)) - n\partial_\nu f'(n) = 2\ \mathrm{tr}[J^\mu F_{\mu\nu}]. \tag{55}$$

The left-hand side is as in Eq. (19) while the right-hand side describes the non-Abelian Lorentz force acting on the charged fluid.

Apart from the gauge field contribution, the energy-momentum tensor for the Lagrange density (45) is the same as in Eq. (23) when Eq. (52) is used to eliminate $\mathrm{tr}\,Kg^{-1}D_\mu g$

$$T^{\mu\nu} = -g^{\mu\nu}[nf'(n) - f(n)] + \frac{u^\mu u^\nu}{c^2} nf'(n). \tag{56}$$

Just as in the Abelian case, the divergence of $T^{\mu\nu}$ entails two independent parts: one proportional to u^ν and the other orthogonal to it:

$$\partial_\mu T^{\mu\nu} = \partial_\mu(nu^\mu)\frac{u^\nu f'(n)}{c^2} + n\left[\frac{u^\mu}{c^2}\partial_\mu(u^\nu f'(n)) - \partial^\nu f'(n)\right]. \tag{57}$$

The first vanishes by the virtue of Eq. (1) and the rest is evaluated from Euler equation (55), leaving

$$\partial_\mu T^{\mu\nu} = 2\ \mathrm{tr}\,[J_\mu F^{\mu\nu}] \tag{58}$$

which is canceled by the divergence of the gauge-field energy-momentum tensor

$$\partial_\mu T^{\mu\nu}_{gauge} = -2\ \mathrm{tr}\,[J_\mu F^{\mu\nu}]. \tag{59}$$

Thus energy-momentum conservation reproduces Abelian current conservation (1) and the non-Abelian Euler force equation (56), but the Wong equation (41) has to be enforced additionally. This is achieved by our Lagrangian (45).

We record the nonrelativistic limit of Eq. (55); using Eqs. (20)–(22), we find that the nonrelativistic limit for the spatial component of Eq. (55) gives the Euler equation with a non-Abelian Lorentz force

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \mathit{force} + Q^a\mathbf{E}_a + \frac{\mathbf{v}}{c}\times Q^a\mathbf{B}_a \tag{60}$$

where *force* is the pressure force coming from the potential V (and is therefore Abelian in nature), while non-Abelian force terms involve the chromoelectric and chromomagnetic fields

$$E_a^i = cF_{0i}^a, \quad B_a^i = -\frac{c}{2}\epsilon_{ijk}F_{jk}^a. \tag{61}$$

It is seen that our non-Abelian fluid moves effectively in a single direction specified by $\mathbf{j} = \rho\mathbf{v}$. Nevertheless, it experiences a non-Abelian Lorentz force. In the next section we present a generalization wherein the non-Abelian fluid develops several independent directions of motion.

C. Generalization

A generalization of our Lagrange density (45) that will give rise to several fluid components carrying various densities and moving with various velocities is obtained by choosing several directions in the Lie algebra, $K_{(s)}$, and coupling to different Abelian currents [10]

$$\mathcal{L}=\sum_{s=1}^{r} j^{\mu}_{(s)}2\,\mathrm{tr}[K_{(s)}g^{-1}D_{\mu}g] - f(n_{(1)},n_{(2)},\ldots,n_{(r)})+\mathcal{L}_{gauge}$$

$$j^{\mu}_{(s)}=(c\rho_{(s)},\rho_{(s)}\mathbf{v}_{(s)})=n_{(s)}u^{\mu}_{(s)} \tag{62}$$

$$u^{\mu}_{(s)}u_{(s)\mu}=c^2$$

$$n_{(s)}=\sqrt{j^{\mu}_{(s)}j_{(s)\mu}/c^2}.$$

(Sums over s are indicated explicitly; the summation convention does not apply to s.) Evidently the current which couples to the gauge potential is now

$$J^{\mu}=\sum_{s=1}^{r} Q_{(s)}j^{\mu}_{(s)}, \quad \text{with} \quad Q_{(s)}=gK_{(s)}g^{-1}. \tag{63}$$

Arbitrary variation of g ensures that the expression in Eq. (63) is covariantly conserved, but we also need the conservation of individual $j^{\mu}_{(s)}$. This is achieved by choosing special forms for $\delta_{(s)}g=gK_{(s)}\lambda_{(s)}$ which will work provided that different $K_{(s)}$ commute (see Appendix A). Therefore, we choose the $K_{(s)}$ to belong to the Cartan subalgebra of the Lie algebra and the total number of different channels equals the rank r of the group. With this choice for the $K_{(s)}$, special variations of g ensure

$$\partial_{\mu}j^{\mu}_{(s)}=\partial_t\rho_{(s)}+\nabla\cdot(\rho_{(s)}\mathbf{v}_{(s)})=0. \tag{64}$$

The Wong equation which follows from the conservation of the non-Abelian current now reads

$$\sum_{s=1}^{r} j^{\mu}_{(s)}D_{\mu}Q_{(s)}=0. \tag{65}$$

Varying the individual $j^{\mu}_{(s)}$ produces the Bernoulli equations

$$2\,\mathrm{tr}[Q_{(s)}(D_{\mu}g)g^{-1}]=\frac{u_{\mu}}{c^2}f^{(s)}$$

$$\text{where} \quad f^{(s)}\equiv\frac{\partial}{\partial n_{(s)}}f(n_{(1)},n_{(2)},\ldots,n_{(r)}). \tag{66}$$

Again, the curl of the above can be cast in the form

$$\frac{1}{c^2}\{\partial^{\mu}(u^{\nu}_{(s)}f^{(s)})-\partial^{\nu}(u^{\mu}_{(s)}f^{(s)})\} =2\,\mathrm{tr}[(D^{\mu}Q_{(s)})(D^{\nu}g)g^{-1}]+2\,\mathrm{tr}[Q_{(s)}F^{\mu\nu}]. \tag{67}$$

When contracted with $j^{\mu}_{(s)}=n_{(s)}u^{\mu}_{(s)}$, this leaves

$$\frac{n_{(s)}u^{\mu}_{(s)}}{c^2}\partial_{\mu}(u^{\nu}_{(s)}f^{(s)})-n_{(s)}\partial^{\nu}f^{(s)} =2\,\mathrm{tr}[j^{\mu}_{(s)}(D_{\mu}Q_{(s)})(D^{\nu}g)g^{-1}] +2\,\mathrm{tr}[j_{\mu(s)}Q_{(s)}F^{\mu\nu}]. \tag{68}$$

However, unlike in the single channel case, the right-hand side does not simplify since $j_{\mu(s)}Q_{(s)}$ cannot be replaced by J_{μ} because the latter requires summing over s. Also the first right-hand term in Eq. (68) does not vanish since Eq. (65) requires summation over s.

The energy-momentum tensor is

$$T^{\mu\nu}=-g^{\mu\nu}\left(\sum_{s=1}^{r} n_{(s)}f^{(s)}-f\right)+\sum_{s=1}^{r}\frac{u^{\mu}_{(s)}u^{\nu}_{(s)}}{c^2}n_{(s)}f^{(s)}. \tag{69}$$

Its divergence of course reproduces Eq. (57)

$$\partial_{\mu}T^{\mu\nu}=\sum_{s=1}^{r}\left\{(\partial_{\mu}(n_{(s)}u^{\mu}_{(s)}))\frac{u^{\nu}_{(s)}f^{(s)}}{c^2} +n_{(s)}\left[\frac{u^{\mu}_{(s)}}{c^2}\partial_{\mu}(u^{\nu}_{(s)}f^{(s)})-\partial^{\nu}f^{(s)}\right]\right\}. \tag{70}$$

The first term in the curly brackets vanishes according to Eq. (64) and the remainder is evaluated from Eq. (68) as

$$\sum_{s=1}^{r}(2\,\mathrm{tr}[(j^{\mu}_{(s)}D_{\mu}Q_{(s)})(D^{\nu}g)g^{-1}]+2\,\mathrm{tr}[j_{\mu(s)}Q_{(s)}F^{\mu\nu}]).$$

Since now we are summing over all channels, it follows from Eqs. (63) and (65) that, as before,

$$\partial_{\mu}T^{\mu\nu}=2\,\mathrm{tr}[J_{\mu}F^{\mu\nu}]. \tag{71}$$

A more transparent picture of what is happening is given if the dynamical potential separates

$$f(n_{(1)},\ldots,n_{(r)})=\sum_{s=1}^{r} f_{(s)}(n_{(s)}) \tag{72}$$

$$f^{(s)}=f'_{(s)}. \tag{73}$$

Then the left-hand side of Eq. (68) refers only to variables labeled s, while the right-hand side may be rewritten with the help of Eqs. (62) and (65) to give

$$\frac{n_{(s)}u^{\mu}_{(s)}}{c^2}\partial_{\mu}(u^{\nu}_{(s)}f'_{(s)})-n_{(s)}\partial^{\nu}f'_{(s)} =2\,\mathrm{tr}[J_{\mu}F^{\mu\nu}]-2\sum_{s'\neq s}^{r}\mathrm{tr}\{j_{\mu(s')}[Q_{(s')}F^{\mu\nu} +(D^{\mu}Q_{(s')})(D^{\nu}g)g^{-1}]\}. \tag{74}$$

Thus in addition to the Lorentz force, there are forces arising from the other channels $s' \neq s$. Note that for separated dynamics as in Eq. (72), the energy-momentum tensor also separates

$$T^{\mu\nu} = \sum_{s=1}^{r} T^{\mu\nu}_{(s)} = \sum_{s=1}^{r} \left\{ -g^{\mu\nu}[n_{(s)}f'_{(s)} - f_{(s)}(n_{(s)})] + \frac{u^{\mu}_{(s)}u^{\nu}_{(s)}}{c^2} n_{(s)} f'_{(s)} \right\} \tag{75}$$

but the divergence of individual $T^{\mu\nu}_{(s)}$ does not vanish. This reflects that energy is exchanged between the different channels and with the gauge field, as is also evident from the equation of motion (74). It is clear that this fluid moves with r different velocities $\mathbf{v}_{(s)}$.

The single-channel Euler equation (55) is expressed in terms of physically relevant quantities (currents, chromomagnetic fields); the many-channel equation (68) involves, additionally, the gauge group element g. One may simplify that equation by going to special gauge, for example $g=I$, so that the right-hand side of Eq. (68) reduces to

$$2\,\mathrm{tr}[j_{\mu(s)}(D^{\mu}Q_{(s)})(D_{\nu}g)g^{-1}] + 2\,\mathrm{tr}[j^{\mu}_{(s)}Q_{(s)}F_{\mu\nu}] = 2\,\mathrm{tr}[K_{(s)}j^{\mu}_{(s)}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})] \tag{76}$$

while the Wong equation (65) becomes

$$\sum_{s=1}^{r} j^{\mu}_{(s)}[A_{\mu}, K_{(s)}] = 0. \tag{77}$$

It is interesting that in this gauge the non-linear terms in $F^{\mu\nu}$ disappear.

IV. NON-ABELIAN FLUIDS WITH A FIELD SUBSTRATUM

In the Abelian case, the Madelung parametrization (13) of the Schrödinger equation gives the conventional nonrelativistic Euler equation [even while the forces are derived from a potential that depends on density and (unconventionally) on its derivatives] (see Sec. II). We are therefore led to examine a Madelung-like construction for a non-Abelian, "colored" Schrödinger equation

$$i\hbar\, \partial_t \psi = -\frac{\hbar^2}{2m}\nabla^2 \psi. \tag{78}$$

We consider only the free, nonrelativistic case, and the "non-Abelian" structure resides solely in the fact that the ψ is a multi-component object, transforming under some representation of a group. The color degrees of freedom also lead to the conserved color current

$$J^{\mu}_{a} = (c\rho_a, \mathbf{J}_a), \quad \rho_a = i\psi^{\dagger}T^{a}\psi, \quad \mathbf{J}_a = \frac{\hbar}{m}\mathrm{Re}\,\psi^{\dagger}T^{a}\nabla\psi. \tag{79}$$

Of course, the singlet current

$$j^{\mu} = (c\rho, \mathbf{j}), \quad \rho = \psi^{\dagger}\psi, \quad \mathbf{j} = \frac{\hbar}{m}\mathrm{Im}\,\psi^{\dagger}\nabla\psi \tag{80}$$

is also conserved. For definiteness and simplicity, we shall henceforth assume that the group is $SU(2)$ and that the representation is the fundamental one: $T^a = \sigma^a/(2i)$, $\{T^a, T^b\} = -\delta^{ab}/2$. We shall also set the mass m and Planck's constant $\hbar$ to unity. The non-Abelian analog of the Madelung decomposition (13) is

$$\psi = \sqrt{\rho}\, g u \tag{81}$$

where ρ is the scalar $\psi^{\dagger}\psi$, g is a group element, and u is a constant vector that points in a fixed direction [e.g., for $SU(2)$ u could be $\binom{1}{0}$, then $iu^{\dagger}T^{a}u = \delta^{a3}/2$]. The singlet density is ρ, while the singlet current $\mathbf{j}$ is

$$\mathbf{j} = \rho\mathbf{v}, \quad \mathbf{v} \equiv -iu^{\dagger}g^{-1}\nabla g\, u. \tag{82}$$

With the decomposition (81), the color density (79) becomes

$$\rho_a = Q_a\rho, \quad Q_a = iu^{\dagger}g^{-1}T^{a}g\, u = iR_{ab}u^{\dagger}T^{b}u = R_{ab}t^{b}/2 \tag{83}$$

where R_{ab} is in the adjoint representation of the group and the unit vector t^a is defined as $t^a/2 = iu^{\dagger}T^{a}u$. On the other hand, the color current reads

$$\mathbf{J}_a = \frac{1}{2}\rho R_{ab}u^{\dagger}(T^{b}g^{-1}\nabla g + g^{-1}\nabla g\, T^{b})u, \tag{84}$$

which with the introduction of

$$g^{-1}\nabla g \equiv -2\mathbf{v}^{a}T^{a} \tag{85}$$

$$\mathbf{v} = \mathbf{v}^{a}t^{a} \tag{86}$$

becomes

$$\mathbf{J_a} = \frac{\rho}{2}R_{ab}\mathbf{v}^{b}. \tag{87}$$

Unlike the Abelian model, the vorticity is nonvanishing

$$\nabla\times\mathbf{v}^{a} = \epsilon^{abc}\mathbf{v}^{b}\times\mathbf{v}^{c}. \tag{88}$$

A difference between the Madelung approach and the previous particle based one is that the color current is not proportional to the singlet current. Equation (87) may be written as

$$\mathbf{J}_a = Q_a\rho\mathbf{v} + \frac{\rho}{2}R_{ab}\mathbf{v}^{b}_{\perp} \tag{89}$$

where the "orthogonal" velocity $\mathbf{v}^{a}_{\perp}$ is defined as

$$\mathbf{v}_\perp^a = (\delta^{ab} - t^a t^b)\mathbf{v}^b. \tag{90}$$

Equation (88) shows that color current possesses components that are orthogonal to the singlet current.

In Appendix D we derive for the $SU(2)$ case the decomposition of the Schrödinger equation with the parametrization (81). Two equations emerge: one regains the conservation of the Abelian current (1) and the other is the "Bernoulli" equation

$$(g^{-1}\partial_t g)^a = \left[\mathbf{v}^b \cdot \mathbf{v}^b - \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}\right] t^a + \frac{1}{\rho}\nabla \cdot (\rho \epsilon^{abc} \mathbf{v}^b t^c). \tag{91}$$

It is further verified that the covariant conservation of the color current is enforced by both Eqs. (1) and (91). However, there is no Wong equation because the color current is not proportional to the conserved singlet current. Finally, using the identity, which is a consequence of the definition (85)

$$\partial_t \mathbf{v}^a = -\frac{1}{2}\nabla (g^{-1}\partial_t g)^a + \epsilon^{abc}\mathbf{v}^b (g^{-1}\partial_t g)^c \tag{92}$$

one can deduce an Euler equation for $\partial_t \mathbf{v}^a$ from Eq. (91).

We record the energy and momentum density

$$\mathcal{E} = \frac{1}{2}\nabla \psi^\dagger \cdot \nabla \psi = \frac{1}{2}\rho \mathbf{v}^a \cdot \mathbf{v}^a + \frac{\nabla \rho \cdot \nabla \rho}{8\rho} \tag{93}$$

$$\boldsymbol{\mathcal{P}} = \frac{i}{2}(\nabla \psi^\dagger \psi - \psi^\dagger \nabla \psi) = \rho \mathbf{v}. \tag{94}$$

Both parallel and orthogonal components of the velocity contribute to the energy density but only the parallel component $\mathbf{v}$ contributes to the momentum density. It is clear that within the present approach the fluid color flows in every direction in the group space, but the mass density is carried by the unique velocity $\mathbf{v}$. This is in contrast to our previous approach where all motion is in a single direction (Sec. III B) or at most in the directions of the Cartan elements of the Lie algebra (Sec. III C).

The difference between the two approaches is best seen from a comparison of Lagrangians. For the color Schrödinger theory in the Madelung representation

$$\mathcal{L}_{Schr\ddot{o}dinger} = \frac{i}{2}(\psi^\dagger \partial_t \psi - \partial_t \psi^\dagger \psi) - \frac{1}{2}\nabla \psi^\dagger \cdot \nabla \psi \tag{95}$$

$$= i\rho u^\dagger g^{-1}\partial_t g\, u - \frac{1}{2}\rho \mathbf{v}^a \cdot \mathbf{v}^a - \frac{(\nabla \rho)^2}{8\rho}. \tag{96}$$

With $u \otimes u^\dagger \equiv I/2 - 2iK$, the free part of the above reads

$$\mathcal{L}^0_{Schr\ddot{o}dinger} = \rho 2 \,\mathrm{tr}[K g^{-1}\partial_t g] - \frac{1}{2}\rho \mathbf{v}^a \cdot \mathbf{v}^a. \tag{97}$$

On the other hand, the free part of the Lagrange density (45) in the nonrelativistic limit is

$$\begin{aligned}\mathcal{L}^0 &= \rho 2 \,\mathrm{tr}[K g^{-1}\partial_t g] + \rho \mathbf{v}\cdot 2\,\mathrm{tr}[K g^{-1}\nabla g] - \sqrt{\rho^2(c^2 - v^2)} \\ &\approx \rho 2 \,\mathrm{tr}[K g^{-1}\partial_t g] + \rho \mathbf{v}\cdot 2\,\mathrm{tr}[K g^{-1}\nabla g] - \rho c^2 + \frac{1}{2}\rho v^2 \\ &= \rho 2 \,\mathrm{tr}[K g^{-1}\partial_t g] - \frac{1}{2}\rho v^2 - \rho c^2\end{aligned} \tag{98}$$

where we have used $\mathbf{v} = -2\,\mathrm{tr}[K g^{-1}\nabla g]$, which follows upon the variation of $\mathbf{v}$, in the next-to-last equality above. Thus the canonical 1-form is the same for both models while the difference resides in the velocity dependence of their respective Hamiltonians. Only the singlet $\mathbf{v}$ enters Eq. (98) while the Madelung construction uses the group vector $\mathbf{v}^a$.

Finally, note that while the Euler equation, which emerges when Eqs. (91) and (92) are combined, intricately couples all directions of the fluid velocity $\mathbf{v}^a$, it does admit the simple solution $\mathbf{v}^a = \mathbf{v} t^a$, with $\mathbf{v}$ obeying the Abelian equations that arise from Eqs. (12)–(14).

V. DISCUSSION

In this paper we have presented in Sec. III two distinct non-Abelian generalizations of ordinary particle based Abelian fluid dynamics. Both versions use a fluid generalization of the Kirillov-Kostant form that naturally encodes the algebra of charge densities in Eq. (51), which is needed if the charge density is to be identified as the generator of non-Abelian symmetry transformations. In Sec. III C we generalized the first version, given in Sec. III B so that the density is specified in terms of a set of Abelian densities equal in number to the rank r of the Lie algebra, rather than a single density as in the first version. Since the charge density at a point in the fluid is an element of the Lie algebra, diagonalization shows that an invariant specification must use r eigenvalues. Alternatively, we may use the r Casimir invariants of ρ_a at a point to characterize it. Therefore the appearance of r Abelian currents in the Lagrangian is entirely natural. This is also in accord with what happens with the Kirillov-Kostant form where our $j^\mu_{(s)}$ are replaced by the fixed weights of a representation of the Lie algebra and lead to that representation upon quantization.

The two versions also differ in the formula for the current. Our first version, which is mathematically more concise, gives the non-Abelian Eckart decomposition, Eq. (49), while the second version does not allow the factorization of the current into a non-Abelian charge density and an Abelian velocity, rather it is a sum of such factorized expressions—a generalized Eckart decomposition as in Eq. (50). The Eckart decomposition shows that we can choose a local Lorentz frame for which a given fluid element can be brought to rest. The charge density is then related to the charge carried by this element. By contrast, in the second version, with a generalized Eckart decomposition, we see that even if we choose a frame where one Lie algebra component of the velocity is zero, the other color velocities need not be. Thus the latter applies to a situation where color separating flows can occur. Physically, it is not yet clear what kinematic regimes of a quark-gluon plasma, for example, would admit or require such flows.

We note that the currents we have obtained are the non-Abelian analogues of the irrotational part of the Abelian current, even though the vorticity is nonvanishing. The other components can be easily incorporated, if needed, by generalizing the Lagrangian in Eq. (62) as

$$\mathcal{L}=\sum_{s=1}^{r} j^{\mu}_{(s)}\{2\,\mathrm{tr}[K_{(s)}g^{-1}D_{\mu}g]+a_{\mu(s)}\} - f(n_{(1)},n_{(2)},\ldots,n_{(r)})+\mathcal{L}_{gauge} \tag{99}$$

where $a_{\mu(s)}$ is given by

$$a_{\mu(s)}=\alpha_{(s)}\partial_{\mu}\beta_{(s)}. \tag{100}$$

The final fluid equations remain unchanged.

We have also derived in Sec. IV a field-based fluid mechanics by extending the Madelung construction to the non-Abelian situation. Here the ‘‘Euler’’ equations are much less appealing because they involve velocities in all group directions. We know of no compelling physical reason for preferring this field-based model over the particle-based one, even though in the Abelian case it coincides with the particle-based construction.

ACKNOWLEDGMENTS

This work was supported in part by DOE grants Nos. DE-FC02-94-ER40818, DE-FG02-91-ER40676, by NSF grant number PHY-0070883 and by a PSC-CUNY grant.

APPENDIX A: VARIATIONS OF *g*

We determine the variation of

$$I^0=\int dt dr \sum_{s=1}^{r} j^{\mu}_{(s)}2\,\mathrm{tr}\,K_{(s)}g^{-1}D_{\mu}g \tag{A1}$$

when g is varied either arbitrarily or in the specific manner

$$g^{-1}\delta g=K_{(s')}\lambda \tag{A2}$$

where λ is an arbitrary function on space-time. This will provide the needed results (1) and (30) for the single channel situation as well as (64) and (65) for many channels.

Recall the definitions $Q_{(s)}=gK_{(s)}g^{-1}$ and $D_{\mu}g=\partial_{\mu}g+A_{\mu}g$, which implies $D_{\mu}g^{-1}=\partial_{\mu}g^{-1}-g^{-1}A_{\mu}$. First, the variation of $g^{-1}D_{\mu}g$ is established

$$\delta(g^{-1}D_{\mu}g)=-g^{-1}\delta g g^{-1}D_{\mu}g+g^{-1}D_{\mu}\delta g. \tag{A3}$$

To evaluate the last term, note that $D_{\mu}\delta g=D_{\mu}(gg^{-1}\delta g)=(D_{\mu}g)g^{-1}\delta g+gD_{\mu}(g^{-1}\delta g)$. Thus

$$\delta(g^{-1}D_{\mu}g)=\partial_{\mu}(g^{-1}\delta g)+[g^{-1}D_{\mu}g,g^{-1}\delta g]. \tag{A4}$$

Inserting Eq. (A4) into the variation of I^0 in Eq. (A1), integrating by parts, and rearranging the trace with $K_{(s)}$, gives

$$\delta I^0=-\int dt dr\sum_{s=1}^{r}(\partial_{\mu}j^{\mu}_{(s)}2\,\mathrm{tr}\,K_{(s)}g^{-1}\delta g + j^{\mu}_{(s)}2\,\mathrm{tr}[g^{-1}D_{\mu}g,K_{(s)}]g^{-1}\delta g). \tag{A5}$$

Considering first arbitrary variations: the vanishing of δI^0 requires

$$\sum_{s=1}^{r}(\partial_{\mu}j^{\mu}_{(s)}K_{(s)}+j^{\mu}_{(s)}[g^{-1}D_{\mu}g,K_{(s)}])=0 \tag{A6}$$

or, after sandwiching the above between $g\ldots g^{-1}$,

$$\sum_{s=1}^{r}\{\partial_{\mu}j^{\mu}_{(s)}Q_{(s)}+j^{\mu}_{(s)}[(D_{\mu}g)g^{-1},Q_{(s)}]\}=0. \tag{A7}$$

Finally we verify that

$$[D_{\mu}gg^{-1},Q_{(s)}]=D_{\mu}Q_{(s)}, \tag{A8}$$

so that the desired results (30) and (65) follow

$$\sum_{s=1}^{r}(\partial_{\mu}j^{\mu}_{(s)}Q_{(s)}+j^{\mu}_{(s)}D_{\mu}Q_{(s)})=D_{\mu}\left(\sum_{s=1}^{r}j^{\mu}_{(s)}Q_{(s)}\right)=D_{\mu}J^{\mu}=0. \tag{A9}$$

Next we consider the specific variation (A2) and separate the sum (A5) into the term $s=s'$ and $s\neq s'$. After a rearrangement of the last term in Eq. (A5), we get

$$\begin{aligned}\delta I^0=-\int dt\,dr\big(&\partial_{\mu}j^{\mu}_{(s')}2\,\mathrm{tr}\,K_{(s')}K_{(s')}\lambda\\ &+j^{\mu}_{(s')}2\,\mathrm{tr}\,g^{-1}D_{\mu}g[K_{(s')},K_{(s')}]\lambda)\\ &+\sum_{s\neq s'}(\partial_{\mu}j^{\mu}_{(s)}2\,\mathrm{tr}\,K_{(s)}K_{(s')}\lambda\\ &+j^{\mu}_{(s)}2\,\mathrm{tr}\,g^{-1}D_{\mu}g[K_{(s)},K_{(s')}]\lambda\big).\end{aligned} \tag{A10}$$

The first commutator vanishes; so does the second when $K_{(s)}$ and $K_{(s')}$ commute, i.e., when they belong to the Cartan subalgebra. Also $2\,\mathrm{tr}\,K_{(s)}K_{(s')}=-K^a_{(s)}K^a_{(s')}$; for $s'=s$ this is constant, while for $s'\neq s$ it vanishes when it is arranged that distinct elements of the Cartan algebra are selected. Thus for stationary variations $j^{\mu}_{(s)}$ must be conserved, and Eq. (1) as well as Eq. (30) are validated.

APPENDIX B: CHARGE DENSITY ALGEBRA

The portion of the Lagrange density (45) that determines the Poisson bracket is

$$\mathcal{L}_{canonical}=\rho 2\,\mathrm{tr}\,Kg^{-1}\partial_t g=\rho 2\,\mathrm{tr}\,Q\partial_t g g^{-1}. \tag{B1}$$

With a parametrization of the group element, e.g., $g(\varphi)=e^{T^a\varphi_a}$, one sees that $(\partial_t g)g^{-1}$ has the form $-\partial_t\varphi_a C^a_b(\varphi)T^b$, where the non-singular matrix $C^a_b(\varphi)$ is defined by

$$C_b^a(\varphi)T^b = -\frac{\partial g(\varphi)}{\partial \varphi_a} g^{-1}(\varphi). \tag{B2}$$

Thus

$$\mathcal{L}_{canonical} = \rho \partial_t \varphi_a C_b^a Q_b = \partial_t \varphi_a C_b^a \rho_b \tag{B3}$$

and the momentum conjugate to φ_a is

$$\Pi^a = C_b^a \rho_b . \tag{B4}$$

With the inverse to C_b^a defined as c_a^b,

$$\rho_a = c_b^a \Pi^b. \tag{B5}$$

The non-Abelian charge density ρ_a is a function of $(t,\mathbf{r})$ and for Eq. (51) we need the bracket with another density evaluated at $(t,\mathbf{r}')$. Since the dependence of c_b^a on φ involves no spatial derivatives of φ, it is clear that the brackets will be local in $\mathbf{r}-\mathbf{r}'$, just as is the bracket between φ and Π,

$$\begin{aligned}\{\rho_a(t,\mathbf{r}),\rho_b(t,\mathbf{r}')\} &= \left(c_{b'}^b \frac{\partial c_{a'}^a}{\partial \varphi_{b'}} \Pi^{a'} - a \leftrightarrow b \right) \delta(\mathbf{r}-\mathbf{r}') \\ &= \left(-c_{b'}^b c_{c'}^a \frac{\partial C_{c''}^{c'}}{\partial \varphi_{b'}} c_{a'}^{c''} \Pi^{a'} - a \leftrightarrow b \right) \\ &\quad \times \delta(\mathbf{r}-\mathbf{r}') \\ &= \left(-c_{b'}^b c_{c'}^a \frac{\partial C_{c''}^{c'}}{\partial \varphi_{b'}} \rho_{c''} - a \leftrightarrow b \right) \delta(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{B6}$$

To evaluate the derivative with respect to φ, return to Eq. (B2) and observe

$$\begin{aligned}\frac{\partial C_{c''}^{c'}}{\partial \varphi_{b'}} &= \frac{\partial}{\partial \varphi_{b'}} \left(2 \operatorname{tr} \frac{\partial g}{\partial \varphi_{c'}} g^{-1} T^{c''} \right) \\ &= 2 \operatorname{tr} \left(\frac{\partial^2 g}{\partial \varphi_{b'} \partial \varphi_{c'}} g^{-1} - C_{d'}^{c'} T^{d'} C_{d''}^{b'} T^{d''} \right) T^{c''}.\end{aligned} \tag{B7}$$

The first term in the parentheses is symmetric in (b',c'); when inserted in Eq. (B6) it produces a symmetric contribution in (a,b) and does not contribute when antisymmetrization in (a,b) is effected. What is left establishes Eq. (45):

$$\begin{aligned}&\{\rho_a(t,\mathbf{r}),\rho_b(t,\mathbf{r}')\} \\ &\quad = (c_{b'}^b c_{c'}^a C_{d'}^{c'} C_{d''}^{b'} 2 \operatorname{tr} T^{d'} T^{d''} T^{c''} \rho_{c''} - a \leftrightarrow b)\, \delta(\mathbf{r}-\mathbf{r}') \\ &\quad = -(2 \operatorname{tr} T^a T^b T^{c''} \rho_{c''} - a \leftrightarrow b)\, \delta(\mathbf{r}-\mathbf{r}') \\ &\quad = -2 \operatorname{tr} f^{abd} T^d T^{c''} \rho_{c''} \delta(\mathbf{r}-\mathbf{r}') = f^{abc} \rho_c(t,\mathbf{r})\, \delta(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{B8}$$

APPENDIX C: MANIPULATING EQ. (53)

Observe that the first term in Eq. (53) equals

$$\begin{aligned}&\partial_\mu 2 \operatorname{tr}[Q(D_\nu g)g^{-1}] \\ &\quad = 2 \operatorname{tr}[(D_\mu Q)(D_\nu g)g^{-1} + Q(D_\mu D_\nu g)g^{-1} \\ &\qquad - Q(D_\nu g)g^{-1}(D_\mu g)g^{-1}].\end{aligned} \tag{C1}$$

The first term on the right-hand side is rewritten with the help of Eq. (A8) and combined with the last term, leaving

$$2 \operatorname{tr}[Q(D_\mu D_\nu g)g^{-1} - Q(D_\mu g)g^{-1}(D_\nu g)g^{-1}].$$

After antisymmetrization in (μ,ν), the left-hand side of Eq. (53) reads

$$\begin{aligned}&2 \operatorname{tr} Q\{([D_\mu, D_\nu]g)g^{-1} - [(D_\mu g)g^{-1},(D_\nu g)g^{-1}]\} \\ &\quad = 2 \operatorname{tr}\{QF_{\mu\nu} - [Q,(D_\mu g)g^{-1}](D_\nu g)g^{-1}\}.\end{aligned} \tag{C2}$$

When Eq. (A8) is used again, Eq. (C2) becomes the left-hand side of Eq. (54).

APPENDIX D: NON-ABELIAN MADELUNG PARAMETRIZATION

When Eq. (81) is inserted into Eq. (78), and use is made of the definition (85), we find in the $SU(2)$ case

$$\begin{aligned}&\frac{1}{2} i \partial_t \rho\, u + i\rho(g^{-1}\partial_t g)^a T^a u \\ &\quad = -\frac{1}{2}\sqrt{\rho}\nabla^2\sqrt{\rho}\, u + \nabla(\rho \mathbf{v}^a) T^a u + \frac{1}{2}\rho \mathbf{v}^a \cdot \mathbf{v}^a u.\end{aligned} \tag{D1}$$

Next Eq. (D1) is premultiplied by $u^\dagger$, where it implies

$$i\partial_t \rho + \rho(g^{-1}\partial_t g)^a t^a = -\sqrt{\rho}\nabla^2\sqrt{\rho} - i\nabla(\rho \mathbf{v}^a t^a) + \rho \mathbf{v}^a \cdot \mathbf{v}^a. \tag{D2}$$

The imaginary part reproduces the continuity equation for the singlet current (82), while the real part gives

$$(g^{-1}\partial_t g)^a t^a = -\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho} + \mathbf{v}^a \cdot \mathbf{v}^a. \tag{D3}$$

To obtain further information, we premultiply Eq. (D1) with $u^\dagger T^b$. This gives

$$\begin{aligned}&\partial_t \rho t^b - i\rho(g^{-1}\partial_t g)^b + \rho(g^{-1}\partial_t g)^a \epsilon^{bac} t^c \\ &\quad = i\sqrt{\rho}\nabla^2\sqrt{\rho} t^b - \nabla\cdot(\rho \mathbf{v}^b) - i\nabla\cdot(\rho \mathbf{v}^a) \\ &\qquad \times \epsilon^{bac} t^c - i\rho \mathbf{v}^a \cdot \mathbf{v}^a t^b.\end{aligned} \tag{D4}$$

The imaginary part gives Eq. (91) while the real part is identically satisfied by virtue of Eqs. (1) and (91).

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[6] For a discussion of the Eckart decomposition, as well as other forms, see S.R. de Groot, W.A. van Leeuwen, and Ch.G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980), p. 10

[7] When comparison is made with the usual expression for the energy-momentum tensor of a relativistic fluid, we see that $nf'(n)-f(n)$ is the pressure and $nf'(n)$ is the pressure summed with the energy density, i.e., $f(n)$ is the energy density; see Ref. [3].

[8] S.K. Wong, Nuovo Cimento A **65**, 689 (1970).

[9] A. Balachandran, S. Borchardt, and A. Stern, Phys. Rev. D **17**, 3247 (1978); A. Balachandran, G. Marmo, B.-S. Skagerstam, and A. Stern, *Gauge Symmetries and Fibre Bundles*, Lecture Notes in Physics Vol. 188 (Springer-Verlag, Berlin, 1982). The canonical analysis presented in our paper (Appendix B) follows the method of D. Bak, R. Jackiw, and S.-Y. Pi, Phys. Rev. D **49**, 6778 (1994) (Appendix).

[10] In a previous paper, we presented a similar candidate Lagrangian for non-Abelian fluids, except that the $K_{(s)}$ comprised all generators for a subgroup H of the group G to which g belongs, with the additional requirement that G/H be a symmetric space. This construction is based on a non-Abelian version of the Clebsch parametrization. But the equations that follow do not have a clear physical interpretation, so we have not pursued this approach. R. Jackiw, V.P. Nair, and S.-Y. Pi, Phys. Rev. D **62**, 085018 (2000).