Black hole scalar hair in asymptotically anti–de Sitter spacetimes

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(Received 3 September 2002; published 31 January 2003)

The unexpected discovery of hairy black hole solutions in theories with scalar fields simply by considering asymptotically anti–de Sitter (AdS) space, rather than asymptotically flat boundary conditions is analyzed in a way that exhibits in a clear manner the differences between the two situations. It is shown that the trivial Schwarzschild–anti–de Sitter space becomes unstable in some of these situations, and the possible relevance of this fact for the AdS conformal field theory conjecture is pointed out.

DOI: 10.1103/PhysRevD.67.024038 PACS number(s): 04.70.Bw

I. INTRODUCTION

The falsehood of the no-hair conjecture for stationary black holes is hardly even disputed these days as the list of counterexamples has become ever larger: Einstein-Yang-Mills [1], Einstein-Skyrme [2], Einstein-Yang-Mills-dilaton [4], Einstein-Yang-Mills-Higgs [3], and Einstein–non-Abelian-Procca [3] fields. In some subcommunities the idea seems to be holding out that a modified version that applies only to stable black holes could remain valid despite the fact that for some of the examples above some claims of stable nontrivial solutions exist in the literature.

One place where it seemed for a while that there was hope for a restricted form of the conjecture was the scalar field arena. Here we had the original no-hair theorems of Bekenstein [5] covering the case of minimally coupled scalar fields with convex potentials, other theorems dealing with the case of minimally coupled fields with arbitrary potentials were obtained in [6,7]. The so-called Bronikov-Melnikov-Bocharova-Bekenstein black hole "solution" [8], which corresponds to a spherical symmetric extremal black hole with a scalar field conformally coupled to gravity, seemed to represent a discrete example of scalar hair, as it was shown [9] that there are no other static spherically symmetric black hole solutions in this theory. Later on, it was shown that this configuration, which presents a divergence of the scalar field at the horizon, cannot be considered as a regular black hole solution because the energy-momentum tensor is ill defined at the horizon [10]. Finally it has been shown that if one demands that the scalar field be bounded throughout the static region, then there are no solutions at all [11].

For more general cases of nonminimal coupling, there are results [12] showing that under the assumption that certain "conformal factors" does not vanish or blow up, there are no nontrivial black hole solutions. Next, there is a result by [13] that does not rely on such an assumption, and which considers the existence of static, spherically symmetric black hole solutions in theories in which the sign of the nonminimal coupling constant is negative (the only case not covered by other theorems). There it is shown that under certain suppositions about the form of the energy-momentum flux, there are no nontrivial solutions. In [14] it is argued that these suppositions are not fully justifiable, and numerical evidence is given against the existence of such a black hole that does not rely on these assumptions.

It is therefore a rather unexpected development that hairy black hole solutions have now been found in both theories with minimal [15] as well as nonminimal [16] coupled scalar fields simply by considering asymptotically anti–de Sitter space, rather than asymptotically flat boundary conditions. Moreover, these papers have strong indications that, under certain conditions, the new solutions are stable.

The purpose of this paper is to analyze the situation regarding the asymptotically anti–de Sitter case, in light of existing results for the asymptotically flat case, discuss the points where the differences are relevant and give a simple explanation of some of the features of the new solutions, and point to some surprising conjectures that can be directly inferred from this understanding. The method used in this part is a generalization of one that was successfully employed in deriving a general characterization of hairy black holes in a wide range of theories [17].

An added reason for interest in the asymptotically anti–de Sitter (AdS) case, and one we briefly touch in this paper is the important place such spacetimes have acquired in view of the AdS conformal field theory (CFT) conjecture. In fact, we will argue that the results and conjectures that are pointed to in this work indicate a difficulty for the notion that the AdS/CFT idea can have as general a validity as it is normally deemed to have.

II. SCALAR HAIR AND ASYMPTOTIC CONDITIONS

We will restrict our consideration to the minimally coupled case as the emergence of hair does not rely at all on the more complicated nonminimal couplings. Thus we will consider a theory given by the action

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$$S=\int \sqrt{-g}d^4x\left[\frac{1}{16\pi}(R-2\Lambda)-1/2(\nabla\phi)^2-V(\phi)\right], \tag{1}$$

where ϕ is a scalar field and V its potential, R is the scalar curvature, and Λ is the true cosmological constant (by which we mean that the minimum of the scalar potential has been set to 0, and any nonzero part has been absorbed into Λ). Now we restrict attention to static spherically symmetric regular black holes whose exterior we parametrize as

$$ds^2=-e^{-2\delta}\mu dt^2+\mu^{-1}dr^2+r^2d\Omega^2, \tag{2}$$

where μ and δ are functions of r. Note that δ can be thought of as representing an additional redshift, beyond the one that could be inferred from the geometry of the static hypersurfaces (i.e., those that are normal to the Killing field). The condition for the presence of a regular horizon at r_H requires the vanishing of μ there. It is customary to parametrize μ as

$$\mu(r)=1-\frac{2m(r)}{r}+\lambda r^2, \tag{3}$$

where the asymptotic geometry is controlled by the parameter λ (i.e., $\lambda=0$ for the asymptotically flat case, $\lambda>0$ for the asymptotically anti–de Sitter case, and $\lambda<0$ for the asymptotically de Sitter case). Einstein's equations give, in this case,

$$\mu'=8\pi rT^tt+\frac{1-\mu}{r}, \quad \delta'=\frac{4\pi r}{\mu}(T^tt-T^rr), \tag{4}$$

where the prime stands for differentiation with respect to r. The scalar field equation can be written as

$$\mu\phi''+[(1/r)(\mu+1)+4\pi r(T^tt+T^rr)]\phi'-\frac{\partial V}{\partial\phi}=0. \tag{5}$$

We must point out that the above formulas refer to the total energy-momentum tensor $T_{\mu\nu}$, which is related to the energy momentum of the scalar field $T_{\mu\nu}(\phi)$ as $T_{\mu\nu}=T_{\mu\nu}(\phi)-g_{\mu\nu}\Lambda/(8\pi)$.

The main tool of our analysis is simply the conservation of the r component of the total energy-momentum tensor $T^{\mu}{}_{r;\mu}=0$, which, through the use of Einstein's equations, can be written as [17]

$$e^{\delta}(e^{-\delta}T^rr)'=\frac{1}{2\mu r}[(T^tt-T^rr)+\mu(2T-3T^tt-5T^rr)], \tag{6}$$

where T stands for the trace of the stress energy tensor.

The energy-momentum tensor of the scalar field by itself satisfies then

$$e^{\delta}[e^{-\delta}T(\phi)^r{}_r]'=\frac{1}{2\mu r}[\{1+r^2(-\Lambda)\}\{T(\phi)^t{}_t-T(\phi)^r{}_r\}+\mu\{2T(\phi)-3T(\phi)^t{}_t-5T(\phi)^r{}_r\}]. \tag{7}$$

Here we can review the reasons behind the fact that there is no nontrivial scalar field in the exterior of such black holes in the asymptotically flat case with $\Lambda=0$. First, the regularity at the horizon requires that mixed components $T(\phi)^{\mu}_{\nu}$ must be bounded at the horizon, since the scalar $T(\phi)^{\mu}_{\nu}T(\phi)^{\nu}{}_{\mu}$ is in this case a sum of non-negative terms. Next, we note that the vanishing of μ at the horizon indicates that on the horizon $T(\phi)^r_r=T(\phi)^t_t$, which is negative definite as follows from the weak energy condition (WEC), which is satisfied, in particular, by minimally coupled scalar fields (in fact, in our case we have $T(\phi)^r_r=1/2\mu(\phi')^2-V$ and $T(\phi)^t_t=T(\phi)^{\theta}_{\theta}=T(\phi)^{\varphi}_{\varphi}=-1/2\mu(\phi')^2-V$). Next, it follows from the WEC that $(T^tt-T^rr)<0$, and from the fact that for the situation at hand the combination $2T(\phi)-3T(\phi)^t{}_t-5T(\phi)^r{}_r$ is $-3\mu(\phi')^2$ and thus nonpositive, that the left hand side of Eq. (7) is nonpositive, and thus that $e^{-\delta}T^rr(\phi)$ is a decreasing function of r. It is thus impossible for this function to approach zero as would be required from asymptotic flatness, the boundary condition that is relevant in this case. The point is that if we consider now asymptotically anti–de Sitter boundary conditions and a negative cosmological constant, two aspects of the above discussion remain unchanged: $e^{-\delta}T(\phi)^r_r$ is negative definite at the horizon, and it is a decreasing function of r. Thus the reason behind the possibility of the scalar black hole hair in such a case is the fact that in the anti–de Sitter case one can allow $T(\phi)^r_r$ to go to a finite (and negative) constant value at infinity, which results in an effective cosmological constant that differs from the true cosmological constant of the theory. In fact we can now restate the result of the above analysis for the asymptotically anti–de Sitter case as follows.

Theorem. There are no nontrivial static and spherically symmetric black hole solutions in the asymptotically anti–de Sitter case in which the asymptotic behavior corresponds to the anti–de Sitter spacetime with the true cosmological constant.

In other words, the asymptotically anti–de Sitter region corresponds to one where the effective cosmological constant is

$$\Lambda_{eff}=\Lambda+8\pi V(\phi_{\infty}). \tag{8}$$

This is in essence the difference between the asymptotically flat $\Lambda=0$ case vs the asymptotically anti–de Sitter case, $\Lambda\neq 0$ case, i.e., the fact that in the former case we must require V to go to 0 at infinity, while in the latter case any nonzero asymptotic value of V can be absorbed into the effective cosmological constant. The theorem above in fact ensures that such an asymptotic value is in fact nonzero and that such absorption cannot be done without. Note that for a nontrivial black hole, $V(\phi_{\infty})>V(\phi_{r_H})\geqslant 0$, thus $\Lambda_{eff}>\Lambda$ and the asymptotic behavior of the spacetime in then less anti–de-Sitter-like than would have been expected from the actual value of the true cosmological constant.

Moreover, as one is interested in situations in which the scalar field converges to a finite value at infinity, we can look at the scalar field equations and note that a necessary condition for such behavior is that the field should go to an extremum of the potential at infinity. Thus, our general result that $T(\phi)_r^r$ must be a decreasing function, together with the fact that in this regime it coincides with V, suggests that the extremum of V must be approached from below at infinity, and thus, that such an extremum must be a maxima. In fact, assuming that the scalar field converges to a finite limit ϕ_∞ at infinity, and that at this point the potential takes a finite value, we write the asymptotic solution as $\phi=\phi_\infty+f(r)$, with $f\rightarrow 0$ at ∞. The asymptotic form of the scalar field equation is

$$\lambda r^2 f''+4\lambda r f'-\left.\frac{\partial V}{\partial \phi}\right|_{\phi_\infty}-\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi_\infty} f=0. \tag{9}$$

From here we see that $(\partial V/\partial\phi)|_{\phi_\infty}=0$. The solution of this equation that goes to zero at infinity is of the form $f=1/r^\beta$ with $\beta>0$. Substituting in Eq. (9) one concludes that

$$\beta=\frac{3}{2}\pm\sqrt{\frac{9}{4}+\frac{1}{\lambda}\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi_\infty}}. \tag{10}$$

On the other hand, from the Einstein equations we have that $m'\sim r^4 f'^2$, so the convergence of m requires that $\Re(\beta)>3/2$ and thus the type of oscillating behavior suggested in [15] cannot occur.

If $(\partial^2 V/\partial\phi^2)|_{\phi_\infty}>0$ one of the roots in Eq. (10) makes ϕ divergent, requiring a fine tuning to avoid this divergence. So, although it is not possible to rule out a solution in this case, we are going to consider the cases in which the scalar field goes to a local maximum at infinity, i.e., $0>(\partial^2 V/\partial\phi^2)|_{\phi_\infty}>-\frac{9}{4}\lambda$, which are in fact the cases in which solutions have been found.

There are several interesting points that come out of this analysis: First we note that we can choose the cosmological constant to be such that the effective cosmological constant vanishes, leading to scalar field hair for black holes in the asymptotically flat context (the price paid for this possibility is the introduction of a fine-tuned cosmological constant). The next point concerns the issue of stability. This, as already mentioned, has been considered important, in the hope to salvage something of the no-hair conjecture. The point is that, in these theories, the Schwarzschild–anti-de Sitter black holes are also trivial solutions, and thus, one could hope that if, as indicated by the available evidence (see [15]), the new, nontrivial black hole solutions are stable, there would seem be a clear violation of even the weakened version (i.e., the version dealing solely with stable black holes) of the no-hair conjecture. The first issue that comes to mind is what is the meaning of stability in the asymptotically anti–de Sitter context. Normally, what one means by stability, in principle, is the following: Given initial data corresponding to the configuration in question, are there small perturbations of these that grow without bounds with evolution in time? The point is that, as the anti–de Sitter spacetime is not globally hyperbolic (i.e., has no Cauchy hypersurfaces) there is, in principle, no well posed initial value formulation that would allow one to investigate such a question, so there is no possible meaningful answer to it, and thus no meaning to the question. The only way to go around this problem seems to be to fix boundary conditions at infinity throughout the time of evolution so as to obtain a well posed initial value problem. Then the issue of stability relates to situations in which we consider small perturbations as in the previous discussion, but keep among other things the value of the scalar field fixed at infinity. It is in this regard that the new black hole solutions could possibly be stable. We will assume from here on that such stability is in fact verified for these solutions. Now let us ask ourselves whether such stability is indeed surprising or not. The first thing we note is that, as mentioned before, at infinity the scalar field is sitting at a local maximum of the potential, and thus, that the stability is intimately connected with the fact that we are dealing with a problem of evolution with fixed conditions at infinity, for otherwise our intuition tells us that under perturbations the field would tend to roll down the potential.

What lies behind the stability of the new stable configurations ought to be, then, that they represent the configurations of "minimal mass" (see [21] and references therein for a formal definition of mass in this context and comparison with alternative ones) among those that have a given black hole area[1] and fixed value of the scalar field at infinity. Assuming that this is the case, the following conjecture naturally follows: For such situations in which the nontrivial black hole is stable, the trivial solution with similar boundary conditions, i.e., the standard Schwarzschild–anti-de Sitter solution with the same boundary conditions (with the scalar field frozen at the top of the potential throughout spacetime) should be unstable. In particular, we can consider the situation in which a fine tuning has made the effective cosmological constant equal to zero, and then, the plain old Schwarzschild solution should be unstable. This situation is analogous to the case of the magnetically charged Reisner-Nostrom solution, which is stable within the Einstein-Maxwell theory, but is unstable within the Einstein-Yang-Mills theory [19].

Finally, we note that in [15], stable as well as unstable nontrivial solutions were found. What lies behind the difference in these situations? It is natural to assume that it has to do with a change in the sign of the mass difference between the two solutions with the same horizon area and asymptotic value of the scalar field. We note that in the situation at hand such a difference can have either sign depending on the details of the scalar potential. In fact let us compare $M_2(r_H)$, the mass of a nontrivial static black hole of radius r_H, with $M_1(r_H)$, the mass of the corresponding Schwarzschild–anti–de Sitter black hole of the same radius. [By black hole radius, we mean $r_H=\sqrt{A/4\pi}$, where A is the horizon area. And by mass we mean the Hamiltonian mass as defined in [21], which in the present situation can be evaluated simply by $M=\lim_{r\rightarrow\infty}m(r)$].

In the latter case the solution is just given by setting $\phi\equiv\phi_\infty$, $\delta\equiv 0$, and $\mu(r)=1-2M_1/r+\lambda r^2$ with M_1 the corresponding mass of the black hole of radius r_H, so

[1]We are assuming a generalization of the ideas presented in [18].

$$M_1(r_H)=\frac{r_H}{2}(1+\lambda r_H^2). \tag{11}$$

In the case of the nontrivial black hole, the mass is obtained from the equation for m' that follows from Eqs. (4) and (3):

$$m'=-4\pi r^2 T_t^t+(3/2)\lambda r^2=(r^2/2)[(3\lambda+\Lambda)+8\pi V(\phi)+4\pi\mu(\phi')^2]. \tag{12}$$

First, we note that the requirement that $m'\to 0$ at ∞ implies that

$$\lambda=-(1/3)[\Lambda+8\pi V(\phi_\infty)]=-1/3\Lambda_{eff}. \tag{13}$$

Next, the vanishing of μ at the horizon requires that $m(r_H)=r_H/2(1+\lambda r_H^2)=M_1(r_H)$, so we can write, using Eq. (13), the mass of the nontrivial black hole as

$$M_2=m(r_H)+4\pi\int_{r_H}^{\infty} r^2[V(\phi)-V(\phi_\infty)+(1/2)\mu(\phi')^2]dr. \tag{14}$$

Thus the sign of M_2-M_1 depends on the integral, which could have either sign depending on the details of the potential and the horizon radius. Note that this is in contrast with the situation that arises, say, in the Einstein-Yang-Mills theory and its hairy black holes in the asymptotically flat context, where the mass of the nontrivial black hole is

$$M_2=m(r_H)+4\pi\int_{r_H}^{\infty} r^2[(1/r^2)V(w)+(1/2)\mu(w')^2]dr, \tag{15}$$

where w parametrizes the Yang-Mills field (as in [1]) and the term $V(w)=(1/2)(1-w^2)^2$, which arises from the self-interaction of the non-Abelian fields, plays the role of an effective—and non-negative—potential in this situation. It is clear that in this case the mass of the hairy black hole is larger than that of the Schwarzschild solution with the same horizon area. In fact it should be possible to numerically test whether the change in stability is associated with the change in the sign of this integral, and we expect to do this in the near future.

Finally, a note regarding the no-hair conjecture and the nature of the counterexample obtained in [15]. The standard understanding is that there are hairy black holes, if, within a specific theory, the boundary conditions and charges at infinity are not sufficient to uniquely specify a stationary black hole solution. If one were to (not advocated by the present authors, but apparently advocated by the authors of [15]) only consider stable black holes in this context, then in order to say that one has found hair, it is not enough to show that the new solutions are stable, one must also ensure that the trivial solution remains stable as well. In fact, using the result of the analysis of [22], we can easily prove that, for certain values of the parameters, the Schwarzschild–anti-de Sitter solution will be unstable in this context, and thus the issue of the violation of the weakened version of the no-hair conjecture in the scalar field arena would be far from settled.

The perturbations around the Schwarzschild–anti-de Sitter black hole with $\phi\equiv\phi_\infty$ are described by

$$\begin{aligned}\mu(t,r)&=\mu_0(r)+\epsilon\mu_1(t,r),\\ \delta(t,r)&=\epsilon\delta_1(t,r),\\ \phi(t,r)&=\phi_\infty+\epsilon\phi_1(t,r),\\ \frac{\partial V}{\partial\phi}&=-\epsilon\phi_1(t,r)\alpha^2,\end{aligned} \tag{16}$$

where $\alpha^2=-(\partial^2 V/\partial\phi^2)|_{\phi_\infty}$ and $\mu_0(r)=1-2M/r+\lambda r^2$. The first order perturbed equation for the scalar field is

$$\ddot{\phi}_1=\mu_0\left[\mu_0\phi_1''+\left(\frac{2}{r}\mu_0+\mu_0'\right)\phi_1'+\alpha^2\phi_1\right] \tag{17}$$

and the first order perturbed Einstein equations are identically satisfied by $\delta_1=0$ and $\mu_1=0$.

Equation (17) can be written as $\ddot{\phi}_1=-A\phi_1$, where $A=-D^aD_a+V$ and D_a is the covariant derivative associated with the auxiliary spatial metric

$${}^{(3)}ds^2=dx^2+r(x)^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{18}$$

where

$$x(r)=\int_{r_H}^{r} dr'\left(1-\frac{2m}{r'}+\lambda r'^2\right)^{-1}. \tag{19}$$

Note that when $r\to\infty$, x converges to a finite constant that we denote by c. In this way we can write

$$D^aD_a=\frac{d^2}{dx^2}+\frac{2\mu_0(x)}{r(x)}\frac{d}{dx}. \tag{20}$$

As mentioned in [22], if ψ is a vector of the Hilbert space L^2 with inner product

$$\langle\psi_1,\psi_2\rangle=\int_0^{2\pi}\int_0^{\pi}\int_0^{c} r^2\psi_1\psi_2\sin\theta dx d\theta d\varphi \tag{21}$$

such that $\langle\psi,A\psi\rangle<0$, then the configuration is unstable.

If we choose $\psi=P(x)/r(x)^n$ with $P(x)$ being any (finite order) polynomial and $n\geqslant 1$, we obtain a finite norm element of L^2. If we take, for instance, $n=1$, then

$$\langle\psi,A\psi\rangle=-4\pi\int_0^{c} dx P(x)\left[\frac{d^2P(x)}{dx^2}+\mu_0 P(x)\times\left(\alpha^2-\frac{2M}{r^3}-2\lambda\right)\right]. \tag{22}$$

Thus, if $\alpha^2>2M/r_H^3+2\lambda$, and we take P to be any polynomial on x which is positive definite and has a positive definite second derivative in the interval $(0,c)$, then $\langle\psi,A\psi\rangle\leqslant 0$, showing that the configuration is unstable.

III. CONCLUSION

We have carefully analyzed the reasons behind the possibility of scalar hair in the asymptotically anti–de Sitter case, comparing it with the situation in the asymptotically flat case. We have discussed also the issue of stability and found a very simple explanation, which in fact points to the instability within these theories and boundary conditions of the usual Schwarzschild–anti–de Sitter solution. This work has dealt with the minimal coupled case; its extension to the nonminimal coupled case is trivial if we can perform a conformal transformation (i.e., if the required conformal factor can be shown to be nowhere vanishing); in the nontrivial cases it is hindered by the fact that in such a case the control provided by the WEC over the signs of the various terms in Eq. (7) is lost. We now briefly note [20] that, according to the conjecture of the AdS/CFT correspondence, the Schwarzschild–anti–de Sitter solution of the theory in the bulk should correspond to a thermal state of the conformal theory in the anti–de Sitter "boundary." But, as the black hole solution is unstable, so should be the corresponding thermal state, and it seems very difficult to envision what could possibly be meant by a thermal state (by definition, an equilibrium state involving fluctuations) that is unstable. Needless to say, such issues should be further investigated, and our point in mentioning them here is to note how the study of hairy black holes can have implications in other, apparently disconnected subjects.

ACKNOWLEDGMENTS

D.S. wishes to thank A. Ashtekar, A. Sen, and U. Nucamendi for helpful discussions. This work was in part supported by DGAPA-UNAM through Grant No. IN 112401 and by CONACyT through grant 32272-E. J.A.G. acknowledges support by CONACyT through grant No. 149945. D.S. acknowledges partial support by the Eberly Endowment and thanks Penn State University for its hospitality.

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