Spontaneous symmetry breaking and reflectionless scattering data

M. Bordag*
University of Leipzig, Institute for Theoretical Physics, Augustusplatz 10/11, 04109 Leipzig, Germany

A. Yurov†
Kaliningrad State University, Theoretical Physics Department, 14, Al. Nevsky St., Kaliningrad 236041, Russia
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We consider the question of which potentials in the action of a (1+1) dimensional scalar field theory allowing for spontaneous symmetry breaking have quantum fluctuations corresponding to reflectionless scattering data. The general problem of restoration from known scattering data is formulated and a number of explicit examples are given. Only certain sets of reflectionless scattering data correspond to symmetry breaking and all restored potentials are similar either to the Φ^4 model or to the sine-Gordon model.

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I. INTRODUCTION

Quantum corrections to classical solutions such as kinks [1,2] and spontaneous symmetry breaking are a field of intensive study and have applications in many branches of theoretical physics ranging from the standard model to solid state. Recent interest appeared from some subtleties connected with supersymmetry [3]. A number of models are usually considered in this connection. The most popular ones are the Φ^4 model and the sine-Gordon model. They result in a scattering problem for the quantum fluctuations with reflectionless potentials. As a result calculations of quantum corrections to the mass become very explicit. In the present paper we investigate the question of which models result in a reflectionless scattering potential. The surprising result is that all of them are very similar to those mentioned above.

The setup of the problem is as follows. We consider a scalar field $\Phi(x,t)$ in (1+1) dimensions with the action

$$S[\Phi]=\frac{1}{2}\int dxdt[(\partial_\mu\Phi)^2+U(\Phi)^2]. \tag{1}$$

If the squared potential $U^2(\Phi)^2$ has two (or more) minima of equal depth, spontaneous symmetry breaking occurs and topological nontrivial kink solutions $\Phi_k(x)$ exist. In order to calculate the quantum fluctuations $\eta(x,t)$ in the background of the kink one has to solve the scattering problem for the potential $V(x)$ which appears from the second derivative $\delta^2 S[\Phi_k]/\delta\Phi_k^2(x)$ of the action [see Eq. (7) below]. In simple models such as those mentioned above this potential $V(x)$ is reflectionless.

In the present paper we try to describe all potentials $U(\Phi)$ in Eq. (1) that correspond to a reflectionless scattering potential $V(x)$ and calculate the corresponding classical energy E_{class} and the quantum energy E_0 which is the ground state energy of the field η in the background of $\Phi_k(x)$.

In calculating these quantities it is usually assumed that the potential $U(\Phi)$ is given. After that one solves the scattering problem related to $V(x)$ and calculates the energies E_{class} and E_0. In Ref. [4] the inverse approach was proposed. One starts from the solution of the scattering problem given in terms of the so called scattering data $\{r(k),\beta_i,\kappa_i\}$ known since [5] to be in a one-to-one correspondence with the potential $V(x)$ (for a representation of these questions see [6] and references therein). Here $r(k)$ is the reflection coefficient, κ_i are the bound state energies, and β_i are numbers connected with the normalization of the bound state wave functions. As shown in [4] the ground state energy can be expressed in a simple way in terms of the scattering data, even including the necessary ultraviolet renormalization [see Eq. (10) below]. In order to find the classical energy one has to restore the potential $V(x)$ from the scattering data. This is the so called inverse scattering problem which was solved in terms of certain integral equations (see, again, [6]). In this way, by solving the inverse scattering problem the classical energy can be calculated from the scattering data. In [4] it was shown how this procedure works on the simplest example of reflectionless $[r(k)=0]$ scattering data containing only one bound state.

In the present paper we use this inverse approach to describe all potentials $U(\Phi)$ corresponding to reflectionless scattering data and having topologically nontrivial solutions, allowing in this way for spontaneous symmetry breaking. It turns out that not all scattering data correspond to such potentials $U(\Phi)$ but only certain classes. So we can formulate the reconstruction problem as finding the mapping between scattering data and potentials $U(\Phi)$ allowing for spontaneous symmetry breaking.

The so called rational scattering data deserve special consideration. Here the reflection coefficient $r(k)$ is a rational function of k and thus given by a finite number of parameters. For a rational $r(k)$ the inverse scattering problem is known to have an explicit algebraic solution (in a similar way as in the reflectionless case) and the classical energy can then be obtained by integration. In addition, the rational scattering data form a dense subset in the set of all scattering data. In this way, the inverse approach may provide an approximation scheme for the general case.

It should be mentioned that quantum corrections to soliton solutions have been extensively studied in the past. Special

*Email addres: Michael.Bordag@itp.uni-leipzig.de
†Email address: artyom.yurov@mail.ru

attention had been paid to models allowing for a solution of the equations for the quantum fluctuations in terms of known special functions and especially for explicit solutions. Also, a reconstruction problem was considered earlier. Thus in [7] it was shown that models whose quantum fluctuations can be expressed in terms of hypergeometric functions are exhausted by the Φ^4 and the sine-Gordon models. Higher order polynomial interactions result in differential equations with more than two regular singularities. A particularly interesting paper is [8] where the reconstruction problem was considered from the group theoretical point of view and potentials are reconstructed starting from zero modes.

The paper is organized as follows. In the next section we consider soliton potentials providing completely explicit formulas. In the third section we consider scattering data given by two bound states. In the fourth section we show how this can be generalized to the general reflectionless case. Conclusions are given in the last section. We use units with $\hbar = c = 1$.

II. FORMULATION OF THE RECONSTRUCTION PROBLEM

We consider a scalar field Φ with action $S[\Phi]$, Eq. (1), in 1+1 dimensions. Static solutions $\Phi(x)$ are subject to the equation of motion $\Phi''(x) = U(\Phi)U'(\Phi)$ where the prime denotes differentiation with respect to the argument. We assume that $U^2(\Phi)$ has at least two minima of equal depth and we are free to denote two neighboring ones by $\pm\Phi_{\rm vac}$. These fields, $\Phi(x) = \pm\Phi_{\rm vac}$, are the vacuum solutions. In case $\Phi_{\rm vac} \neq 0$ there exist topological nontrivial solutions $\Phi_k(x)$ called kink solutions which interpolate between the vacuum solutions by means of $\Phi_k(x \to \pm\infty) = \pm\Phi_{\rm vac}$. These solutions obey the Bogomol'nyi equations

$$\Phi_k'(x) = U(\Phi_k(x)) \tag{2}$$

and have the classical energy

$$E_{\rm class} = \frac{1}{2}\int_{-\infty}^{\infty} dx\{[\Phi_k'(x)]^2 + U^2(\Phi_k(x))\} \tag{3}$$

which by means of Eq. (2) can be written in the form

$$E_{\rm class} = \int_{-\infty}^{\infty} dx\, U^2(\Phi_k(x)). \tag{4}$$

In order to have a finite energy of the kink we must assume that the potential $U(\Phi)$ is zero in its minima.

The quantization of the scalar field in the background of the kink solution by means of the shift

$$\Phi(x,t) = \Phi_k(x) + \eta(x,t) \tag{5}$$

delivers in the Gaussian approximation the action

$$S_{\rm fluct}[\eta] = \frac{1}{2}\int dx\,dt\,\eta(x,t)[\partial_t^2 - \partial_x^2 + \mu^2 + V(x)]\eta(x,t) \tag{6}$$

for the fluctuations where the potential $V(x)$ results from

$$\frac{1}{2}\left.\frac{\delta^2 U^2(\Phi)}{\delta\Phi^2}\right|_{\Phi=\Phi_k} = [U'(\Phi)]^2 + U(\Phi)U''(\Phi) \equiv \mu^2 + V(x). \tag{7}$$

Here μ is defined from demanding $V(x\to\infty) = 0$ and has the meaning of being the mass of the fluctuating field $\eta(x,t)$.

The one loop quantum corrections to the energy are given by a functional determinant. For a static background they can be formulated equivalently in terms of the ground state energy E_0 of $\eta(x,t)$ in the background of the kink,

$$E_0 = \frac{1}{2}\sum_{(n)} \epsilon_{(n)}, \tag{8}$$

where the $\epsilon_{(n)}$ are the one particle energies of the fluctuations. They are eigenvalues of the corresponding Schrödinger equation

$$[-\partial_x^2 + \mu^2 + V(x)]\eta_{(n)}(x) = \epsilon_{(n)}^2 \eta_{(n)}(x). \tag{9}$$

Here, the index (n) denotes the spectrum of the operator in the left-hand side of Eq. (9). In fact, Eq. (8) defines E_0 only symbolically. One has to subtract the Minkowski space contribution and perform the ultraviolet renormalization. These procedures are by now well known. We follow here the treatment in [4]. For a discussion of the relations to different renormalization schemes we refer to [9] where, for instance, the equivalence of the subtraction scheme based on the heat kernel expansion and the mass renormalization with the "no tadpole condition" was shown.

In terms of the scattering data the renormalized ground state energy can be written in the form [10]

$$E_0 = \frac{-1}{4\pi^2}\int_0^\infty \frac{dq\,q}{\sqrt{\mu^2+q^2}} \log\frac{q+\sqrt{\mu^2+q^2}}{\sqrt{\mu^2+q^2}-q} \log\frac{1}{1-r(q)^2} - \frac{1}{\pi}\sum_{i=1}^{N}\left(\kappa_i - \sqrt{\mu^2-\kappa_i^2}\arcsin\frac{\kappa_i}{\mu}\right). \tag{10}$$

Here, the κ_i are the binding energies of the bound states in the potential $V(x)$,

$$[-\partial_x^2 + V(x)]\eta_i(x) = -\kappa_i^2\eta_i(x), \tag{11}$$

where the $\eta_i(x)$ are the corresponding eigenfunctions. These are bound state wave functions and they are normalizable, $\int_{-\infty}^{\infty} dx\,\eta_i^2(x) < \infty$. The function $r(k)$ is the reflection coefficient and both κ_i and $r(k)$ belong to the scattering data. It should be underlined that in E_0, Eq. (10), the ultraviolet divergences are subtracted. This results in this quite simple form because the heat kernel coefficients can be expressed in terms of the scattering data.[1] A nice consequence that can be read off from this formula is that the ground state energy is always negative.

[1]This is related to the fact that here the heat kernel coefficients are just the conservation laws of the Korteweg–de Vries equation.

As mentioned in the Introduction, the problem of calculating quantum corrections can be inverted. One starts from the scttering data and by means of Eq. (10) the quantum corrections can be obtained by simple integration. The price one has to pay is a more complicated procedure to obtain the classical energy. One has to solve the inverse scattering problem, i.e., one has to reconstruct the potential $V(x)$ from the scattering data. This problem was intensively studied in connection with the solution of nonlinear evolution equations in the 1970s. The last step in this procedure is then to restore the potential $U(\Phi)$ from $V(x)$ using Eq. (7) and finally to calculate the classical energy from Eq. (4).

In following this general procedure we make use of Eq. (7) and the Bogomol'nyi equation (2). Differentiating Eq. (2) twice with respect to x we obtain

$$\Phi'''(x)=\{[U'(x)]^2+U(x)U''(x)\}\Phi'(x). \tag{12}$$

By means of Eq. (7) and with the notation $\eta(x):=\Phi'(x)$ we rewrite this equation in the form

$$[-\partial_x^2+V(x)]\eta(x)=-\mu^2\eta(x). \tag{13}$$

This equation shows that the derivative of the kink is a bound state solution of the scattering problem associated with the potential $V(x)$ and that the mass μ of the fluctuating field $\eta(x,t)$ in Eq. (6) is the corresponding binding energy, i.e., one of the κ_i's in the scattering data. Note that $\eta(x)$ in Eq. (13) cannot be a scattering solution because in that case μ^2 would be negative. The decrease of $\eta(x)$ for $x\rightarrow\pm\infty$ is by means of

$$\int_{-\infty}^{\infty}dx\,\eta(x)=\int_{-\infty}^{\infty}dx\,\frac{d}{dx}\Phi_k(x)=\Phi_k(\infty)-\Phi_k(-\infty)=2\Phi_{\rm vac} \tag{14}$$

connected with a finite vacuum solution.

In this way, if we know $\eta(x)$, the field $\Phi(x)$ is given by

$$\Phi_k(x)=-\Phi_{\rm vac}+\int_{-\infty}^{x}d\xi\,\eta(\xi) \tag{15}$$

and we restored $\Phi_k(x)$ from $\eta(x)$. The potential $U(\Phi)$ can be restored simply as

$$U(\Phi_k(x))=\eta(x). \tag{16}$$

Note that the potential $U(\Phi)$ can be restored only from the ground state wave function of the scattering potential $V(x)$ because it is only this function which does not have zeros. If $\eta(x)$ vanishes for some finite x, the function $U(\Phi_k(x))$ will do so, in contradiction with our assumption that two neighboring zeros correspond to $x\rightarrow\pm\infty$.

In this way, by means of Eqs. (15) and (16) we obtained a parametric representation of the potential $U(\Phi)$ in terms of the ground state wave function $\eta(x)$. We note that this representation covers the region with $\Phi\in[-\Phi_{\rm vac},\Phi_{\rm vac}]$. How to go beyond this we consider in the following sections.

There is a freedom in the parametric representation Eqs. (15),(16). The ground state wave function $\eta(x)$ which we obtain as a solution of the inverse scattering problem is determined up to a multiplicative factor, which has the meaning of the normalization of $\eta(x)$ only. So we are free to multiply the function $\eta(x)$ by a constant, $\eta(x)\rightarrow\alpha\eta(x)$. After that we can assume $\eta(x)$ to be normalized, $\int_{-\infty}^{\infty}dx\,\eta(x)=1$. In doing so we express α from Eq. (14) as

$$\alpha=2\Phi_{\rm vac}.$$

In this way the freedom in the normalization of $\eta(x)$ is expressed in terms of $\Phi_{\rm vac}$. After this rescaling we rewrite Eqs. (15) and (16) in the final form

$$\Phi_k(x)=-\Phi_{\rm vac}+2\Phi_{\rm vac}\int_{-\infty}^{x}d\xi\quad\eta(\xi) \tag{17}$$

and

$$U(\Phi_k(x))=2\Phi_{\rm vac}\quad\eta(x). \tag{18}$$

Using Eq. (18) we obtain from Eq. (4) the classical energy which is the quantity we are interested in,

$$E_{\rm class}=4\Phi_{\rm vac}^2\int_{-\infty}^{\infty}dx\,\eta^2(x). \tag{19}$$

By the pair of equations (10) and (19), we obtain the final expressions relating the complete energy

$$E=E_{\rm class}+E_0 \tag{20}$$

to the scattering data.

However, it should be noticed that this is merely a formal solution. We restored $U(\Phi)$ for a restricted range of Φ only. We have to construct a continuation to all values of Φ which must deliver a single valued function $U(\Phi)$ having the necessary extrema in order to allow for spontaneous symmetry breaking. The investigation of this property is the main difficulty in the restoration problem.

We conclude this section with a discussion of the free parameters. First of all there are the scattering data which constitute a set of independent parameters. Second, we have the vacuum solution $\Phi_{\rm vac}$, which is in fact the condensate of the field Φ. As seen from the above formulas there is no further freedom in the restoration process. Together with the uniqueness of the restoration of $\eta(x)$ from the scattering data the above mentioned parameters are the only independent ones. As for the dimensions we note that $\Phi_{\rm vac}$ is dimensionless (we work in 1+1 dimensions) and that the bound state levels κ_i have the dimension of a mass. For reflectionless scattering data these are the only dimensional parameters and a rescaling $\kappa_i\rightarrow\lambda\kappa_i$ results in $E\rightarrow\lambda E$. In the rest of this paper we put the mass scale equal to 1.

III. RECONSTRUCTION FROM SOLITON POTENTIALS

In this section we consider the case of reflectionless scattering data $[r(k)=0]$ given by N bound states with energy levels

$$\kappa_i=i\quad(i=1,2,\ldots,N). \tag{21}$$

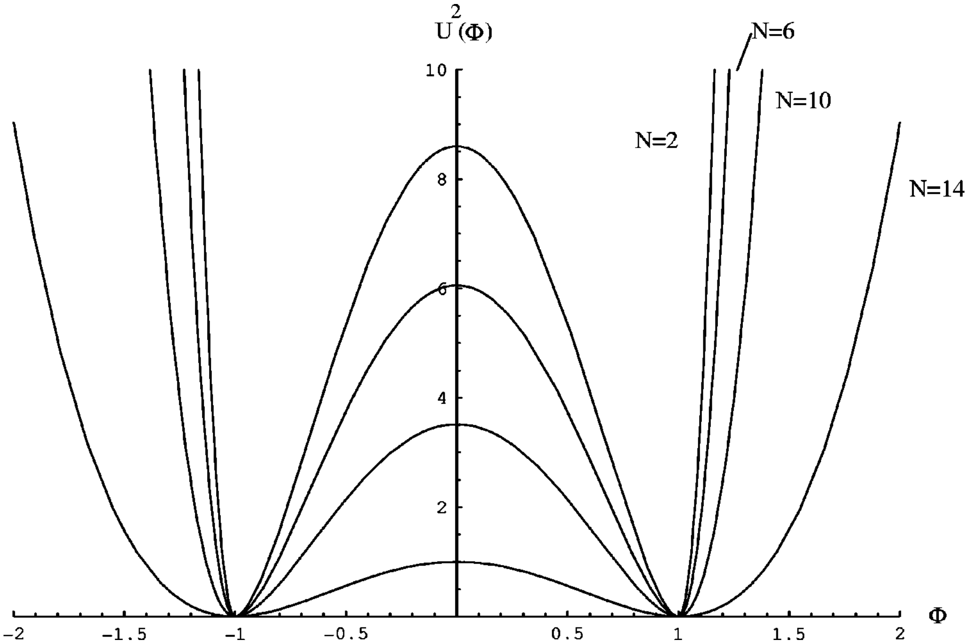

FIG. 1. The squared potential $U^2(\Phi)$ reconstructed from a soliton potential with an even number of bound states $N=2,6,10,14$ and $\Phi_{\rm vac}=1$.

Here the ground state is that with number $i=N$. The potential $V(x)$ belonging to these scattering data is well known,

$$V(x)=\frac{-N(N+1)}{\cosh^2 x}. \tag{22}$$

The solutions $\eta(x)$ of Eq. (11) are well known too. The ground state wave function reads

$$\eta(x)=\frac{1/\gamma_N}{\cosh^N x} \tag{23}$$

and the corresponding eigenvalue is $\kappa_N=N$. The normalization factor γ_N is defined from $\int_{-\infty}^{\infty}dx\,\eta(x)=1$ and will be calculated later in Eq. (29). We call these $V(x)$ soliton potentials because they are related to the soliton solutions of the Korteweg–de Vries equation.

Now, in order to solve the restoration problem we first consider even N. Here it is useful to change the variable in Eq. (17) according to

$$x=\operatorname{arctanh} t. \tag{24}$$

We introduce the notation $\Phi(t)=\Phi(x(t))$. After that the integral over ξ in Eq. (17) can be calculated easily and we arrive at

$$\begin{aligned}\Phi(t)&=-\Phi_{\rm vac}+\frac{2\Phi_{\rm vac}}{\gamma_N}\int_{-1}^{t}\frac{d\tau}{1-\tau^2}(1-\tau^2)^{N/2}\\&=-\Phi_{\rm vac}+\frac{2\Phi_{\rm vac}}{\gamma_N}\sum_{i=0}^{N/2-1}\binom{N/2-1}{i}\frac{(-1)^i}{2i+1}\\&\quad\times(t^{2i+1}+1),\end{aligned} \tag{25}$$

$$U(\Phi(t))=\frac{2\Phi_{\rm vac}}{\gamma_N}(1-t^2)^{N/2}. \tag{26}$$

Now we observe that for $t\in[-1,1]$, or equivalently for $x\in(-\infty,\infty)$, we restored just the kink solution $\Phi_k(t)$ and the potential $U(\Phi_k(t))$ in a parametric representation. In this way we know $U(\Phi)$ for $\Phi\in[-\Phi_{\rm vac},\Phi_{\rm vac}]$. However, the parametrization (24) together with the explicit formulas (25) and (26) allow us to go beyond the region $t\in[-1,1]$. Simply, we have to consider Eqs. (25) and (26) for $|t|>1$. For that t, the variable x becomes complex but $\Phi(t)$ and $U(\Phi(t))$ remain real. We have to ensure that $t\in(-\infty,\infty)$ covers the whole range $\Phi\in(-\infty,\infty)$ and that the resulting $U(\Phi)$ is a single valued function. For this end we consider the derivative

$$\frac{d\Phi(t)}{dt}=\frac{2\Phi_{\rm vac}}{\gamma_N}(1-t^2)^{N/2-1}.$$

It may change its sign at $t=\pm1$. If it changes it sign the function $\Phi(t)$ is not monotonic and, as a consequence, $U(\Phi)$ is not single valued. If, on the contrary, there is no change in the sign, $\Phi(t)$ is monotonic. Finally, from the remark that $\Phi(t)$ is a polynomial in t the coverage of the whole region for Φ follows. This is the case for $N=2(2s+1)$ $(s=1,2,\dots)$.[2] From Eq. (26) it is seen that $U(\Phi)$ is in that case a function with two minima as in the Φ^4 model. For large Φ, the asymptotic behavior is

$$U(\Phi)\underset{\Phi\to\infty}{\sim}\Phi^{N/(N-1)}.$$

[2]Note that these are not all even N. For instance, for $N=4$, we have from Eq. (25) $\Phi(t)=\Phi_{\rm vac}(7/6+t-t^3/3)$ which is clearly not monotonic, hence the corresponding $U(\Phi)$ is not single valued.

Some examples for $U(\Phi)$ are shown in Fig. 1.

For $N=2$ we reobtain the Φ^4 model. Here the explicit formulas read

$$\Phi(t)=\Phi_{\rm vac}t,$$

$$U(\Phi(t))=\Phi_{\rm vac}\ \ (1-t^2),$$

which can be trivially resolved,

$$U(\Phi)=\Phi_{\rm vac}\left[1-\left(\frac{\Phi^2}{\Phi_{\rm vac}}\right)^2\right].$$

The next example is $N=6$. Here the parametric representation reads

$$\Phi(t)=\frac{1}{8}\Phi_{\rm vac}t(15-10t^2+3t^4),$$

$$U(\Phi(t))=\frac{15}{8}\Phi_{\rm vac}\ \ (1-t^2)^3,$$

which for $t\in(-\infty,\infty)$ defines the complete dependence $U(\Phi)$. However, as can be seen, there is no explicit expression for $U(\Phi)$. Only the inverse function can be given explicitly,

$$\Phi(U)=\Phi(t)\Big|_{t=\sqrt{1-(8U/15\Phi_{\rm vac})^{1/3}}},$$

where the branches have to be chosen accordingly (the parametric representation is much simpler).

In this example we see explicitly how the continuation beyond the initial region works. The reason that it works at all is that we assumed the potential $U(\Phi)$ to be a function of Φ and not a more general object like, for instance, a functional.

Now we turn to odd N. Here it is useful to change the variable x to θ according to

$$\frac{1}{\cosh x}=\cos\theta. \tag{27}$$

We obtain again an explicit parametric representation,

$$\Phi(\theta)=\frac{\Phi_{\rm vac}}{\gamma_N}\binom{N-1}{(N-1)/2}\frac{\theta}{2^{N-1}}+\frac{2\Phi_{\rm vac}}{\gamma_N}\times\sum_{k=0}^{(N-3)/2}\frac{1}{2^{2k-1}}\binom{N-1}{k}\frac{\sin(N-1-2k)\theta}{N-1-2k},$$

$$U(\Phi(\theta))=\frac{\Phi_{\rm vac}}{\gamma_N}\cos^N\theta. \tag{28}$$

The region $x\in(-\infty,\infty)$ corresponds to $\theta\in[-\pi/2,\pi/2]$ and Eq. (28) gives for that θ the kink solution $\Phi_k(\theta)=\Phi_k(x(\theta))$. Again, we obtain from this explicit parametric representation all Φ by going beyond this region to $|\theta|>\pi/2$. From Eqs. (28) and (28) it is obvious that $U(\Phi)$ defined in this way is a single valued function. It has neighboring zeros located at $\Phi=\pm\Phi_{\rm vac}$. It is a periodic function with period $2\Phi_{\rm vac}$. So we see that for each odd N the restoration delivers a periodic potential $U(\Phi)$. For $N=1$ we note that

$$\Phi(\theta)=\frac{2\Phi_{\rm vac}}{\pi}\theta,$$

$$U(\Phi(\theta))=\frac{2\Phi_{\rm vac}}{\pi}\cos\theta,$$

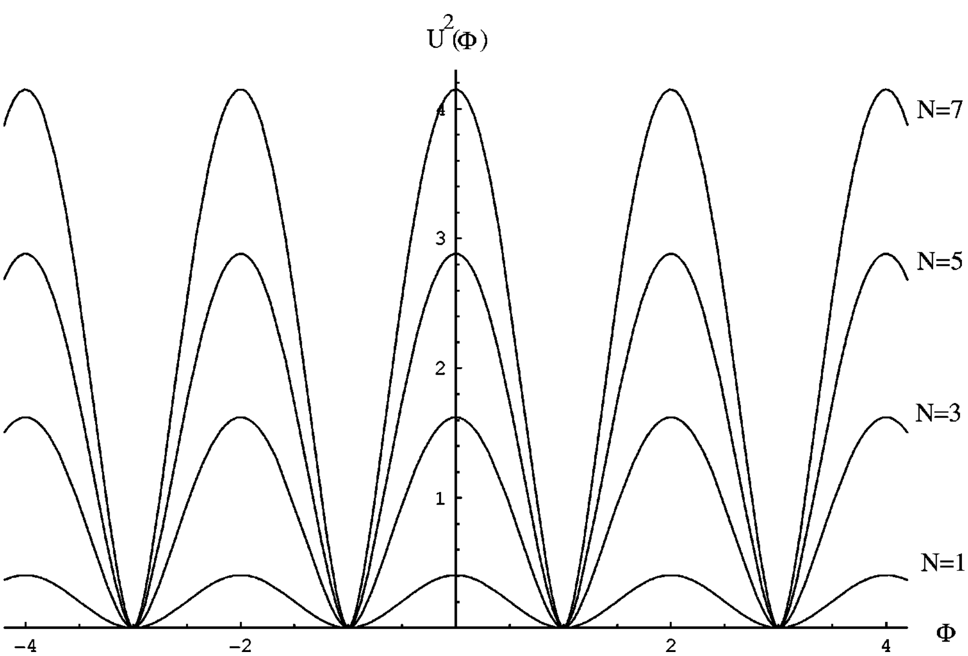

FIG. 2. The squared potential $U^2(\Phi)$ reconstructed from a soliton potential with an odd number of bound states $N=1,3,5,7$ and $\Phi_{\rm vac}=1$.

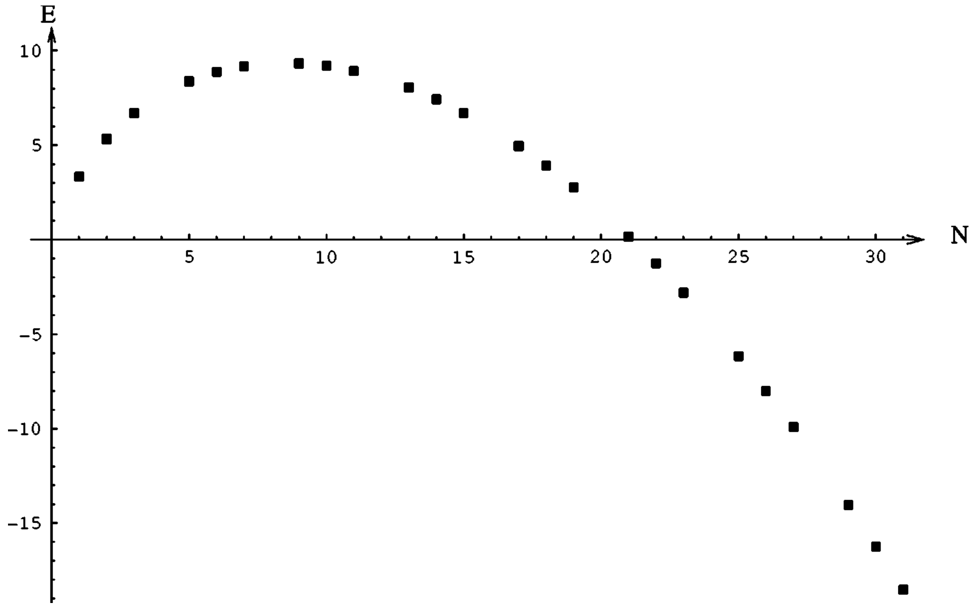

FIG. 3. The complete energy for soliton potentials with N bound states; the value of the condensate is $\Phi_{\rm vac}=1.5$.

which can be resolved to $U(\Phi)=(2\Phi_{\rm vac}/\pi)\cos(\pi\Phi/2\Phi_{\rm vac})$, which is the sine-Gordon model. For $N=3$ we obtain

$$\Phi(\theta)=\frac{\Phi_{\rm vac}}{\pi}[2\theta+\sin(2\theta)],$$

$$U(\Phi(\theta))=\frac{4\Phi_{\rm vac}}{\pi}\cos^3\theta.$$

Again, there is an explicit expression for $\Phi(U)$ but not for $U(\Phi)$. Examples for some of the first odd N are given in Fig. 2.

It remains to calculate the corresponding energies. The normalization factor γ_N in Eq. (23) can be calculated explicitly,

$$\gamma_N=\int_{-\infty}^{\infty}dx\,\frac{1}{\cosh^N x}=\frac{\sqrt{\pi}\,\Gamma(N/2)}{\Gamma((N+1)/2)}. \tag{29}$$

The asymptotics for large N is $\gamma_n\sim\sqrt{\pi/(2N)}$. Further, we note that

$$\int_{-\infty}^{\infty}dx\,\eta^2(x)=\gamma_{2N}.$$

In this way we obtain

$$E_{\rm class}=4\Phi_{\rm vac}^2\frac{\gamma_{2N}}{(\gamma_N)^2} \tag{30}$$

and

$$E_0=-\frac{1}{\pi}\sum_{i=1}^{N}\left(i-\sqrt{N^2-i^2}\arcsin\frac{i}{N}\right). \tag{31}$$

As mentioned in [10], the renormalized vacuum energy is always negative in 1+1 dimensions, which can be checked for Eq. (31) easily. The classical energy is of course positive so that these two contributions to the complete energy compete. For any finite N, which prevails depends on $\Phi_{\rm vac}$. For large $\Phi_{\rm vac}$, which corresponds to a weak coupling, we have positive complete energy, whereas for large N the quantum energy grows faster than the classical one. This is shown in Fig. 3.

IV. RECONSTRUCTION FROM TWO BOUND STATES

In this section we consider reflectionless scattering data consisting of two bound states,

$$\kappa_1=N_1,$$

$$\kappa_2=N_2,\quad \text{(ground state)},$$

assuming $N_2>N_1$. The ground state wave function reads

$$\eta(x)=\frac{2\cosh(N_1x)}{(N_2-N_1)\cosh[(N_2+N_1)x]+(N_2+N_1)\cosh[(N_2-N_1)x]} \tag{32}$$

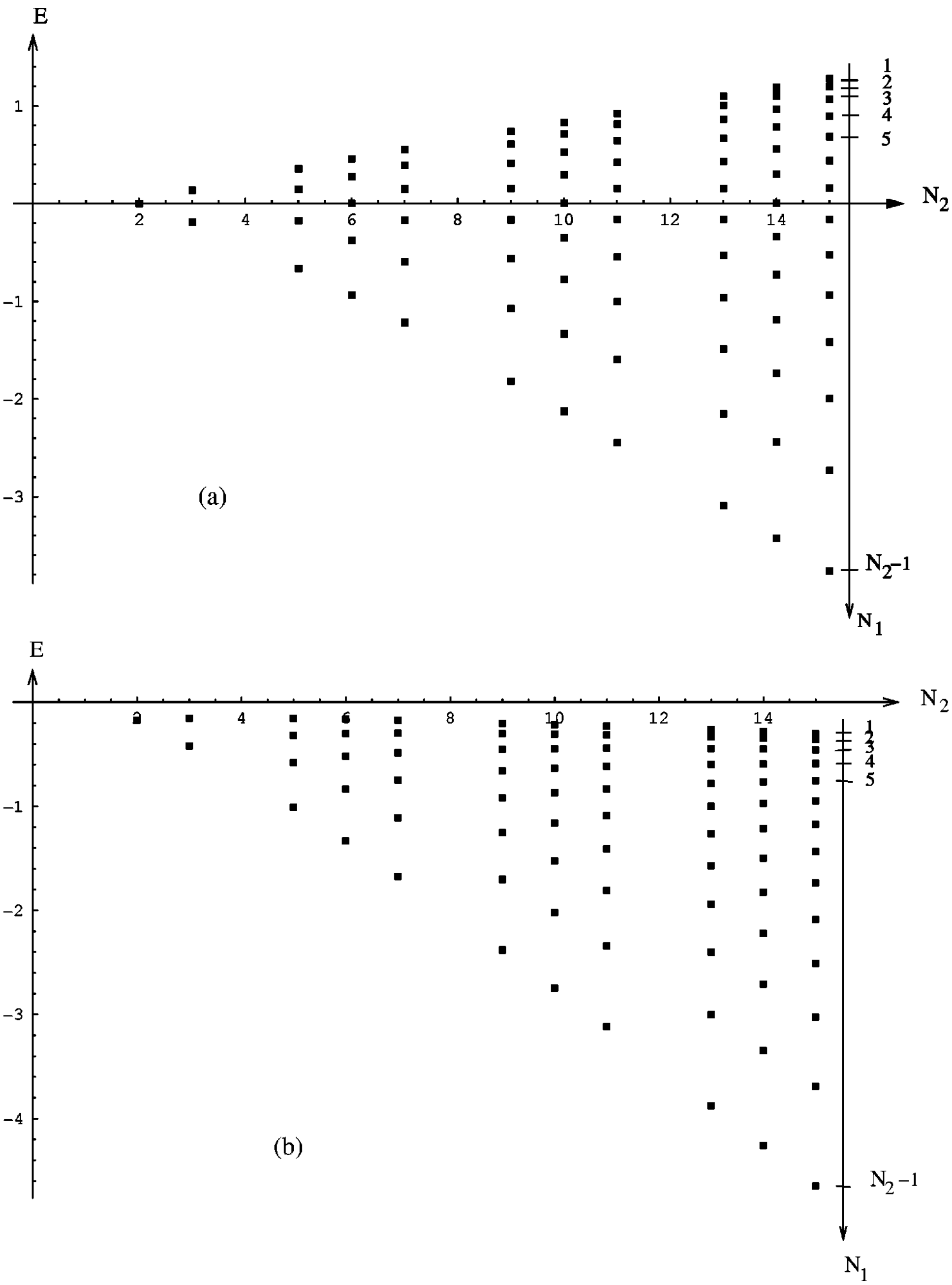

FIG. 4. The complete energy for potentials restored from two bound states, the value of the condensate is (a), $\Phi_{\rm vac}=0.5$ and (b), $\Phi_{\rm vac}=0.45$.

(up to the normalization factor). By means of Eqs. (17) and (18) we restore $U(\Phi_k(x))$ and $\Phi_k(x)$. In this way we obtain information on $U(\Phi)$ for $\Phi\in[-\Phi_{\rm vac},\Phi_{\rm vac}]$. To go beyond this region we used in the preceding section some specific parametrization. In fact we made an analytic continuation to complex x. Indeed, for $|t|>1$ we note for the first parametrization, Eq. (24),

$$x=\frac{1}{2}\ln\frac{1+1/t}{1-1/t}\pm i\frac{\pi}{2}, \tag{33}$$

and for the second one, Eq. (27), for $\theta\in[\pi/2,3\pi/2]$ (where $\cos\theta<0$)

$$x=\ln\left(\frac{-1}{\cos\theta}-\sqrt{\frac{1}{\cos^2\theta}-1}\right)\pm i\pi. \tag{34}$$

Here the signs of the imaginary parts depend on which side we bypass the corresponding branch point. Led by these examples we consider $\eta(x+iy)$ (with real x and y). Now we have to ensure that both U and Φ are real. Because Φ contains an additional integration as compared to U we need $\eta(x+iy)$ to be real for all x. Hence, only shifts parallel to the real axis are allowed. From the structure of η, Eq. (32), it is clear that this may happen only if N_1 and N_2 are integer numbers and if we take the shift in multiples of $i\pi/2$. In general, rational numbers are possible too. But the denominators can be removed by a rescaling of x, i.e., they can be

TABLE I. Allowed (1) and forbidden (0) combinations of the bound state levels for four bound states. This is independent of the ground state level, N_4.

N_2	N_1 (even N_3) 1	2	3	4	N_1 (odd N_3) 1	2	3	4
2	1				1			
3	0	0			1	0		
4	1	0	1		1	1	1	
5	0	0	0	0	1	0	1	0

absorbed into the mass scale. In this way we see that the two parametrizations introduced in the preceding section provide just the required continuation.

As already mentioned we have to ensure that the parametrizations provide monotonic functions $\Phi(t)[\Phi(\theta)]$ which cover the whole range $\Phi\in(-\infty,\infty)$. First we check the monotonicity. For that task we consider the derivative of Φ with respect to the parameter. In the first parametrization we note that $dx/dt=1/(1-t^2)$ and obtain

$$\frac{d\Phi(t)}{dt}=\frac{\eta(x(t))}{1-t^2} \tag{35}$$

which must have a definite sign. A change in the sign may occur only in passing through $t=1$, i.e., when going through $x\to\infty$. Using

$$U(x)\underset{x\to\infty}{\sim} e^{-N_2 x} \tag{36}$$

and

$$x(t)\underset{t\to 1}{\sim} -\frac{1}{2}\ln(1-t)$$

we obtain

$$\frac{d\Phi(t)}{dt}\underset{t\to 1}{\sim}(1-t)^{N_2/2-1}. \tag{37}$$

This derivative is non-negative for $t>1$ too only if $N_2=2(2s+1)$ $(s=0,1,2,\dots)$.

In the second parametrization we have to investigate the behavior at $\theta=\pi/2$. By means of $dx/d\theta=1/\cos\theta$ and

$$x(\theta)\underset{\theta\to(\pi/2)}{\sim} -\ln\left(\frac{\pi}{2}-\theta\right)$$

we obtain

$$\frac{d\Phi(\theta)}{d\theta}=\frac{U(x(\theta))}{\cos\theta}\underset{t\to 1}{\sim}\left(\frac{\pi}{2}-\theta\right)^{N_2-1} \tag{38}$$

which is positive for $\theta>\pi/2$ for odd N_2, $N_2=2s+1$ $(s=0,1,2,\dots)$.

In this way we arrive at the result that for each second even N_2 by the first parametrization and for each odd N_2 by the second parametrization a monotonic function $\Phi(t)$ $\times[\Phi(\theta)]$ appears. It remains to check that the whole region $\Phi\in(-\infty,\infty)$ is covered. For the second parametrization this is indeed the case simply by periodicity. However, for the first one this turns out not to be the case for all even N_2. To check this we note that for $t\to\infty$ the real part of x returns to zero as follows from Eq. (33). In $\eta(x)$, Eq. (32), after $x\to x+iy$, the cosh functions in the denominator turn into $\pm$sinh functions of the corresponding arguments. As a consequence, for $x\to 0$ there may be a cancellation of the contributions linear in x. It is just this cancellation that lets $U(x)$ increase. It can be checked that this cancellation happens just for $N_2=2(2s+1)$, i.e., for the values we selected from the sign of the derivative, and not for the other even N_2. There is no restriction on N_1. As a result we obtain that the potential $U(\Phi)$ is again similar to that in the Φ^4 model; its asymptotic behavior is $U(\Phi)\sim_{\Phi\to\infty}\Phi^2$.

The classical energy can be calculated using Eq. (19). However there is no such simple explicit formula as in Sec. III. The results are shown in Fig. 4. As seen, it depends on the value of the condensate which contribution prevails. For N_1 close to N_2, for any fixed value of the condensate, the energy becomes negative for sufficiently large N_2.

V. RECONSTRUCTION FROM A GENERAL REFLECTIONLESS POTENTIAL

In this section we consider a general reflectionless potential. Is is given by M bound states with energies $\kappa_i=N_i$ $(i=1,2,\dots,M)$. We assume $N_1<N_2<\cdots<N_M$. The wave function of the ground state (its energy is N_M) can be obtained from the inverse scattering method or by Darboux transformation. It is a quotient

$$\eta(x)=\frac{P}{Q},$$

where P is a monomial in $\cosh[(N_1\pm N_2\pm\cdots\pm N_{M-1})x]$ and Q is a monomial in $\cosh[(N_1\pm N_2\pm\cdots\pm N_M)x]$. Q contains the ground state energy $\kappa_M=N_M$ and P does not. Following the discussion in the preceding section we conclude that all N_i must be integer. For the behavior at $x\to\infty$ from the eigenvalue largest in modulus

$$\eta(x)\underset{x\to\infty}{\sim} e^{-N_M x}$$

follows. Again, we conclude that for $N_M=2(2s+1)(s=0,1,2,\dots)$ using the first parametrization, Eq. (24), we obtain a monotonic function $\Phi(t)$ and that for odd N_M the second parametrization does the job. Whereas the second parametrization covers the whole region of Φ by periodicity, the first does this only for certain sets of numbers $N_1,N_2,\dots,N_{M-1}$. Here it seems too hard or even impossible to give a general rule other than in special cases. So, for example, for three bound states $(M=3)$ and a ground state energy $N_3=2(2s+1)$, the energy of the second level N_2

must be an odd number and that of the first level N_1 an even number. This is a conjecture from considering N_3 explicitly up to 20. For four bound states ($M=4$) some allowed combinations are shown in Table I.

The general behavior of $U(\Phi)$ is the same as seen before. When the ground state energy is an even number, a potential as in the Φ^4 model appears, and for an odd number it is periodic. It seems that for reflectionless scattering data no other behavior of $U(\Phi)$ is possible.

VI. CONCLUSIONS

We formulated the reconstruction problem for how to get the potential $U(\Phi)$ allowing for spontaneous symmetry breaking in the action, Eq. (1), for a scalar field in 1+1 dimensions from the scattering data related to the quantum fluctuation in the background of the corresponding kink solution. We considered reflectionless scattering data and solved the reconstruction problem explicitly for some classes, for soliton potentials, and for two bound states. We gave a conjecture for the general reflectionless case. It states that $U(\Phi)$ reconstructed from reflectionless scattering data can be only like a Φ^4 potential, i.e., with two minima, or periodic as in the sine-Gordon model. It would be interesting to give a proof of this conjecture.

There are more questions left. The scattering data include for each bound state (with level κ_i) a constant β_i which is related to the normalization of the corresponding wave function and the reflection coefficient $r(k)$. So it would be interesting to investigate the dependence on β_i. Here a natural conjecture is that there are no restrictions on these parameters. As the ground state energy is independent of the β_i's [see Eq. (10)] no qualitative change in the results is to be expected. A quite different picture should appear if we allow for nonvanishing reflection coefficients $r(k)$. Here the general situation with no specific restrictions on $U(\Phi)$ must take place, for instance, $U(\Phi)$ with three or more minima must be possible. Taking $r(k)$ as a rational function and hence given by a finite number of parameters is an interesting special case.

We wrote down the formulas for the classical (quantum) energies in terms of the scattering data (the ground state wave functions). In the considered examples it is seen that, in dependence on the free parameters the complete energy may take both signs. In general, by an increase of the bound state energies the quantum energy (it is negative) grows faster than the classical one and the complete energy becomes increasingly negative.

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