1/4 BPS dyonic calorons

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We explore the 1/4 Bogomol'nyi-Prasad-Sommerfield (BPS) configurations of the supersymmetric gauge theories on $R^{1+3}\times S^1$. The BPS bound for energy and the BPS equations are obtained and the characteristics of the BPS solutions are studied. These BPS configurations describe electrically charged calorons, which are constituted of dyons and carry linear momentum along the compact direction. We carry out various approaches to the single caloron case in the theory of the SU(2) gauge group.

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I. INTRODUCTION

It has been known for some time that instantons on $R^3\times S^1$, or so-called calorons, can be considered to be made of magnetic monopoles when there is a nontrivial Wilson loop which breaks the gauge symmetry to its Abelian subgroups [1–3]. For the SU(N) gauge group, there are N different kinds of fundamental monopoles, each corresponding to roots in the extended Dynkin diagram. The relation between instantons and magnetic monopoles can also be understood by exploring the five-dimensional Yang-Mills theories which appear as the low energy Lagrangian on parallel D4-branes and its T-dual version. In their $N=4$ supersymmetric version on $R^{1+3}\times S^1$, instantons appear as 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) objects. The low energy dynamics of calorons or instantons is given by the metric on the moduli space of caloron solutions.

There has been some work done some time ago on 1/4 BPS dyons on $N=4$ supersymmetric Yang-Mills theories on R^{1+3}, which can arise when several Higgs fields take expectation values [4,5]. In the five-dimensional Yang-Mills theories, there are five Higgs fields and they can take nontrivial expectation values, in addition to the nontrivial Wilson loop along the compact circle. In these theories the 1/4 BPS and non-BPS configurations are also possible. In this paper we explore these 1/4 BPS configurations. In particular, we work out a single 1/4 BPS dyonic caloron case in the SU(2) gauge theory.

More recently there has been some work on dyonic or electrically charged instantons in five-dimensional field theory [6–8]. As in the four-dimensional theory, these dyonic instantons are 1/4 BPS instead of 1/2 BPS as in the Yang-Mills theories with 16 supersymmetries. These dyonic instantons also carry nontrivial angular momentum. They become calorons when the space is compactified. As we will show in this paper, BPS calorons come with richer characteristics.

Usually we consider a BPS configuration to be at rest. However, they remain BPS when the configuration is Lorentz boosted. As our space is compactified along a circle, a BPS configuration can carry nonzero linear momentum along the circle. However, the linear momentum is not topological and cannot be expressed as a boundary term in general.

The 1/4 BPS dyons can be understood as a planar web of fundamental strings and D-strings connecting parallel D3-branes in type IIB theory [9]. Similarly, the 1/4 BPS dyonic calorons have the string web picture. We explore this in our simple model.

The low energy dynamics of magnetic monopoles can be approached by moduli space dynamics. When an additional scalar field is turned on, its effect can be incorporated as a potential term. It was shown in [10] that the BPS configuration of this low energy dynamics corresponds to the 1/4 BPS dyonic configurations. From this correspondence one can read the electric charge of dyons for a given set of moduli parameters. This result can also be found directly from the field theory analysis. We consider the low energy dynamics of 1/4 BPS dyonic calorons and work out these results in detail in the SU(2) case.

The plan of this paper is as follows. In Sec. II, we find the BPS bound on the energy functional. In Sec. III, we find the BPS equations that are satisfied by the configurations saturating the BPS bound. In Sec. IV, we find the BPS caloron configurations, which can be regarded as composed of monopoles and dyons. In Sec. V, we study the SU(2) gauge group case in detail. In particular, we relate our 1/4 BPS configuration to the string web picture. In Sec. VI, we conclude with some remarks.

II. THE BPS BOUND

The underlying spacetime is chosen to be five dimensional, with one of the space dimensions being compactified to a circle. The coordinates x^M where $M=0,\ldots,4$ are split into the time coordinate x^0 and space coordinates x^μ with $\mu=1,2,3,4$. The compactified coordinate x^4 has the finite range

$$0\leq x^4<\beta. \tag{1}$$

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We consider only the periodic gauge and scalar fields. The allowed gauge transforms are those which leave the gauge fields periodic: small gauge transformations whose gauge functions are periodic, and large gauge transformations whose gauge functions are multivalued. While our considerations can easily be generalized to an arbitrary semisimple gauge group, we focus on the SU(N) gauge group for simplicity. We consider the Hermitian generators T^a in the N-dimensional fundamental representation with normalization $\mathrm{tr}\, T^aT^b=\delta^{ab}/2$. The gauge field is then $A_M=A_M^aT^a$.

The Lagrangian we start with is

$$\mathcal{L}=\frac{1}{e^2}\mathrm{tr}\left(-\frac{1}{2}F_{MN}F^{MN}+D_M\phi_I D^M\phi_I-\sum_{I<J}(-i[\phi_I,\phi_J])^2\right), \tag{2}$$

where $D_M\phi_I=\partial_M\phi_i-i[A_M,\phi_I]$ and e^2 is a five-dimensional coupling constant of length dimension. We decompose the Higgs field into one component and the rest:

$$\phi_I=a_I\phi+\zeta_I, \tag{3}$$

where a_I is a unit vector in five dimensions and ζ_I is orthogonal to a_I. The Gauss law is

$$D_iE_i+D_4F_{04}-i[\phi,D_0\phi]-i[\zeta_I,D_0\zeta_I]=0, \tag{4}$$

where $E_i=F_{0i}$ with $i=1,2,3$.

The energy density is given by

$$\mathcal{E}=\frac{1}{e^2}\mathrm{tr}\,(E_i^2+B_i^2+F_{04}^2+F_{i4}^2+D_0\phi^2+D_i\phi^2+D_4\phi^2)+\mathcal{E}_\zeta, \tag{5}$$

where $B_i=\frac{1}{2}\epsilon_{ijk}F_{jk}$ and

$$e^2\mathcal{E}_\zeta=\mathrm{tr}\left((D_0\zeta_I)^2+(D_\mu\zeta_I)^2-\sum_I[\phi,\zeta_I]^2-\sum_{I<J}[\zeta_I,\zeta_J]^2\right). \tag{6}$$

The energy density can be written as

$$\begin{aligned}\mathcal{E}=\frac{1}{e^2}\mathrm{tr}\{&(E_i+F_{4i}\sin\alpha-D_i\phi\cos\alpha)^2+(B_i-F_{4i}\cos\alpha\\&-D_i\phi\sin\alpha)^2+(F_{04}-D_4\phi\cos\alpha)^2+(D_0\phi\\&+D_4\phi\sin\alpha)^2\}+2\cos\alpha[\mathrm{tr}\;B_iF_{4i}+\partial_i\,\mathrm{tr}\,(E_i\phi)]\\&+2\sin\alpha\{\partial_i\,\mathrm{tr}\,(B_i\phi)-\mathrm{tr}\,(E_iF_{4i}+D_0\phi D_4\phi\\&+D_0\zeta_ID_4\zeta_I)\}+\tilde{\mathcal{E}}_\zeta,\end{aligned} \tag{7}$$

where

$$\begin{aligned}e^2\tilde{\mathcal{E}}_\zeta=&\,\mathrm{tr}\,(D_0\zeta_I-i[\phi,\zeta_I]\cos\alpha+D_4\zeta_I\sin\alpha)^2\\&+\mathrm{tr}\,(D_4\zeta_I\cos\alpha+i[\phi,\zeta_I]\sin\alpha)^2+(D_i\zeta_I)^2\\&-\sum_{I<J}[\zeta_I,\zeta_J]^2.\end{aligned} \tag{8}$$

In the above we have used the Gauss law and the single-valuedness of the fields in x_4.

We introduce four conserved charges:

$$Q^E=\frac{2}{e^2}\int d^4x\,\partial_i\,\mathrm{tr}\,(E_i\phi), \tag{9}$$

$$Q^M=\frac{2}{e^2}\int d^4x\,\partial_i\,\mathrm{tr}\,(B_i\phi), \tag{10}$$

$$P^4=-\frac{2}{e^2}\int d^4x\;\mathrm{tr}\,(E_iF_{4i}+D_0\phi D_4\phi+D_0\zeta_ID_4\zeta_I), \tag{11}$$

$$\mathcal{T}=\frac{8\pi^2}{e^2}\nu_P, \tag{12}$$

where d^4x is the volume element of the four-dimensional space. The linear momentum along the circle P^4 is conserved but is not topological. The rest of them are topological. In particular, $\mathcal{T}$ is related to the Pontriyagin index by

$$\nu_P=\frac{1}{8\pi^2}\int d^4x2\;\mathrm{tr}\,(B_iF_{4i}). \tag{13}$$

The bound on the energy functional $H=\int d^4x\mathcal{E}$ is then

$$H\geqslant(\mathcal{T}+Q^E)\cos\alpha+(Q^M+P^4)\sin\alpha \tag{14}$$

for any angle α. The maximum possible value of the right side is obtained when

$$\cos\alpha=(\mathcal{T}+Q^E)/\sqrt{(\mathcal{T}+Q^E)^2+(Q^M+P^4)^2}, \tag{15}$$

$$\sin\alpha=(Q^M+P_4)/\sqrt{(\mathcal{T}+Q^E)^2+(Q^M+P^4)^2}. \tag{16}$$

Then the strictest energy bound is

$$H\geqslant Z_\pm=\sqrt{(\mathcal{T}\pm Q^E)^2+(Q^M\pm P^4)^2}, \tag{17}$$

which is the so-called BPS energy bound. Z_- is obtained by changing the sign of the ϕ field.

III. BPS EQUATIONS

This energy bound is saturated, say, $H=Z_+$, by the configurations that satisfy the following BPS equations:

$$B_i=F_{4i}\cos\alpha+D_i\phi\sin\alpha, \tag{18}$$

$$E_i=-F_{4i}\sin\alpha+D_i\phi\cos\alpha, \tag{19}$$

$$F_{04}=D_4\phi\cos\alpha, \tag{20}$$

$$D_0\phi=-D_4\phi\sin\alpha, \tag{21}$$

and the Gauss law (4). In addition, there are conditions for the ζ_I field,

$$D_0\zeta_I-i[\phi,\zeta_I]\cos\alpha+D_4\zeta_I\sin\alpha=0, \tag{22}$$

$$D_4\zeta_I\cos\alpha+i[\phi,\zeta_I]\sin\alpha=0, \tag{23}$$

$$D_i\zeta_I=0, \tag{24}$$

$$[\zeta_I,\zeta_J]=0. \tag{25}$$

The above BPS equations seem complicated to solve. However, a considerable simplification can be made by noticing that Eq. (18) can be written as a self-dual equation. Let us introduce a new coordinate

$$\tilde{x}^4=\frac{1}{\cos\alpha}x^4, \tag{26}$$

and a new fourth component gauge field

$$\tilde{A}_4=A_4\cos\alpha-\phi\sin\alpha. \tag{27}$$

Then Eq. (18) becomes

$$B_i=\tilde{\partial}_4A_i-D_i\tilde{A}_4=\tilde{F}_{4i}. \tag{28}$$

This is the self-dual equation for calorons on $R^3\times\tilde{S}^1$, where $\tilde{x}^4\in[0,\beta/\cos\alpha]$. Since the fields are periodic under x^4, they are period under $\tilde{x}^4$. If $\alpha=\pi/2$, the above method fails. However, the BPS equations in this case become those of 1/2 BPS monopoles with a single scalar field ϕ and have been studied extensively. (Not all 1/4 BPS configurations of the theory are described by the above BPS equation. For those configurations, two Higgs fields are involved and $A_4=0$, and so the compactified direction does not play any role.)

Introducing

$$\tilde{D}_4=\tilde{\partial}_4-i\tilde{A}_4, \tag{29}$$

Eq. (20) becomes $F_{04}=\tilde{D}_4\phi$ and Eq. (21) becomes $D_0\phi=-\tilde{D}_4\phi\tan\alpha$. Taking the covariant divergence of Eq. (18), we get

$$D_iF_{4i}\cos\alpha+D_i^2\phi\sin\alpha=0. \tag{30}$$

Using the above relations with Eqs. (22),(23), Eq. (19) and the Gauss law (4) can be put in a single equation,

$$D_i^2\phi+\tilde{D}_4^2\phi-[\zeta_I,[\zeta_I,\phi]]=0. \tag{31}$$

Equation (23) can be put in the form $\tilde{D}_4\zeta_I=0$. Since we are interested in the 1/4 BPS configuration such that the vacuum expectation values of ζ_I vanish ($\langle\zeta_I\rangle=0$) and $D_i\zeta_I=\tilde{D}_4\zeta_I=0$, ζ_I should vanish. Hence, we can drop the ζ_I fields from the further discussion. Thus, Eq. (31) becomes

$$D_i^2\phi+\tilde{D}_4^2\phi=0. \tag{32}$$

The topological charge of the new variables

$$\tilde{\mathcal{T}}=\frac{2}{e^2}\int d^4\tilde{x}\,\mathrm{tr}\,B_i\tilde{F}_{4i}=\mathcal{T}-Q^M\tan\alpha. \tag{33}$$

Note that $\tilde{\mathcal{T}}$ and $\mathcal{T}$ need not be integer numbers for the generic configuration. Another topological quantity appearing naturally here is

$$\tilde{Q}^E=\frac{2}{e^2}\int d^4\tilde{x}\tilde{\partial}_\mu\,\mathrm{tr}\;\phi\tilde{D}_\mu\phi=Q^E+Q^M\tan\alpha \tag{34}$$

for BPS configurations.

A. The case where $\alpha=0$

What is the reason behind this simplification of BPS equation? Let us first consider the $\alpha=0$ case. The BPS equations are considerably simplified:

$$B_i=F_{4i}, \tag{35}$$

$$E_i=D_i\phi, \tag{36}$$

$$F_{04}=D_4\phi, \tag{37}$$

$$D_0\phi=0. \tag{38}$$

In the gauge $A_0=-\phi$, the field configurations become time independent. In particular, Eqs. (37) and (38) are automatically satisfied. The fields A_μ satisfy the self-dual equation and the Gauss law constraint can be put in the simple form

$$D_\mu^2\phi=0. \tag{39}$$

From the above simplified BPS equations, we can see that $P^4=-Q_M$ from Eqs. (11),(10), which is consistent with the $\alpha=0$ picture in the BPS bound (17).

B. Lorentz boost along x^4

Let us start with the $\alpha=0$ case. We call its spacetime coordinates $\tilde{x}_M$ and its BPS field configurations $\tilde{A}_M$ and $\tilde{\phi}$. We can Lorentz boost this coordinate to get a new coordinate (x^4,t) such that

$$\tilde{t}=\frac{t}{\cos\alpha}-x^4\tan\alpha, \tag{40}$$

$$\tilde{x}^4=-t\tan\alpha+\frac{x^4}{\cos\alpha}. \tag{41}$$

Note that $1/(\cos\alpha)^2-(\tan\alpha)^2=1$ so that it is a Lorentz boost. The spatial coordinates x^i remain unchanged. We start with the compact radius $0<\tilde{x}^4<\beta/\cos\alpha$, and so the compact radius of x^4 becomes β.

The A_0 and A_4 fields transform as

$$\tilde{A}_0 = \frac{A_0}{\cos\alpha} + A_4 \tan\alpha, \tag{42}$$

$$\tilde{A}_4 = A_0 \tan\alpha + \frac{A_4}{\cos\alpha}, \tag{43}$$

and the ϕ and A_i fields are invariant. We work in the gauge $\tilde{A}_0 = \tilde{\phi} = \phi$. The BPS equations (35)–(38) of the $\alpha=0$ case become the old BPS equations (18)–(21) in terms of the new variables.

What is interesting about this Lorentz transformation is that $\mathcal{T}+Q^E$ is like the rest mass and P^4+Q^M is like the four-momentum of the Lorentz boost. [The Lorentz boost interpretation of Eqs. (33) and (34) shows that while $\mathcal{T}$ and Q^E transform nontrivially $\mathcal{T}+Q^E$ remains invariant like the rest mass.] Thus when $\alpha=0$ the momentum $P^4+Q^M=0$ as noted before. Of course our Lorentz boost is not an exact symmetry as the interval of space changes and is a kind of combination of a Lorentz boost and rescaling of the x^4 coordinate. However, this shows that the BPS configuration with nonzero α can be obtained from the BPS configurations with $\alpha=0$. This is exactly what we have seen in Eqs. (28) and (32). While the BPS configuration of the $\alpha=0$ case can be chosen to be time independent, this is not true for those of $\alpha \neq 0$.

IV. MONOPOLES AND CALORONS

The solutions of the primary BPS equation (28) are identical to the self-dual equations for the caloron. Since we are interested in the case where $\alpha \neq \pi/2$ and it can be obtained by a Lorentz boost from the case $\alpha=0$, we focus mainly on the $\alpha=0$ case. The first one (35) is the self-dual equation of A_μ on $\mathrm{R}^3\times \mathrm{S}^1$. The second BPS equation is the zero eigenvalue equation (39) for ϕ, around the solution of the first BPS equation.

Let us first consider the solution of the primary BPS equation. This is the BPS equation for the 1/2 BPS configurations. First of all we need the boundary condition for A_4. The vacuum expectation value of A_4 is single valued and takes the form

$$\langle A_4 \rangle = \mathrm{diag}(h_1, h_2, \ldots, h_N) = \mathbf{h}\cdot\mathbf{H}, \tag{44}$$

where $\Sigma_a h_a = 0$ and by gauge choice

$$h_1 < h_2 < \cdots < h_N < h_1 + \frac{2\pi}{\beta}. \tag{45}$$

This leads to a nontrivial Wilson loop $P\exp(i\int dx^4 A_4)$ and the symmetry is spontaneously broken to $\mathrm{U}(1)^{N-1}$. If any of two h_a's coincide, the gauge symmetry will have unbroken non-Abelian symmetry. While this possibility is quite interesting by itself, we will not pursue this direction in the present paper.

As shown in Ref. [1], the general solutions of the self-dual equation (35) describe the superpositions of N fundamental monopoles corresponding to the simple roots $\boldsymbol{\beta}_i, i = 1, \ldots, N-1$ and the lowest negative root $\boldsymbol{\beta}_0$. These roots form the extended Dynkin diagram. The topological charge ν_P of each type of monopole is fractional and takes the fractional value

$$\mu_r = \frac{\beta}{2\pi}(h_{r+1} - h_r), \quad r = 1, \ldots, N-1,$$
$$\mu_0 = 1 - \frac{\beta}{2\pi}(h_N - h_1). \tag{46}$$

While all monopoles are on an equal footing, the magnetic monopoles of $\boldsymbol{\beta}_0$ are called Kaluza-Klein monopoles as they have intrinsic x^4 dependence on the gauge (45).

Thus the general solution of the primary BPS equation is characterized by the N nonnegative integers n_r, each of which is the number of $\boldsymbol{\beta}_r$ monopoles. The total topological charge is then

$$\nu_P = \sum_0^{N-1} n_r \mu_r, \tag{47}$$

and the magnetic charge obtained from the asymptotic of $B_i = (r_i/r^3)\mathbf{g}\cdot\mathbf{H}$ is given by

$$\mathbf{g} = 4\pi \sum_0^{N-1} n_r \boldsymbol{\beta}_r. \tag{48}$$

The topological charge and the magnetic charge are N charges together and so determine the monopole numbers n_r, $r = 0, \ldots, N$, uniquely. The total number of partons or monopoles is

$$N_m = \sum_0^{N-1} n_r, \tag{49}$$

and the total number of zero modes of the first BPS equation is $4N_m$. A single caloron or instanton can be regarded as composed of N distinct fundamental monopoles, whose topological charge is 1 and whose magnetic charge is equal to zero. The number of zero modes of a single instanton is then $4N$, as expected from the index theorem. For a given set $\{n_r\}$ of monopoles, the solution of the first BPS equation is uniquely determined by the moduli parameters. The dimension of the moduli space of these configurations is $4N_m$. The general method to solve the first BPS equation is the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction, as detailed in Ref. [3]. Thus, the solutions of the primary BPS equations are identical to 1/2 BPS configurations.

The secondary BPS equation can be regarded as the gauge zero modes of the primary BPS equation. The linear fluctuations δA_μ should satisfy the linearized BPS equation and the gauge fixing condition

$$\epsilon_{ijk} D_j \delta A_k = D_4 \delta A_i - D_i \delta A_4,$$
$$D_\mu \delta A_\mu = 0. \tag{50}$$

When the linear fluctuation is due to the gauge zero mode, $\delta A_\mu = D_\mu \Lambda$, the linearized BPS equation is automatically satisfied and the gauge fixing condition becomes

$$D_\mu^2 \Lambda = 0, \tag{51}$$

which is identical to the secondary BPS equation (39). Since there are $N-1$ unbroken U(1) symmetries, there will be $N-1$ linearly independent solutions of the second BPS equation. The general method for solving the second BPS equation is given in the Appendix of Ref. [4].

The BPS equation (39) gives the electric field in terms of the solution ϕ. The solution of the second BPS equation is again uniquely determined for a given monopole background and the asymptotic value

$$\langle \phi \rangle = (a_1, a_2, \ldots, a_N) = \mathbf{a}\cdot \mathbf{H}, \quad \sum_{i=1}^{N} a_i = 0. \tag{52}$$

Thus, the moduli of the 1/2 BPS configurations determine the solution of the secondary BPS equation uniquely. From the asymptotics of the field ϕ

$$\phi = \langle \phi \rangle + \frac{1}{4\pi r}\mathbf{q}\cdot \mathbf{H} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{53}$$

we can read the electric field and so the electric charge Q^E.

However, the story is more complicated. There are N distinct monopoles, each of which can carry its own electric fields. Thus for a group of $\boldsymbol{\beta}_r$ monopoles there will be total $q_r \boldsymbol{\beta}_r$ electric charge and the electric charge is determined by the asymptotics of the ϕ field as

$$\mathbf{q} = \sum_{0}^{N-1} q_r \boldsymbol{\beta}_r . \tag{54}$$

Similarly to the magnetic charge, the asymptotics alone cannot decide each q_r since $\boldsymbol{\beta}_r$ is not independent although it determines the relative electric charges. The electric charge $\mathbf{q}$ is determined by the moduli parameters of the magnetic background and the asymptotic value $\mathbf{a}$.

In addition to the $N-1$ global U(1) Abelian symmetries of the gauge group SU(N), there is an additional U(1) related to the translation along the x^4 direction. Thus there are N global U(1) charges, the $N-1$ electric charge and the linear momentum P^4, which in turn determines the N parameters q_r. The linear momentum P^4 is not topological and so it is hard to see its relation to q_r explicitly. When the constituent monopoles are well separated from each other, one may be able to assign electric charge and linear momentum to each monopole. In particular, when $\alpha = 0$, the linear momentum (11) for very isolated β_r monopoles of electric charge q_r carries linear momentum $q_r \mu_r$. Then the total linear momentum becomes $\Sigma_r q_r \mu_r$, which should be $-Q^M$ due to Eq. (16). This leads to an additional relation between q_r's. The configuration with nonzero α can be obtained from the Lorentz boost of the above case.

V. LOW ENERGY DYNAMICS OF CALORONS

From the consideration in the previous sections, one can see that 1/4 BPS dyonic calorons can be constructed of fundamental dyons. The low energy dynamics of 1/4 BPS configurations has been explored in recent years [10,11]. It was shown that low energy dynamics is possible for 1/2 BPS configurations. The kinetic energy is given by the moduli space metric of the 1/2 BPS configurations. The additional Higgs field appearing in the 1/4 BPS configuration contributes to the potential in the low energy Lagrangian. The form of the potential is given by the sum of the square norm of the Killing vectors. For the low energy dynamics of fundamental objects one has to specify what class of 1/2 BPS configurations one starts with. For example, one can start from the case where $\langle A_4 \rangle \neq 0$ and $\langle \phi \rangle = 0$. In this case one starts with the constituent fundamental monopoles of calorons. Then we consider the low energy dynamics of monopoles with small but nonzero ϕ expectation value. This is the case we will focus on in this work.

There are other cases which we will not explore here. With $\langle A_4 \rangle = 0$ and $\langle \phi \rangle \neq 0$, one start with the the 1/2 BPS monopoles without any Kaluza-Klein monopoles, which would make calorons. It would be interesting to consider the low energy dynamics of these monopoles with small but nonzero $\langle A_4 \rangle$. One could consider more complicated 1/2 BPS configurations with $\langle A_4 \rangle \propto \langle \phi \rangle$ but without Kaluza-Klein monopoles.

The moduli space dynamics of dyonic instantons on R^4 has been studied before in Refs. [16,17]. It is somewhat simpler than our case as there is no symmetry breaking due to the A_4 expectation value.

A. Caloron moduli space

We start with the caloron case with $\langle \phi \rangle = 0$. Its low energy dynamics can be described by the caloron moduli space metric. The kinetic energy due to the spatial motion and electric charge are much smaller than the rest mass of the monopoles. We consider the modification of the low energy dynamics when $\langle \phi \rangle \neq 0$ but very small. Thus the angle α is very small. The $\alpha \neq 0$ case can be obtained by the Lorentz transformation and rescaling of the x^4 coordinate. As α is very small, it will become an infinitesimal Galilean transformation along the x^4 direction. Under this transformation the electric field transforms in a more complicated way as one can see from Eq. (19). The moduli space metric gets split into that for the center of mass motion and that for the relative motion. Thus we just focus on the case $\alpha = 0$.

In this case the solution of the primary BPS equation is identical to the 1/2 BPS caloron solutions. We assume that all n_r are positive and so all constituent monopoles are interacting. In this case, the 1/2 BPS configuration has an intrinsic x^4 dependence which cannot be gauged away.

The dimension of the moduli space is $4N_m$ and the moduli space coordinates are z^M with $M = 1, \ldots, 4N_m$. The moduli space metric can be obtained from the study of the linear fluctuations around the 1/2 BPS configurations $A_\mu(x,z)$,

$$\delta_M A_\mu = \frac{\partial A_\mu}{\partial z^M} - D_\mu \epsilon_M , \tag{55}$$

which satisfies Eq. (50). The moduli space metric is

$$g_{MN}(z)=2\int d^4x\,\mathrm{tr}\,(\delta_M A_\mu \delta_N A_\mu). \tag{56}$$

For instanton or caloron solutions of SU(N) gauge theory, the moduli space is $4N$ dimensional and has orbifold singularity at the point where all monopoles come together and the caloron collapses. This moduli space metric was obtained using the constituent monopole picture and the nature of the singularity is identified as $\mathrm{R}^{4N}/\mathrm{Z}_N$ [1]. It is also known that the singularity is resolved if we turn on the noncommutativity [12].

The effective Lagrangian for the low energy dynamics can be generically written as

$$L_K=\frac{1}{2e^2}g_{MN}\dot{z}^M\dot{z}^N. \tag{57}$$

There is a natural $N=4$ supersymmetric generalization of the above Lagrangian. Since there are N fundamental monopoles in calorons, instead of $N-1$ for SU(N) gauge theory, there are N conserved U(1) symmetries, each leading to the $q_r\boldsymbol{\beta}_r$ electric charges on $\boldsymbol{\beta}_r$ monopoles. This matches the field theoretic symmetries; $N-1$ of them are made of unbroken Abelian subgroups of SU(N) and one is for the translation along the x^4 direction.

To understand the dynamics better, let us split the center of mass motion and the relative motion. The Lagrangian (57) becomes the sum of the Lagrangian $L_{K\,c.m.}$ for the center of mass motion and L_{Krel} for the relative motion.

As the position of individual monopole is not well defined when two identical monopoles come together, it is hard to express the center of mass position in general. However one can argue that the charge for the central U(1) should be

$$q_{c.m.}=\frac{1}{\nu_P}\sum_{r=0}^{n} q_r\mu_r. \tag{58}$$

This is true at large separation. (See, for example, Ref. [13].) Also $q_{c.m.}$ should be a linear combination of q_r and is conserved, and so the coefficient should be independent of the monopole positions. We would like to identify $P_4=(2\pi/e^2)\nu_P q_{c.m.}$ at large separation. As we consider the $\alpha=0$ case, the center of mass charge is constrained to be $-Q_M=-(1/e^2)\mathbf{a}\cdot\mathbf{g}$. The nonzero α case is obtained by an infinitesimal Galilean transformation along the x^4 direction. Equation (16) implies $P_4+Q_M=(8\pi^2/e^2)\nu_P\alpha$ with ν_P in Eq. (47). The Killing vector on the moduli space corresponding to the x^4 translation is $K_{c.m.}=\partial/\partial\psi_{c.m.}$ with the center of mass angle variable $\psi_{c.m.}$.

The relative charges arise as the Noether charges for the $N-1$ Killing vectors $K_r^M\partial_M$ with $r=1,2,\ldots,N-1$ on the moduli space, which correspond to the $N-1$ unbroken generators of SU(N). For each of these Killing vectors, there is a cyclic coordinate ψ_r.

We can separate the moduli space coordinates z^M into $(\mathbf{r}_{c.m.},\psi_{c.m.})$ for the center of mass motion and (y^i,ψ^r) with $r=1,2,\ldots,N-1$ for the relative motion. The kinetic energy (57) also get split into $L_{c.m.}(\mathbf{r}_{c.m.},\psi_{c.m.})$ for the center of mass motion and $L_{rel}(y^i,\psi^r)$ for the relative motion. The center of mass motion is free and trivial. The relative part L_{rel} is independent of the cyclic coordinates ψ_r and can be expressed as

$$L_{rel}=\frac{1}{2e^2}h_{ij}(y)\dot{y}^i\dot{y}^j+\frac{1}{2e^2}L_{rs}(y)[\dot{\psi}^r+w_i^r(y)\dot{y}^i][\dot{\psi}^s+w_j^s(y)\dot{y}^j]. \tag{59}$$

The $N-1$ conserved Abelian charges of the above Lagrangian are

$$t_r=\frac{1}{e^2}L_{rs}(y)[\dot{\psi}^s+w_i^s(y)\dot{y}^i]. \tag{60}$$

B. The potential

For a given 1/2 BPS caloron background specified by its moduli parameters, we introduce the scalar field ϕ which takes a nonzero but very small expectation value. The scalar field takes the lowest possible energy when it satisfies $D_\mu^2\phi=0$. This scalar field in turn modifies the caloron dynamics at the second order in ϕ. In particular, its asymptotic behavior is given by Eqs. (52) and (53); especially $\mathbf{q}=\Sigma_r(q_r-q_0)\boldsymbol{\beta}_r=\Sigma_{r=1}q_r^{rel}\boldsymbol{\beta}_r$ depends on $\mathbf{a}$ and the caloron moduli parameters z^M. At this moment $\mathbf{q}\cdot\mathbf{H}$ is not the electric charge but simply the $1/r$ piece of the asymptotic ϕ.

The solution of $D_\mu^2\phi=0$ depends on the caloron moduli parameters. As ϕ satisfies the same equation (51) as the global gauge transformation parameters, we can interpret

$$D_\mu\phi=\mathbf{a}\cdot\mathbf{K}^M\delta_M A_\mu \tag{61}$$

where $\mathbf{K}^m\partial_M=\Sigma_{r=1}^{N-1}\boldsymbol{\beta}_r K_r^M\partial_M$ are $N-1$ Killing vectors for the $N-1$ U(1) gauge generators. As the global gauge transformations change the cyclic coordinates ψ^r of the relative motion, we choose these cyclic coordinates such that

$$K_r^M\frac{\partial}{\partial z^M}=\frac{e^2}{\beta}\frac{\partial}{\partial\psi^r}. \tag{62}$$

Then the part of the relative moduli space metric becomes $L_{rs}(y)=(e^4/\beta^2)g_{MN}K_r^M K_s^N$. The induced potential energy [14,10] is then given by

$$\begin{aligned}\mathcal{U}&=\frac{1}{e^2}\int d^4x\,\mathrm{tr}\,(D_\mu\phi)^2\\&=\frac{\beta}{2e^2}\mathbf{q}\cdot\mathbf{a}\\&=\frac{1}{2e^2}g_{MN}\mathbf{a}\cdot\mathbf{K}^M\mathbf{a}\cdot\mathbf{K}^N\\&=\frac{\beta^2}{2e^6}L_{rs}a^r a^s\end{aligned} \tag{63}$$

where $\mathbf{a}=\Sigma_r a^r \boldsymbol{\lambda}_r$ with the fundamental weights satisfying $\boldsymbol{\lambda}_r \cdot \boldsymbol{\beta}_s = \delta_{rs}$. The Tong formula [14] fixes the asymptotic value $\mathbf{q}$ explicitly in terms of the moduli space metric and the asymptotic value

$$\beta \mathbf{q} = g_{MN} \mathbf{a} \cdot \mathbf{K}^M \mathbf{K}^N \tag{64}$$

$$= \frac{\beta^2}{e^4} L_{rs} a^r \boldsymbol{\beta}_s . \tag{65}$$

The Hamiltonian $H_{rel}=L_{rel}+\mathcal{U}$ has a BPS bound because

$$H = \frac{1}{2e^2} h_{ij}(y) \dot{y}^i \dot{y}^j + \frac{e^2}{2} L^{rs}(y) \left(t_r \mp \frac{\beta}{e^4} L_{rr'} a^{r'} \right) \times \left(t_s \mp \frac{\beta}{e^4} L_{ss'} a^{s'} \right) \pm \frac{\beta}{e^2} q_r a^r, \tag{66}$$

which is saturated when $\dot{y}^i = 0$ and

$$t_r = q_r^{rel} \equiv \frac{\beta}{e^4} L_{rs} a^s. \tag{67}$$

The energy for the BPS configuration is then

$$\frac{\beta}{e^2} a^r q_r^{rel} = \frac{\beta^2}{e^6} L_{rs}(y) a^r a^s. \tag{68}$$

The above results for the electric charge and energy for the BPS configuration match exactly those from field theory [Eq. (54)] with $\alpha=0$, and also the above BPS energy plus the rest mass $\mathcal{T}$ becomes the field theoretic BPS energy (17).

VI. THE SU(2) CASE

We explore in more detail the SU(2) case. This shows the above description of 1/4 BPS configurations in a more concrete form. The first BPS equation describes the self-dual calorons, and can be solved by the ADHMN method. The calorons in the SU(2) case have been studied in detail before [3,2]. Using the small and large gauge transformations, we choose the expectation value

$$\langle A_4 \rangle = -\frac{v}{2} \sigma^3 = \mathrm{diag}(h_1, h_2) \tag{69}$$

with $0 < v \leq \pi/\beta$. There are two fundamental monopoles of opposite magnetic charge. One is the ordinary BPS monopole of topological charge $\mu_1 = \beta v / 2\pi$. Another is the Kaluza-Klein KK monopole of opposite magnetic charge and topological index $\mu_0 = 1 - \mu_1$. There is no force between these distinct monopoles as the magnetic attraction is canceled by the Higgs repulsion. A single caloron of a unit Pontriyagin index is made of one of these monopoles, so that the net magnetic charge is equal to zero. A single caloron configuration has been obtained explicitly. This complicated configuration describes a nonlinear superposition of two distinct monopoles.

Consider the $\alpha=0$ case. The solution of the first BPS equation is characterized by the two non-negative integers (n_0, n_1). The lowest root is $\boldsymbol{\beta}_0 = -\boldsymbol{\beta}_1$ and so the magnetic charge is

$$\mathbf{g} = 4\pi(-n_0 + n_1)\boldsymbol{\beta}_1 \tag{70}$$

and the total topological charge is

$$\nu_R = n_0 \mu_0 + n_1 \mu_1 . \tag{71}$$

For a single caloron $n_0 = n_1 = 1$, the solution of the primary BPS equation has been found by the ADHMN method and was shown to be a superposition of two these monopoles. Furthermore, the moduli space of the 1/2 BPS configurations was shown to be eight dimensional such that the center of mass part is flat space and the relative part is Taub–Newman-Unti-Tamburino (NUT) space.

A. Second BPS equation

The secondary BPS equation around the first BPS equation can also be found explicitly by the method summarized in the Appendix of Ref. [4]. Here we briefly outline the solution following that reference. The Nahm data for the SU(2) one-caloron case are given by [3,2]

$$\mathbf{T}_0 = -\mathbf{a}_0, \quad \mathbf{T}_1 = -\mathbf{a}_1, \tag{72}$$

where $\mathbf{a}_{0,1}$ are constant vectors representing the positions of constituent monopoles. We can put $\mathbf{a}_{0,1}$ by spatial rotation and translation at the z axis such that $\mathbf{a}_0^i = -(R/2)\delta^{i3}, \mathbf{a}_1^i = (R/2)\delta^{i3}$. In ADHMN formalism the covariant Laplacian for the adjoint scalar field ϕ is replaced by the equation for the function $p(t)$:

$$\ddot{p}(t) - W(t)p(t) + \Lambda(t) = 0, \tag{73}$$

where $W(t)$ and $\Lambda(t)$ are given by

$$W(t) \equiv \mathrm{tr}_2 \sum_{a=1}^{2} \delta(t-h_a) w_a^\dagger w_a ,$$
$$\Lambda = \mathrm{tr}_2 \sum_{a=1}^{2} \delta(t-h_a) w_a^\dagger \langle \phi \rangle w_a . \tag{74}$$

For the given Nahm data, $w_{1,2}$ are

$$w_1 = (\sqrt{2R}, 0), \quad w_2 = (0, \sqrt{2R}), \tag{75}$$

which is determined by the jumping condition

$$\mathbf{T}(h_a+) - \mathbf{T}(h_a-) = \frac{1}{2} \mathrm{tr}_2(\sigma^i w_a^\dagger w_a). \tag{76}$$

The solution of Eq. (73) for these data and the boundary value of the adjoint scalar field ϕ

$$\langle \phi \rangle = -\frac{\eta}{2}\sigma^3 = \mathbf{a} \cdot \mathbf{H}, \quad \mathbf{a} \equiv \eta \boldsymbol{\beta}_1 \tag{77}$$

is given by

$$p(t)=\begin{cases} p_0(t+\pi/\beta), & t\in[-\pi/\beta,-v/2],\\ p_1 t, & t\in[-v/2,v/2],\\ p_0(t-\pi/\beta), & t\in[v/2,\pi/\beta],\end{cases} \tag{78}$$

$$p_0=-\eta\frac{(\beta/2\pi)vR}{1+[1-(\beta/2\pi)v]vR}, \tag{79}$$

$$p_1=\eta\frac{[1-(\beta/2\pi)v]R}{1+[1-(\beta/2\pi)v]vR}. \tag{80}$$

Then we can construct the adjoint scalar field by

$$\phi=N^{-1/2}\langle\phi\rangle N^{-1/2}+N^{-1/2}\int dt u^\dagger p(t)uN^{-1/2}, \tag{81}$$

where N and $u(t)$ are the functions given in Ref. [3]. It is straightforward to obtain the ϕ field in terms of the functions given in Ref. [3] and extract the asymptotic behavior of ϕ.

From the asymptotic behavior, we get

$$\phi=-\eta\frac{\sigma_3}{2}\left(1-\frac{R}{r}\,\frac{\beta}{\beta+2\pi\mu_0\mu_1 R}\right)+\mathcal{O}\left(\frac{1}{r^2}\right), \tag{82}$$

and thus the electric charge is given by

$$Q^E=\frac{4\pi}{e^2}\,\frac{\eta^2\beta^2R}{\beta+2\pi\mu_0\mu_1R}. \tag{83}$$

B. Moduli space metric and Tong's method

The moduli space of the SU(2) one caloron is eight dimensional and this eight-dimensional moduli space can be split into the center of motion space and the relative motion space, and their metrics are given by [1–3,16]

$$ds^2_{c.m.}=8\pi^2\{(d\mathbf{X})^2+(d\chi_{c.m.})^2\}, \tag{84}$$

$$ds^2_{\rm rel}=8\pi^2\mu_0\mu_1\left\{\left(1+\frac{\bar r}{r}\right)d\mathbf{r}^2+\bar{r}^2 \times\left(1+\frac{\bar r}{r}\right)^{-1}[d\psi+\mathbf{w}(\mathbf{r})\cdot d\mathbf{r}]^2\right\}, \tag{85}$$

where $\bar r=\beta/2\pi\mu_0\mu_1$. The relative metric ds^2_{rel} is the one for the well-known Taub-NUT space. For this case

$$\frac{\beta}{e^2}\mathbf{q}=\frac{1}{e^2}g_{\psi\psi}(\mathbf{a}\cdot\mathbf{K}^\psi)\mathbf{K}^\psi. \tag{86}$$

Since $q_\psi=q_1-q_0$, $\mathbf{q}=q_\psi\boldsymbol{\beta}_1$, and $\mathbf{K}^\psi=\boldsymbol{\beta}_1$, one can see that

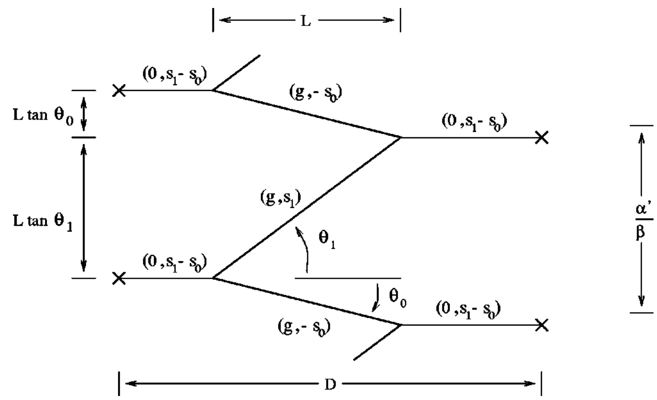

FIG. 1. String web picture of SU(2) single caloron case.

$$\frac{\beta}{e^2}q_\psi=\frac{1}{e^2}g_{\psi\psi}(\mathbf{a}\cdot\mathbf{K}^\psi)=\frac{\eta}{e^2}g_{\psi\psi} \tag{87}$$

$$=\frac{4\pi}{e^2}\,\frac{\eta\beta^2R}{\beta+2\pi\mu_0\mu_1R}. \tag{88}$$

This gives us the electric charge Q^E

$$Q^E=\frac{\beta}{e^2}\mathbf{a}\cdot\mathbf{q}=\frac{4\pi}{e^2}\,\frac{\eta^2\beta^2R}{\beta+2\pi\mu_0\mu_1R}, \tag{89}$$

which is the identical expression with that obtained in the previous section.

The low energy dynamics of this configuration is described by the free c.m. part and the relative Taub-NUT metric with potential term $U=\frac{1}{2}Q^E$. Note that the field theory BPS energy bound (17) is saturated by $\mathcal{T}=8\pi^2/e^2$ and the above Q^E.

C. String picture

A dyon in maximally supersymmetric four-dimensional Yang-Mills theory appears as fundamental and a D-string composition connecting D3-branes. The positions of D3-branes are specified by the adjoint scalar vacuum expectation values and the total energy of strings connecting D3-branes matches the field theoretic energy of dyons [9]. In the caloron case the string picture appears after the T dual is taken and then the vacuum expectation value of A_4 specifies the position of D3-branes.

With the tension for a single fundamental string $T=1/2\pi\alpha'$, the tension for the D-string is gT with $g=4\pi\beta/e^2$ in terms of the field theory parameters. The tension of $s=\beta q/e^2$ fundamental strings on a single D-string is $\sqrt{g^2+s^2}T$, where q is the field theory electric charge.

A single dyonic caloron in the SU(2) gauge group can be represented in the string picture by the T-dual transformation. Both A_4 and ϕ expectation values denote the position of the D3-branes in the compact dual space and transverse x^5 direction given by the ϕ field. The detail of the string picture of this configuration is given in Fig. 1.

This string configuration is 1/4 BPS when there is a tension balance at the string junctions [15], which implies that

$$\sin\theta_1 = \frac{g}{\sqrt{g^2+s_1^2}}, \tag{90}$$

$$\sin\theta_0 = \frac{g}{\sqrt{g^2+s_0^2}}, \tag{91}$$

where $s_i = (\beta/e^2)q_i$.

Let us consider the total energy of the above string configuration, which is the sum of the energy, that is, the tension times the length, of the individual string segments. The total string energy is then

$$E = (s_1 - s_0)T(D-L) + \sqrt{g^2+s_1^2}T\frac{L}{\cos\theta_1} + \sqrt{g^2+s_0^2}T\frac{L}{\cos\theta_0}. \tag{92}$$

This can be written as

$$E = gTL(\tan\theta_1 + \tan\theta_0) + (s_1 - s_0)TD \tag{93}$$

which is what we expect for the BPS energy from field theory since we can identify the string theory parameters with the field theory ones as $TD = \eta$, $\beta TL \tan\theta_i = 2\pi\mu_i$ $(i=0,1)$. The critical charge appears when $L = D$ or

$$\Delta\tilde{q}_c \equiv (s_1 - s_0)_c = \frac{\beta T}{2\pi\mu_1\mu_0} gD, \tag{94}$$

which is identical with $\lim_{R\to\infty}(q_1 - q_0)\beta/e^2$ from the field theory result (89).

VII. CONCLUSION

In this paper we investigated the 1/4 BPS configuration on $R^{3+1}\times S^1$ in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. The generic BPS bound and BPS equations are obtained when P^4, Q^E, Q^M, and $\mathcal{T}$ are turned on. The BPS equations are more complicated than for the 1/4 BPS configuration on R^{3+1}. Nevertheless, BPS equations can be simplified by a suitable transformation, resulting in two equations. Then, the static solution for the 1/4 BPS configuration can be obtained by solving first the primary equation, which is the much studied 1/2 BPS caloron equation on $R^3\times S^1$ and next the secondary equation on this 1/2 BPS caloron background. This is quite similar to the approach for the 1/4 BPS dyonic state in four-dimensional field theory [12].

A new feature in the 1/4 BPS dyonic caloron case compared to the 1/4 BPS dyon is the appearance of a topological charge $\mathcal{T}$ and a four-momentum P^4 in the BPS bound. In the theory of the SU(2) gauge group, there are no 1/4 BPS dyons in four dimensions but there are 1/4 BPS caloron configurations in five dimensions. In the infinite β limit these dyonic calorons become the dyonic instantons that were studied before [6,7].

We also considered the low energy dynamics of this 1/4 BPS configuration and it turns out that it is described by the same type of nonlinear σ-model Lagrangian with 1/4 BPS dyons. But the potential term is more complicated because of the existence of P^4 and Q^M.

There are several directions for further study. First, it would be interesting to understand the $\alpha\neq 0$ case from the low energy dynamics viewpoint. We argued that it is related to the center of mass motion of the low energy Lagrangian.

Second, it would be interesting to study the quantum spectrum of this configuration. After quantization the translational generator along the x^4 direction takes discrete integer values. However, the momentum P^4 can take a fractional value. Thus, it would be interesting to elucidate this point.

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